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THE THETA MULTIPLIER FOR NUMBER FIELDS VIA $\mathfrak{p}$-ADIC PLANES

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Given any number field and Dirichlet character for that field, we construct a theta function on a restricted product of $\mathfrak{p}$-adic half-planes. An explicit formula for the theta multiplier is obtained through local calculations.

1. Introduction.

In [8] Tate showed how zeta functions associated to grossencharacters of number fields and their functional equations could be developed from local considerations. Since that work is really an adaptation to an adelic setting of Hecke's [1] original approach to proving the validity of the functional equations, a transformation formula (Tate's Riemann-Roch Theorem) for what is essentially an adelic theta function plays a crucial role. However, the theta function is developed as a function of a single adelic variable since that is all that is needed for the intended application.

On the other hand, the classical theta functions (of real variables) which motivated that work, are often extended to functions defined on (products of) upper half-planes, giving modular forms. As in [6] in the rational case, Dirichlet characters can be incorporated into the definition of the theta function, giving forms on congruence subgroups. (In Tate's work this was not necessary since he integrates the theta functions against the characters to obtain the zeta functions.)

A. Schwartz [5], in the rational case, extended the classical theta function with trivial character defined on the upper half-plane to a function defined on a product of the upper half-plane and a p-adic "half-plane." While the existence of the theta multiplier is shown in that paper through local calculations, Schwartz returned to the classical theta function in order to deduce an explicit formula for it.

In this paper we associate to any number field and Dirichlet character a theta function of an appropriate adelic variable and establish its transformation properties under a discrete theta group. The theta multiplier is developed entirely by piecing together local calculations in an adelic argument. We follow the approach taken in [7], [4], and [5], where $\mathfrak{p}$-adic planes are used as the domains of $\mathfrak{p}$-adic variables, as a convenient substitute for the tree associated to SL_2. While those papers avoid explicit mention of adeles, their use here simplifies matters, since a set of generators for the theta group is easier to specify in the adelic setting.

Of course implicit in the transformation formulas for the theta functions developed here are the transformation formulas for the theta functions on the complex upper half-plane associated to the rational field, as well as the results in [5]. We briefly indicate how these can be recovered.

2. Preliminaries.

Let K be a number field and for any prime $\mathfrak{p}$ of K let $K_\mathfrak{p}$ be its completion with respect to $\mathfrak{p}$. For finite $\mathfrak{p}$, $O_\mathfrak{p}$ denotes the ring of $\mathfrak{p}$-adic integers and $U_\mathfrak{p}$ the group of $\mathfrak{p}$-adic units. The $\mathfrak{p}$-adic norm is normalized so that $|x|_\mathfrak{p} = N(\mathfrak{p})^{-l}$ where $N(\mathfrak{p})$ is the number of elements in $O_\mathfrak{p}/\mathfrak{p}$ and $l = ord_\mathfrak{p}(x)$. π will be used to denote a generator of $\mathfrak{p}$ in $O_\mathfrak{p}$. (It will also be used to denote the real number usually so named; in any given expression the appropriate meaning should be clear from the context.) $\delta_\mathfrak{p}$ will denote the different of $K_\mathfrak{p}$, by which in an abuse of notation we may mean either an ideal or a generator of that ideal. By δ we denote the global different of K.

The adele ring of K is denoted by $V = V_K$. K is embedded in V on the diagonal so that K froms a discrete additive subgroup of V with compact quotient. Similarly, the idele group of K is denoted by $J = J_K$, with $K^\times$ embedded in J on the diagonal. We view $K_\mathfrak{p}^\times$ as embedded in J via $x \mapsto (1, \dots, 1, x, 1, \dots)$.

We now summarize some background material from several sources in order to fix our terminology.

Characters ([3], [8]).

By a character $\mathbf{c}$ we will always mean a continuous, multiplicative map of J into the complex unit circle that is trivial on $K^\times$. If $a = (\dots, a_\mathfrak{p}, \dots) \in J$, then there is a factorization $\mathbf{c}(a) = \prod_\mathfrak{p} \mathbf{c}_\mathfrak{p}(a_\mathfrak{p})$ of $\mathbf{c}$ into local characters where $\mathbf{c}_\mathfrak{p}$ denotes the restriction of $\mathbf{c}$ to $K_\mathfrak{p}^\times$.

For finite $\mathfrak{p}$, if $\mathbf{c}_\mathfrak{p}$ is further restricted to the units $U_\mathfrak{p} \subset K_\mathfrak{p}$, then since $\mathbf{c}_\mathfrak{p}$ is continuous, $\mathbf{c}_\mathfrak{p}(1 + \mathfrak{p}^n) = 1$ for some n. Choosing the minimal such n, or 0 if $\mathbf{c}_\mathfrak{p}$ is trivial on all of $U_\mathfrak{p}$, we call $\mathfrak{f}_\mathfrak{p} = \mathfrak{p}^n$ the conductor of $\mathbf{c}_\mathfrak{p}$, and $n_\mathfrak{p} = n$ the ramification degree of $\mathbf{c}_\mathfrak{p}$. For all but finitely many $\mathfrak{p}$ the ramification degree is 0.

For real $\mathfrak{p}$, $\mathbf{c}_\mathfrak{p}(a_\mathfrak{p}) = \operatorname{sgn}(a_\mathfrak{p})^n |a_\mathfrak{p}|_\mathfrak{p}^s$ for some $n \in \{0, 1\}$ and s purely imaginary. For complex $\mathfrak{p}$, $\mathbf{c}_\mathfrak{p}(a_\mathfrak{p}) = a_\mathfrak{p}^n |a_\mathfrak{p}|_\mathfrak{p}^{s-n}$ for some integer n and s purely imaginary. In either archimedean case we call $n_\mathfrak{p} = n$ the ramification degree of $\mathbf{c}_\mathfrak{p}$.

The conductor of $\mathbf{c}$ is defined to be $\mathfrak{f} = \prod_{\mathfrak{p} \text{ finite}} \mathfrak{f}_{\mathfrak{p}}$.

If a character only takes on a discrete set of values, then it is called a Dirichlet character. A character is a Dirichlet character if and only if at each archimedean prime $\mathfrak{p}$ the complex parameter s appearing in the local character $\mathbf{c}_{\mathfrak{p}}$ can be taken to be 0, and if at the complex primes the ramification degree is 0. At real primes the ramification degree may be non-zero.

Given a Dirichlet character $\mathbf{c}$, let S be the set of all archimedean primes together with those finite primes dividing the conductor of $\mathbf{c}$. Let I^S denote the group of fractional ideals prime to all elements of S. Then $\mathbf{c}$ gives rise to a character of I^S as follows: If $\mathfrak{a} = \prod_{\mathfrak{p} \notin S} \mathfrak{p}^{e_{\mathfrak{p}}} \in I^S$, let $a \in J$ be chosen so that

$$a_{\mathfrak{p}} = \begin{cases} 1 & \text{if } \mathfrak{p} \in S \\ \pi^{e_{\mathfrak{p}}} & \text{if } \mathfrak{p} \notin S \end{cases}$$

where π denotes any generator of $\mathfrak{p}$ in $O_{\mathfrak{p}}$. Set $\chi_{\mathbf{c}}(\mathfrak{a}) = \mathbf{c}(a)$. $\chi_{\mathbf{c}}$ is the grossencharacter associated to $\mathbf{c}$. Note that $\chi_{\mathbf{c}}$ will be trivial on the subgroup of I^S composed of principal ideals with totally positive generators congruent to 1 modulo $\mathfrak{f}$. More generally, if $x \in K$ and $x \in U_{\mathfrak{p}}$ for all $\mathfrak{p} \mid \mathfrak{f}$ then

$$\chi_{\mathbf{c}}((x)) = \prod_{\substack{\mathfrak{p} \nmid \mathfrak{f} \\ \mathfrak{p} \text{ finite}}} \mathbf{c}_{\mathfrak{p}}(x_{\mathfrak{p}}) = \prod_{\mathfrak{p} \mid \mathfrak{f}} \overline{\mathbf{c}_{\mathfrak{p}}}(x_{\mathfrak{p}}) \prod_{\mathfrak{p} \text{ real}} \operatorname{sgn}(x_{\mathfrak{p}})^{n_{\mathfrak{p}}}.$$

Hyperbolic Spaces ([7], [4]).

For each prime $\mathfrak{p}$ of K we define a "hyperbolic space" and a group which acts on it.

For real primes $\mathfrak{p}$, $\mathfrak{H}_{\mathfrak{p}} = \{x + iy \mid x, y \in \mathbb{R},\ y > 0\}$ is the usual complex upper-half-plane. Let $G_{\mathfrak{p}} = SL_2(\mathbb{R})$ be the group of 2 by 2 real matrices with determinant 1.

For complex primes $\mathfrak{p}$, $\mathfrak{H}_{\mathfrak{p}} = \{x + ky \mid x \in \mathbb{C},\ y > 0\}$ is the quaternionic upper-half-space. For $z \in \mathfrak{H}_{\mathfrak{p}}$, $\mathbf{N}_{\mathfrak{p}}(z) = z\bar{z} = x_1^2 + x_2^2 + y^2$ is the quaternionic norm and $|z|_{\mathfrak{p}} = \sqrt{\mathbf{N}_{\mathfrak{p}}(z)}$ is the quaternionic absolute value. Let $G_{\mathfrak{p}} = SL_2(\mathbb{C})$ be the group of 2 by 2 complex matrices with determinant 1.

For finite $\mathfrak{p}$, $\mathfrak{H}_{\mathfrak{p}} = \{x + I_{\mathfrak{p}}y \mid x, y \in K_{\mathfrak{p}},\ y \neq 0,\ ord_{\mathfrak{p}}(y\delta_{\mathfrak{p}}) \text{ is even}\}$ where $I_{\mathfrak{p}}$ is chosen so that $\{1, I_{\mathfrak{p}}\}$ is a basis over $O_{\mathfrak{p}}$ for the integers of the unique unramified extension of $K_{\mathfrak{p}}$ of degree 2. While $I_{\mathfrak{p}}$ is not uniquely determined, for each finite $\mathfrak{p}$ of K we choose an $I_{\mathfrak{p}}$ and fix it for the remainder of this paper. $\mathfrak{H}_{\mathfrak{p}}$ is endowed with the subspace topology it inherits from the field $K_{\mathfrak{p}}(I_{\mathfrak{p}})$. For $z \in \mathfrak{H}_{\mathfrak{p}}$, $\mathbf{N}_{\mathfrak{p}}(z) = (x + I_{\mathfrak{p}}y)(x + \bar{I}_{\mathfrak{p}}y)$ denotes the norm of z from

$K_\mathfrak{p}(I_\mathfrak{p})$ to $K_\mathfrak{p}$, and $|z|_\mathfrak{p} = \sqrt{|\mathbf{N}_\mathfrak{p}(z)|_\mathfrak{p}}$ is the usual $\mathfrak{p}$-adic absolute value. To each point $z = x + I_\mathfrak{p} y \in \mathfrak{H}_\mathfrak{p}$ we associate a neighborhood

$$M_z = \left\{ u + I_\mathfrak{p} v \;\middle|\; \frac{u-x}{y} \in O_\mathfrak{p},\ \frac{v}{y} \in U_\mathfrak{p} \right\}.$$

$\mathfrak{H}_\mathfrak{p}$ is the disjoint union of neighborhoods of this form. Let $G_\mathfrak{p} = SL_2(K_\mathfrak{p})$.

With these definitions, for any $\mathfrak{p}$ there is an action of $G_\mathfrak{p}$ on $\mathfrak{H}_\mathfrak{p}$ via Möbius transformations; given $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_\mathfrak{p}$ and $z \in \mathfrak{H}_\mathfrak{p}$ we define

$$A \circ z = (az + b)(cz + d)^{-1}.$$

Note that for complex $\mathfrak{p}$, where z is a quaternion, the order in this expression is important.

For finite $\mathfrak{p}$, one checks, as in [4], that this definition gives an action on the set of neighborhoods $\{M_z \mid z \in \mathfrak{H}_\mathfrak{p}\}$, with $A \circ M_z = M_{A \circ z}$.

Lemma 2.1. $SL_2(O_\mathfrak{p}) \circ I_\mathfrak{p} = M_{I_\mathfrak{p}}$.

Proof. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(O_\mathfrak{p})$, then letting a bar denote the nontrivial automorphism of $K_\mathfrak{p}(I_\mathfrak{p})$ over $K_\mathfrak{p}$,

$$A \circ I_\mathfrak{p} = \frac{(bd + bc(I_\mathfrak{p} + \bar{I}_\mathfrak{p}) + acI_\mathfrak{p}\bar{I}_\mathfrak{p}) + I_\mathfrak{p}}{(cI_\mathfrak{p} + d)(c\bar{I}_\mathfrak{p} + d)}.$$

But $(cI_\mathfrak{p} + d)(c\bar{I}_\mathfrak{p} + d)$ is the norm of a unit in $K_\mathfrak{p}(I_\mathfrak{p})$, hence is in $U_\mathfrak{p}$, and $bd + bc(I_\mathfrak{p} + \bar{I}_\mathfrak{p}) + acI_\mathfrak{p}\bar{I}_\mathfrak{p} \in O_\mathfrak{p}$. Thus $A \circ I_\mathfrak{p} \in M_{I_\mathfrak{p}}$.

