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THE EQUIVALENCE PROBLEM FOR HIGHER-CODIMENSIONAL CR STRUCTURES

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The equivalence problem for CR structures can be viewed as a special case of the equivalence problem for G-structure. This paper uses Cartan's methods (in modernized form) to show that a CR manifold of codimension 3 or greater with suitably generic Levi form admits a canonical connection on a reduced structure bundle whose group is isomorphic to the multiplicative group of complex numbers. As corollaries, it follows that the CR manifold admits a canonical affine connection, and consequently that the automorphisms of the CR manifold constitute a Lie group.

The most difficult technical step is to construct a smooth moduli space for generic vector-valued hermitian forms, which is tied to the CR manifold via the Levi map. The techniques used to construct this space are drawn from the classical invariant theory of complex projective hypersurfaces.

1. Introduction.

A CR structure on a manifold M is customarily defined in terms of a complex distribution — that is, a subbundle of the complexified tangent bundle of M — but it is a simple matter to recast the definition in terms of a complex G-structure — that is, a subbundle of the principal bundle of complex coframes. In this form, the equivalence problem for CR structures can be approached by way of Cartan's method of studying the equivalence of geometric structures in general. In essence, this is the approach taken to codimension 1 CR structures in [**CM**] and to codimension 2 CR structures in [**M**].

In this paper, we apply Cartan's method to CR structures of codimension c and CR dimension n with $n > c^2$. We show that if the Levi form satisfies certain mild conditions given in Definition 2.6, crudely summarized as "the Levi form is generic, and its type doesn't vary too much", then the structure bundle can be reduced twice, resulting in a new G-structure group isomorphic to $\mathbb{C}^*$, the multiplicative group of complex numbers (Theorem 3.1). Then, exploiting the fact that the first prolongation of the Lie algebra of this group is trivial, we define a canonical connection on this reduced bundle (Theorem 4.1). Finally, we obtain several corollaries, including the facts that this connection determines an affine connection (Corollary 4.3) and that the automorphisms of the CR manifold constitute a Lie group (Corollary 4.2).

It is a basic fact that for a CR structure of any codimension the structure group of the defining bundle is of infinite type. Therefore, the equivalence problem can not be solved by standard prolongations alone: reductions are required. (The three cases — codimension 1, codimension 2 and codimension 3 or greater — differ greatly in the type of reductions available.) Tanaka ([**T1**]-[**T4**]) takes a different approach to the difficulty of infinite type, developing an alternative scheme of prolongation, uniformly applicable in all codimensions. However, his method requires the assumption that the Levi form be of constant type. This assumption is much more stringent than ours for generic Levi forms, since we allow the type to vary over a suitable open set (however, unlike ours, his method does handle Levi forms of non-generic constant type).

The details of this paper are as follows. In Section 2 we set up the problem, in Section 3 we carry out the reductions, and in Section 4 we determine the connection. The first reduction in Section 3 relies on a theorem about the smoothness of the moduli space of vector valued hermitian forms which is easy to state but difficult to prove: a self-contained proof is given in Section 5 and Section 6, using techniques from the classical invariant theory of complex projective hypersurfaces. Finally, in Section 7 we discuss some open questions. As for prerequisites, no prior knowledge of CR structures is needed; however, some familiarity with the standard method for dealing with the equivalence problem of G-structures — in particular, facility with computations with moving coframes — is assumed.

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2. Formulation of the problem.

To begin, we establish a few notational conventions. The letter M denotes a fixed C^∞ manifold of dimension $2n+c$. When indices are used, the Einstein summation convention applies. Indices such as i, j, k range from 1 to n; indices such as α, β, γ range from 1 to c.

We need little from the theory of CR structures beyond the standard definitions of a CR structure and its Levi form. (For background, see [**B**] or [**J**].)

Definition 2.1. A *CR structure of dimension n and codimension c* is a rank n complex subbundle $\mathcal{H} \subset \mathbb{C} \otimes TM$ with the following properties:

(a) $\mathcal{H} \cap \overline{\mathcal{H}}$ is the zero subbundle;

(b) $[\mathcal{H}, \mathcal{H}] \subset \mathcal{H}$.

From here on, $\mathcal{H}$ denotes a fixed CR structure of dimension n and codimension c.

Definition 2.2. The *Levi form* of $\mathcal{H}$ is the bundle map

$$L\colon \mathcal{H} \times \mathcal{H} \to \mathbb{C} \otimes TM/\left(\mathcal{H} \oplus \overline{\mathcal{H}}\right)$$

defined by

$$L(X,Y) = i\pi\left[\overline{X}, Y\right],$$

for all sections X and Y of $\mathcal{H}$, where $\pi\colon \mathbb{C}\otimes TM \to \mathbb{C}\otimes TM/(\mathcal{H}\oplus\overline{\mathcal{H}})$ is the natural projection. (It is easy to verify that L is well-defined.)

For the purpose of studying the equivalence problem for CR structures, it is convenient to recast these definitions in terms of G-structures. (For background on G-structures, see [**G**] or [**S**].) The first step in this process is the construction of the defining structure bundle consisting of suitably adapted coframes.

Definition 2.3. Let $p \in M$. An adapted coframe at p is a frame $(\theta^1, \theta^2, \dots, \theta^c; \omega^1, \omega^2, \dots, \omega^n; \omega^{\overline{1}}, \omega^{\overline{2}}, \dots, \omega^{\overline{n}})$ of the complex cotangent bundle of M at p, where

(a) each form θ^α is real;

(b) $\omega^{\overline{i}} = \overline{\omega}^i$;

(c) the $(n+c)$-tuple $(\theta^1, \theta^2, \dots, \theta^c; \omega^{\overline{1}}, \omega^{\overline{2}}, \dots, \omega^{\overline{n}})$ frames the annihilator of $\mathcal{H}_p$ (and consequently $(\omega^1, \omega^2, \dots, \omega^n)$ frames the dual of $\mathcal{H}_p$).

Since the adapted coframe $(\theta^1, \theta^2, \dots, \theta^c; \omega^1, \omega^2, \dots, \omega^n; \omega^{\overline{1}}, \omega^{\overline{2}}, \dots, \omega^{\overline{n}})$ is determined by the $\mathbb{R}^c$-valued 1-form $\theta = (\theta^1, \theta^2, \dots, \theta^c)$ and the $\mathbb{C}^n$-valued 1-form $\omega = (\omega^1, \omega^2, \dots, \omega^n)$, we denote it simply as $\langle\theta;\omega\rangle_p$.

If $\langle\theta;\omega\rangle_p$ and $\langle\theta';\omega'\rangle_p$ are two adapted coframes at the same point p, then there exist unique matrices $P \in GL(c,\mathbb{R})$, $Q \in GL(n,\mathbb{C})$ and $v \in \mathrm{Hom}(\mathbb{R}^c, \mathbb{C}^n)$ such that

$$\begin{aligned} \theta' &= P\theta \\ \omega' &= Q\omega + v\theta. \end{aligned} \tag{2.1}$$

Therefore, the set of all adapted coframes constitutes a reduction of the bundle of complex coframes of M to a subbundle $\mathcal{S}$ with structure group S, where S denotes the group of all matrices in $GL(2n+c,\mathbb{C})$ with block form

$$\mathrm{Block}\,(P,Q,v) = \begin{pmatrix} P & 0 & 0 \\ v & Q & 0 \\ \overline{v} & 0 & \overline{Q} \end{pmatrix}.$$

We call $\mathcal{S}$ the defining structural bundle of the CR structure $\mathcal{H}$.

A section of $\mathcal{S}$ defined on an open subset $U \subset M$ is called an adapted moving coframe and denoted as $\langle\theta;\omega\rangle_U$, or simply $\langle\theta;\omega\rangle$ when convenient. The exterior derivatives of θ and ω can be expanded as follows:

$$\begin{aligned} d\theta^\alpha &= ih^\alpha_{j\overline{k}}\omega^j \wedge \omega^{\overline{k}} + a^\alpha_{\beta j}\theta^\beta \wedge \omega^j + a^\alpha_{\beta\overline{j}}\theta^\beta \wedge \omega^{\overline{j}} \\ &\quad + b^\alpha_{\beta\gamma}\theta^\beta \wedge \theta^\gamma + s^\alpha_{jk}\omega^i \wedge \omega^k + s^\alpha_{\overline{jk}}\omega^{\overline{j}} \wedge \omega^{\overline{k}} \\ d\omega^i &= c^i_{jk}\omega^j \wedge \omega^k + d^i_{j\overline{k}}\omega^j \wedge \omega^{\overline{k}} + t^i_{\overline{jk}}\omega^{\overline{j}} \wedge \omega^{\overline{k}} \\ &\quad + e^i_{\alpha j}\theta^\alpha \wedge \omega^j + f^i_{\alpha\overline{j}}\theta^\alpha \wedge \omega^{\overline{j}} + g^i_{\beta\gamma}\theta^\beta \wedge \theta^\gamma \end{aligned} \tag{2.2}$$

where $b^\alpha_{\gamma\beta} = -b^\alpha_{\beta\gamma}$, $s^\alpha_{kj} = -s^\alpha_{jk}$, $c^i_{kj} = -c^i_{jk}$, $t^i_{\overline{kj}} = -t^i_{\overline{jk}}$, and $g^i_{\gamma\beta} = -g^i_{\beta\gamma}$, and moreover, since θ^α is real, $a^\alpha_{\beta\overline{j}} = \overline{a^\alpha_{\beta j}}$, $s^\alpha_{\overline{jk}} = \overline{s^\alpha_{jk}}$ and $h^\alpha_{k\overline{j}} = \overline{h^\alpha_{j\overline{k}}}$.

Since $\mathcal{H}$ is determined by the vanishing of the forms θ and ω, the integrability condition $[\mathcal{H},\mathcal{H}] \subset \mathcal{H}$ is equivalent to the requirement that the coefficients $s^\alpha_{\overline{jk}}$ and $t^i_{\overline{jk}}$ vanish identically. (By conjugation, s^α_{jk} vanishes as well.) Moreover, the coefficients $h^\alpha_{j\overline{k}}$ express the Levi form in terms of coordinates.

