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ASYMPTOTICS FOR SINGULAR VECTORS IN VERMA MODULES OVER THE VIRASORO ALGEBRA

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For the singular vector in the reducible Verma module over the Virasoro algebra the terms of extremal degrees in the parameter are explicitly found in this article.

1. Introduction and statement of the main result.

The Virasoro algebra Vir is the infinitedimensional complex Lie algebra with the basis $\{e_i(i \in \mathbf{Z}),\, z\}$ and the commutator operator

$$\begin{aligned}[e_i, z] =& 0,\\ [e_i, e_j] =& (j-i)e_{i+j} + \delta_{-i,j}\frac{j^3-j}{12}z.\end{aligned}$$

(Sometimes the generators of the Virasoro algebra are denoted by L_i; to translate our results into these notations one should put $e_i = -L_{-i}$.)

The Lie algebra Vir is graded, $\mathrm{Vir} = \oplus_{k\in\mathbf{Z}} \mathrm{Vir}_k$ $(\deg e_i = i, \deg z = 0)$, and has the Cartan decomposition $\mathrm{Vir} = N_- \oplus H \oplus N_+$ with $N_- = \oplus_{k>0} \mathrm{Vir}_k$ is spanned by e_i with $i > 0$, $N_+ = \oplus_{k<0} \mathrm{Vir}_k$ is spanned by e_i with $i < 0$, and $H = \mathrm{Vir}_0$ is spanned by e_0 and z. Clearly N_-, H, N_+ are Lie subalgebras of Vir, and N_-, N_+ are nilpotent (in the sense that the intersection $\cap_k N_\pm^{(k)}$ of subspaces of $N_\pm$ spanned by k–fold is zero) commutators while H is Abelian.

For $h, c \in \mathbb{C}$ the Verma module $M(h, c)$ over Vir is defined by the formula

$$M(h,c) = \mathrm{Ind}_{N_+\oplus H}^{\mathrm{Vir}}\, \chi_{h,c}$$

where $\chi_{h,c}$ denotes the one–dimensional $(N_+ \oplus H)$–module with $e_0 = h, z = c, e_i = 0$ for $i < 0$ and Ind is the inducing operator. As an $U(N_-)$–module $M(h, c)$ is a free module with one generator, which we denote by v. Hence as a linear space $M(h, c)$ is spanned (for any h, c) by the vectors $e_{i_1} \dots e_{i_r} v$, $i_1 \geq \dots \geq i_r \geq 1$. The module $M(h, c)$ is graded, $M(h, c) = \oplus_{k\geq 0} M_k(h, c)$, by $\deg(e_{i_1} \dots e_{i_r} v) = i_1 + \dots + i_r$; in particular, $M_0(h, c) = \mathbb{C}v$, $\dim M_k(h, c) = \mathbf{p}(k)$ where $\mathbf{p}$ is the partition function. Evidently, $zw = cw$ for any $w \in M(h, c)$, and if $\deg w = d$ then $e_0 w = (h + d)w$.

A non–zero element w of an arbitrary Vir–module N is called a *singular vector of the type* (h, c) if $e_0 w = hw, zw = cw$ and $e_i w = 0$ for $i < 0$. For example, $v \in M(h,c)$ is a singular vector of the type (h,c). It is easy to see that any non–zero submodule N of $M(h,c)$ contains a singular vector. Moreover, for some $k \geq 0$ there exists a singular vector (of the type $(h+k,c)$) in $N \cap M(h,c)$, and $N \subset \oplus_{l \geq k} M_l(h,c)$; if $N \neq M(h,c)$, then $k \neq 0$, and vice versa. Thus *the module $M(h,c)$ is reducible if and only if it contains a singular vector not in $\mathbb{C}v$.*

The two major problems in the Virasoro representation theory are: (i) For which h,c is the module $M(h,c)$ reducible? (ii) What are the singular vectors in a reducible $M(h,c)$?

The solution of the first problem is provided by the following well known Kac theorem.