Conversely, if $z = x + I_\mathfrak{p} y \in M_{I_\mathfrak{p}}$, then, since $y \in U_\mathfrak{p}$, and the norm map from the units of the unramified extension $K_\mathfrak{p}(I_\mathfrak{p})$ is surjective onto the units of $K_\mathfrak{p}$, there is some unit $cI_\mathfrak{p} + d \in K_\mathfrak{p}(I_\mathfrak{p})$ with norm y^{-1}. Since $cI_\mathfrak{p} + d$ is a unit, both c and d are integral, with at least one being a unit. Thus we can find an a and b such that $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(O_\mathfrak{p})$ and $B \circ I_\mathfrak{p} = t + I_\mathfrak{p} y$ for some $t \in O_\mathfrak{p}$. Finally, letting $A = \begin{pmatrix} 1 & x-t \\ 0 & 1 \end{pmatrix} B$, we have $A \circ I_\mathfrak{p} = z$. ☐

For all $\mathfrak{p}$ for which $ord_\mathfrak{p}(\delta_\mathfrak{p})$ is even this lemma allows the tree $G_\mathfrak{p}/SL_2(O_\mathfrak{p})$ to be identified with $\{M_z \mid z \in \mathfrak{H}_\mathfrak{p}\}$ via

$$A\, SL_2(O_\mathfrak{p}) \longleftrightarrow A \circ M_{I_\mathfrak{p}} = M_{A \circ I_\mathfrak{p}}.$$

For those $\mathfrak{p}$ with $ord_{\mathfrak{p}}(\delta_{\mathfrak{p}})$ odd, note that

$$\begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix} SL_2(O_{\mathfrak{p}}) \begin{pmatrix} \pi^{-1} & 0 \\ 0 & 1 \end{pmatrix} \circ I_{\mathfrak{p}}\pi = M_{I_{\mathfrak{p}}\pi}$$

and so $\{M_z \mid z \in \mathfrak{H}_{\mathfrak{p}}\}$ is identified with the tree

$$G_{\mathfrak{p}} / \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix} SL_2(O_{\mathfrak{p}}) \begin{pmatrix} \pi^{-1} & 0 \\ 0 & 1 \end{pmatrix}$$

in the same way. In either case, the action of $G_{\mathfrak{p}}$ on $\{M_z\}$ via Möbius transformations is then identified with the action of $G_{\mathfrak{p}}$ on the tree by left multiplication on the cosets.

To any integral ideal $\mathfrak{n}$ of K we associate an adelic hyperbolic space

$$\mathfrak{H}(\mathfrak{n}) = \prod_{\mathfrak{p} \nmid \mathfrak{n}}{}' \mathfrak{H}_{\mathfrak{p}}$$

that has components for all primes *not* dividing $\mathfrak{n}$, and hence for any $\mathfrak{n}$ has components for all the infinite primes, and all but finitely many of the finite primes. The accent on the product symbol denotes that it is a restricted direct product, in the sense that for any element of $\mathfrak{H}(\mathfrak{n})$, all but finitely many of the components are in $M_{I_{\mathfrak{p}}}$. One must note that $\delta_{\mathfrak{p}} = 1$, and hence $I_{\mathfrak{p}} \in \mathfrak{H}_{\mathfrak{p}}$, for all but finitely many $\mathfrak{p}$.

$SL_2(K)$ acts on $\mathfrak{H}(\mathfrak{n})$ by acting on each component via the Möbius transformation defined above. Since any $A \in SL_2(K)$ is in $SL_2(O_{\mathfrak{p}})$ for all but finitely many $\mathfrak{p}$, and elements of $SL_2(O_{\mathfrak{p}})$ map $M_{I_{\mathfrak{p}}}$ onto itself, A does in fact send $\mathfrak{H}(\mathfrak{n})$ to itself.

Fourier Analysis ([8]).

For $\mathfrak{p}$ any prime of K, define an additive homomorphism

$$\Phi_{\mathfrak{p}} : K_{\mathfrak{p}} \longrightarrow \mathbb{R}/\mathbb{Z}$$

so that $\Phi_{\mathfrak{p}}$ is the composition of the sequence of maps

$$K_{\mathfrak{p}} \xrightarrow{Tr^{K_{\mathfrak{p}}}_{\mathbb{Q}_p}} \mathbb{Q}_p \xrightarrow{\phi_1} \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\phi_2} \mathbb{R}/\mathbb{Z}.$$

Here $\mathbb{Q}_p$ is the completion of the rationals with respect to the rational prime p (possibly ∞) over which $\mathfrak{p}$ lies and ϕ_1 is the obvious quotient map. If $\mathfrak{p}$ is infinite, then ϕ_2 is the identity, while if $\mathfrak{p}$ is finite, then ϕ_2 sends an element of $\mathbb{Q}_p/\mathbb{Z}_p \simeq \mathbb{Q}/\mathbb{Z}$ into $\mathbb{R}/\mathbb{Z}$ via first multiplying by -1 and then utilizing the inclusion of $\mathbb{Q}$ into $\mathbb{R}$.

The additive group $K_\mathfrak{p}$ is self-dual under the identification of $\nu \in K_\mathfrak{p}$ with the character $x \mapsto \exp(2\pi i \Phi_\mathfrak{p}(\nu x))$. Furthermore, for finite $\mathfrak{p}$, this character is trivial on $O_\mathfrak{p}$ if and only if $\nu \in \delta_\mathfrak{p}^{-1}$, the local inverse different.

Definition. The Trace of an adele $x \in V$ is $Tr(x) = \sum_\mathfrak{p} \Phi_\mathfrak{p}(x_\mathfrak{p}) \in \mathbb{R}/\mathbb{Z}$.

Note that $Tr(x) = 0$ for any $x \in K$.

The additive group of the adeles V is self-dual under the identification of a vector $\nu \in V$ with the character $x \mapsto \exp(2\pi i Tr(\nu x))$.

We fix the following normalizations of the additive Haar measure on $K_\mathfrak{p}$. For $\mathfrak{p}$ real, $dt_\mathfrak{p}$ is the ordinary Lebesgue measure on $\mathbb{R}$. For $\mathfrak{p}$ complex, $dt_\mathfrak{p}$ is twice the Lebesgue measure on $\mathbb{C} \cong \mathbb{R}^2$. For finite $\mathfrak{p}$, $dt_\mathfrak{p}$ is such that the measure of $O_\mathfrak{p}$ is $|\delta_\mathfrak{p}|_\mathfrak{p}^{\frac{1}{2}}$. Let $dt = \prod_\mathfrak{p} dt_\mathfrak{p}$ be the product measure on V.

Fourier transforms of functions $f_\mathfrak{p}\ K_\mathfrak{p} \to \mathbb{C}$ are defined by

$$\hat{f}_\mathfrak{p}(s) = \int_{K_\mathfrak{p}} f_\mathfrak{p}(t) e^{2\pi i \Phi_\mathfrak{p}(ts)} dt_\mathfrak{p}$$

for $s \in K_\mathfrak{p}$. Then, due to the choice of measures, the Fourier inversion formula is simply

$$\hat{\hat{f}}_\mathfrak{p}(t) = f_\mathfrak{p}(-t)$$

with suitable restrictions on $f_\mathfrak{p}$ so that the integrals converge. Similarly, Fourier transforms of functions $f\ V \to \mathbb{C}$ are defined by

$$\hat{f}(s) = \int_V f(t) e^{2\pi i Tr(ts)} dt$$

for $s \in V$. Again,

$$\hat{\hat{f}}(t) = f(-t)$$

with suitable restrictions on f so that the integrals converge. Finally, if $f(t) = \prod_\mathfrak{p} f_\mathfrak{p}(t_\mathfrak{p})$, then $\hat{f} = \prod_\mathfrak{p} \hat{f}_\mathfrak{p}$.

Proposition 2.2 (Poisson Summation). *With appropriate restrictions on $f : V \to \mathbb{C}$ to ensure convergence,*

$$\sum_{\nu \in K} f(\nu) = \sum_{\nu \in K} \hat{f}(\nu).$$

3. Local Calculations.

For this section and the next fix a choice of a number field K and a Dirichlet character $\mathbf{c}$ with conductor $\mathfrak{f}$.

For each prime $\mathfrak{p}$ we define specific functions $f_\mathfrak{p}$ and compute their Fourier transforms. In most cases $f_\mathfrak{p}$ will be a function on $K_\mathfrak{p} \times \mathfrak{H}_\mathfrak{p}$. However, if $\mathfrak{p} \mid 2$ or if $\mathfrak{p}$ is finite and $\mathbf{c}_\mathfrak{p}$ ramified, then $f_\mathfrak{p}$ will be a function on $K_\mathfrak{p}$ alone. We emphasize that the choice of $f_\mathfrak{p}$ depends on the ramification degree of $\mathbf{c}_\mathfrak{p}$ for real and finite primes.

$\mathfrak{p}$ real.

Definition. For $t \in K_\mathfrak{p} = \mathbb{R}$, $n \in \{0, 1\}$ the ramification degree of $\mathbf{c}_\mathfrak{p}$, and $z \in \mathfrak{H}_\mathfrak{p}$, let

$$f_\mathfrak{p}(t, z) = f_{\mathfrak{p},n}(t, z) = t^n e^{2\pi i t^2 z}.$$

Proposition 3.1.

$$\hat{f}_{\mathfrak{p},n}(s, z) = \frac{i^n}{\sqrt{-2iz}^{2n+1}} f_{\mathfrak{p},n}\left(s, \frac{-1}{4z}\right) \tag{3.1}$$

where $\sqrt{w}$ is the principal value determined by $-\frac{\pi}{2} < \arg(\sqrt{w}) \leq \frac{\pi}{2}$.

Proof. For $n = 0$,

$$\hat{f}_{\mathfrak{p},0}(s, z) = \int_{K_\mathfrak{p}} e^{2\pi i t^2 z} e^{2\pi i t s} dt.$$

As is well known,

$$\hat{f}_{\mathfrak{p},0}(s, iy) = \frac{1}{\sqrt{2y}} f_{\mathfrak{p},0}\left(s, \frac{i}{4y}\right)$$

for $y \in \mathbb{R}$ so by analytic continuation (3.1) holds for $z \in \mathfrak{H}_\mathfrak{p}$ if $n = 0$.

By differentiating (3.1) in the $n = 0$ case with respect to s we obtain the $n = 1$ case. □

$\mathfrak{p}$ complex.

Definition. For $t = t_1 + it_2 \in K_\mathfrak{p} = \mathbb{C}$ and $z = x + ky = x_1 + ix_2 + ky \in \mathfrak{H}_\mathfrak{p}$, let

$$f_\mathfrak{p}(t, z) = e^{-4\pi |t|^2 y} e^{2\pi i \Phi_\mathfrak{p}(t^2 x)}.$$

Proposition 3.2.

$$\hat{f}_\mathfrak{p}(s, z) = \frac{1}{|2z|_\mathfrak{p}} f_p\left(s, \frac{-1}{4z}\right). \tag{3.2}$$

Proof. For $s = s_1 + is_2 \in K_\mathfrak{p}$,

$$\hat{f}_\mathfrak{p}(s,z) = \int_{K_\mathfrak{p}} e^{-4\pi|t|^2 y} e^{2\pi i(t^2 x + \bar{t}^2 \bar{x})} e^{2\pi i(ts + \bar{t}\bar{s})} dt.$$

In order to evaluate this integral, we recall the generalization of (3.1) to several dimensions. Let A be an $n \times n$ symmetric matrix with positive definite imaginary part. For $u \in \mathbb{R}^n$, let $A[u] = u^T A u$. Then for $v \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} e^{2\pi i A[u]} e^{2\pi i u^T v} du = \frac{1}{\sqrt{\det(-2iA)}} e^{2\pi i (-4A)^{-1}[v]} \tag{3.3}$$

where du denotes the usual Euclidean measure on $\mathbb{R}^n$ and the square root is given by analytic continuation from the principal value when A is purely imaginary.

In particular, set

$$A = A(z) = \begin{pmatrix} x_1 + iy & -x_2 \\ -x_2 & -x_1 + iy \end{pmatrix}$$

$$u = \sqrt{2}(t_1, t_2)^T$$
$$v = \sqrt{2}(s_1, -s_2)^T$$

so that $f_\mathfrak{p}(t,z) = e^{2\pi i A[u]}$.