If $\langle\theta';\omega'\rangle_U$ is another moving coframe on U, related to $\langle\theta;\omega\rangle_U$ by the equations

$$\begin{aligned} \theta' &= P\theta \\ \omega' &= Q\omega + v\theta, \end{aligned}$$

where P, Q and v now matrix-valued functions on U, then

$$h'^\alpha_{j\overline{k}} = P^\alpha_\beta \left(Q^{-1}\right)^u_j \left(Q^{-1}\right)^{\overline{v}}_{\overline{k}} h^\beta_{u\overline{v}} \tag{2.3}$$

where, of course, $(Q^{-1})^{\overline{v}}_{\overline{k}} = (\overline{Q}^{-1})^v_k$. Motivated by this transformation law, we denote the real vector space of c-tuples of $n \times n$ hermitian matrices by $\mathrm{Herm}(n,c)$ (or simply Herm, when n and c are understood) and define an action of the group $G = GL(c,\mathbb{R}) \times GL(n,\mathbb{C})$ on Herm as follows: If $h \in$ Herm has components $h^\alpha_{j\overline{k}}$ then $h' = (P,Q)\cdot h$ has components $h'^\alpha_{j\overline{k}} = P^\alpha_\beta(Q^{-1})^u_j(Q^{-1})^{\overline{v}}_{\overline{k}}h^\beta_{u\overline{v}}$. The previous computations show that independently of choice of moving coframe, the Levi form determines a map $\mathcal{L}\colon M \to \mathrm{Herm}/G$, which we call the *Levi map*: indeed, the moving coframes $\langle\theta;\omega\rangle_U$ and $\langle\theta';\omega'\rangle_U$ recast the Levi form as maps $L_{\langle\theta;\omega\rangle}$ and $L_{\langle\theta';\omega'\rangle}$ from U to Herm sending the point p to the c-tuples

$$\left(h^1_{j\overline{k}}(p),\ldots,h^c_{j\overline{k}}(p)\right) \quad \text{and} \quad \left(h'^1_{j\overline{k}}(p),\ldots,h'^c_{j\overline{k}}(p)\right),$$

and the action of G on Herm has been defined so as to ensure that these c-tuples lie in the same orbit.

(It is worth noting that the original definition of the Levi form is already independent of coordinates. However, the domain and range of the Levi form vary from point to point on M. In order to be able to compare the Levi forms at different points, one is led as above to the Herm$/G$-valued Levi map.)

The following theorem, which asserts the existence of a natural smooth structure on an open subset of Herm$/G$, is foundational for the rest of the paper. A self-contained proof using techniques from the classical invariant theory of complex projective hypersurfaces is given below in Section 5 and Section 6.

Theorem 2.4. *Let $G = GL(c, \mathbb{R}) \times GL(n, \mathbb{C})$, let $K \subset G$ be the subgroup consisting of all pairs $(|z|^2 I_c, zI_n)$ for z in the complex multiplicative group $\mathbb{C}^*$, and suppose that $c > 2$ and $n > c^2$. There exists a non-empty G-invariant open subset $Z^\# \subset \mathrm{Herm}(n, c)$ whose image Z by the projection $\rho\colon \mathrm{Herm} \to \mathrm{Herm}/G$ can be given a smooth structure in such a way that $Z^\# \to Z$ is a principal bundle with structure group G/K.*

Remark 2.5. The sets $Z^\#$ and Z can be described explicitly in terms of matrices and their determinants; in particular, every point in Z is the image of some point in $Z^\#$ whose component matrices are invertible. This explicit description is given below in Theorem 6.1, an expanded version of Theorem 2.4. The proof is long and intricate (see Sections 5 and 6), but uses no highly specialized results; the restriction on n, required by this proof, can probably be relaxed considerably, but we will need the assumption that $n > 1$ in the proof of Theorem 4.1.

Definition 2.6. An open set $\mathcal{U} \subset Z$ is *tractable* if it admits a section $\sigma\colon \mathcal{U} \to Z^\#$ whose component matrices are invertible. A fixed choice of such a section σ is called a *canonical section*, its image $\sigma(\mathcal{U}) \subset p^{-1}(\mathcal{U}) \subset$ Herm is called the *canonical slice*, and an element of the canonical slice is called a *canonical vector-valued hermitian form* (*canonical form* for short).

Since $Z^\# \to Z$ is a principal bundle, every sufficiently small open subset of Z admits a section, so, in view of Remark 2.5(b), every point of Z belongs to some tractable open subset.

Definition 2.7. Let $\mathcal{U}$ be a tractable set. A CR structure is *tractable of type $\mathcal{U}$* if its Levi map $\mathcal{L}\colon M \to \mathrm{Herm}/G$ is valued in $\mathcal{U}$.

More informally, a CR structure is tractable if its Levi form does not vary too much. The existence of tractable CR structures of any given type is established by the familiar examples of quadrics, which have constant Levi maps (see [B]); deformations of quadrics provide numerous examples of tractable CR structures with varying Levi maps.

3. Reducing the structure bundle.

From here on, $\mathcal{U}$ denotes a fixed tractable set, σ denotes a fixed canonical section, and the CR structure $\mathcal{H}$ on M is assumed to be tractable of type $\mathcal{U}$. The goal of this section is to prove the following result.

Theorem 3.1. *The structure bundle $\mathcal{S}$ on the tractable* CR *structure $\mathcal{H}$ of type $\mathcal{U}$ can be reduced by means of the canonical section σ to a subbundle $\mathcal{R}$ with group $R \subset S$ consisting of all matrices of the form* $\mathrm{Block}(P, Q, v)$, *where $v = 0$ and $(P, Q) \in K$ — that is, $P = |z|^2 I_c$ and $Q = zI_n$ for some z in the complex multiplicative group $\mathbb{C}^*$.*

As a first step, we carry out a preliminary reduction.

Lemma 3.2. *The structure bundle $\mathcal{S}$ of the tractable* CR *structure $\mathcal{H}$ of type $\mathcal{U}$ can be reduced by means of the canonical section σ to a subbundle $\mathcal{T}$ with group $T \subset S$ consisting of all matrices of the form* $\mathrm{Block}(P, Q, v)$, *where $(P, Q) \in K$.*

As a mnemonic aid, note that three bundles in question are $\mathcal{S}$ for structural, $\mathcal{R}$ for reduced, and $\mathcal{T}$ for temporary (since we use $\mathcal{T}$ only as a step in obtaining $\mathcal{R}$), and that the corresponding structural groups are S, R and T.

The reductions from $\mathcal{S}$ to $\mathcal{T}$ and from $\mathcal{T}$ to $\mathcal{R}$ both follow from an analysis of the tautology forms Θ, Ω and $\overline{\Omega}$ defined on $\mathcal{S}$ as follows: If X is a vector tangent to $\mathcal{S}$ at the point $\langle\theta;\omega\rangle_p$, then

$$\Theta(X) = \theta(\pi_* X), \qquad \Omega(X) = \omega(\pi_* X), \quad \text{and} \quad \overline{\Omega}(X) = \overline{\omega}(\pi_* X),$$

where $\pi \colon \mathcal{S} \to M$ is the natural projection.

We recall the moving coframe $\langle\theta;\omega\rangle_U$ from the preceding section, and reproduce its structure Equations (2.2), suppressing the terms we now know to be zero:

(3.1)

$$\begin{aligned} d\theta^\alpha &= i h^\alpha_{j\bar{k}} \omega^j \wedge \omega^{\bar{k}} + a^\alpha_{\beta j} \theta^\beta \wedge \omega^j + a^\alpha_{\beta \bar{j}} \theta^\beta \wedge \omega^{\bar{j}} + b^\alpha_{\beta\gamma} \theta^\beta \wedge \theta^\gamma \\ d\omega^i &= c^i_{jk} \omega^j \wedge \omega^k + d^i_{j\bar{k}} \omega^j \wedge \omega^{\bar{k}} + e^i_{\alpha j} \theta^\alpha \wedge \omega^j + f^i_{\alpha\bar{j}} \theta^\alpha \wedge \omega^{\bar{j}} + g^i_{\beta\gamma} \theta^\beta \wedge \theta^\gamma \end{aligned}$$

where $b^\alpha_{\gamma\beta} = -b^\alpha_{\beta\gamma}$, $c^i_{kj} = -c^i_{jk}$, $g^i_{\gamma\beta} = -g^i_{\beta\gamma}$, $a^\alpha_{\beta\bar{j}} = \overline{a^\alpha_{\beta j}}$ and $h^\alpha_{k\bar{j}} = \overline{h^\alpha_{j\bar{k}}}$. By standard abuse of notation, we denote these forms and functions defined on U and their pullbacks to $\mathcal{S}_U$ by the same symbols.