For $k,l \in \mathbf{Z}$ denote by $\Phi(k,l)$ the curve in the plane $\mathbb{C}^2(h,c)$ given by the parametric equations

$$\begin{aligned} h = h_{k,l}(t) &= \frac{1-k^2}{4}\,t + \frac{1-kl}{2} + \frac{1-l^2}{4}\,t^{-1}, \\ c = c(t) &= 6t + 13 + 6t^{-1} \end{aligned}$$

$(t \in \mathbb{C} - 0)$. It is the straight line $h = \dfrac{c-1}{24}(1-k^2)$ if $k = l$ and an irreducible second order curve otherwise. Obviously $\Phi(k,l) = \Phi(l,k)$ (with the parameter change $t \mapsto t^{-1}$); otherwise all the curves $\Phi(k,l)$ are different.

Theorem 1.1 (see [**K**], [**FF1**])**.** *The module $M(h,c)$ is reducible if and only if (h,c) belongs to the union of the curves $\Phi(k,l)$ with $k,l > 0$. If $(h,c) \in \Phi(k,l)$, $k,l > 0$ and $(h,c) \notin \Phi(k',l')$ for $k',l' > 0$, $k'l' < kl$ then $M(h,c)$ has a singular vector in $M_{kl}(h,c)$ and has no singular vectors in $M_s(h,c)$ with $s < kl$.*

In particular, for $k,l > 0$ and generic t the module $M(h_{k,l}(t), c(t))$ contains a singular vector of the degree kl. Actually it is true for all t; moreover, the vector has the form $S_{k,l}(t)v$ where

$$S_{k,l}(t) = e_1^{kl} + \sum_{\substack{i_1+\dots+i_r=kl \\ kl>i_1\geq\dots\geq i_r\geq 1}} P_{k,l}^{i_1\dots i_r}(t)\, e_{i_1}\dots e_{i_r},$$
$$P_{k,l}^{i_1\dots i_r}(t) \in \mathbb{C}\left[t,t^{-1}\right].$$

(This fact may be regarded as well known; a proof of it is contained, for example in [**F**].)

It is known also (see [**FF3**]) that any singular vector in $M(h,c)$ is proportional either to $S_{k,l}(t)v$ with $(h,c) \in \Phi(k,l)$ or to $S_{k',l'}(t)S_{k,l}(t)v$ with $(h,c) \in \Phi(k,l)$ and $(h+kl,c) \in \Phi(k',l')$. This shows that the problem of the explicit description of singular vectors is reduced to the problem of the explicit description of $S_{k,l}(t)$.

It is easy to see that $S_{k,l}(t) = S_{l,k}(t^{-1})$; it is also not difficult to calculate $S_{k,l}(t)$ for small values of k, l. In particular,

$$\begin{aligned}
&S_{1,1}(t) = e_1,\\
&S_{2,1}(t) = e_1^2 + te_2,\\
&S_{3,1}(t) = e_1^3 + 4te_2e_1 + (4t^2 + 2t)e_3,\\
&S_{4,1}(t) = e_1^4 + 10te_2e_1^2 + 9t^2e_2^2 + (24t^2 + 10t)e_3e_1 + (36t^3 + 24t^2 + 6t)e_4,\\
&S_{2,2}(t) = e_1^4 + 2ue_2e_1^2 + (u^2 - 4)e_2^2 + (2u + 6)e_3e_1 + (3u + 6)e_4,\ u = t + t^{-1}.
\end{aligned}$$

Some partial results on $S_{k,l}(t)$ were obtained by Feigin and the second author in [**FF2**]. One of them is formulated below (see Proposition 2.1).

An explicit formula for $S_{k,l}(t)$ was obtained in the case $l = 1$ by Benoit and Saint–Aubin [**BS**] and in general case by Bauer, di Francesco, Itzykson and Zuber [**BFIZ**]. But these formulas do not seem to be explicit enough to give any expression for the polynomials $P_{k,l}^{i_1\dots i_r}(t)$.

The main result of this article is the following:

Theorem 1.2.

$$S_{k,l}(t) = (k-1)!^{2l} e_k^l t^{(k-1)l} + \dots + (l-1)!^{2k} e_l^k t^{-(l-1)k},$$

where "..." denotes the terms of intermediate degrees in t.

In particular, the highest and the lowest degrees of $S_{k,l}(t)$ in t are equal, respectively, to $(k-1)l$ and $-(l-1)k$; this fact is essential for [**FF3**] and was used there more or less without proof. (An elucidation of some details of some proofs in [**FF3**] an a generalization of some results of that work is contained in [**A**].)

