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STANDARD INVARIANTS FOR CROSSED PRODUCTS INCLUSIONS OF FACTORS

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When $N \subset M$ is an inclusion of factors with finite index and a group G acts on $N \subset M$, we compare the standard invariants of $N \subset M$ and the crossed product inclusion $N \rtimes G \subset M \rtimes G$. The cases when G is a discrete group and when G is a locally compact abelian group are separately considered. Applying to a common crossed product decomposition, we obtain comparison results between the type II and type III standard invariants of an inclusion of type III factors.

0. Introduction.

Since Jones [**21**] initiated the index theory of type II_1 subfactors, a great progress has been made particularly on classification of hyperfinite type II_1 subfactors (see [**35**, **37-39**] and also [**20**, **23**] for example). For any inclusion $N \subset M$ of type II_1 factors with finite index, we have the Jones tower $N \subset M \subset M_1 \subset M_2 \subset \cdots$ and the sequence of higher relative commutants $\{M' \cap M_n \subset N' \cap M_n\}_{n \geq 0}$ constituting the canonical commuting squares [**39**]. The (dual) principal graph $\Gamma_{N,M}$ and the standard vector $\vec{s}$ are derived from $\{M' \cap M_n\}_{n \geq 0}$. The sequence $\{M' \cap M_n \subset N' \cap M_n\}_{n \geq 0}$ together with $(\Gamma_{N,M}, \vec{s}\,)$ is called the standard invariant (or the paragroup) of $N \subset M$ and denoted by $\mathcal{G}_{N,M}$. An axiomatic approach to paragroups was studied by Ocneanu [**35**]. Popa [**37-39**] proved that $\mathcal{G}_{N,M}$ is a complete invariant for the isomorphism class of $N \subset M$ if $N \subset M$ is a strongly amenable inclusion of hyperfinite type II_1 factors. Several characterizations of the strong amenability of $\mathcal{G}_{N,M}$ were established in [**39**]. (See Propositions 1.5 and 1.6 below.) Furthermore, it is known [**16**] that the subexponential growth of $(\Gamma_{N,M}, \vec{s}\,)$ implies the strong amenability of $\mathcal{G}_{N,M}$.

In the course of classifying hyperfinite type II_1 subfactors with small indices, it has been observed that symmetries on principal graphs (or paragroup symmetries) play a vital role typically in orbifold constructions (see [**4**, **8**, **10**, **20**, **23**]). A paragroup symmetry is an action of a (finite) group G on the principal graph and Ocneanu's connection made from an inclusion $N \subset M$. When this symmetry can extend to the subfactor level, the crossed product inclusion $N \rtimes G \subset M \rtimes G$ arises and its standard invariant is the quotient of $\mathcal{G}_{N,M}$ by G-symmetry; for example, the graphs D_{2n} and $D_n^{(1)}$ are obtained from $\mathbf{Z}_2$-symmetries on A_{4n-3} and $A_{2n-5}^{(1)}$, respectively (see [23, 20]). Thus, the crossed product construction is sometimes useful to get subfactors with new principal graphs from old ones. From this viewpoint, it would be important to compare the standard invariants of $N \subset M$ and $N \rtimes G \subset M \rtimes G$ in various aspects.

On the other hand, the notion of index was generalized to an arbitrary inclusion of factors (more precisely to a conditional expectation onto a subfactor) in several ways such as the Kosaki index [24], the best constant of Pimsner-Popa inequality [36], etc. Sector theory developed by Longo [32, 33] and Izumi [17] is quite useful particularly in the type III index theory. Similarly to the type II_1 case, the standard invariant $\mathcal{G}_{N,M}$ for an inclusion $N \subset M$ of type III factors can be defined by taking the Jones tower iterated by the minimal conditional expectation [12, 13, 32]. When $N \subset M$ admits a common decomposition, i.e. $(N \subset M) \cong (\tilde{N} \rtimes_\theta \mathbf{R} \subset \tilde{M} \rtimes_\theta \mathbf{R})$ or $(\tilde{N} \rtimes_\theta \mathbf{Z} \subset \tilde{M} \rtimes_\theta \mathbf{Z})$, $\tilde{N} \subset \tilde{M}$ being a type II_∞ inclusion with θ a trace-scaling action, we can consider the type II standard invariant $\mathcal{G}_{\tilde{N},\tilde{M}}$ besides the original type III invariant $\mathcal{G}_{N,M}$. In [18] (also [19]), the difference between the type II and type III principal graphs was characterized in terms of modular automorphisms by using the sector technique. This phenomenon is another typical example of graph change under taking crossed products. The coincidence of type II and type III graphs is necessary for a type III inclusion $N \subset M$ to split as $(N \subset M) \cong (B \otimes L \subset A \otimes L)$ with a type II_1 inclusion $B \subset A$. But it is also sufficient in some cases (see e.g. [29, 22]). A big progress in this direction is found in Popa's recent work [42].