The moving coframe $\langle\theta;\omega\rangle_U$ determines an identification of $\mathcal{S}_U$ with $U\times S$: The coframe $\langle\theta';\omega'\rangle_p$ related to $\langle\theta;\omega\rangle_p$ by Equation (2.1) is identified with the ordered pair $(p, \mathrm{Block}(P,Q,v))$. Therefore, there are well-defined coordinate functions

$$P\colon \mathcal{S}_U \to GL(c,\mathbb{R}), \quad Q\colon \mathcal{S}_U \to GL(n,\mathbb{C}) \quad \text{and} \quad v\colon \mathcal{S}_U \to \mathrm{Hom}(\mathbb{R}^c,\mathbb{C}^n).$$

In terms of these functions, on $\mathcal{S}_U$ the tautology forms Θ and Ω are related to the pullbacks θ and ω as follows:

$$\begin{aligned} \Theta &= P\theta \\ \Omega &= Q\omega + v\theta. \end{aligned} \tag{3.2}$$

Differentiation of the first of these equations shows that

$$d\Theta = dP\wedge\theta + Pd\theta = \left(dP\cdot P^{-1}\right)\wedge\Theta + Pd\theta; \tag{3.3}$$

routine computations using Equations (3.1)-(3.3) show that

$$\begin{aligned} d\Theta^\alpha = \left(dP\cdot P^{-1}\right)^\alpha_\beta \wedge\Theta^\beta + iH^\alpha_{j\overline{k}}\Omega^j\wedge\Omega^{\overline{k}} \\ + A^\alpha_{\beta j}\Theta^\beta\wedge\Omega^j + A^\alpha_{\beta\overline{j}}\Theta^\beta\wedge\Omega^{\overline{j}} + B^\alpha_{\beta\gamma}\Theta^\beta\wedge\Theta^\gamma \end{aligned} \tag{3.4}$$

where $H^\alpha_{j\overline{k}}$, $A^\alpha_{\beta j}$, $A^\alpha_{\beta\overline{j}}$ and $B^\alpha_{\beta\gamma}$ are functions on $\mathcal{S}_U$ that can be written explicitly in terms of P, Q, v, $h^\alpha_{j\overline{k}}$, $a^\alpha_{\beta j}$, $a^\alpha_{\beta\overline{j}}$ and $b^\alpha_{\beta\gamma}$. For instance,

$$H^\alpha_{j\overline{k}} = P^\alpha_\beta \left(Q^{-1}\right)^u_j \left(Q^{-1}\right)^{\overline{v}}_{\overline{k}} h^\beta_{u\overline{v}}, \tag{3.5}$$

where, as before, $(Q^{-1})^{\overline{v}}_{\overline{k}} = (\overline{Q}^{-1})^v_k$.

Repeating this process, starting with the moving coframe $\langle\theta';\omega'\rangle_U$, we obtain new coordinate functions

$$P'\colon \mathcal{S}_U \to GL(c,\mathbb{R}), \quad Q'\colon \mathcal{S}_U \to GL(n,\mathbb{C}), \quad \text{and} \quad v'\colon \mathcal{S}_U \to \mathrm{Hom}(\mathbb{R}^c,\mathbb{C}^n)$$

and find that

$$\begin{aligned} d\Theta^\alpha = \left(dP'\cdot P'^{-1}\right)^\alpha_\beta \wedge\Theta^\beta + iH'^\alpha_{j\overline{k}}\Omega^j\wedge\Omega^{\overline{k}} \\ + A'^\alpha_{\beta j}\Theta^\beta\wedge\Omega^j + A'^\alpha_{\beta\overline{j}}\Theta^\beta\wedge\Omega^{\overline{j}} + B'^\alpha_{\beta\gamma}\Theta^\beta\wedge\Theta^\gamma. \end{aligned} \tag{3.6}$$

Elimination of $d\Theta^\alpha$ from Equations (3.4) and (3.6) yields the equation

$$\begin{aligned} 0 = &\left(dP\cdot P^{-1} - dP'\cdot P'^{-1}\right)^\alpha_\beta \wedge\Theta^\beta \\ &+ i\left(H^\alpha_{j\overline{k}} - H'^\alpha_{j\overline{k}}\right)\Omega^j\wedge\Omega^{\overline{k}} + \left(A^\alpha_{\beta j} - A'^\alpha_{\beta j}\right)\Theta^\beta\wedge\Omega^j \\ &+ \left(A^\alpha_{\beta\overline{j}} - A'^\alpha_{\beta\overline{j}}\right)\Theta^\beta\wedge\Omega^{\overline{j}} + \left(B^\alpha_{\beta\gamma} - B'^\alpha_{\beta\gamma}\right)\Theta^\beta\wedge\Theta^\gamma \end{aligned} \tag{3.7}$$

which plays an important role below.

Proof of Lemma 3.2. The functions $H^{\alpha}_{j\bar{k}}$ determine a map $H\colon \mathcal{S}_U \to$Herm, which by assumption of tractability is valued in the pre-image $\rho^{-1}(\mathcal{U})$. By Equation (3.5), H maps each fibre of $\mathcal{S}_U$ onto an entire orbit in $\rho^{-1}(\mathcal{U})$, so each fibre of $\mathcal{S}_U$ contains points that are mapped to canonical forms in Herm. In fact, routine verifications show that these points constitute a reduction of $\mathcal{S}_U$ to a subbundle $\mathcal{T}_U$ with structure group T. This reduction is independent of the initial choice of moving coframe, since by Equation (3.7), the functions $H^{\alpha}_{j\bar{k}}$ and $H'^{\alpha}_{j\bar{k}}$ determined by the moving coframes $\langle\theta;\omega\rangle_U$, and $\langle\theta';\omega'\rangle_U$, are equal. Therefore, if this construction is carried out for each set in some open cover of M, then the resulting locally defined subbundles piece together to yield a global subbundle $\mathcal{T} \subset \mathcal{S}$ with structure group T. $\square$

Proof of Theorem 3.1. We now restrict the tautology forms to the subbundle $\mathcal{T}$, and take $\langle\theta;\omega\rangle_U$ to be a section of $\mathcal{T}_U$, which is equivalent to requiring that the map $h\colon U \to$Herm determined by the functions $h^{\alpha}_{j\bar{k}}$ in Equation (3.1) be valued in the canonical slice. Mimicking the previous argument, we use $\langle\theta;\omega\rangle_U$, to identify $\mathcal{T}_U$ with $U \times T$, thereby obtaining coordinate functions P, Q and v on $\mathcal{T}_U$. Since Block(P, Q, v) is valued in T, it follows that $P = |z|^2 I_c$ and $Q = zI_n$ for some function $z\colon \mathcal{T}_U \to \mathbb{C}^*$.

Using Equations (3.1)-(3.4) in this new setting, we find that

$$d\Theta^{\alpha} = d\log|z|^2 \wedge \Theta^{\alpha} + iH^{\alpha}_{j\bar{k}}\Omega^j \wedge \Omega^{\bar{k}} + A^{\alpha}_{\beta j}\Theta^{\beta} \wedge \Omega^j + A^{\alpha}_{\beta\bar{j}}\Theta^{\beta} \wedge \Omega^{\bar{j}} + B^{\alpha}_{\beta\gamma}\Theta^{\beta} \wedge \Theta^{\gamma} \tag{3.8}$$

where $A^{\alpha}_{\beta j}$, $A^{\alpha}_{\beta\bar{j}}$ and $B^{\alpha}_{\beta\gamma}$ are functions on $\mathcal{T}_U$ with the following properties:

(a) $A^{\alpha}_{\beta j}$ and $A^{\alpha}_{\beta\bar{j}}$ are conjugate;

(b) $B^{\alpha}_{\beta\gamma} = -B^{\alpha}_{\gamma\beta}$;

(c) $A^{\alpha}_{\beta j} = ih^{\alpha}_{j\bar{k}}v^{\bar{k}}_{\beta} + z^{-1}a^{\alpha}_{\beta j}$ (writing $v^{\bar{k}}_{\beta}$ for $\overline{v^{k}_{\beta}}$).

Again mimicking the previous argument, we repeat this process starting with the moving coframe $\langle\theta';\omega'\rangle_U$ and obtain new coordinates P', Q' and v' (with P' and Q' determined by a scalar function z') and a corresponding structure equation

$$d\Theta^{\alpha} = d\log|z'|^2 \wedge \Theta^{\alpha} + iH'^{\alpha}_{j\bar{k}}\Omega^j \wedge \Omega^{\bar{k}} + A'^{\alpha}_{\beta j}\Theta^{\beta} \wedge \Omega^j + A'^{\alpha}_{\beta\bar{j}}\Theta^{\beta} \wedge \Omega^{\bar{j}} + B'^{\alpha}_{\beta\gamma}\Theta^{\beta} \wedge \Theta^{\gamma}. \tag{3.9}$$

Elimination of $d\Theta^{\alpha}$ from (3.8) and (3.9) yield the following refinement of (3.7):

(3.10)
$$\begin{aligned} 0 &= \left(d\log|z|^2 - d\log|z'|^2\right)\wedge\Theta^\alpha \\ &\quad + i\left(H^\alpha_{j\bar k} - H'^\alpha_{j\bar k}\right)\Omega^j\wedge\Omega^{\bar k} + \left(A^\alpha_{\beta j} - A'^\alpha_{\beta j}\right)\Theta^\beta\wedge\Omega^j \\ &\quad + \left(A^\alpha_{\beta\bar j} - A'^\alpha_{\beta\bar j}\right)\Theta^\beta\wedge\Omega^{\bar j} + \left(B^\alpha_{\beta\gamma} - B'^\alpha_{\beta\gamma}\right)\Theta^\beta\wedge\Theta^\gamma. \end{aligned}$$

It follows immediately that $A^\alpha_{\beta j}$ equals $A'^\alpha_{\beta j}$, provided that α and β are distinct. In particular, the functions $A^1_{2j}, A^2_{3j}, \ldots, A^{c-1}_{cj}, A^c_{1j}$ are defined on $\mathcal{T}_U$, independently of the initial choice of moving coframe. Therefore, since by assumption of canonicity each matrix $h^\alpha_{j\bar k}$ is invertible, (c) implies that on the subset $\mathcal{R}\subset\mathcal{T}_U$ defined by equations $A^1_{2j}=0$, $A^2_{3j}=0,\ldots,$ $A^{c-1}_{cj}=0$, $A^c_{1j}=0$, for each $v^{\bar k}_\beta$ is a function of $h^\alpha_{j\bar k}$, $a^\alpha_{\beta j}$ and z. Consequently, $\mathcal{R}_U$ is a subbundle of $\mathcal{T}_U$ with structure group R. As with the previous reduction, it is possible to piece together such local subbundles to obtain a global subbundle $\mathcal{R}$ with structure group R. $\square$

4. The connection.

Straightforward computations show that the first prolongation of $\mathfrak{R}$, the Lie algebra of the group R, is trivial. Consequently, in light of the general theory of G-structures, it is reasonable to attempt to define an e-structure – that is, a global parallelism – on the bundle $\mathcal{R}$ by means of further analysis of the structure equations. The following theorem describes one such parallelism, which turns out to be a connection.