2. Proof of Theorem 1.2.

2.1. Preliminary calculation. Let $\mathcal{F}_{\lambda,\mu}$ be the Vir–module spanned over $\mathbb{C}$ by $f_j,\ j \in \mathbf{Z}$ with the action of Vir

$$zf_j = 0,\ e_if_j = (j + \mu - (i+1)\lambda)f_{i+j}.$$

The following is proved in [**FF2**].

Proposition 2.1. *In* $\mathcal{F}_{\lambda,\mu}$

$$S_{k,l}(t)f_0 = P_{k,l}(t;\lambda,\mu)f_{kl},$$

where

$$P_{k,l}(t;\lambda,\mu)^2 = \prod_{\substack{0\le u<k\\0\le v<l}} R_{k,l,u,v}(t;\lambda,\mu),$$

$$\begin{aligned}R_{k,l,u,v}&(t;\lambda,\mu) = \nu^2\\ &+\nu\big[(2u(k-1-u)+k-1)t+kl-(k-1-2u)(l-1-2v)-1\\ &\qquad\qquad+(2v(l-1-v)+l-1)t^{-1}\big]\\ &-\lambda\big[(k-1-2u)^2t+2(k-1-2u)(l-1-2v)+(l-1-2v)^2t^{-1}\big]\\ &+\frac{(ut+v)((u+1)t+(v+1))((k-u)t+(l-v))((k-1-u)t+(l-1-v))}{t^2}\end{aligned}$$

where, in turn, $\nu = \mu - 2\lambda$.

It is not hard to extract from the last formula the terms of the extreme degrees in t.

Proposition 2.2. *In* $\mathcal{F}_{\lambda,\mu}$

$$\begin{aligned}S_{k,l}(t)f_0 = \Bigg[&(k-1)!^{2l}t^{(k-1)l}\prod_{v=0}^{l-1}(\mu-(k+1)\lambda+kv)+\cdots\\ &+(l-1)!^{2k}t^{-(l-1)k}\prod_{u=0}^{k-1}(\mu-(l+1)\lambda+lu)\Bigg]f_{kl},\end{aligned}$$

where "..." denotes the terms of intermediate degrees in t. *In other words,*

$$S_{k,l}(t)f_0 = \left[(k-1)!^{2l}t^{(k-1)l}e_k^l+\cdots+(l-1)!^{2k}t^{-(l-1)k}e_l^k\right]f_0.$$

Proof. Direct calculation. □

Proposition 2.2 gives a good motivation for Theorem 1.2. We will also use it in the proof of Theorem 1.2 (though one can avoid it).

2.2. Main lemma: An estimate for the degree of $P_{k,l}^{i_1,\dots,i_r}(t)$. For a polynomial $P\in\mathbb{C}[t,t^{-1}]$ we denote by $d_+(P)$ and $d_-(P)$ the highest degree and the lowest degree of P in t; in other words,

$$d_+\left(\sum a_it^i\right) = \sup\{i|a_i\neq 0\},\quad d_-\left(\sum a_it^i\right) = \inf\{i|a_i\neq 0\}.$$

Proposition 2.2 shows that

(1)
$$\max_{i_1+\dots+i_r=kl} d_+\left(P_{k,l}^{i_1,\dots,i_r}(t)\right) \geq (l-1)k,$$
$$\min_{i_1+\dots+i_r=kl} d_-\left(P_{k,l}^{i_1,\dots,i_r}(t)\right) \leq -(k-1)l.$$

For a positive integer j put

$$\varphi(j) = j - 1 - \left[\frac{j-1}{k}\right].$$

Lemma 2.3. $d_+\left(P_{k,l}^{i_1,\dots,i_r}(t)\right) \leq \sum_{s=1}^{r} \varphi(i_s).$

The proof is contained in 2.3–2.6.

2.3. Properties of the function φ. For an integer s denote by $\rho(s)$ such integer that $1 \leq \rho(\beta) \leq k$, $s \equiv \rho(s) \mod k$. We will need the following properties of φ.