Note that while $z \mapsto A(z)$ is closely related a standard representation of the quaternions in $GL_2(\mathbb{C})$ it is not quite a restriction of a representation of the quaternions to $\mathfrak{H}_\mathfrak{p}$. Nonetheless, $\det A(z) = -\mathbf{N}_\mathfrak{p}(z)$ and

$$A\left(\frac{-1}{z}\right) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \left(-A(z)^{-1}\right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Using the above choices of A, u, and v in (3.3) yield (3.2). The factor of $\sqrt{2}$ in u accounts for the fact that $dt = 2du$. □

$\mathfrak{p}$ finite. As in the real case, we'll have to consider several different functions, depending on the ramification of the local character $\mathbf{c}_\mathfrak{p}$ and whether $\mathfrak{p} \mid 2$. First we assume $\mathbf{c}_\mathfrak{p}$ is unramified and $\mathfrak{p} \nmid 2$.

Definition. If $\mathfrak{p} \nmid 2$ and $\mathfrak{f}_\mathfrak{p} = 1$, then for $t \in K_\mathfrak{p}$ and $z = x + I_\mathfrak{p} y \in \mathfrak{H}_\mathfrak{p}$, let

$$f_\mathfrak{p}(t,z) = W_\mathfrak{p}(\delta_\mathfrak{p} t^2 y) e^{2\pi i \Phi_\mathfrak{p}(t^2 x)}$$

where

$$W_\mathfrak{p}(v) = \begin{cases} 1 & \text{if } v \in O_\mathfrak{p} \\ 0 & \text{otherwise} \end{cases}.$$

Note that the definition of $f_\mathfrak{p}$ is independent of the choice of a generator for the local different. Also, this function is locally constant in z; its value is unchanged as z ranges through any set of the form $M_w \subset \mathfrak{H}_\mathfrak{p}$. Thus in what follows we can, and will, vary z within such a set in order to simplify expressions.

Proposition 3.3. *With $z = x + I_\mathfrak{p} y \in \mathfrak{H}_\mathfrak{p}$, let $n = ord_\mathfrak{p}(y\delta_\mathfrak{p})/2 \in \mathbb{Z}$. If $\mathfrak{p} \nmid 2$ and $\mathfrak{f}_\mathfrak{p} = 1$ then*

$$\hat{f}_\mathfrak{p}(s, z) = G_\mathfrak{p}(z)\frac{1}{\sqrt{|2z|_\mathfrak{p}}} f_\mathfrak{p}\left(s, \frac{-1}{4z}\right)$$

where

$$G_\mathfrak{p}(z) = \sqrt{\frac{|z|_\mathfrak{p}}{|y\delta_\mathfrak{p}|_\mathfrak{p}}} \int_{O_\mathfrak{p}} e^{2\pi i \Phi_\mathfrak{p}(xt^2\pi^{-2n})} dt.$$

Proof. For $s \in K_\mathfrak{p}$,

$$\hat{f}_\mathfrak{p}(s, z) = \int_{K_\mathfrak{p}} W_\mathfrak{p}(\delta_\mathfrak{p} t^2 y) e^{2\pi i \Phi_\mathfrak{p}(t^2 x)} e^{2\pi i \Phi_\mathfrak{p}(ts)} dt.$$

After some preliminary simplification, the evaluation of this integral will be broken into several cases which will proceed essentially as in [5].

Using $ord_\mathfrak{p}(y\delta_\mathfrak{p}) = 2n$,

$$\hat{f}_\mathfrak{p}(s, z) = \int_{K_\mathfrak{p}} W_\mathfrak{p}(t^2\pi^{2n}) e^{2\pi i \Phi_\mathfrak{p}(t^2 x)} e^{2\pi i \Phi_\mathfrak{p}(ts)} dt.$$

Substituting $t\pi^{-n}$ in place of t yields

$$\begin{aligned} \hat{f}_\mathfrak{p}(s, z) &= |\pi|_\mathfrak{p}^{-n} \int_{K_\mathfrak{p}} W_\mathfrak{p}(t^2) e^{2\pi i \Phi_\mathfrak{p}(t^2 x\pi^{-2n})} e^{2\pi i \Phi_\mathfrak{p}(ts\pi^{-n})} dt \\ &= |\pi|_\mathfrak{p}^{-n} \int_{O_\mathfrak{p}} e^{2\pi i \Phi_\mathfrak{p}(t^2 x\pi^{-2n})} e^{2\pi i \Phi_\mathfrak{p}(ts\pi^{-n})} dt. \end{aligned} \tag{3.5}$$

First consider the case $|x|_\mathfrak{p} \leq |y|_\mathfrak{p}$. By the local constancy of the original integrand for $z \in M_w$, we may assume $x = 0$. So

$$\hat{f}_\mathfrak{p}(s, z) = |\pi|_\mathfrak{p}^{-n} \int_{O_\mathfrak{p}} e^{2\pi i \Phi_\mathfrak{p}(ts\pi^{-n})} dt.$$

Replacing t with $t + \alpha$ for any $\alpha \in O_\mathfrak{p}$ gives

$$\hat{f}_\mathfrak{p}(s, z) = e^{2\pi i \Phi_\mathfrak{p}(\alpha s\pi^{-n})} |\pi|_\mathfrak{p}^{-n} \int_{O_\mathfrak{p}} e^{2\pi i \Phi_\mathfrak{p}(ts\pi^{-n})} dt.$$

Comparing this to the previous equation shows the integral must be zero unless $s\pi^{-n} \in \delta_{\mathfrak{p}}^{-1}$ so that the extra factor is 1. However, if $s\pi^{-n} \in \delta_{\mathfrak{p}}^{-1}$, then the integrand is identically 1, and we have

$$\hat{f}_{\mathfrak{p}}(s,z) = \begin{cases} |\delta_{\mathfrak{p}}|_{\mathfrak{p}}^{\frac{1}{2}} |\pi|_{\mathfrak{p}}^{-n} & \text{if } s\pi^{-n} \in \delta_{\mathfrak{p}}^{-1} \\ 0 & \text{otherwise} \end{cases}.$$

But $s\pi^{-n} \in \delta_{\mathfrak{p}}^{-1}$ if and only if $\delta_{\mathfrak{p}} s^2 \frac{1}{y} \in O_{\mathfrak{p}}$, so

(3.6)
$$\hat{f}_{\mathfrak{p}}(s,z) = |\delta_{\mathfrak{p}}|^{\frac{1}{2}} |\pi|^{-n} W_{\mathfrak{p}}\left(\delta_{\mathfrak{p}} s^2 \frac{1}{y}\right) = \frac{1}{\sqrt{|y|_{\mathfrak{p}}}} f_{\mathfrak{p}}\left(s, \frac{I_{\mathfrak{p}}}{y}\right) = \frac{1}{\sqrt{|2z|_{\mathfrak{p}}}} f_{\mathfrak{p}}\left(s, \frac{-1}{4z}\right).$$

Since $G_{\mathfrak{p}}(z) = 1$ in this case, we have proved the formula.

The case of $|x|_{\mathfrak{p}} > |y|_{\mathfrak{p}}$ will be more complicated. Since $yx^{-1} \in O_{\mathfrak{p}}$, replacing t with $t + \alpha yx^{-1}$ in (3.5) for any $\alpha \in O_{\mathfrak{p}}$ gives

$$\hat{f}_{\mathfrak{p}}(s,z) = |\pi|_{\mathfrak{p}}^{-n} \int_{O_{\mathfrak{p}}} e^{2\pi i \Phi_{\mathfrak{p}}((t+\alpha\frac{y}{x})^2 x\pi^{-2n})} e^{2\pi i \Phi_{\mathfrak{p}}((t+\alpha\frac{y}{x})s\pi^{-n})} dt.$$

But

$$\left(t + \alpha\frac{y}{x}\right)^2 x\pi^{-2n} = t^2 x\pi^{-2n} + 2t\alpha y\pi^{-2n} + \alpha^2 \frac{y}{x} y\pi^{-2n}$$

where $y\pi^{-2n} \in \delta_{\mathfrak{p}}^{-1}$. So

$$\hat{f}_{\mathfrak{p}}(s,z) = |\pi|_{\mathfrak{p}}^{-n} e^{2\pi i \Phi_{\mathfrak{p}}(\alpha\frac{y}{x}s\pi^{-n})} \int_{O_{\mathfrak{p}}} e^{2\pi i \Phi_{\mathfrak{p}}(t^2 x\pi^{-2n})} e^{2\pi i \Phi_{\mathfrak{p}}(ts\pi^{-n})} dt.$$

Comparing this to equation (3.5) shows the integral is zero, unless $yx^{-1}s\pi^{-n} \in \delta_{\mathfrak{p}}^{-1}$, or equivalently, unless $\delta_{\mathfrak{p}} s^2 y x^{-2} \in O_{\mathfrak{p}}$. Since $x^2(4\mathbf{N}_{\mathfrak{p}}(z))^{-1} \in U_{\mathfrak{p}}$, the integral is zero unless $\delta_{\mathfrak{p}} s^2 y (4\mathbf{N}_{\mathfrak{p}}(z))^{-1} \in O_{\mathfrak{p}}$.

If the integral is non-zero, then completing the square in the exponential gives

$$\hat{f}_{\mathfrak{p}}(s,z) = |\pi|_{\mathfrak{p}}^{-n} e^{2\pi i \Phi_{\mathfrak{p}}\left(-\frac{1}{x}\frac{s^2}{4}\right)} \int_{O_{\mathfrak{p}}} e^{2\pi i \Phi_{\mathfrak{p}}\left(\frac{1}{x}\left(tx\pi^{-n} + \frac{s}{2}\right)^2\right)} dt. \tag{3.7}$$

However

$$\left(\frac{1}{x} - \frac{x + (I_{\mathfrak{p}} + \bar{I}_{\mathfrak{p}})y}{\mathbf{N}_{\mathfrak{p}}(z)}\right)\frac{s^2}{4} = \frac{I_{\mathfrak{p}}\bar{I}_{\mathfrak{p}}}{4}\frac{y}{x} s^2 \frac{y}{\mathbf{N}_{\mathfrak{p}}(z)} \in \delta_{\mathfrak{p}}^{-1}$$

since $\mathfrak{p} \nmid 2$. Therefore we can replace the $\frac{1}{x}$ in the first exponential in (3.7) with $\frac{x+(I_\mathfrak{p}+\bar{I_\mathfrak{p}})y}{\mathbf{N}_\mathfrak{p}(z)}$ to get

$$\hat{f}_\mathfrak{p}(s,z) = |\pi|_\mathfrak{p}^{-n} e^{2\pi i \Phi_\mathfrak{p}\left(\frac{-x-(I_\mathfrak{p}+\bar{I_\mathfrak{p}})y}{\mathbf{N}_\mathfrak{p}(z)} \frac{s^2}{4}\right)} \int_{O_\mathfrak{p}} e^{2\pi i \Phi_\mathfrak{p}(\frac{1}{x}(tx\pi^{-n}+\frac{s}{2})^2)} dt.$$

Since $\frac{s\pi^n}{2x} \in O_\mathfrak{p}$, substituting $t - \frac{s\pi^n}{2x}$ for t yields

$$\hat{f}_\mathfrak{p}(s,z) = |\pi|_\mathfrak{p}^{-n} e^{2\pi i \Phi_\mathfrak{p}\left(\frac{-x-(I_\mathfrak{p}+\bar{I_\mathfrak{p}})y}{\mathbf{N}_\mathfrak{p}(z)} \frac{s^2}{4}\right)} \int_{O_\mathfrak{p}} e^{2\pi i \Phi_\mathfrak{p}(xt^2\pi^{-2n})} dt.$$

Therefore we've shown that

$$\begin{aligned}
\hat{f}_\mathfrak{p}(s,z) &= |\pi|_\mathfrak{p}^{-n} \int_{O_\mathfrak{p}} e^{2\pi i \Phi_\mathfrak{p}(xt^2\pi^{-2n})} dt \; W_\mathfrak{p}\left(\delta_\mathfrak{p} s^2 \frac{y}{4\mathbf{N}_\mathfrak{p}(z)}\right) e^{2\pi i \Phi_\mathfrak{p}\left(\frac{-x-(I_\mathfrak{p}+\bar{I_\mathfrak{p}})y}{\mathbf{N}_\mathfrak{p}(z)} \frac{s^2}{4}\right)} \\
&= |\pi|_\mathfrak{p}^{-n} \sqrt{\frac{|y\delta_\mathfrak{p}|_\mathfrak{p}}{|z|_\mathfrak{p}}} G_\mathfrak{p}(z) W_\mathfrak{p}\left(\delta_\mathfrak{p} s^2 \frac{y}{4\mathbf{N}_\mathfrak{p}(z)}\right) e^{2\pi i \Phi_\mathfrak{p}\left(\frac{-x-(I_\mathfrak{p}+\bar{I_\mathfrak{p}})y}{\mathbf{N}_\mathfrak{p}(z)} \frac{s^2}{4}\right)} \\
&= |\pi|_\mathfrak{p}^{-n} \sqrt{\frac{|y\delta_\mathfrak{p}|_\mathfrak{p}}{|z|_\mathfrak{p}}} G_\mathfrak{p}(z) f\left(s, \frac{-1}{4z}\right).
\end{aligned}$$

Finally, we need only note that

$$|\pi|_\mathfrak{p}^{-n} \sqrt{\frac{|y\delta_\mathfrak{p}|_\mathfrak{p}}{|z|_\mathfrak{p}}} = |\pi|_\mathfrak{p}^{-n} \sqrt{\frac{|\pi|_\mathfrak{p}^{2n}}{|z|_\mathfrak{p}}} = \frac{1}{\sqrt{|2z|_\mathfrak{p}}}$$

to complete the proof. □

We can simplify further by noting that $G_\mathfrak{p}(z)$ is really a normalized quadratic Gauss sum.