Theorem 4.1. *Let $\mathcal{R}$ be the R-bundle determined by the* CR *structure $\mathcal{H}$ (assumed tractable of type $\mathcal{U}$) and the canonical section σ, and let Θ and Ω be tautology forms on $\mathcal{R}$. For each vector v in the Lie algebra $\mathfrak{R}$, let v^* denote the associated vertical vector field on the bundle $\mathcal{R}$.*

There exist unique 1-forms Π^α_β and Γ^i_j and unique functions $H^\alpha_{j\bar k}$, $A^\alpha_{\beta j}$, $A^\alpha_{\beta\bar j}$, $B^\alpha_{\beta\gamma}$, C^i_{jk}, $D^i_{j\bar k}$, $E^i_{\alpha j}$, $F^i_{\alpha\bar j}$ and $G^i_{\beta\gamma}$, all defined on $\mathcal{R}$, that satisfy the following conditions:

(a) *the forms $\Pi, \Gamma, \Theta, \Omega$ and $\overline{\Omega}$ frame the bundle $\mathcal{R}$;*

(b) $\mathrm{Block}(\Pi,\Gamma,0)$ *is valued in the Lie algebra $\mathfrak{R}$, and maps v^* to v for each v in $\mathfrak{R}$;*

(c) $d\Theta^\alpha = \Pi^\alpha_\beta\wedge\Theta^\beta + iH^\alpha_{j\bar k}\Omega^j\wedge\Omega^{\bar k} + A^\alpha_{\beta j}\Theta^\beta\wedge\Omega^j + A^\alpha_{\beta\bar j}\Theta^\beta\wedge\Omega^{\bar j} + B^\alpha_{\beta\gamma}\Theta^\beta\wedge\Theta^\gamma$;

(d) $d\Omega^i = \Gamma^i_j\wedge\Omega^j + C^i_{jk}\Omega^j\wedge\Omega^k + D^i_{j\bar k}\Omega^j\wedge\Omega^{\bar k} + E^i_{\alpha j}\Theta^\alpha\wedge\Omega^j + F^i_{\alpha\bar j}\Theta^\alpha\wedge\Omega^{\bar j} + G^i_{\beta\gamma}\Theta^\beta\wedge\Theta^\gamma$;

(e) $B^\alpha_{\gamma\beta} = -B^\alpha_{\beta\gamma}$, $C^i_{kj} = -C^i_{jk}$, $G^i_{\gamma\beta} = -G^i_{\beta\gamma}$;

(f) $H^\alpha_{k\bar{j}} = \overline{H^\alpha_{j\bar{k}}}$, $A^\alpha_{\beta\bar{j}} = \overline{A^\alpha_{\beta j}}$, $B^\alpha_{\beta\gamma} = \overline{B^\alpha_{\beta\gamma}}$;

(g) *the map* $(H^1_{j\bar{k}}, \ldots, H^c_{j\bar{k}}) \colon \mathcal{R} \to \mathrm{Herm}$ *is valued in the canonical slice;*

(h) $A^1_{2j} = 0, A^2_{3j} = 0, \ldots, A^{c-1}_{cj} = 0, A^c_{1j} = 0$;

(i) *the contractions* $C^i_{ik}, D^i_{i\bar{k}}$ *and* $E^i_{\alpha i}$ *all vanish.*

Moreover, the $\mathfrak{R}$*-valued form* $\mathrm{Block}(\Pi, \Gamma, 0)$ *is a connection.*

Proof. Suppose that two sets of forms and functions, primed and unprimed, satisfy conditions (a)-(h). Then by (a) and (b), the $\mathfrak{R}$-valued 1-form $\mathrm{Block}(\Pi - \Pi', \Gamma - \Gamma', 0)$ can be expressed in terms of the tautology forms. More precisely,

$$\Pi - \Pi' = (\tau + \overline{\tau}) I_c \quad \text{and} \quad \Gamma - \Gamma' = \tau I_n,$$

where

$$\tau = \lambda_\gamma \Theta^\gamma + \mu_j \Omega^j + \eta_{\bar{k}} \Omega^{\bar{k}}$$

and λ_γ, μ_j and $\eta_{\bar{k}}$ are complex functions defined on the bundle $\mathcal{R}$. The relations between the primed and unprimed functions can be computed from equations (c) and (d). In particular, we find that:

$$\begin{aligned} C^i_{jk} &= C'^i_{jk} - \frac{1}{2}\mu_k \delta^i_j + \frac{1}{2}\mu_j \delta^i_k; \\ D^i_{j\bar{k}} &= D'^i_{j\bar{k}} - \eta_{\bar{k}} \delta^i_j; \\ E^i_{\alpha j} &= E'^i_{\alpha j} + \lambda_\alpha \delta^i_j. \end{aligned}$$

Consequently, taking contradictions shows that

$$\begin{aligned} C^i_{ik} &= C'^i_{ik} + \frac{1}{2}(1-n)\mu_k; \\ D^i_{i\bar{k}} &= D'^i_{i\bar{k}} - n\eta_{\bar{k}}; \\ E^i_{\alpha i} &= E'^i_{\alpha i} + n\lambda_\alpha. \end{aligned} \tag{4.1}$$

If condition (i) is met by both the primed and unprimed functions, equations (4.1) imply that $\tau = 0$ (we use here the assumption that $n > 1$, which was part of the definition of tractability), and the uniqueness assertion of the theorem follows. Because of this uniqueness, it suffices to prove existence locally. Moreover, Equations (4.1) show that independently of any further assumptions on the primed functions, there is a unique choice of τ that yields unprimed functions satisfying condition (i). Therefore, it suffices to prove local existence of primed forms and functions satisfying conditions (a)-(h).

To do this, take $\langle\theta;\omega\rangle_U$ to be a local section of $\mathcal{R}$ and use it to identify $\mathcal{R}_U$ with $U \times R$, thereby obtaining coordinate functions P and Q on $\mathcal{R}_U$, with $P = |z|^2 I_c$ and $Q = zI_n$ for some function $z\colon \mathcal{R}_U \to \mathbb{C}^*$. In terms of these coordinates, $\Theta = P\theta$ and $\Omega = Q\omega$. Differentiating these equations and using Equation (3.1), we find that conditions (a)-(f) can be met by taking $\Pi' = (d\log|z|^2)I_c$ and $\Gamma' = (z^{-1}dz)I_n$. The fact that conditions (g) and (h) are satisfied is clear from the proof of Theorem 3.1.

Finally, to prove that $\mathrm{Block}(\Pi, \Gamma, 0)$ is a connection, we must simply check that it transforms properly under the action of the structure group R. The requisite computations are routine. □

Corollary 4.2. *The automorphisms of a tractable* CR *structure constitute a Lie group.*

Proof. This follows from Theorem 4.1 and a well-known theorem of Kobayashi (see, e.g., [**K**, p. 15]) which states that the automorphisms of an e-structure constitute a Lie group. □

Corollary 4.2 extends a result of Tanaka [**T2**], applicable to CR structures with constant Levi map.

Corollary 4.3. *A tractable* CR *structure carries a canonical affine connection.*

Proof. The connection on $\mathcal{R}$ described in Theorem 4.1 extends to a connection on the full coframe bundle. □

The prime significance of Corollary 4.3 is that it introduces standard geometric constructs, such as covariant derivatives and geodesics, into the study of CR geometry. It is worth noting as confirmation of the appropriateness of the connection that in the case of quadrics – the simplest nontrivial CR structure – the torsion consists of the Levi form, and the curvature vanishes, as one would expect.

Corollary 4.4. *The complexified tangent bundle of a tractable* CR *manifold decomposes as a direct sum of $2n + c$ complex line bundles; the real tangent bundle decomposes as a direct sum of c real line bundles and n real plane bundles with complex structure.*

Proof. This is a simple translation of Theorem 3.1 from the language of coframes to the language of distributions. □

Corollary 4.4 implies that there are topological obstructions to tractability. For instance, if M is compact, it cannot carry a tractable CR structure unless its characteristic classes satisfy various identities. (For earlier work on relations between compact real submanifolds of complex manifolds and characteristic classes, see [**L**] and [**W**].)

5. Proof of Theorem 2.4: Initial stage.

The argument needed to prove Theorem 2.4 is quite long. In this section, we formulate one part of this argument as Theorem 5.1, a result which is of interest in its own right. In the following section we use this theorem to conclude the proof of Theorem 2.4. Our basic technical tool is the following well-known result in the theory of Lie group actions.

Theorem A. *Let $\Phi\colon H \times N \to N$ be a smooth action of the Lie group H on the manifold N. If this action is free and proper, then the quotient space N/H can be given a smooth structure compatible with the quotient topology in such a way that $N \to N/H$ is a principal bundle with structure group H.*

(Recall that the action Φ is free *if the isotropy group of each $x \in N$ is trivial and is* proper *if the following equivalent conditions hold:*

(1) *If $\{x_j\}$ and $\{h_j\}$ are sequences in N and H respectively such that both $\{x_j\}$ and $\{h_jx_j\}$ coverge in N, then there exists a subsequence of $\{h_j\}$ that converges in H.*
(2) *The map $\Psi\colon (h,x) \to (hx,x)$ is a proper map from $H \times N$ to $N \times N$ – i.e. the Ψ-pre-image of a compact subset of $N \times N$ is compact.)*

Proof. See [**AM**], Section 4.1, especially Exercise 4.1M. □

We begin by establishing some notation and definitions. Let $GL(c,\mathbb{R})$ act on the polynomial ring $\mathbb{R}[T] = \mathbb{R}[t_1,\dots,t_c]$ in the usual way: $Pf(T) = f(TP)$ for all $P \in GL(c,\mathbb{R})$ and $f \in \mathbb{R}[T]$, where we view T as a row vector $T = (t_1,\dots,t_c)$. For technical convenience, we extend this action to an action of G on $\mathbb{R}[T]$ as follows:

$$(A,P)f(T) = |\det A|^{-2} f(TP).$$

The homogeneous polynomials of a given degree j form a vector subspace $\mathbb{R}_j[T]$, invariant under the actions of both $GL(c,\mathbb{R})$ and G. Denote the projectivization of this space by $\mathbb{PR}_j[T]$, and observe that the actions of G and $GL(c,\mathbb{R})$ on $\mathbb{R}_j[T]$ induce actions of G, $GL(c,\mathbb{R})$ and $PGL(c,\mathbb{R})$ on $\mathbb{PR}_j[T]$.