(1) For any s, t

$$\varphi(s+t) \geq \varphi(s) + \varphi(t) \geq \varphi(s+t-1);$$

more precisely,

$$\varphi(s+t) - \varphi(s) - \varphi(t) = \begin{cases} 1 & \text{if } \rho(s)+\rho(t) \leq k, \\ 0 & \text{if } \rho(s)+\rho(t) > k, \end{cases}$$
$$\varphi(s) + \varphi(t) - \varphi(s+t-1) = \begin{cases} 0 & \text{if } \rho(s)+\rho(t) \leq k+1, \\ 1 & \text{if } \rho(s)+\rho(t) > k+1. \end{cases}$$

(2) For any u and $s_1, \dots, s_u$

$$\varphi(s_1 + \dots + s_u + 1) \geq \varphi(s_1) + \cdots \varphi(s_u);$$

the equality holds if and only if $u = 1$ and $s_1 \equiv 0 \mod k$.

The properties (1), (2) are checked immediately.

2.4. Induction. Following [F] we arrange the sets $(i_1, \dots, i_r)$ with $i_1 + \dots + i_r = kl$ in the inverse lexicographical order: For $(i_1, \dots, i_r)$, $(i'_1, \dots, i'_{r'})$ with $i'_1 \geq \dots \geq i'_{r'}$, $i_1 \geq \dots \geq i_r$ we say that $(i'_1, \dots, i'_{r'}) \prec (i_1, \dots, i_r)$ if for some u

$$i'_{r'} = i_r,\ i'_{r'-1} = i_{r-1}, \dots i'_{r'-u-1} = i_{r-u-1},\ i'_{r'-u} < i_{r-u}.$$

Thus,

$$(1, \dots, 1) \prec (2, 1, \dots, 1) \prec \dots \prec (kl).$$

We prove Lemma 2.3 by induction. Obviously, $d_+(P_{k,l}^{1,\dots,1}) = d_+(1) = 0 = \sum\varphi(1)$; suppose that the inequality holds for all $P_{k,l}^{i_1,\dots,i_r}$ with $(i'_1,\dots,i'_{r'}) \prec (i_1,\dots,i_r)$ and prove it for $P_{k,l}^{i_1,\dots,i_r}$.

We suppose that $i_1 \geq \dots \geq i_u > 1,\ i_{u+1} = \dots = i_r = 1$; that is our monomial is $e_{i_1}\dots e_{i_u}e_1^{r-u}$.

2.5. Case 1: $i_u \not\equiv 1 \mod k$**.** Since $S_{k,l}(t)v$ is a singular vector then

$$e_{-(i_u-1)}S_{k,l}(t)v = 0.$$

We find the coefficient at $e_{i_1}\cdots e_{i_{u-1}}e_1^{r-u-1}$ in the left side of the last equality and equate this coefficient to 0. This coefficient is a linear combination of the form $\sum a_{i'_1\dots i'_{r'}}P_{k,l}^{i'_1\dots i'_{r'}}(t)$ where $a_{i'_1\dots i'_{r'}}$ is a polynomial in t with $d_+ \leq 1,\ d_- \geq -1$. This linear combination involves $P_{k,l}^{i_1\dots i_r}(t)$ with a non–zero constant coefficient (namely $2i_u - 1$) and some $P_{k,l}^{i'_1\dots i'_{r'}}(t)$'s with $(i'_1,\dots,i'_{r'}) \prec (i_1,\dots,i_r)$. For all these $P_{k,l}^{i'_1\dots i'_{r'}}(t)$'s we have, by the induction hypothesis, $d_+\left(P_{k,l}^{i'_1\dots i'_{r'}}(t)\right) \leq \sum\varphi(i'_{s'})$. Hence we must check that for each of our $P_{k,l}^{i'_1\dots i'_{r'}}(t)$'s either $\sum\varphi(i'_{s'}) \leq \sum\varphi(i_s)$ and $d_+(a_{i'_1\dots i'_{r'}}) = 0$, or $\sum\varphi(i'_{s'}) < \sum\varphi(i_s)$ (for always $d_+(a_{i'_1\dots i'_{r'}}) \leq 1$).