Proposition 3.4. *Let $z = x + I_\mathfrak{p} y \in \mathfrak{H}_\mathfrak{p}$. If $|x|_\mathfrak{p} > |y|_\mathfrak{p}$, write $x = \epsilon\pi^l$ with $\epsilon \in U_\mathfrak{p}$. Then*

$$G_\mathfrak{p}(z) = \begin{cases} 1 & \text{if } |x|_\mathfrak{p} \le |y|_\mathfrak{p} \\ 1 & \text{if } |x|_\mathfrak{p} > |y|_\mathfrak{p} \text{ and } \mathrm{ord}_\mathfrak{p}(x\delta_\mathfrak{p}) \text{ is even} \\ \left(\frac{\epsilon}{\mathfrak{p}}\right)\mu_\pi i & \text{if } |x|_\mathfrak{p} > |y|_\mathfrak{p} \text{ and } \mathrm{ord}_\mathfrak{p}(x\delta_\mathfrak{p}) \text{ is odd} \end{cases}$$

where $\left(\frac{\epsilon}{\mathfrak{p}}\right)$ is the quadratic residue symbol and $\mu_\pi i$ is a complex number with $\mu_\pi i^2 = \left(\frac{-1}{\mathfrak{p}}\right)$. Although $\mu_\pi i$ depends on the choice of uniformizer π, $G_\mathfrak{p}(z)$ does not.

Proof. If $|x|_\mathfrak{p} \le |y|_\mathfrak{p}$ the result is clear.

If $|x|_\mathfrak{p} > |y|_\mathfrak{p}$, then in preparation for simplifying $G_\mathfrak{p}$, consider the integral

$$H(m,\epsilon) = \int_{O_\mathfrak{p}} e^{2\pi i \Phi_\mathfrak{p}\left(\frac{\epsilon t^2}{\pi^{m+d}}\right)} dt$$

where $\epsilon \in U_\mathfrak{p}$, $d = ord_\mathfrak{p} \delta_\mathfrak{p}$, and $m \geq 0$ is an integer. (Although the notation does not indicate it, H also depends on the choice of the uniformizer π.) If $m \geq 2$, replacing t with $t + \alpha\pi^{m-1}$ for any $\alpha \in O_\mathfrak{p}$ gives

$$\begin{aligned} H(m,\epsilon) &= \int_{O_\mathfrak{p}} e^{2\pi i \Phi_\mathfrak{p}\left(\frac{\epsilon t^2}{\pi^{m+d}}\right)} e^{2\pi i \Phi_\mathfrak{p}\left(\frac{2\epsilon\alpha t}{\pi^{1+d}}\right)} e^{2\pi i \Phi_\mathfrak{p}\left(\frac{\epsilon\alpha^2\pi^{m-2}}{\pi^{d}}\right)} dt \\ &= \int_{O_\mathfrak{p}} e^{2\pi i \Phi_\mathfrak{p}\left(\frac{\epsilon t^2}{\pi^{m+d}}\right)} e^{2\pi i \Phi_\mathfrak{p}\left(\frac{2\epsilon\alpha t}{\pi^{1+d}}\right)} dt. \end{aligned}$$

Since this expression for H is valid for any $\alpha \in O_\mathfrak{p}$, averaging it over $O_\mathfrak{p}$ yields

$$\begin{aligned} H(m,\epsilon) &= |\delta_\mathfrak{p}|_\mathfrak{p}^{-\frac{1}{2}} \int_{O_\mathfrak{p}} \int_{O_\mathfrak{p}} e^{2\pi i \Phi_\mathfrak{p}\left(\frac{\epsilon t^2}{\pi^{m+d}}\right)} e^{2\pi i \Phi_\mathfrak{p}\left(\frac{2\epsilon\alpha t}{\pi^{1+d}}\right)} dt\, d\alpha \\ &= |\delta_\mathfrak{p}|_\mathfrak{p}^{-\frac{1}{2}} \int_{O_\mathfrak{p}} e^{2\pi i \Phi_\mathfrak{p}\left(\frac{\epsilon t^2}{\pi^{m+d}}\right)} \int_{O_\mathfrak{p}} e^{2\pi i \Phi_\mathfrak{p}\left(\frac{2\epsilon\alpha t}{\pi^{1+d}}\right)} d\alpha\, dt. \end{aligned}$$

Considering only the inner integral and replacing α with $\alpha+\beta$ for any $\beta \in O_\mathfrak{p}$ shows

$$\int_{O_\mathfrak{p}} e^{2\pi i \Phi_\mathfrak{p}\left(\frac{2\epsilon\alpha t}{\pi^{1+d}}\right)} d\alpha = e^{2\pi i \Phi_\mathfrak{p}\left(\frac{2\epsilon\beta t}{\pi^{1+d}}\right)} \int_{O_\mathfrak{p}} e^{2\pi i \Phi_\mathfrak{p}\left(\frac{2\epsilon\alpha t}{\pi^{1+d}}\right)} d\alpha.$$

This equality implies the integral must be 0 unless $t \in \mathfrak{p}$, in which case the integrand is 1 so the integral is $|\delta_\mathfrak{p}|_\mathfrak{p}^{\frac{1}{2}}$. Therefore

$$\begin{aligned} H(m,\epsilon) &= \int_{\mathfrak{p}} e^{2\pi i \Phi_\mathfrak{p}\left(\frac{\epsilon t^2}{\pi^{m+d}}\right)} dt \\ &= |\pi| \int_{O_\mathfrak{p}} e^{2\pi i \Phi_\mathfrak{p}\left(\frac{\epsilon \pi^2 t^2}{\pi^{m+d}}\right)} dt \\ &= |\pi| H(m-2,\epsilon). \end{aligned}$$

This formula reduces evaluating $H(m,\epsilon)$ to the cases $m = 0$ and 1. While clearly $H(0,\epsilon) = |\delta_\mathfrak{p}|_\mathfrak{p}^{\frac{1}{2}}$, the $m = 1$ case requires more work.

$$\begin{aligned} H(1,\epsilon) &= \sum_{\substack{\alpha \in O_\mathfrak{p} \\ \bmod \mathfrak{p}}} \int_{\alpha+\mathfrak{p}} e^{2\pi i \Phi_\mathfrak{p}\left(\frac{\epsilon t^2}{\pi^{1+d}}\right)} dt \\ &= \sum_{\substack{\alpha \in O_\mathfrak{p} \\ \bmod \mathfrak{p}}} |\pi|_\mathfrak{p} \int_{O_\mathfrak{p}} e^{2\pi i \Phi_\mathfrak{p}\left(\frac{\epsilon(\alpha+\pi t)^2}{\pi^{1+d}}\right)} dt \end{aligned}$$

$$= |\pi|_{\mathfrak{p}} |\delta_{\mathfrak{p}}|^{\frac{1}{2}} \sum_{\substack{\alpha \in O_{\mathfrak{p}} \\ \bmod \mathfrak{p}}} e^{2\pi i \Phi_{\mathfrak{p}}\left(\frac{\epsilon\alpha^2}{\pi^{1+d}}\right)}$$

$$= |\pi|_{\mathfrak{p}} |\delta_{\mathfrak{p}}|^{\frac{1}{2}} \sum_{\substack{\alpha \in O_{\mathfrak{p}} \\ \bmod \mathfrak{p}}} \left(\left(\frac{\alpha}{\mathfrak{p}}\right) + 1 \right) e^{2\pi i \Phi_{\mathfrak{p}}\left(\frac{\epsilon\alpha}{\pi^{1+d}}\right)}$$

where $\left(\frac{\alpha}{\mathfrak{p}}\right)$ is the quadratic residue symbol so that $\left(\frac{\alpha}{\mathfrak{p}}\right) + 1$ is the number of solutions to the congruence $x^2 \equiv \alpha \mod \mathfrak{p}$. Since $\epsilon \in U_{\mathfrak{p}}$

$$\sum_{\substack{\alpha \in O_{\mathfrak{p}} \\ \bmod \mathfrak{p}}} e^{2\pi i \Phi_{\mathfrak{p}}\left(\frac{\epsilon\alpha}{\pi^{1+d}}\right)} = 0$$

so

$$H(1,\epsilon) = |\pi|_{\mathfrak{p}} |\delta_{\mathfrak{p}}|^{\frac{1}{2}} \sum_{\substack{\alpha \in O_{\mathfrak{p}} \\ \bmod \mathfrak{p}}} \left(\frac{\alpha}{\mathfrak{p}}\right) e^{2\pi i \Phi_{\mathfrak{p}}\left(\frac{\epsilon\alpha}{\pi^{1+d}}\right)}.$$

Replacing α with $\epsilon^{-1}\alpha$ shows $H(1,\epsilon) = \left(\frac{\epsilon}{\mathfrak{p}}\right) H(1,1)$. Thus

$$H(m,\epsilon) = \begin{cases} |\delta_{\mathfrak{p}}\pi^m|_{\mathfrak{p}}^{\frac{1}{2}} & \text{if } m \text{ is even} \\ |\pi^{(m-1)}|_{\mathfrak{p}}^{\frac{1}{2}} \left(\frac{\epsilon}{\mathfrak{p}}\right) H(1,1) & \text{if } m \text{ is odd} \end{cases}.$$

We reiterate that the complex number $H(1,1)$ depends on the choice of π; changing π by multiplication by a unit ϵ multiplies $H(1,1)$ by $\left(\frac{\epsilon}{\mathfrak{p}}\right)$.

Nonetheless, the square of H(1,1) is easily found by

$$\begin{aligned} \left(\frac{-1}{\mathfrak{p}}\right) H(1,1)^2 &= H(1,1)H(1,-1) \\ &= \int_{O_{\mathfrak{p}}} e^{2\pi i \Phi_{\mathfrak{p}}\left(\frac{t^2}{\pi^{1+d}}\right)} dt \int_{O_{\mathfrak{p}}} e^{-2\pi i \Phi_{\mathfrak{p}}\left(\frac{s^2}{\pi^{1+d}}\right)} ds \\ &= \int_{O_{\mathfrak{p}}} \int_{O_{\mathfrak{p}}} e^{2\pi i \Phi_{\mathfrak{p}}\left(\frac{(t-s)(t+s)}{\pi^{1+d}}\right)} dt\ ds. \end{aligned}$$

A change of variables (since $\mathfrak{p} \nmid 2$)then gives

$$\left(\frac{-1}{\mathfrak{p}}\right) H(1,1)^2 = \int_{O_{\mathfrak{p}}} \int_{O_{\mathfrak{p}}} e^{2\pi i \Phi_{\mathfrak{p}}\left(\frac{ts}{\pi^{1+d}}\right)} dt\ ds.$$

Replacing t with $t+\alpha$ for any $\alpha \in O_{\mathfrak{p}}$ shows the inner integral is

$$\int_{O_{\mathfrak{p}}} e^{2\pi i \Phi_{\mathfrak{p}}\left(\frac{ts}{\pi^{1+d}}\right)} dt = e^{2\pi i \Phi_{\mathfrak{p}}\left(\frac{\alpha s}{\pi^{1+d}}\right)} \int_{O_{\mathfrak{p}}} e^{2\pi i \Phi_{\mathfrak{p}}\left(\frac{ts}{\pi^{1+d}}\right)} dt.$$

Thus the inner integral is 0 unless $s \in \mathfrak{p}$, and in that case, since the integrand is 1, the inner integral gives $|\delta_\mathfrak{p}|_\mathfrak{p}^{\frac{1}{2}}$. Therefore

$$H(1,1)^2 = \left(\frac{-1}{\mathfrak{p}}\right) |\delta_\mathfrak{p}|_\mathfrak{p}^{\frac{1}{2}} \int_\mathfrak{p} ds = \left(\frac{-1}{\mathfrak{p}}\right) |\pi|_\mathfrak{p} |\delta_\mathfrak{p}|_\mathfrak{p}.$$

To apply this the the evaluation of $G_\mathfrak{p}(z)$, write $x = \epsilon\pi^l$ where $\epsilon \in U_\mathfrak{p}$. Recall that $ord_\mathfrak{p}(y\delta_\mathfrak{p}) = 2n$, so that $ord_\mathfrak{p}(y) = 2n - d$. Then

$$\begin{aligned}
G_\mathfrak{p}(z) &= \sqrt{\frac{|z|_\mathfrak{p}}{|y\delta_\mathfrak{p}|_\mathfrak{p}}} \int_{O_\mathfrak{p}} e^{2\pi i \Phi_\mathfrak{p}\left(\frac{\epsilon t^2}{\pi^{2n-l}}\right)} dt \\
&= \sqrt{\frac{|z|_\mathfrak{p}}{|y\delta_\mathfrak{p}|_\mathfrak{p}}} H(2n-l-d, \epsilon) \\
&= |\pi|^{(l-2n)/2} \begin{cases} |\pi|^{(2n-l)/2} & \text{if } l+d \text{ is even} \\ |\pi|^{(2n-l-d-1)/2} \left(\frac{\epsilon}{\mathfrak{p}}\right) H(1,1) & \text{if } l+d \text{ is odd} \end{cases} \\
&= \begin{cases} 1 & \text{if } l+d \text{ is even} \\ \left(\frac{\epsilon}{\mathfrak{p}}\right) \frac{H(1,1)}{\sqrt{|\pi\delta_\mathfrak{p}|_\mathfrak{p}}} & \text{if } l+d \text{ is odd.} \end{cases}
\end{aligned}$$

Let $\mu_\pi i = \frac{H(1,1)}{\sqrt{|\pi\delta_\mathfrak{p}|_\mathfrak{p}}}$. □

Now we consider a different function that will be used when the local character $\mathbf{c}_\mathfrak{p}$ is ramified. Unlike the unramified case, in this case the function will not depend on a parameter from $\mathfrak{H}_\mathfrak{p}$.