Given a hermitian form $h \in \mathrm{Herm}$, viewed as c-tuple of hermitian matrices $(H^1, H^2,\dots,H^c)$, define a polynomial

$$D_h^{\#}(T) = \det(t_1H^1 + t_2H^2 \ldots + t_cH^c).$$

Clearly, this polynomial has real coefficients and is homogeneous of degree n unless it vanishes identically. Consequently, we have defined a map

$$D^{\#}\colon \operatorname{Herm} \to \mathbb{R}_n[T] \cup \{0\},$$

which in turn determines a map

$$D\colon \operatorname{Herm}/G \to (\mathbb{R}_n[T] \cup \{0\})\,/G.$$

The following assertions are easily verified.

(1) The space $X = (\mathbb{R}_n[T] \cup \{0\})/G$ is the disjoint union of the closed singleton $X_0 = \{0\}/G$ and the dense open subset $X_1 = \mathbb{R}_n[T]/G$.

(2) The subset X_1 is homeomorphic to the quotient $\mathbb{PR}_n[T]/PGL(c,\mathbb{R})$.

We now use Theorem A to give an open subset of X_1 a smooth structure.

Theorem 5.1. *Suppose that $c > 2$ and $n > c^2$, and let $Y^{\#}$ comprise all points $y \in \mathbb{PR}_n[T]$ with the following properties:*

(i) *the isotropy group of y in $PGL(c,\mathbb{R})$ is trivial;*

(ii) *y has no points of multiplicity $c+1$ – that is, if the polynomial $f(T) \in \mathbb{R}_n[T]$ projects to y, then there exists no point in $\mathbb{CP}^{c-1}$ at which $f(T)$ vanishes simultaneously with all of its partial derivatives of order c or less.*

Let Y denote the quotient space $Y^{\#}/PGL(c,\mathbb{R})$, endowed with the quotient topology. Then

(1) *$Y^{\#}$ is a $PGL(c,\mathbb{R})$-invariant open subset of $\mathbb{PR}_n[T]$;*

(2) *$PGL(c,\mathbb{R})$ acts freely and properly on $Y^{\#}$;*

(3) *the set of polynomials in $\mathbb{R}_n[T]$ that project into $Y^{\#}$ pulls back via $D^{\#}$ to a nonempty G-invariant open subset $\mathcal{O}$ of* Herm .

(4) *Y is an open subset of the spaces X_1 and X.*

(5) *Y can be made into a manifold in such a way that $Y^{\#} \to Y$ is a principal bundle with structure group $PGL(c,\mathbb{R})$.*

(6) *The D-image of* Herm *intersects Y non-trivially.*

Remark. If our goal were simply to prove the existence of some open subset of X with a smooth structure, we could probably appeal to known results from geometric invariant theory (see [**MF**]). However, in order for this subset to have any relevance to the invariant theory of hermitian forms and CR geometry, it must intersect the D-image of Herm/G non-trivially – hence the need for a more specific theorem and proof.

Proof. Statement (4) follows immediately from definitions; (5) is a consequence of Theorem A and (1) and (2); (6) follows from (3). The proof of the first three statements is more difficult, and requires several lemmas.

Lemma 5.2. *$Y^\#$ is not empty. In fact, there exists a hermitian form $h \in \mathcal{O}$ – that is, a hermitian form $h \in$ Herm such that the polynomial $D^\#(h) \in \mathbb{R}_n[T]$ projects into $Y^\#$.*

Proof. Recall that $c > 2$ and $n > c^2$. It follows immediately from the definition of $D^\#(h)$ that if h is any form with diagonal component matrices, then $D^\#(h)$ splits into linear factors. For a generic choice of h, this n-fold product of linear factors will have no symmetries or multiplicities, so it will project into $Y^\#$. □

For technical convenience, we consider $Y^\#$ as a subject of a larger set $\mathcal{Y}^\#$.

Definition 5.3. $\mathcal{Y}^\#$ is the set of all points $y \in \mathbb{PR}_n[T]$ with the following properties:

(i) The isotropy group of y in $PGL(c, \mathbb{R})$ is a Lie group of dimension zero (recall that when a Lie group acts smoothly, all isotropy groups are closed subgroups, and hence Lie groups);

(ii) y has no points of multiplicity $c + 1$.

Lemma 5.4. *$\mathcal{Y}^\#$ is a dense open $PGL(c, \mathbb{R})$-invariant subset of $\mathbb{PR}_n[T]$.*

Proof. Invariance is clear. By Lemma 5.2, $\mathcal{Y}^\#$ is not empty; therefore, it suffices to show that its complement $\mathfrak{C}$ is a projective subvariety.

Define two subsets of $\mathbb{PR}_n[T]$ as follows: $y \in \mathfrak{C}_a$ if the isotropy group of y in $PGL(c, \mathbb{R})$ has positive dimension, and $y \in \mathfrak{C}_b$ if y has a point of multiplicity $c + 1$. Clearly, $\mathfrak{C}$ is the union of $\mathfrak{C}_a$ and $\mathfrak{C}_b$, so it suffices to show that each of these sets is a projective subvariety.

Let $f(T) \in \mathbb{R}_n[T]$ represent the point $y \in \mathbb{PR}_n[T]$. It is easy to show that $y \in \mathfrak{C}_a$ if and only if the isotropy group of $f(T)$ in $GL(c, \mathbb{R})$ has positive dimension, or equivalently, if and only if there exists a non-constant curve $P\colon (-\varepsilon, \varepsilon) \to GL(c, \mathbb{R})$ such that $P(0) = I$ and $P(s)f(T) = f(T)$ for all $s \in (-\varepsilon, \varepsilon)$. Differentiating with respect to s and evaluating at 0, we obtain a homogeneous system of linear equations, whose coefficients are linear functions of the coefficients of $f(T)$. Clearly, $y \in \mathfrak{C}_a$ if and only if this system has a non-trivial solution, or equivalently, if and only if all minors of largest possible dimension r vanish. Since each minor is a homogeneous polynomial of degree r in the coefficients of $f(T)$, it follows that $\mathfrak{C}_a$ is a projective subvariety.

Again letting $f(T)$ represent y, we see that $y \in \mathfrak{C}_b$ if and only if $f(T)$ and all of its partial derivatives of order c or less vanish simultaneously at a non-zero point of $\mathbb{C}^c$. It follows from elimination theory (see Theorem 5.7A of chapter 1 of [**H**]) that $f(T)$ has this property if and only if its coefficients satisfy a system of homogeneous polynomial equations. Thus, $\mathfrak{C}_b$ is a projective subvariety. □

Lemma 5.5. *$PGL(c,\mathbb{R})$ acts properly on $\mathcal{Y}^{\#}$.*

The proof of this lemma is difficult. We postpone it temporarily, and go on to finish the proof of Theorem 5.1.

Proof of Theorem 5.5 (*conclusion*).

(1) By Lemma 5.5, $PGL(c,\mathbb{R})$ acts properly on $\mathcal{Y}^{\#}$, so all isotropy groups are compact; by definition of $\mathcal{Y}^{\#}$, all isotropy groups are Lie groups of dimension 0. Therefore, all isotropy groups are finite. It is easy to show that if a smooth Lie group action is proper and all of the isotropy groups are finite (but not necessarily of the same order), then the points with isotropy groups of least order form an open set. Therefore, the points in $\mathcal{Y}^{\#}$ with isotropy groups of least order form an open subset; by Lemma 5.2, this open subset is $Y^{\#}$. Invariance is obvious.

(2) By definition, $PGL(c,\mathbb{R})$ acts freely on $Y^{\#}$; by Lemma 5.5, $PGL(c,\mathbb{R})$ acts properly on $\mathcal{Y}^{\#}$, *a fortiori* on the subset $Y^{\#}$.

(3) This follows from Lemma 5.2 and the continuity of $D^{\#}$.

□

Thus (assuming Lemma 5.5) we have proved Theorem 5.1. In fact, our arguments also establish the following variation.

Corollary 5.6. *There exists a dense open $PGL(c,\mathbb{R})$-invariant subset $\mathcal{Y}^{\#}_{\circ} \subset \mathcal{Y}^{\#}$, whose quotient $\mathcal{Y}_{\circ} = \mathcal{Y}^{\#}_{\circ}/PGL(c,\mathbb{R})$ can be given a smooth structure in such a way that the projection $\mathcal{Y}^{\#}_{\circ} \to \mathcal{Y}_{\circ}$ is a submersion.*

Proof. By Lemma 5.5, $PGL(c,\mathbb{R})$ acts properly on $\mathcal{Y}^{\#}$. As noted in the proof of part 1 of Theorem 5.1, it follows from the definition of $\mathcal{Y}^{\#}$ and properness that all isotropy groups are finite. However, it is not hard to strengthen Theorem A to show that if a Lie group H acts smoothly and properly on a manifold N and all of the isotropy groups are finite (but not necessarily of the same order), then there exists a dense open subset of M/G which carries a natural smooth structure – i.e., a smooth structure with respect to which the corresponding projection is a submersion. □

The essential difference between this corollary and the first three statements of Theorem 5.1 is that $\mathcal{Y}_{\circ}$ is open and dense in $\mathbb{PR}_n[T]/PGL(c,\mathbb{R})$, while Y is merely open; we pay for this improvement by losing the clean bundle-theoretic property given in the fifth statement of theorem.