There are three possibilities for $e_{i_1}\dots e_{i_{u-1}}e_1^{r-u-1}v$ to appear in $e_{-(i_u-1)}e_{i'_1}\dots e_{i'_{r'}}v$. The first is that $e_{-(i_u-1)}$ interacts (forms the commutator) with some $e_{i'_v}$ with $i'_v > i_u - 1$. Then we get $e_{i'_1}\dots e_{i'_v-(i_u-1)}\dots e_{i'_{r'}}v$ with some non–zero constant coefficient; the subscripts $i'_1,\dots,i'_v-(i_u-1),\dots,i'_{r'}$ may go in a wrong order, in which case we need to transpose some of them which may result in some of the numbers summing up and more constant factor arising. Thus in this case the set $i'_1,\dots,i'_{r'}$ becomes $i_1,\dots,i_{v-1},i_v+i_u-1,i_{v+1},\dots,i_r$ after a permutation and summing up some successive numbers. In virtue of 2.3.1 one has

$$\sum_{s'=1}^{r'}\varphi\left(i'_{s'}\right) \leq \sum_{s\neq u,v}\varphi(i_s) + \varphi(i_v+i_u-1) \leq \sum_{s=1}^{r}\varphi(i_s);$$

and we have seen that $a_{i'_1,\dots,i'_{r'}} = \text{const}$, hence $d_+(a_{i'_1,\dots,i'_{r'}}) = 0$.

The second possibility is that $e_{-(i_u-1)}$ interacts with some $e_{i'_{s'}}$'s with $i'_{s'} \leq i_u - 1$ and eventually becomes e_1. In this case the set $i'_1,\dots,i'_{r'}$ has the form $i_1,\dots,i_{u-1},\ k_1,\dots,k_l,\ 1,\dots,1$ with $k_1+\dots+k_l = i_u$ and again $a_{i'_1,\dots,i'_{r'}} = \text{const}$. We apply 2.3.1 again and get

$$\sum_{s'=1}^{r'}\varphi\left(i'_{s'}\right) = \sum_{s\neq u}\varphi(i_s) + \sum_{t=1}^{l}\varphi(k_t) \leq \sum_{s=1}^{r}\varphi(i_s). \tag{2}$$

The third possibility is that $e_{-(i_u-1)}$ again interacts with some $e_{i'_{s'}}$'s with $i'_{s'} \le i_u - 1$ and eventually becomes e_0. In this case the set $i'_1, \dots, i'_{r'}$ again has the form $i_1, \dots, i_{u-1}, k_1, \dots, k_l, 1 \dots, 1$ but with $k_1 + \dots + k_l = i_u - 1$ (now it is possible that some of k_j's are equal to 1, but it makes actually no difference). The important difference between this case and the previous one is that $a_{i'_1,\dots,i'_{r'}}$ generally is not a constant any more: it is a linear function of h and c. Hence $d_+(a_{i'_1,\dots,i'_{r'}}) \le 1$ (and normally $= 1$). In virtue of 2.3.2 (and the fact that $\varphi(1) = 0$) one has again (2), but the equality in (2) holds only if $l = 1$ and $k_1 \equiv 0 \mod k$. This imply $i_u = k_1 + 1 \equiv 1 \mod k$ which contradicts to the assumption of Case 1. Hence

$$\sum_{s'=1}^{r'} \varphi\left(i'_{s'}\right) < \sum_{s=1}^{r} \varphi(i_s).$$

2.6. Case 2: $i_u \equiv 1 \mod k$**.** Let $i_u = mk + 1$. In this case instead of the equality $e_{-(i_u-1)} S_{k,l}(t)v = 0$ we consider the equality

$$e_{-k}^m S_{k,l}(t)v = 0.$$

Again the left hand side involves $P_{k,l}^{i_1 \dots i_r}$ with a non–zero constant coefficient and involves some other $P_{k,l}^{i'_1 \dots i'_{r'}}$'s with $(i'_1, \dots, i'_{r'}) \prec (i_1, \dots, i_r)$. The proof proceeds precisely as above with the exception of the case $(i'_1, \dots, i'_{r'}) = (i_1, \dots, i_{u-1}, i_u - 1, 1, \dots, 1)$. Since $i_u - 1 \equiv 0 \mod k$ the equality (2) becomes an equality. But now $d_+(a_{i'_1,\dots,i'_{r'}}) = 0$. Indeed, $a_{i'_1,\dots,i'_{r'}}$ is the product of several constants and the polynomial $q(t)$ from