Definition. If $\mathfrak{f}_\mathfrak{p} \neq 1$, then let

$$f_\mathfrak{p}(t) = \begin{cases} \mathbf{c}_\mathfrak{p}(t) & \text{if } t \in U_\mathfrak{p} \\ 0 & \text{otherwise} \end{cases}.$$

Proposition 3.5. *If $\mathfrak{f}_\mathfrak{p} = \mathfrak{p}^n$ and $\delta_\mathfrak{p} = \mathfrak{p}^d$, let $q \in K_\mathfrak{p}$ be any element such that $ord_\mathfrak{p} q = n + d$. Then*

$$\hat{f}_\mathfrak{p}(s) = |q|_\mathfrak{p}^{\frac{1}{2}} G_\mathfrak{p}(\mathbf{c}_\mathfrak{p}, q) \overline{f_\mathfrak{p}}(sq)$$

where $G_\mathfrak{p}$, the normalized Gauss sum associated to $\mathbf{c}_\mathfrak{p}$ and a choice of q, is defined by

$$G_\mathfrak{p}(\mathbf{c}_\mathfrak{p}, q) = |\mathfrak{f}_\mathfrak{p}|_\mathfrak{p}^{\frac{1}{2}} \sum_{\alpha \in U_\mathfrak{p}/(1+\mathfrak{f}_\mathfrak{p})} \mathbf{c}_\mathfrak{p}(\alpha) e^{2\pi i \Phi_\mathfrak{p}(\alpha q^{-1})}.$$

Proof.

$$\hat{f}_{\mathfrak{p}}(s) = \int_{U_{\mathfrak{p}}} \mathbf{c}_{\mathfrak{p}}(t) e^{2\pi i \Phi_{\mathfrak{p}}(ts)} dt.$$

Replacing t by $t + \alpha\pi^n$ for any $\alpha \in O_{\mathfrak{p}}$ gives

$$\hat{f}_{\mathfrak{p}}(s) = e^{2\pi i \Phi_{\mathfrak{p}}(\alpha s \pi^n)} \int_{U_{\mathfrak{p}}} \mathbf{c}_{\mathfrak{p}}(t) e^{2\pi i \Phi_{\mathfrak{p}}(ts)} dt$$

Thus $\hat{f}_{\mathfrak{p}}(s) = 0$ unless $s\pi^n \in \delta_{\mathfrak{p}}^{-1}$.

Now choose an $\epsilon \equiv 1 \mod \mathfrak{p}^{n-1}$ with $\mathbf{c}_{\mathfrak{p}}(\epsilon) \neq 1$. Writing $\epsilon = 1 + \alpha\pi^{n-1}$ with $\alpha \in U_{\mathfrak{p}}$, and replacing t with $t\epsilon$ yields

$$\begin{aligned} \hat{f}_{\mathfrak{p}}(s) = \int_{U_{\mathfrak{p}}} \mathbf{c}_{\mathfrak{p}}(t) e^{2\pi i \Phi_{\mathfrak{p}}(ts)} dt &= \int_{U_{\mathfrak{p}}} \mathbf{c}_{\mathfrak{p}}(t\epsilon) e^{2\pi i \Phi_{\mathfrak{p}}(t\epsilon s)} dt \\ &= \mathbf{c}_{\mathfrak{p}}(\epsilon) \int_{U_{\mathfrak{p}}} \mathbf{c}_{\mathfrak{p}}(t) e^{2\pi i \Phi_{\mathfrak{p}}(ts)} e^{2\pi i \Phi_{\mathfrak{p}}(t\alpha\pi^{n-1}s)} dt. \end{aligned}$$

Now if $s\pi^{n-1} \in \delta_{\mathfrak{p}}^{-1}$, then $e^{2\pi i \Phi_{\mathfrak{p}}(t\alpha\pi^{n-1}s)} = 1$ for all $t \in U_{\mathfrak{p}}$, so in this case we have

$$\int_{U_{\mathfrak{p}}} \mathbf{c}_{\mathfrak{p}}(t) e^{2\pi i \Phi_{\mathfrak{p}}(ts)} dt = \mathbf{c}_{\mathfrak{p}}(\epsilon) \int_{U_{\mathfrak{p}}} \mathbf{c}_{\mathfrak{p}}(t) e^{2\pi i \Phi_{\mathfrak{p}}(ts)} dt.$$

Since $\mathbf{c}_{\mathfrak{p}}(\epsilon) \neq 1$, we thus see that $\hat{f}_{\mathfrak{p}}(s) = 0$ unless $s\pi^n\delta_{\mathfrak{p}} \in U_{\mathfrak{p}}$.

We now can express $\hat{f}_{\mathfrak{p}}(s)$ for $s\pi^n\delta_{\mathfrak{p}} \in U_{\mathfrak{p}}$ as a Gauss sum associated to $\mathbf{c}_{\mathfrak{p}}$. Thus

$$\begin{aligned} \hat{f}_{\mathfrak{p}}(s) &= \int_{U_{\mathfrak{p}}} \mathbf{c}_{\mathfrak{p}}(t) e^{2\pi i \Phi_{\mathfrak{p}}(ts)} dt \\ &= \sum_{\alpha \in U_{\mathfrak{p}}/(1+\mathfrak{f}_{\mathfrak{p}})} \int_{\alpha + \mathfrak{f}_{\mathfrak{p}}} \mathbf{c}_{\mathfrak{p}}(t) e^{2\pi i \Phi_{\mathfrak{p}}(ts)} dt \\ &= \sum_{\alpha \in U_{\mathfrak{p}}/(1+\mathfrak{f}_{\mathfrak{p}})} \mathbf{c}_{\mathfrak{p}}(\alpha) e^{2\pi i \Phi_{\mathfrak{p}}(\alpha s)} \int_{\mathfrak{f}_{\mathfrak{p}}} e^{2\pi i \Phi_{\mathfrak{p}}(ts)} dt \\ &= \sum_{\alpha \in U_{\mathfrak{p}}/(1+\mathfrak{f}_{\mathfrak{p}})} \mathbf{c}_{\mathfrak{p}}(\alpha) e^{2\pi i \Phi_{\mathfrak{p}}(\alpha s)} |\mathfrak{f}_{\mathfrak{p}}|_{\mathfrak{p}} \int_{O_{\mathfrak{p}}} e^{2\pi i \Phi_{\mathfrak{p}}(t\pi^n s)} dt \\ &= \sum_{\alpha \in U_{\mathfrak{p}}/(1+\mathfrak{f}_{\mathfrak{p}})} \mathbf{c}_{\mathfrak{p}}(\alpha) e^{2\pi i \Phi_{\mathfrak{p}}(\alpha s)} |\mathfrak{f}_{\mathfrak{p}}|_{\mathfrak{p}} |\delta_{\mathfrak{p}}|_{\mathfrak{p}}^{\frac{1}{2}} \\ &= |\mathfrak{f}_{\mathfrak{p}}^2 \delta_{\mathfrak{p}}|_{\mathfrak{p}}^{\frac{1}{2}} \sum_{\alpha \in U_{\mathfrak{p}}/(1+\mathfrak{f}_{\mathfrak{p}})} \mathbf{c}_{\mathfrak{p}}(\alpha) e^{2\pi i \Phi_{\mathfrak{p}}(\alpha s)}. \end{aligned}$$

With $ord_{\mathfrak{p}} q = n + d$ so that $sq \in U_{\mathfrak{p}}$, replacing α by $\alpha(sq)^{-1}$ in the sum yields

$$\begin{aligned}\hat{f}_{\mathfrak{p}}(s) &= \overline{\mathbf{c}_{\mathfrak{p}}}(sq)|\mathfrak{f}_{\mathfrak{p}}^2\delta_{\mathfrak{p}}|_{\mathfrak{p}}^{\frac{1}{2}} \sum_{\alpha\in U_{\mathfrak{p}}/(1+\mathfrak{f}_{\mathfrak{p}})} \mathbf{c}_{\mathfrak{p}}(\alpha)e^{2\pi i\Phi_{\mathfrak{p}}(\alpha q^{-1})} \\ &= |q|_{\mathfrak{p}}^{\frac{1}{2}} G_{\mathfrak{p}}(\mathbf{c}_{\mathfrak{p}}, q)\overline{f_{\mathfrak{p}}}(sq).\end{aligned}$$

□

The factor in front of the sum in the definition of $G_{\mathfrak{p}}$ ensures the first part of the following.

Proposition 3.6.

a. $|G_{\mathfrak{p}}(\mathbf{c}_{\mathfrak{p}}, q)| = 1$.

b. $G_{\mathfrak{p}}(\mathbf{c}_{\mathfrak{p}}, \epsilon q) = \mathbf{c}_{\mathfrak{p}}(\epsilon)G_{\mathfrak{p}}(\mathbf{c}_{\mathfrak{p}}, q)$ *for* $\epsilon \in U_{\mathfrak{p}}$.

c. $\overline{G_{\mathfrak{p}}(\mathbf{c}_{\mathfrak{p}}, q)} = G_{\mathfrak{p}}(\overline{\mathbf{c}_{\mathfrak{p}}}, -q)$.

Proof. First, (b) is easily seen by changing the index of summation in the definition of $G_{\mathfrak{p}}$, and (c) is also immediately clear.

For (a), the double Fourier transform of $f_{\mathfrak{p}}$ is computed by applying the last proposition:

$$\begin{aligned}\hat{\hat{f}}_{\mathfrak{p}}(s) &= |q|_{\mathfrak{p}}^{\frac{1}{2}} G_{\mathfrak{p}}(\mathbf{c}_{\mathfrak{p}}, q)\widehat{\overline{f_{\mathfrak{p}}}(sq)} \\ &= |q|_{\mathfrak{p}}^{\frac{1}{2}} G_{\mathfrak{p}}(\mathbf{c}_{\mathfrak{p}}, q)|q|_{\mathfrak{p}}^{-1}\widehat{\overline{f_{\mathfrak{p}}}}\left(\frac{s}{q}\right).\end{aligned}$$

Applying the previous proposition again, but using $-q$ instead of q, gives

$$\begin{aligned}\hat{\hat{f}}_{\mathfrak{p}}(s) &= |q|_{\mathfrak{p}}^{\frac{1}{2}} G_{\mathfrak{p}}(\mathbf{c}_{\mathfrak{p}}, q)|q|_{\mathfrak{p}}^{-1}|q|_{\mathfrak{p}}^{\frac{1}{2}} G_{\mathfrak{p}}(\overline{\mathbf{c}_{\mathfrak{p}}}, -q)f_{\mathfrak{p}}(-s) \\ &= G_{\mathfrak{p}}(\mathbf{c}_{\mathfrak{p}}, q)G_{\mathfrak{p}}(\overline{\mathbf{c}_{\mathfrak{p}}}, -q)f_{\mathfrak{p}}(-s).\end{aligned}$$

Combining this result with the Fourier inversion formula, and using (c) completes the proof. □

The final case we must consider is when $\mathbf{c}_{\mathfrak{p}}$ is unramified and $\mathfrak{p} \mid 2$.