It remains to prove Lemma 5.5, which we obtain as an immediate corollary of the following two results.

Proposition 5.7. *Let D be the subgroup of $SL(c,\mathbb{R})$ comprising all diagonal matrices with positive eigenvalues. Consider the actions of $D, SL(c,\mathbb{R})$ and $PGL(c,\mathbb{R})$ on the space $\mathcal{Y}^{\#}$.*

(1) *$PGL(c,\mathbb{R})$ acts properly if $SL(c,\mathbb{R})$ acts properly.*

(2) *$SL(c,\mathbb{R})$ acts properly if D acts properly.*

Theorem 5.8. *The group D acts properly on the space $\mathcal{Y}^{\#}$.*

Remark 5.9. In proving these and subsequent results concerning proper actions, we shall repeatedly "pass to subsequences". Recall that the action of a Lie group H on a manifold N is proper if and only if for all sequences $\{x_j\}$ in N and $\{h_j\}$ in H such both $\{x_j\}$ and $\{h_j x_j\}$ converge in N, there exists a subsequence of $\{h_j\}$ that converges in H. Therefore, in order to prove properness it suffices to pass to subsequences $\{\tilde{x}_j\}$ and $\{\tilde{h}_j\}$ of $\{x_j\}$ and $\{h_j\}$ that satisfy specified conditions, and then to show that some subsequence of $\{\tilde{h}_j\}$ converges in H. In fact, without loss of generality, we may simplify notation by assuming that the original sequences satisfy these specified conditions.

Proof of Proposition 5.7. (1) Consider the following commutative diagram

$$\begin{array}{ccc} SL(c,\mathbb{R}) \times \mathcal{Y}^{\#} & \rightarrow & \mathcal{Y}^{\#} \times \mathcal{Y}^{\#} \\ \downarrow & \nearrow & \\ PGL(c,\mathbb{R}) \times \mathcal{Y}^{\#} & & \end{array}$$

The horizontal and diagonal maps are determined by the actions of $SL(c,\mathbb{R})$ and $PGL(c,\mathbb{R})$ on $\mathcal{Y}^{\#}$ (see Theorem A for details); the vertical map is the product of the natural projection and the identity. Suppose that $SL(c,\mathbb{R})$ acts properly. Then the horizontal map is proper (again, see Theorem A), and, since the vertical map is continuous, a simple diagram chase shows that the diagonal map is proper. Therefore, $PGL(c,\mathbb{R})$ acts properly on $\mathcal{Y}^{\#}$.

(2) Assuming that D acts properly on $\mathcal{Y}^{\#}$, let $\{y_j\}$ and $\{g_j\}$ be sequences in $\mathcal{Y}^{\#}$ and $SL(c,\mathbb{R})$ respectively, and suppose that $y_j \to y$ and $g_j y_j \to y'$ for some $y, y' \in \mathcal{Y}^{\#}$. By Lie theory (Cartan decomposition) or direct computation (see §9 of [**Z**], especially Exercise 3), each g_j can be factored as $B_j d_j C_j$, where $d_j \in D$ and B_j and C_j are in the proper orthogonal group. Since this group is compact, by passing to subsequences we may assume that $B_j \to B$ and $C_j \to C$ for some proper orthogonal matrices C and D. By assumption, $y_j \to y$ and $B_j d_j C_j y_j \to y'$; consequently, $C_j y_j \to Cy$ and $d_j C_j y_j \to B^{-1}y'$. Since D acts properly, by passing to subsequences we may assume that $\{d_j\}$ converges to some $d \in D$. Clearly, BdC is the limit of a subsequence of our original sequence $\{g_j\}$. Therefore, $SL(c,\mathbb{R})$ acts properly. $\square$

Before proving Theorem 5.8, we establish some conventions, and prove a simple lemma. A polynomial $f(T)$ is written in the usual multi-index notation as $f(T) = \sum f_I T^I$, where the summation extends over all multi-indices $(i_1, i_2, \dots, i_c)$ of non-negative integers that sum to n, and T^I stands for $(t_1)^{i_1}(t_2)^{i_2} \cdots (t_c)^{i_c}$. Given a matrix $d = \operatorname{diag}(d_1, d_2, \dots, d_c)$, we write d_I for $(d_1)^{i_1}(d_2)^{i_2} \cdots (d_c)^{i_c}$. Clearly, if d acts on $f(T)$, yielding a new polynomial $f^*(T)$, then $f^*_I = d_I f_I$ for each multi-index I. When dealing with sequences in D, we shall write the j-th element as $d(j) = \operatorname{diag}(d_1(j), d_2(j), \dots, d_c(j))$; for the sake of symmetry, we shall also write the j-th element of a sequence in $\mathcal{Y}^\#$ as $y(j)$ rather than y_j.

Lemma 5.10. *If $f(T) = \sum f_I T^I$ projects to a point $y \in \mathcal{Y}^\#$, then*

(1) *there exists a multi-index I with $i_1 \geq n - c$ such that $f_I \neq 0$;*

(2) *there exists a multi-index I with $i_c \geq n - c$ such that $f_I \neq 0$.*

Proof. (1) Suppose that $f_I = 0$ for every multi-index I with $i_1 \geq n - c$. Then $f(T)$ and all of its partial derivatives of order c or less vanish at the point $(1, 0, 0, \dots 0)$, which contradicts the fact that $y \in \mathcal{Y}^\#$.

(2) The proof is similar, using the point $(0, 0, \dots, 0, 1)$ instead of $(1, 0, 0, \dots 0)$. □

Proof of Theorem 5.8. Let $\{y(j)\}$ and $\{d(j)\}$ be sequences in $\mathcal{Y}^\#$ and D, and suppose that

$$(*) \qquad y(j) \to y \quad \text{and} \quad d(j)y(j) \to y'$$

for some $y, y' \in \mathcal{Y}^\#$. We must find a subsequence of $\{d(j)\}$ that converges in D.

By passing to subsequences, we may assume that for each $i = 1, 2, \dots, n$, the sequence $\{d(j)\}$ is positive and monotonic, with limit $d_i \in [0, \infty]$, and that for some permutation $(r_1, r_2, \dots, r_c)$ of $(1, 2, \dots, c)$

$$d_{r_1}(j) \geq d_{r_2}(j) \geq \cdots \geq d_{r_c}(j) > 0 \quad \text{for all} \quad j.$$

For convenience, we assume that this permutation is trivial; in general the argument differs only by notation. Thus,

$$(**) \qquad d_1(j) \geq d_2(j) \geq \cdots \geq d_c(j) > 0 \quad \text{for all} \quad j.$$

Since each matrix $d(j)$ has determinant 1, we see that either $\operatorname{diag}(d_1, d_2, \dots, d_c)$ is an element of D – in which case we are done – or $d_1 = \infty$ and $d_c = 0$. We assume the latter and derive a contradiction.

Represent the points $y(j)$ and y of $(*)$ by polynomials $\sum f_I(j) T^I$ and $\sum f_I T^I$ chosen so that $f_I(j) \to f_I$ for each multi-index I, and let $s_I(j) =$

$d_I(j)f_I(j)$. Clearly, the polynomial $\sum s_I(j)T^I$ represents the point $d(j)y(j)$; however, so does any non-zero scalar multiple. The remainder of our argument is based on this fact.

Passing to subsequences, we may assume that each sequence $\{s_I(j)\}$ is monotonic, with limit $s_I \in [-\infty, \infty]$. (This limit s_I need not be uniquely determined by the original sequence, but this does not matter – we simply make a choice.) Those multi-indices I for which $|s_I| = \infty$ constitute a set $\mathfrak{I}$.

Claim 1. Since $d_1 = \infty$ and $d_c = 0$,
(i) $d_I(j) \to \infty$ for all I with $i_1 \geq n - c$;
(ii) $d_I(j) \to 0$ for all I with $i_c \geq n - c$.

Claim 2. Since $d_1 = \infty$ and $d_c = 0$, by passing to subsequences we may assume that there exists a multi-index $I^* \in \mathfrak{I}$ such that
(i) $s_{I^*}(j) \neq 0$ for $j = 1, 2, \dots$;
(ii) each sequence $\{s_I(j)/s_{I^*}(j)\}$ converges to some $f'_I \in [-1, 1]$. (Clearly, $f'_{I^*} = 1$.)

Assuming the validity of these claims for now, we derive the desired contradiction. On the one hand, by part (i) of Claim 2, the polynomial $\sum(s_I(j)/s_{I^*}(j))T^I$ is a non-zero scalar multiple of $\sum s_I(j)T^I$ and consequently (as mentioned above) projects to the point $d(j)y(j)$. It follows from $(*)$ and part (ii) of Claim 2 that the polynomial $\sum f'_I T^I$ is non-zero and projects to the point y'. Since $y \in \mathcal{Y}^\#$, Lemma 5.10 implies that there exists some multi-index I with $i_c \geq n - c$ such that $f'_I \neq 0$. On the other hand, $f'_I = \lim s_I(j)/s_{I^*}(j)$. Since $I^* \in \mathfrak{I}$, it follows by definition that $\lim |s_{I^*}(j)| = \infty$. Moreover, $d_I(j) \to 0$ by part (ii) of Claim 1, and $f_I(j) \to f_I$ by definition, so $\{s_I(j)\} = \{d_I(j)f_I(j)\}$ converges to zero. Consequently, $f'_I = 0$ – a contradiction.

Thus, all that remains is to prove Claims 1 and 2.