$$[e_{-k}, e_k] e_1^s v = q(t) e_1^s v$$

$(s = r - u + 1)$. Since $e_0 e_1^s v = (e_1^s e_0 + s e_1^s)v$, we have

$$[e_{-k}, e_k] e_1^s v = \left(2k e_0 + \frac{k^3 - k}{12} z\right) e_1^s v = \left(2k(h(t) + s) + \frac{k^3 - k}{12} c(t)\right) e_1^s v$$
$$= \left(2k\left(\frac{1 - k^2}{4} t + \frac{1 - kl}{2} + \frac{1 - l^2}{4} t^{-1}\right) + 2ks + \frac{k^3 - k}{12}(6t + 13 + 6t^{-1})\right) e_1^s,$$

and the coefficient in terms with t is equal to

$$2k \cdot \frac{1 - k^2}{4} + \frac{k^3 - k}{12} \cdot 6 = 0.$$

Hence $d_+(q(t)) = 0$.

Lemma 2.3 is proved. □

Remark. The last calculation shows also that if $j \ne k$ and $[e_{-j}, e_j] e_1^s v = q(t) e_1^s v$ then $d_+(q(t)) = 1$.

2.7. One more calculation.

Lemma 2.4.
$$\max_{i_1+\dots+i_r=kl} \sum_{s=1}^{r} \varphi(i_s) = (k-1)l;$$
this maximum is attained precisely when all i_s are divisible by k.

Proof. Let $i_s = m_s k - l_s,\ 0 \le l_s < k$. We have $kl = \sum i_s = k\sum m_s - \sum l_s$, hence $\sum m_s \ge l$ and the equality holds only if all $l_s = 0$. Furthermore,
$$\varphi(i_s) = m_s k - l_s - 1 - \left[\frac{i_s - 1}{k}\right] = m_s k - l_s - m_s$$
and hence
$$\sum \varphi(i_s) = \sum (m_s k - l_s) - \sum m_s = \sum i_s - \sum m_s = kl - \sum m_s \le kl - l,$$
and the equality holds only if all $l_s = 0$, that is all i_s are divisible by k. Lemma 2.4 is proved. □

2.8. End of the proof of Theorem 1.2. Lemma 2.4 shows that $d_+\left(P_{k,l}^{i_1\dots i_r}(t)\right) \le (k-1)l$, which shows, together with formula (1), that
$$\max_{i_1+\dots+i_r=kl} d_+\left(P_{k,l}^{i_1\dots i_r}(t)\right) = (l-1)k$$
and the equality holds if and only if $(i_1,\dots,i_r) = (m_1 k,\dots,m_r k)$ with $m_1,\dots,m_r$ being integers. We prove that actually it holds only if $r = l$ and $m_1 = \dots = m_l = 1$. Assume that, on the contrary, $d_+\left(P_{k,l}^{m_1 k,\dots,m_r k}(t)\right) = (k-1)l$ for some other set $m_1,\dots,m_r$; assume that for some $j > 1$
$$d_+\left(P_{k,l}^{m_1 k,\dots,m_r k}(t)\right) < (k-1)l$$
if $1 < m_s < j$ for some m_s and
$$d_+\left(P_{k,l}^{m_1 k,\dots,m_r k}(t)\right) = (k-1)l$$
for some $m_1,\dots,m_r$ with $m_1 \ge \dots \ge m_u = j$, $m_{u+1} = \dots = m_r = 1$. Then we use the remark at the end of 2.6 and see that the left hand side of the equality
$$e_{-jk} S_{k,l}(t) v = 0$$
contains the term $e_{m_1 k} \dots e_{m_{u-1} k} e_{m_{u+1} k} \dots e_{m_r k} v \cdot t^{(k-1)l+1}$ with a non–zero coefficient, which is impossible.

Thus the only term of degree $(k-1)l$ in $S_{k,l}(t)$ is $Ae_k^l t^{(k-1)l}$ where A is a coefficient. Proposition 2.2 shows now that $A = (k-1)!^{2l}$.

Theorem 1.2 is proved. □

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