Definition. If $\mathfrak{p} \mid 2$ and $\mathfrak{f}_{\mathfrak{p}} = 1$ let

$$f_{\mathfrak{p}}(t) = \begin{cases} 1 & \text{if } t \in O_{\mathfrak{p}} \\ 0 & \text{if } t \notin O_{\mathfrak{p}} \end{cases}.$$

Proposition 3.7.

$$\hat{f}_{\mathfrak{p}}(s) = |\delta_{\mathfrak{p}}|^{\frac{1}{2}} f_{\mathfrak{p}}(\delta_{\mathfrak{p}} s).$$

Proof.

$$\hat{f}_{\mathfrak{p}}(s) = \int_{O_{\mathfrak{p}}} e^{2\pi i \Phi_{\mathfrak{p}}(ts)} dt.$$

Replacing t with $t + \alpha$ for any $\alpha \in O_{\mathfrak{p}}$ shows

$$\hat{f}_{\mathfrak{p}}(s) = e^{2\pi i \Phi_{\mathfrak{p}}(\alpha s)} \int_{O_{\mathfrak{p}}} e^{2\pi i \Phi_{\mathfrak{p}}(ts)} dt.$$

Therefore the integral is 0 unless $s \in \delta_{\mathfrak{p}}^{-1}$, and in that case the integrand is 1. □

We note for convenience that the statement of Proposition 3.7 may be subsumed into that of Proposition 3.5. That is, for all $\mathfrak{p}|2\mathfrak{f}$, if $ord_{\mathfrak{p}}(q) = ord_{\mathfrak{p}}(\mathfrak{f}_{\mathfrak{p}}\delta_{\mathfrak{p}})$ then

$$\hat{f}_{\mathfrak{p}}(s) = |q|^{\frac{1}{2}} G_{\mathfrak{p}}(\mathbf{c}_{\mathfrak{p}}, q)\overline{f_{\mathfrak{p}}}(sq)$$

where we interpret $G_{\mathfrak{p}}(\mathbf{c}_{\mathfrak{p}}, q)$ as 1 in the case where $\mathfrak{p}|2$ but $\mathfrak{p} \nmid \mathfrak{f}$.

4. The Theta Series.

For our fixed Dirichlet character $\mathbf{c}$ with conductor $\mathfrak{f}$, define $f : K \times \mathfrak{H}(2\mathfrak{f}) \to \mathbb{C}$ by

$$f(\nu, z) = \prod_{\mathfrak{p}|2f} f_{\mathfrak{p}}(\nu_{\mathfrak{p}}) \prod_{\mathfrak{p}\nmid 2\mathfrak{f}} f_{\mathfrak{p}}(\nu_{\mathfrak{p}}, z_{\mathfrak{p}})$$

where the $f_{\mathfrak{p}}$ are as given in Section 3 and thus depend on $\mathbf{c}_{\mathfrak{p}}$ for $\mathfrak{p}|\mathfrak{f}$. Note that for any value of ν and z all but finitely many of the terms in this product are 1.

Definition. The theta function associated to the Dirichlet character $\mathbf{c}$ of conductor $\mathfrak{f}$ is the function $\Theta_{\mathbf{c}}(z) : \mathfrak{H}(2\mathfrak{f}) \to \mathbb{C}$ defined by

$$\Theta_{\mathbf{c}}(z) = \sum_{\nu \in K} f(\nu, z).$$

One easily checks that this series converges absolutely and uniformly. As expected, $\Theta_{\mathbf{c}}$ satisfies a number of functional equations. In order to state them we need the following definitions.

Definition. Let $\mathfrak{n}$ be any integral ideal of K. Then $O_{\mathfrak{n}} = \bigcap_{\mathfrak{p}|\mathfrak{n}} (K \cap O_{\mathfrak{p}})$ denotes those elements of K integral at all primes dividing $\mathfrak{n}$ and $U_{\mathfrak{n}} = \bigcap_{\mathfrak{p}|\mathfrak{n}} (K \cap U_{\mathfrak{p}})$ denotes those elements of K that are units at all primes dividing $\mathfrak{n}$. Finally, $\delta_{\mathfrak{n}} = \delta O_{\mathfrak{n}}$ denotes the $O_{\mathfrak{n}}$ ideal generated by the global different of K.

In terms of its prime factorization, $\delta_{\mathfrak{n}}$ is simply δ with any factors prime to $\mathfrak{n}$ deleted.

Note that $O_{\mathfrak{n}}$ is always a principal ideal domain; by eliminating all primes except those finitely many that divide $\mathfrak{n}$, we have essentially collapsed the ideal class group of $O(K)$. In fact, even the narrow class number of $O_{\mathfrak{n}}$ is 1, so for any $O_{\mathfrak{n}}$ ideal we can choose a generator that is totally positive (i.e. positive at all real primes).

Proposition 4.1. $\Theta_{\mathbf{c}}$ *satisfies the functional equations:*
(a) *For* $\epsilon \in U_{2\mathfrak{f}}$,

$$\Theta_{\mathbf{c}}\left(\begin{pmatrix}\epsilon & 0\\ 0 & \epsilon^{-1}\end{pmatrix}\circ z\right) = \chi_{\mathbf{c}}((\epsilon)) \prod_{\mathfrak{p}\text{ real}} |\epsilon_{\mathfrak{p}}|_{\mathfrak{p}}^{-n_{\mathfrak{p}}}\ \Theta_{\mathbf{c}(z)}.$$

(b) *For* $\alpha \in \delta_{2\mathfrak{f}}^{-1}$,

$$\Theta_{\mathbf{c}}\left(\begin{pmatrix}1 & \alpha\\ 0 & 1\end{pmatrix}\circ z\right) = \Theta_{\mathbf{c}}(z).$$

(c) *For any* $\phi \in K$ *that generates the* $O_{2\mathfrak{f}}$ *ideal* $\mathfrak{f}\delta_{2\mathfrak{f}}$ *with* ϕ *totally positive,*

$$\Theta_{\mathbf{c}}\left(\begin{pmatrix}0 & -(2\phi)^{-1}\\ 2\phi & 0\end{pmatrix}\circ z\right) = \prod_{\mathfrak{p}|2\mathfrak{f}} G_{\mathfrak{p}}(\overline{\mathbf{c}_{\mathfrak{p}}}, \phi_{\mathfrak{p}})^{-1} \prod_{\substack{\mathfrak{p}\nmid 2\mathfrak{f}\\ \mathfrak{p}\text{ finite}}} G_{\mathfrak{p}}(z_{\mathfrak{p}})^{-1}\sqrt{|2\phi_{\mathfrak{p}} z_{\mathfrak{p}}|_{\mathfrak{p}}}$$
$$\prod_{\mathfrak{p}\text{ real}} (-i)^{n_{\mathfrak{p}}}\sqrt{-2i\phi_{\mathfrak{p}} z_{\mathfrak{p}}}^{\,2n_{\mathfrak{p}}+1}$$
$$\prod_{\mathfrak{p}\text{ complex}} |2\phi_{\mathfrak{p}} z_{\mathfrak{p}}|_{\mathfrak{p}} \Theta_{\bar{\mathbf{c}}}(z)$$

where the square root is such that $-\frac{\pi}{2} < \arg(w^{1/2}) \leq \frac{\pi}{2}$.

Proof. First, for any $\epsilon \in U_{2\mathfrak{f}}$,

$$\begin{aligned}\Theta_{\mathbf{c}}(z) &= \sum_{\nu\in K} f(\nu, z) = \sum_{\nu\in K} f(\epsilon\nu, z)\\ &= \sum_{\nu\in K}\prod_{\mathfrak{p}|2\mathfrak{f}} f_{\mathfrak{p}}(\epsilon_{\mathfrak{p}}\nu_{\mathfrak{p}}) \prod_{\mathfrak{p}\nmid 2\mathfrak{f}} f_{\mathfrak{p}}(\epsilon_{\mathfrak{p}}\nu_{\mathfrak{p}}, z_{\mathfrak{p}})\\ &= \sum_{\nu\in K}\prod_{\mathfrak{p}|\mathfrak{f}} \mathbf{c}_{\mathfrak{p}}(\epsilon_{\mathfrak{p}}) f_{\mathfrak{p}}(\nu_{\mathfrak{p}}) \prod_{\substack{\mathfrak{p}|2\\ \mathfrak{p}\nmid\mathfrak{f}}} f_{\mathfrak{p}}(\nu_{\mathfrak{p}}) \prod_{\mathfrak{p}\text{ real}} \epsilon_{\mathfrak{p}}^{n_{\mathfrak{p}}} f_{\mathfrak{p}}\left(\nu_{\mathfrak{p}}, \begin{pmatrix}\epsilon_{\mathfrak{p}} & 0\\ 0 & \epsilon_{\mathfrak{p}}^{-1}\end{pmatrix}\circ z_{\mathfrak{p}}\right)\\ &\qquad \prod_{\substack{\mathfrak{p}\nmid 2\mathfrak{f}\\ \mathfrak{p}\text{ not real}}} f_{\mathfrak{p}}\left(\nu_{\mathfrak{p}}, \begin{pmatrix}\epsilon_{\mathfrak{p}} & 0\\ 0 & \epsilon_{\mathfrak{p}}^{-1}\end{pmatrix}\circ z_{\mathfrak{p}}\right)\end{aligned}$$

$$= \prod_{\mathfrak{p}|\mathfrak{f}} \mathbf{c}_\mathfrak{p}(\epsilon_\mathfrak{p}) \prod_{\mathfrak{p} \text{ real}} \epsilon_\mathfrak{p}^{n_\mathfrak{p}} \sum_{\nu \in K} f\left(\nu, \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix} \circ z\right)$$
$$= \chi_{\overline{\mathbf{c}}}((\epsilon)) \prod_{\mathfrak{p} \text{ real}} |\epsilon_\mathfrak{p}|_\mathfrak{p}^{n_\mathfrak{p}} \, \Theta_\mathbf{c}\left(\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix} \circ z\right)$$

which establishes (a).

For (b), let $\alpha \in \delta_{2\mathfrak{f}}^{-1}$ so

$$\Theta_\mathbf{c}(z+\alpha) = \sum_{\nu \in K} f(\nu, z+\alpha) = \sum_{\nu \in K} f(\nu, z) e^{2\pi i \sum_{\mathfrak{p}\nmid 2\mathfrak{f}} \Phi_\mathfrak{p}(\nu_\mathfrak{p}^2 \alpha_\mathfrak{p})}.$$

But

$$\sum_{\mathfrak{p}\nmid 2\mathfrak{f}} \Phi_\mathfrak{p}(\nu_\mathfrak{p}^2 \alpha_\mathfrak{p}) = Tr(\nu^2\alpha) - \sum_{\mathfrak{p}|2\mathfrak{f}} \Phi_\mathfrak{p}(\nu_\mathfrak{p}^2 \alpha_\mathfrak{p}).$$

Since $\nu^2\alpha \in K$, $Tr(\nu^2\alpha) = 0$. Observing that for $\mathfrak{p} \mid 2\mathfrak{f}$, $f_\mathfrak{p}(\nu_\mathfrak{p}) = 0$ unless $\nu_\mathfrak{p} \in O_\mathfrak{p}$, and hence $f(\nu, z) = 0$ unless $\nu \in O_{2\mathfrak{f}}$, we may as well assume $\nu \in O_{2\mathfrak{f}}$. But then since $\alpha \in \delta_{2\mathfrak{f}}^{-1}$, $\nu_\mathfrak{p}^2\alpha_\mathfrak{p} \in \delta_\mathfrak{p}^{-1}$ for any $\mathfrak{p} \mid 2\mathfrak{f}$, so $\Phi_\mathfrak{p}(\nu_\mathfrak{p}^2\alpha_\mathfrak{p}) = 0$. Therefore,

$$\Theta_\mathbf{c}(z+\alpha) = \Theta_\mathbf{c}(z).$$

Finally, (c) follows from Poisson summation and Propositions 3.1, 3.2, 3.3, 3.5, and 3.7 since

$$\Theta_\mathbf{c}(z) = \sum_{\nu \in K} f(\nu, z) = \sum_{\nu \in K} \hat{f}(\nu, z)$$
$$= \sum_{\nu \in K} \prod_{\mathfrak{p}|2\mathfrak{f}} \hat{f}_\mathfrak{p}(\nu) \prod_{\mathfrak{p}\nmid 2\mathfrak{f}} \hat{f}_\mathfrak{p}(\nu, z_\mathfrak{p})$$
$$= \sum_{\nu \in K} \prod_{\mathfrak{p}|2\mathfrak{f}} |\phi_\mathfrak{p}|_\mathfrak{p}^{\frac{1}{2}} G_\mathfrak{p}(\mathbf{c}_\mathfrak{p}, \phi_\mathfrak{p}) \overline{f_\mathfrak{p}}(\nu\phi_\mathfrak{p}) \prod_{\substack{\mathfrak{p}\nmid 2\mathfrak{f} \\ \mathfrak{p} \text{ finite}}} \frac{G_\mathfrak{p}(z_\mathfrak{p})}{\sqrt{|2z_\mathfrak{p}|_\mathfrak{p}}} f_\mathfrak{p}\left(\nu, \frac{-1}{4z_\mathfrak{p}}\right)$$
$$\prod_{\mathfrak{p} \text{ real}} \frac{i^{n_\mathfrak{p}}}{\sqrt{-2iz_\mathfrak{p}}^{2n_\mathfrak{p}+1}} f_\mathfrak{p}\left(\nu, \frac{-1}{4z_\mathfrak{p}}\right) \prod_{\mathfrak{p} \text{ complex}} \frac{1}{|2z_\mathfrak{p}|_\mathfrak{p}} f_\mathfrak{p}\left(\nu, \frac{-1}{4z_\mathfrak{p}}\right)$$