Proof of Claim 1. For any positive integer j,

$$\begin{aligned} d_I(j) &= d_1(j)^{i_1} d_2(j)^{i_2} \cdots d_c(j)^{i_c} \\ &= d_1(j)^{i_1} [d_2(j)/d_c(j)]^{i_2} [d_3(j)/d_c(j)]^{i_3} \cdots [d_{c-1}(j)/d_c(j)]^{i_{c-1}} d_c(j)^{n-i_1}. \end{aligned}$$

(The second equality follows from the fact that $n = i_1 + i_2 + \cdots + i_c$.) Hence, by $(**)$, $d_I(j) \geq d_1(j)^{i_1} d_c(j)^{n-i_1}$. Since $\det d(j) = d_1(j)d_2(j) \cdots d_c(j) = 1$, it also follows from $(**)$ that $d_1(j)^{c-1}d_c(j) \geq 1$. Therefore, by writting

$$d_1(j)^{i_1} d_c(j)^{n-i_1} = \left[d_1(j)^{c-1} d_c(j)\right]^{n-i_1} d_1(j)^{i_1-(c-1)(n-i_1)},$$

we see that $d_I(j) \geq d_1(j)^{i_1-(c-1)(n-i_1)}$. An elementary computation now shows that if $i_1 \geq n - c$, then $i_1 - (c-1)(n - i_1) \geq n - c^2$. Since we have assumed at the onset that $n > c^2$, part (i) of the claim follows immediately. The proof of part (ii) is similar. $\square$

Proof of Claim 2. We begin by showing that $\mathfrak{I}$ is not empty. Indeed, since the polynomial $\sum f_I T^I$ represents the point $y \in \mathcal{Y}^\#$, Lemma 5.10 implies that there exists some index I with $i_1 \geq n-c$ and $f_I \neq 0$. But $s_I(j) = d_I(j) f_I(j)$ by definition, and $d_I(j) \to \infty$ by Claim 1. Consequently, $|s_I(j)| \to \infty$, so $I \in \mathfrak{I}$.

For each positive integer j, let $I(j)$ be the multi-index in $\mathfrak{I}$ defined as follows:

$$|s_{I(j)}(j)| \geq |s_I(j)| \quad \text{for all} \quad I \in \mathfrak{I}.$$

This yields a sequence $\{I(j)\}$ drawn from the finite set $\mathfrak{I}$, so at least one multi-index must occur infinitely often; let I^* be such a multi-index. Since $|s_{I^*}(j)| \to \infty$, by passing to a subsequence we may assume that $s_{I^*}(j) \neq 0$ for any j. Hence, (i) is proved. Moreover, by passing to subsequences we may assume that $|s_{I^*}(j)| \geq |s_I(j)|$ for all $I \in \mathfrak{I}$ and for all j. Equivalently, $|s_I(j)/s_{I^*}(j)| \leq 1$ for all $I \in \mathfrak{I}$ and for all j, so by passing to subsequences we may assume that $s_I(j)/s_{I^*}(j) \to f'_I \in [-1,1]$ for all $I \in \mathfrak{I}$. However, if $I \notin \mathfrak{I}$, then $\{s_I(j)\}$ is bounded and $|s_{I^*}(j)| \to \infty$, so $s_I(j)/s_{I^*}(j) \to 0$. Thus, (ii) is proved. $\square$

6. Proof of Theorem 2.4: Final stage.

In this section we rely heavily on the results and methods of Section 5. The following theorem is the elaboration of Theorem 2.4 promised in Section 2.

Theorem 6.1. *Suppose that $c > 2$ and $n > c^2$, and let $Z^\#$ be the set of all forms $h \in$ Herm with the following properties:*

(i) *The polynomial $D^\#(h)$ projects into $Y^\#$ (see Theorem 5.1 for definitions);*

(ii) *The isotropy group of h in G/K is trivial;*

(iii) *h has no non-zero null vectors – that is, if $h(x,x) = 0$ then $x = 0$.*

Let Z denote the quotient of $Z^\#$ by the action of the group G/K, endowed with the quotient topology. Then

(1) *$Z^\#$ is a non-empty G/K-invariant open subset of* Herm.

(2) *G/K acts freely and properly on $Z^\#$.*

(3) *Z is an open subset of* Herm $/G$.

(4) *Z can be made into a manifold in such a way that $Z^\# \to Z$ is a principal bundle with structure group G/K.*

Lemma 6.2. *$Z^\#$ is not empty.*

Proof. Let $\pi\colon \mathbb{R}_n[T] \to \mathbb{P}\mathbb{R}_n[T]$ be the natural projection. By Theorem 5.1, there is a G-invariant (and hence G/K-invariant) open subset $\mathcal{O} \subset \mathrm{Herm}$ that is mapped into $Y^\#$ by $\pi \circ D^\#$. Building on the proof of Lemma 5.2, we see that $\mathcal{O}$ contains a form h' whose first component matrix is the identity matrix, and whose remaining component matrices are diagonal matrices with distinct non-zero eigenvalues. Since $\mathcal{O}$ is open and $c > 2$, $\mathcal{O}$ contains a form h differing from h' only in that the first row and column of the third component matrix have no zero entries.

If $(A, P) \in G$ is in the isotropy group of h, then P is in the isotropy group of $\pi \circ D^\#(h)$. But $\pi \circ D^\#(h)$ is in $Y^\#$, so its isotropy group in $PGL(c, \mathbb{R})$ is trivial; consequently, $P = sI$ for some non-zero real number s. It follows from the definition of the action of G on Herm that $A^*H^iA = sH^i$ for $i = 1, 2, \ldots, c$. In particular, taking $i = 1$ and using the fact that $H^1 = I$, we see that $A^*A = sI$. Therefore $s > 0$. Let $B = (1/r)A$, where r is the positive square root of s. Clearly, $B^*H^iB = H^i$ for $i = 1, 2, \ldots, c$. Again taking $i = 1$, we see that B is unitary. Next, taking $i = 2$ and recalling that H^2 is a diagonal matrix with n distinct eigenvalues, we see that the standard principal axes theorem implies that $B = \mathrm{diag}(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n})$ where each $\theta_j \in [0, 2\pi)$. Finally, taking $i = 3$ and denoting the j-th entry of the first row of H^3 by q_j, we see that $e^{i(\theta_j - \theta_1)}q_j = q_j$. Since $q_j \neq 0$, it follows that $\theta_j = \theta_1$, so $B = e^{i\theta}I$ for some $\theta \in [0, 2\pi)$. Thus, $P = r^2 I$ and $A = re^{i\theta}I$, so $(A, P) \in K$. The fact that $h \in Z^\#$ is now obvious. □

Imitating the proof of Theorem 5.1, for technical convenience we now view $Z^\#$ as a subset of a larger set $\mathcal{Z}^\#$.

Definition 6.3. $\mathcal{Z}^\#$ is the set obtained by weakening condition (ii) of the definition of $Z^\#$ (see Theorem 6.1) as follows: The isotropy group of h in G/K is a Lie group of dimension zero.

Lemma 6.4. *$\mathcal{Z}^\#$ is a non-empty open G/K-invariant subset of* Herm*.*

Proof. Invariance is clear, and Lemma 6.2 shows that $\mathcal{Z}^\#$ is not empty. Note that $\mathcal{Z}^\# = \mathcal{O} \cap \mathcal{Q} \cap \mathcal{F}$, where $\mathcal{O}$ is the subset of Herm defined in the proof of Lemma 6.2, $\mathcal{Q}$ is the set of all forms with no non-zero vectors, and $\mathcal{F}$ is the set of forms whose isotropy groups in G/K are Lie groups of dimension zero. Since $\mathcal{O}$ is open, in order to show that $\mathcal{Z}^\#$ is open it suffices to show that $\mathcal{Q}$ and $\mathcal{F}$ are open, or equivalently, that their complements $\mathcal{Q}^c$ and $\mathcal{F}^c$ are closed.

Begin by putting an inner product on the vector space V. Let $\{h_j\}$ be a sequence in $\mathcal{Q}^c$ that converges to some $h \in$Herm. Since each form h_j has a non-zero null vector, we may choose a sequence $\{v_j\}$ in the unit ball of V such that $h_j(v_j) = 0$ for all j. Using the compactness of the unit ball and passing to subsequences, we may assume that $\{v_j\}$ converges to some unit vector v. By continuity, $h(v) = 0$, so $h \in \mathcal{Q}^c$. Thus $\mathcal{Q}^c$ is closed.

A form h belongs to $\mathcal{F}^c$ if and only if its isotropy group in G/K has positive dimension, or equivalently, if and only if the dimension of its isotropy group in G is greater than 2. The equations for the Lie algebra of this isotropy group in G constitute a homogeneous linear system, with coefficients that are linear functions of the coordinates of h. Thus, $h \in \mathcal{F}^c$ if and only if every minor of order $r-2$ vanishes, where r is the order of the largest possible minor of the system. The vanishing of these minors imposes polynomial conditions on h, so $\mathcal{F}^c$ is closed. □

Lemma 6.5. *Let $\mathcal{D} = \{A \in GL(n,\mathbb{C})|\ A$ is diagonal, with positive eigenvalues$\}$, and $SL^{\pm}(c,\mathbb{R}) = \{P \in GL(c,\mathbb{R})|\ \det P = \pm 1\}$.*

(1) *If $\mathcal{D} \times \{I\}$ acts properly on $\mathcal{Z}^{\#}$, then so does $GL(n,\mathbb{C}) \times \{I\}$.*
(2) *If $GL(n,\mathbb{C})\times\{I\}$ acts properly on $\mathcal{Z}^{\#}$, then so does $GL(n,\mathbb{C})\times SL(c,\mathbb{R})$.*
(3) *If $GL(n,\mathbb{C}) \times SL(c,\mathbb{R})$ acts properly on $\mathcal{Z}^{\#}$, then so does $GL(n,\mathbb{C}) \times SL^{\pm}(c,\mathbb{R})$.*
(4) *If $GL(n,\mathbb{C}) \times SL^{\pm}(c,\mathbb{R})$ acts properly on $\mathcal{Z}^{\#}$, then so does G/K.*

Proof. Let $\{h_j\}$ and $\{g_j\} = \{(A_j, P_j)\}$ be sequences in $\mathcal{Z}^{\#}$ and G respectively, and suppose that

$$(*) \qquad h_j \to h \ \text{ and } \ g_j h_j \to h' \ \text{ for some } \ h, h' \in \mathcal{Z}^{\#}.$$

(1) Imitate the proof of part 2 Theorem 5.8, using the unitary group in place of the proper orthogonal group.