where for $p|2\mathfrak{f}$, $\phi_\mathfrak{p} \in K_\mathfrak{p}$ is any element with $(\phi_\mathfrak{p}) = \mathfrak{f}_\mathfrak{p}\delta_\mathfrak{p}$. Choosing a single ϕ as in the statement of the proposition which can then serve as all of these $\phi_\mathfrak{p}$, and replacing ν with $\nu\phi^{-1}$ in the summation gives

$$\Theta_\mathbf{c}(z)$$
$$= \sum_{\nu \in K} \prod_{\mathfrak{p}|2\mathfrak{f}} |\phi_\mathfrak{p}|_\mathfrak{p}^{\frac{1}{2}} G_\mathfrak{p}(\mathbf{c}_\mathfrak{p}, \phi_\mathfrak{p}) \overline{f_\mathfrak{p}}(\nu) \prod_{\substack{\mathfrak{p}\nmid 2\mathfrak{f} \\ \mathfrak{p} \text{ finite}}} \frac{G_\mathfrak{p}(z_\mathfrak{p})}{\sqrt{|2z_\mathfrak{p}|_\mathfrak{p}}} f_\mathfrak{p}\left(\nu\phi^{-1}, \frac{-1}{4z_\mathfrak{p}}\right)$$

$$\prod_{\mathfrak{p}\text{ real}} \frac{i^{n_\mathfrak{p}}}{\sqrt{-2iz_\mathfrak{p}}^{2n_\mathfrak{p}+1}} f_\mathfrak{p}\left(\nu\phi^{-1}, \frac{-1}{4z_\mathfrak{p}}\right) \prod_{\mathfrak{p}\text{ complex}} \frac{1}{|2z_\mathfrak{p}|_\mathfrak{p}} f_\mathfrak{p}\left(\nu\phi^{-1}, \frac{-1}{4z_\mathfrak{p}}\right)$$
$$= \prod_{\mathfrak{p}|2\mathfrak{f}} |\phi_\mathfrak{p}|_\mathfrak{p}^{\frac{1}{2}} G_\mathfrak{p}(\mathbf{c}_\mathfrak{p}, \phi_\mathfrak{p}) \prod_{\substack{\mathfrak{p}\nmid 2\mathfrak{f}\\ \mathfrak{p}\text{ finite}}} \frac{G_\mathfrak{p}(z_\mathfrak{p})}{\sqrt{|2z_\mathfrak{p}|_\mathfrak{p}}} \prod_{\mathfrak{p}\text{ real}} \frac{i^{n_\mathfrak{p}}}{\sqrt{-2iz_\mathfrak{p}}^{2n_\mathfrak{p}+1}} \phi_\mathfrak{p}^{-n_\mathfrak{p}} \prod_{\mathfrak{p}\text{ complex}} \frac{1}{|2z_\mathfrak{p}|_\mathfrak{p}}$$
$$\sum_{\nu\in K} \prod_{\mathfrak{p}|2\mathfrak{f}} \overline{f_\mathfrak{p}}(\nu) \prod_{\mathfrak{p}\nmid 2\mathfrak{f}} f_\mathfrak{p}\left(\nu, \begin{pmatrix} 0 & -(2\phi)^{-1} \\ 2\phi & 0 \end{pmatrix} \circ z_\mathfrak{p}\right).$$

Noting that by the product formula, since $\phi_\mathfrak{p} > 0$ for all real $\mathfrak{p}$,

$$\prod_{\mathfrak{p}|2\mathfrak{f}} |\phi_\mathfrak{p}|_\mathfrak{p}^{\frac{1}{2}} = \prod_{\mathfrak{p}\text{ real}} \phi_\mathfrak{p}^{-\frac{1}{2}} \prod_{\mathfrak{p}\text{ complex}} |\phi_\mathfrak{p}|_\mathfrak{p}^{-1} \prod_{\substack{\mathfrak{p}\nmid 2\mathfrak{f}\\ \mathfrak{p}\text{ finite}}} |\phi_\mathfrak{p}|_\mathfrak{p}^{-\frac{1}{2}}$$

we have

$$\Theta_\mathbf{c}(z) = \prod_{\mathfrak{p}|2\mathfrak{f}} G_\mathfrak{p}(\mathbf{c}_\mathfrak{p}, \phi_\mathfrak{p}) \prod_{\substack{\mathfrak{p}\nmid 2\mathfrak{f}\\ \mathfrak{p}\text{ finite}}} \frac{G_\mathfrak{p}(z_\mathfrak{p})}{\sqrt{|2\phi_\mathfrak{p} z_\mathfrak{p}|_\mathfrak{p}}} \prod_{\mathfrak{p}\text{ real}} \frac{i^{n_\mathfrak{p}}}{\sqrt{-2i\phi_\mathfrak{p} z_\mathfrak{p}}^{2n_\mathfrak{p}+1}} \prod_{\mathfrak{p}\text{ complex}} \frac{1}{|2\phi_\mathfrak{p} z_\mathfrak{p}|_\mathfrak{p}}$$
$$\Theta_{\bar{\mathbf{c}}}\left(\begin{pmatrix} 0 & -(2\phi)^{-1} \\ 2\phi & 0 \end{pmatrix} \circ z\right).$$

□

We now turn to investigating discrete groups for which $\Theta_\mathbf{c}$ satisfies a transformation formula.

Definition. If $\mathfrak{n}$ is an integral ideal of $O(K)$, let

$$\Gamma_0(\mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(K) \;\middle|\; a, d \in O_\mathfrak{n},\ c \in \mathfrak{n}\delta_\mathfrak{n},\ b \in \delta_\mathfrak{n}^{-1} \right\}.$$

Let $\Gamma = \Gamma_0(1)$.

Definition. If $\mathfrak{n}$ is an integral ideal of $O(K)$, the theta group $\Gamma_\Theta(\mathfrak{n})$ is the group generated by the following matrices:

$$\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix},\ \epsilon \in U_\mathfrak{n};\ \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix},\ \alpha \in \delta_\mathfrak{n}^{-1};\ \begin{pmatrix} 0 & -\phi^{-1} \\ \phi & 0 \end{pmatrix}$$

where $\phi \in K$ is any generator of the $O_\mathfrak{n}$ ideal $\mathfrak{n}\delta_\mathfrak{n}$ with $\phi_\mathfrak{p} > 0$ for all real $\mathfrak{p}$.

One readily sees that the definition of the theta group is independent of the choice of ϕ.

The relationship between these two groups is rather simple:

Proposition 4.2. $\Gamma_0(\mathfrak{n}^2) = \Gamma_\Theta(\mathfrak{n}) \cap \Gamma$.

Proof. We assume $\mathfrak{n} \neq (1)$, and leave the minor modifications for the $\mathfrak{n} = (1)$ case to the reader.

Letting ϕ denote any element as in the definition of $\Gamma_\Theta(\mathfrak{n})$, if $A \in \Gamma_0(\mathfrak{n}^2)$, then $A = \begin{pmatrix} a & b \\ \phi^2 c & d \end{pmatrix}$ with $a, d \in U_\mathfrak{n}$ and $b, c \in \delta_\mathfrak{n}^{-1}$. But since

$$A = \begin{pmatrix} 1 & bd^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -d^{-1} & 0 \\ 0 & -d \end{pmatrix} \begin{pmatrix} 0 & -\phi^{-1} \\ \phi & 0 \end{pmatrix} \begin{pmatrix} 1 & -cd^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -\phi^{-1} \\ \phi & 0 \end{pmatrix}$$

then $A \in \Gamma_\Theta(\mathfrak{n}) \cap \Gamma$.

Now suppose $A \in \Gamma_\Theta(\mathfrak{n}) \cap \Gamma$ and write A as a product of the generators of $\Gamma_\Theta(\mathfrak{n})$ given in the definition of that group. Letting $W = \begin{pmatrix} 0 & -\phi^{-1} \\ \phi & 0 \end{pmatrix}$, and noting that the other generators in the definition are already in $\Gamma_0(\mathfrak{n}^2)$, we can therefore express A as a product of elements of $\Gamma_0(\mathfrak{n}^2)$ alternating with Ws and W^{-1}s. Since $W^{-1} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} W$, and W is in the normalizer of $\Gamma_0(\mathfrak{n}^2)$, we can further collapse the product: If an even number of Ws and W^{-1}s occur in the original expression for A, then we find $A \in \Gamma_0(\mathfrak{n}^2)$. On the other hand, if an odd number occur, we find $A = BW$ for some $B \in \Gamma_0(\mathfrak{n}^2)$. Since $A \in \Gamma$, this means $W = B^{-1}A \in \Gamma$. That of course is absurd under our assumption $\mathfrak{n} \neq (1)$. □

Proposition 4.1 says $\Theta_\mathbf{c}$ transforms nicely under all elements of $\Gamma_\Theta(2\mathfrak{f})$. For the smaller group $\Gamma_0(4\mathfrak{f}^2)$ the transformation formula is given explicitly by the following proposition.

Proposition 4.3. *If $A \in \Gamma_0(4\mathfrak{f}^2)$ where $\mathfrak{f}$ is the conductor of the Dirichlet character $\mathbf{c}$, then let $\phi \in K$ be any totally positive generator of the $O_{2\mathfrak{f}}$ ideal $\mathfrak{f}\delta_{2\mathfrak{f}}$ and write $A = \begin{pmatrix} a & b \\ 4\phi^2 c & d \end{pmatrix}$. Then*

$$\Theta_\mathbf{c}(A \circ z) = \overline{\chi_\mathbf{c}}((d)) \prod_{\substack{\mathfrak{p} \nmid 2\mathfrak{f} \\ \mathfrak{p} \text{ finite}}} G_\mathfrak{p} \left(-\frac{c}{d} - \frac{1}{4\phi^2 z_\mathfrak{p}} \right)^{-1} G_\mathfrak{p}(z_\mathfrak{p})^{-1} \; j_\mathbf{c}(A, z) \Theta_\mathbf{c}(z)$$

where

$$j_\mathbf{c}(A, z) = \prod_{\substack{\mathfrak{p} \nmid 2\mathfrak{f} \\ \mathfrak{p} \text{ finite}}} |4\phi^2 c z_\mathfrak{p} + d|_\mathfrak{p}^{\frac{1}{2}} \prod_{\mathfrak{p} \text{ real}} (sgn(d)(4\phi^2 c z_\mathfrak{p} + d))^{\frac{2n_\mathfrak{p}+1}{2}} \prod_{\mathfrak{p} \text{ complex}} |4\phi^2 c z_\mathfrak{p} + d|_\mathfrak{p},$$

and where the square root is such that $-\frac{\pi}{2} < \arg(w^{\frac{1}{2}}) \leq \frac{\pi}{2}$.

Proof. Using the factorization

$$A = \begin{pmatrix} 1 & bd^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -d^{-1} & 0 \\ 0 & -d \end{pmatrix} \begin{pmatrix} 0 & -(2\phi)^{-1} \\ 2\phi & 0 \end{pmatrix} \begin{pmatrix} 1 & -cd^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -(2\phi)^{-1} \\ 2\phi & 0 \end{pmatrix}$$

and statements (b) and (a) of Proposition 4.1 we see

$$\Theta_{\mathbf{c}}\left(A \circ z\right) = \chi_{\mathbf{c}}((-d^{-1})) \prod_{\mathfrak{p} \text{ real}} |d|_{\mathfrak{p}}^{n_{\mathfrak{p}}} \\ \Theta_{\mathbf{c}}\left(\begin{pmatrix} 0 & -(2\phi)^{-1} \\ 2\phi & 0 \end{pmatrix} \begin{pmatrix} 1 & -cd^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -(2\phi)^{-1} \\ 2\phi & 0 \end{pmatrix} \circ z\right).$$

Using (c) of Proposition 4.1 yields

$$\Theta_{\mathbf{c}}\left(A \circ z\right) = \chi_{\mathbf{c}}((d^{-1})) \prod_{\mathfrak{p} \text{ real}} |d|_{\mathfrak{p}}^{n_{\mathfrak{p}}} \prod_{\mathfrak{p} | 2\mathfrak{f}} G_{\mathfrak{p}}(\overline{\mathbf{c}_{\mathfrak{p}}}, \phi_{\mathfrak{p}})^{-1} \\ \prod_{\substack{\mathfrak{p} \nmid 2\mathfrak{f} \\ \mathfrak{p} \text{ finite}}} G_{\mathfrak{p}}\left(-\frac{c}{d} - \frac{1}{4\phi^2 z_{\mathfrak{p}}}\right)^{-1} \left|2\phi\left(-\frac{c}{d} - \frac{1}{4\phi^2 z_{\mathfrak{p}}}\right)\right|_{\mathfrak{p}}^{\frac{1}{2}} \\ \prod_{\mathfrak{p} \text{ real}} (-i)^{n_{\mathfrak{p}}} \left(-2i\phi\left(-\frac{c}{d} - \frac{1}{4\phi^2 z_{\mathfrak{p}}}\right)\right)^{\frac{2n_{\mathfrak{p}}+1}{2}} \\ \prod_{\mathfrak{p} \text{ complex}} \left|2\phi\left(-\frac{c}{d} - (2\phi)^{-1}(2\phi z_{\mathfrak{p}})^{-1}\right)\right|_{\mathfrak{p}} \\ \Theta_{\overline{\mathbf{c}}}\left(\begin{pmatrix} 1 & -cd^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -(2\phi)^{-1} \\ 2\phi & 0 \end{pmatrix} \circ z\right).$$

Applying (b) and then (c) of Proposition 4.1 now shows

$$\Theta_{\mathbf{c}}\left(A \circ z\right) = \overline{\chi_{\mathbf{c}}}((d)) \prod_{\mathfrak{p} \text{ real}} |d|_{\mathfrak{p}}^{n_{\mathfrak{p}}} \prod_{\mathfrak{p} | 2\mathfrak{f}} G_{\mathfrak{p}}(\overline{\mathbf{c}_{\mathfrak{p}}}, \phi_{\mathfrak{p}})^{-1} G_{\mathfrak{p}}(\mathbf{c}_{\mathfrak{p}}, \phi_{\mathfrak{p}})^{-1} \\ \prod_{\substack{\mathfrak{p} \nmid 2\mathfrak{f} \\ \mathfrak{p} \text{ finite}}} G_{\mathfrak{p}}\left(-\frac{c}{d} - \frac{1}{4\phi^2 z_{\mathfrak{p}}}\right)^{-1} \left|\frac{2\phi c}{d} + \frac{1}{2\phi z_{\mathfrak{p}}}\right|_{\mathfrak{p}}^{\frac{1}{2}} G_{\mathfrak{p}}(z_{\mathfrak{p}})^{-1} |2\phi z_{\mathfrak{p}}|_{\mathfrak{p}}^{\frac{1}{2}} \\ \prod_{\mathfrak{p} \text{ real}} (-i)^{n_{\mathfrak{p}}} \left(i\left(\frac{2\phi c}{d} + \frac{1}{2\phi z_{\mathfrak{p}}}\right)\right)^{\frac{2n_{\mathfrak{p}}+1}{2}} (-i)^{n_{\mathfrak{p}}} (-2i\phi z_{\mathfrak{p}})^{\frac{2n_{\mathfrak{p}}+1}{2}} \\ \prod_{\mathfrak{p} \text{ complex}} \left|\frac{2\phi c}{d} + (2\phi z_{\mathfrak{p}})^{-1}\right|_{\mathfrak{p}} |2\phi z_{\mathfrak{p}}|_{\mathfrak{p}} \,\Theta_{\mathbf{c}}\left(z\right).$$

Simplifying, and using (b) and (c) of Proposition 3.6, we see

$$\Theta_{\mathbf{c}}(A \circ z) = \overline{\chi_{\mathbf{c}}}((d)) \prod_{\mathfrak{p}|2\mathfrak{f}} \mathbf{c}_{\mathfrak{p}}(-1)$$

$$\prod_{\substack{\mathfrak{p}\nmid 2\mathfrak{f} \\ \mathfrak{p}\text{ finite}}} G_{\mathfrak{p}}\left(-\frac{c}{d} - \frac{1}{4\phi^2 z_{\mathfrak{p}}}\right)^{-1} G_{\mathfrak{p}}(z_{\mathfrak{p}})^{-1} \left|\frac{4\phi^2 c}{d} z_{\mathfrak{p}} + 1\right|_{\mathfrak{p}}^{\frac{1}{2}}$$

$$\prod_{\mathfrak{p}\text{ real}} (-|d|_{\mathfrak{p}})^{n_{\mathfrak{p}}} \left(\frac{4\phi^2 c}{d} z_{\mathfrak{p}} + 1\right)^{\frac{2n_{\mathfrak{p}}+1}{2}} \prod_{\mathfrak{p}\text{ complex}} \left|\frac{4\phi^2 c}{d} z_{\mathfrak{p}} + 1\right|_{\mathfrak{p}} \Theta_{\mathbf{c}}(z).$$

But $\prod_{\mathfrak{p}|2\mathfrak{f}} \mathbf{c}_{\mathfrak{p}}(-1) \prod_{\mathfrak{p}\text{ real}} (-1)^{n_{\mathfrak{p}}} = 1$ and since $d \in U_{2\mathfrak{f}}$,

$$\prod_{\substack{\mathfrak{p}\nmid 2\mathfrak{f} \\ \mathfrak{p}\text{ not complex}}} |d|_{\mathfrak{p}} \prod_{\mathfrak{p}\text{ complex}} |d|_{\mathfrak{p}}^2 = 1,$$

so the last equation becomes

$$\Theta_{\mathbf{c}}(A \circ z) = \overline{\chi_{\mathbf{c}}}((d)) \prod_{\substack{\mathfrak{p}\nmid 2\mathfrak{f} \\ \mathfrak{p}\text{ finite}}} G_{\mathfrak{p}}\left(-\frac{c}{d} - \frac{1}{4\phi^2 z_{\mathfrak{p}}}\right)^{-1} G_{\mathfrak{p}}(z_{\mathfrak{p}})^{-1} j_{\mathbf{c}}(A, z)\Theta_{\mathbf{c}}(z).$$

□

5. $K = \mathbb{Q}$.

We now specialize to the case $K = \mathbb{Q}$ to illustrate how to recover the transformation formula for the theta functions associated to Dirichlet characters mod N as in [6].

For a positive integer N, by a Dirichlet character mod N is meant a function ψ on $\mathbb{Z}$ such that

$$\psi(n) = \begin{cases} 0 & \text{if } (n, N) \neq 1 \\ \psi_0(n \mod N) & \text{if } (n, N) = 1 \end{cases}$$

where $\psi_0 : (\mathbb{Z}/N\mathbb{Z})^{\times} \to \mathbb{C}$ is a homomorphism. ψ is said to be a primitive character mod N if N is the smallest positive integer such that $\psi(n)$ depends only on $n \mod N$ when $(n, N) = 1$.

Corollary 5.1. *Let ψ be a primitive Dirichlet character mod N with $n_\infty \in \{0, 1\}$ such that $\psi(-1) = (-1)^{n_\infty}$. For $z_\infty \in \mathfrak{H}_\infty = \{x + iy | x, y \in \mathbb{R},\ y > 0\}$, let*

$$\theta_\psi(z_\infty) = \sum_{m\in\mathbb{Z}} \psi(m) m^{n_\infty} e^{2\pi i m^2 z_\infty}.$$

If $A = \begin{pmatrix} a & b \\ 4N^2c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ *with* $c \in \mathbb{Z}$*, then*

$$\theta_\psi(A \circ z_\infty) = \psi(d) \left(\frac{c}{d}\right) \epsilon_d^{-1} \left(4N^2 c z_\infty + d\right)^{\frac{2n_\infty+1}{2}} \theta_\psi(z_\infty)$$

where the quadratic residue symbol $(\frac{c}{d})$ *is defined as in* [6] *and*

$$\epsilon_d = \begin{cases} 1 & \textit{if } d \equiv 1 \pmod 4 \\ i & \textit{if } d \equiv 3 \pmod 4 \end{cases}.$$

Proof. ψ gives rise to a grossencharacter $\chi_{\mathbf{c}}$ and a Dirichlet character $\mathbf{c}$ in the sense of Section 2 with $\chi_{\mathbf{c}}((d)) = \overline{\psi}(|d|)$. The conductor of $\mathbf{c}$ is then N. Thus by Proposition 4.3, if A is as stated, then for $z \in \mathfrak{H}(2N)$,

$$\Theta_{\mathbf{c}}(A \circ z) = \overline{\chi_{\mathbf{c}}}((d)) \prod_{\substack{\mathfrak{p} \nmid 2\mathfrak{f} \\ \mathfrak{p} \text{ finite}}} G_p\left(-\frac{c}{d} - \frac{1}{4\phi^2 z_{\mathfrak{p}}}\right)^{-1} G_p(z_{\mathfrak{p}})^{-1} j_{\mathbf{c}}(A, z) \Theta_{\mathbf{c}}(z).$$

For all finite $p \nmid 2N$, choose $z_p = I_p$. Then

$$\Theta_{\mathbf{c}}(z) = \theta_\psi(z_\infty)$$
$$\Theta_{\mathbf{c}}(A \circ z) = \theta_\psi(A \circ z_\infty)$$
$$G_p(z_p) = 1 \text{ for finite } p \nmid 2N$$
$$|4N^2 c z_p + d|_p = 1 \text{ for finite } p \nmid 2N.$$

Thus the transformation formula becomes

$$\theta_\psi(A \circ z_\infty) = \psi(|d|) \prod_{\substack{p \nmid 2N \\ p \text{ finite}}} G_p\left(-\frac{c}{d} - \frac{1}{4N^2 I_p}\right)^{-1} \left(\operatorname{sgn}(d)(4N^2 c z_\infty + d)\right)^{\frac{2n_\infty+1}{2}} \theta_\psi(z_\infty).$$

But from Proposition 3.4, for $p \nmid 2N$,

$$G_p\left(-\frac{c}{d} - \frac{1}{4N^2 I_p}\right) = \begin{cases} 1 & \text{if } ord_p(d) \text{ is even} \\ \left(\frac{-cd^{-1}p^{ord_p d}}{p}\right) \mu_p & \text{if } ord_p(d) \text{ is odd} \end{cases}.$$

Note that we have made a canonical choice of uniformizer for $\mathbb{Q}_p$ by taking $\pi = p$ where $p > 0$ in $\mathbb{Q}$.

As is well known (see [2] for instance), the Gauss sum defining μ_p can be explicitly evaluated to give

$$\mu_p = \epsilon_p \left(\frac{-1}{p}\right).$$

Thus

$$\prod_{\substack{p\nmid 2N \\ p \text{ finite}}} G_p\left(-\frac{c}{d}-\frac{1}{4N^2 I_p}\right)^{-1} = \prod_{\substack{p \text{ such that} \\ ord_{\mathfrak{p}} d \text{ is odd}}} \left(\frac{cd^{-1}p^{ord_p d}}{p}\right)\epsilon_p^{-1}$$

$$= \prod_{\substack{p \text{ such that} \\ ord_{\mathfrak{p}} d \text{ is odd}}} \left(\frac{c}{p}\right)\left(\frac{dp^{-ord_p d}}{p}\right)\epsilon_p^{-1}$$

$$= \left(\frac{c}{|d|}\right) \prod_{\substack{p \text{ such that} \\ ord_{\mathfrak{p}} d \text{ is odd}}} \left(\frac{dp^{-ord_p d}}{p}\right)\epsilon_p^{-1}.$$

We leave as an easy exercise in the use of quadratic reciprocity (most usefully expressed for this purpose by the identity $\left(\frac{a}{b}\right)\left(\frac{b}{a}\right)\epsilon_a\epsilon_b = \epsilon_{ab}$ for positive odd a and b) the proof that for $d > 0$

$$\prod_{\substack{p \text{ such that} \\ ord_{\mathfrak{p}} d \text{ is odd}}} \left(\frac{dp^{-ord_p d}}{p}\right)\epsilon_p^{-1} = \epsilon_d^{-1}.$$

(Of course, to make this development more self-contained, the quadratic reciprocity law (for arbitrary number fields) can be deduced from the properties of the theta functions of this paper by adapting the proofs of Cauchy and Hecke found in [**2**].)

Therefore, for $d > 0$ we have

$$\theta_\psi(A \circ z_\infty) = \psi(d)\left(\frac{c}{d}\right)\epsilon_d^{-1}\left(4N^2 c z_\infty + d\right)^{\frac{2n_\infty+1}{2}}\theta_\psi(z_\infty)$$

as claimed.

The $d < 0$ case is most easily deduced from this by considering $-A$ instead of A.

□

The formula relating $\theta_\psi\left(\frac{-1}{4N^2 z_\infty}\right)$ to $\theta_{\bar\psi}(z_\infty)$ is proved similarly using part (c) of Proposition 4.1 and Proposition 3.6.

The formulae in [**5**] can also be recovered by considering the trivial character and instead of choosing $z_p = I_{\mathfrak{p}}$ for all finite $p \neq 2$, allowing z_p for a single p to remain a free variable.

References

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