(2) Suppose that each $P_j \in SL(c,\mathbb{R})$. Applying the map $\pi \circ D^{\#}$, it follows from $(*)$ that $\pi \circ D^{\#}(h_j) \to \pi \circ D^{\#}(h)$ and $P_j \pi \circ D^{\#}(h_j) \to \pi \circ D^{\#}(h')$. Since $\pi \circ D^{\#}$ maps $\mathcal{Z}^{\#}$ into $\mathcal{Y}^{\#}$, and $SL(c,\mathbb{R})$ acts properly on $\mathcal{Y}^{\#}$ (see Proposition 5.7 and Theorem 5.8), by passing to subsequences we may assume that $\{P_j\}$ converges to some $P \in SL(c,\mathbb{R})$. Hence, $(I,P_j)h_j \to (I,P)h$. Since $g_j h_j = (A_j, I)[(I,P_j)h_j]$, $(*)$ implies that $(A_j, I)[(I,P_j)h_j] \to h'$. Therefore, if $GL(n,\mathbb{C}) \times \{I\}$ acts properly, then by passing to subsequences we may assume that $\{A_j\}$ converges to some $A \in GL(n,\mathbb{C})$. Clearly, (A,P) is the limit of a subsequence of the original sequence $\{g_j\}$.

(3) Suppose that $\{P_j\} \subset SL^{\pm}(c,\mathbb{R})$ and let $J = \mathrm{diag}(-1,1,1,\dots,1)$. By passing to subsequences we may assume that either $\{P_j\} \subset SL(c,\mathbb{R})$ or $\{P_j J\} \subset SL(c,\mathbb{R})$. In the former case the result is trivial, so assume the latter. Then $(*)$ implies that $(I,J)h_j \to (I,J)h$ and $(A_j, P_j J)[(I,J)h_j] \to h'$. If $GL(n,\mathbb{C})\times SL(c,\mathbb{R})$ acts properly, then by passing to subsequences we may

assume that $\{(A_j, P_jJ)\}$ converges to some $(A, P) \in GL(n, \mathbb{C}) \times SL(c, \mathbb{R})$, so $\{(A_j, P_j)\}$ converges to $(A, PJ) \in GL(n, \mathbb{C}) \times SL^{\pm}(c, \mathbb{R})$.
(4) Imitate the proof of Proposition 5.7, using the natural projection of $GL(n, \mathbb{C}) \times SL^{\pm}(c, \mathbb{R})$ onto G/K. □

Lemma 6.6. *$\mathcal{D} \times \{I\}$ acts properly on $\mathcal{Z}^{\#}$.*

Proof. Let $\{h(j)\} \to h$ in $\mathcal{Z}^{\#}$, let $\{A(j)\} = \{\mathrm{diag}(a_1(j), a_2(j), \dots, a_n(j))\}$ be a sequence in $\mathcal{D}$, and suppose that there exists $h' \in \mathcal{Z}^{\#}$ such that $(A(j), I)h(j) \to h'$. By passing to subsequences we may assume that for each $i = 1, 2, \dots, n$ the sequence $\{a_i(j)\}$ converges monotonically to some $a_i \in [0, \infty]$. It follows readily from the relevant definitions that $h'(e_i, e_i) = (a_i)^{-2}h(e_i, e_i)$ for each basis vector e_i of V. Since h and h' have no non-zero null vectors, neither $h'(e_i, e_i)$ nor $h(e_i, e_i)$ is the zero vector in W. It follows that each a_i must be finite and non-zero. Therefore, the matrix $A = \mathrm{diag}(a_1, a_2, \dots, a_n)$ is in $\mathcal{D}$, and (A, I) is the limit of a subsequence of our original sequence $\{(A(j), I)\}$. □

The following corollary is an immediate consequence of Lemmas 6.5 and 6.6.

Corollary 6.7. *G/K acts properly on $\mathcal{Z}^{\#}$.*

Proof of Theorem 6.1. (1) Since G/K acts properly on $\mathcal{Z}^{\#}$, all isotropy groups are compact; by definition of $\mathcal{Z}^{\#}$, all isotropy groups are zero-dimensional Lie groups. Therefore, all isotropy groups are finite. It follows from Proposition 5.7 that the points of $\mathcal{Z}^{\#}$ with smallest isotropy groups form an open subset; by Lemma 6.2 this subset is $Z^{\#}$. Invariance is obvious.
(2) By definition, G/K acts freely on $Z^{\#}$. Since G/K acts properly on $\mathcal{Z}^{\#}$, *a fortiori* it acts properly on the invariant subset $Z^{\#}$.
(3) and (4) follow from (1) and (2) and Theorem A in Section 5. □

7. Future directions.

We conclude with a few observations.

(1) The theory of tractable CR structures would be greatly enhanced by a fully developed theory of canonical forms. Ideally, one would like an analogue of Sylvester's Theorem of Inertia for vector-valued hermitian forms – that is, one would like a complete canonical form for vector-valued hermitian forms, just as one has a canonical diagonal form for scalar-valued hermitian forms. Unfortunatelly, it is not at all clear how to go about deriving such an analogue. However, more modest results might be within reach. For

instance, it is reasonable to conjecture that particular tractable subsets can be described by the non-vanishing of polynomial invariants of vector-valued forms, and explicit formulas for all such invariants are already available in [**GM**]. The whole subject is ripe for investigation.

(2) The space $X = (\mathbb{R}_n[T] \cup \{0\})/GL(c, \mathbb{R})$ considered in Section 5 has a complex analogue $X' = (\mathbb{C}_n[T] \cup \{0\})/GL(c, \mathbb{C})$. There is a natural map $\tau\colon X \to X'$ obtained by composing the inclusion $(\mathbb{R}_n[T] \cup \{0\})/GL(c, \mathbb{R}) \to (\mathbb{C}_n[T] \cup \{0\})/GL(c, \mathbb{R})$ with the projection $(\mathbb{C}_n[T] \cup \{0\})/GL(c, \mathbb{R}) \to (\mathbb{C}_n[T] \cup \{0\})/GL(c, \mathbb{C})$. Consequently, we can consider the composite $\tau \circ D$ rather than D. Clearly, in so doing we lose information. However, there are compensating advantages. For X' decomposes as a closed singleton and a dense open subset homeomorphic to $\mathbb{PC}_n[T]/PGL(c, \mathbb{C})$, and this latter space underlies classical invariant theory and algebraic geometry. Indeed, $\mathbb{PC}_n[T]/PGL(c, \mathbb{C})$ may be viewed invariant-theoretically as a space of complex polynomials modulo complex-linear change of variable, or algebro-geometrically as a spce of degree n hypersurfaces in $\mathbb{CP}^{c-1}$ modulo projective equivalence. Thus, via the composite $\tau \circ D$, classical results in invariant theory and algebraic geometry apply to CR geometry. (Many of these results could be applied by means of the more subtle map D, but those which utilize the hypothesis of an algebraically closed ground field can be applied by means of $\tau \circ D$ only.)

Pursuing the ties with algebraic geometry, we define the *associated variety* S_h of a hermitian form h $\in$Herm to be the image in $\mathbb{P}W^*$ of the subset $C_h \subset W^*$ given as follows:

$$w^* \in C_h \text{ if and only if}$$
$$\text{the complex -valued sesquilinear form } w^* \circ h \text{ is singular.}$$

Clearly, the covector $w^* = a_1 f^1 + a_2 f^2 \cdots + a_c f^c$ belongs to C_h if and only if the coordinate vector $(a_1, a_2, \dots, a_c) \in \mathbb{C}^n$ is a zero of the polynomial $D_h^{\#}(T)$. Therefore, S_h is a projective hypersurface unless $D_h^{\#}(T)$ vanishes identically, in which case S_h is the entire space $\mathbb{P}W^*$.

Working pointwise with the Levi form L of the CR manifold M, we obtain a subset C_L of the bundle $\mathbb{C} \otimes (TM/H)^*$ and its image S_L in the projectivization of this bundle. Via S_L, a great deal of classical algebraic geometry can be brought to bear on CR geometry without the intervention of a generalized hermitian invariant. For instance, for a generic point $p \in M$, the corresponding fibre of S_L is a projective hypersurface and can be decomposed into its smooth points and its singular points of various types. Since the self-conjugate points in the fibre constitute a real projective hypersurface, discrete invariants, based on topology, can also be assigned to each point p. This approach looks particularly promising when $c = 3$, since it facilitates the application of the theory of plane curves and the theory of Riemann surfaces to the study of codimension three CR structures. For instance, consideration of inflection points and Weirstrass points yields (local) sections of the bundle $\mathbb{P}(TM/H)^*$. As the dimension of W increases, for generic h the variety S_h is a hypersurface with singularities: these singularities lead to sections of $\mathbb{P}(TM/H)^*$ for higher-codimensional CR structures. It is important to note that the varieties S_h are linked with determinantal varieties, so specialized techniques are available for their study ([AC]) – indeed, our assertions of smoothness and singularity are easily proved using such techniques.

(3) The invariant theory of vector-valued symmetric forms is quite similar to that of vector-valued hermitian forms (see [**GM**]). Therefore, comparison of standard invariants of the second fundamental form of a submanifold of a Riemannian manifold with Riemannian invariants obtained by the methods of this paper may suggest fruitful interpretations of CR invariants. (It may also shed some light on Riemannian geometry as well – at the very least, it will provide computable scalar-valued relative invariants.)

(4) Since the Levi map of a CR manifold is valued in the quotient space Herm$/G$, every insight into the nature of this space has ramifications for CR geometry. We have made a good beginning by identifying the smooth open subset Z (see Theorem 6.1). However, we still lack a clear understanding of the fine structure of this subset and of the global structure of the entire space.

(5) Finally, we note that although we have concentrated on local CR geometry, our methods are applicable to the study of the global structure of compact CR manifolds. It seems likely that there are significant links between the topology of the manifold and degeneracies and singularities in some of the canonical objects we have described.

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