The notion of strong outerness (or proper outerness) for automorphisms on an inclusion $N \subset M$ was introduced by Choda and Kosaki [6] (also [25, 26]) and independently by Popa [40]. This notion has turned out to play a fundamental role when we study group actions on $N \subset M$. Roughly speaking, an automorphism on $N \subset M$ is strongly outer if and only if it does not appear in the descendant sectors (or bimodules) of $N \subset M$. (See Proposition 1.11 below.)

Also, it should be mentioned that there are many close connections between subfactor theory and entropy theory; for instance, the relation between the index $[M\colon N]$ and the relative entropy $H(M|N)$ [36] (also [13]), the dynamical entropy of the canonical shift [3, 5], the characterization of the strong amenability of $\mathcal{G}_{N,M}$ in terms of the relative entropy [16, 39], etc.

The aim of this paper is to present a rather systematic treatment for the comparison between the standard invariants of $N \subset M$ and $(\tilde{N} \subset \tilde{M}) = (N \rtimes_\alpha G \subset M \rtimes_\alpha G)$. Here, $N \subset M$ is an arbitrary inclusion of factors with finite index and α is an action of a group G on $N \subset M$. Section 1 is a collection of definitions and preliminary results for later use. In Section 2, assume that G is a general discrete group and α is a strongly outer action of G on $N \subset M$. Then α extends to the Jones tower $M_1 \subset M_2 \subset \cdots$ subject to fixing the Jones projections and the Jones tower of $\tilde{N} \subset \tilde{M}$ is $\{\tilde{M}_n = M_n \rtimes_\alpha G\}$. The extended α gives rise to actions on the higher relative commutants $M' \cap M_n$ so that $(M' \cap M_n)^\alpha = \tilde{M}' \cap \tilde{M}_n$. That is, the standard invariant $\mathcal{G}_{\tilde{N},\tilde{M}}$ is the quotient of $\mathcal{G}_{N,M}$ by α-symmetry. This shows that the finite depth, the amenability, and the strong amenability of $\mathcal{G}_{\tilde{N},\tilde{M}}$ imply those of $\mathcal{G}_{N,M}$, respectively. Moreover, when G is a finite group, the growth and the (strong) amenability of $\mathcal{G}_{\tilde{N},\tilde{M}}$ and $\mathcal{G}_{N,M}$ are equivalent. Consequently, we show Winsløw's results [**51**] in a different way.

In Section 3, let G be a locally compact abelian group and α a continuous action of G on $N \subset M$. Assume that $\tilde{N} \subset \tilde{M}$ are factors. Then the dual action $\hat{\alpha}$ extends to the Jones tower $\{\tilde{M}_n = M_n \rtimes_\alpha G\}$ of $\tilde{N} \subset \tilde{M}$, so that $(M' \cap M_n)^\alpha = (\tilde{M}' \cap \tilde{M}_n)^{\hat{\alpha}}$. We can consider the growth of $\alpha|_{\cup_n M' \cap M_n}$ (the Loi part of α) taking the eigenvalues of $\alpha|_{M' \cap M_n}$ into account. It is shown that the growth of $\mathcal{G}_{\tilde{N},\tilde{M}}$ is controlled by those of $\mathcal{G}_{N,M}$ and $\alpha|_{\cup_n M' \cap M_n}$. For instance, if G is $\mathbf{R}$ (or $\mathbf{T}, \mathbf{Z}$) and α_g is strongly outer for any $g \in G \setminus \{e\}$, then $\mathcal{G}_{\tilde{N},\tilde{M}}$ has subexponential growth if and only if so do both $\mathcal{G}_{N,M}$ and $\alpha|_{\cup_n M' \cap M_n}$. In particular, we prove that $\alpha|_{\cup_n M' \cap M_n}$ has polynomial growth at most if $N \subset M$ has finite depth. Finally in Section 4, let $N \subset M$ be an inclusion of type III_1 or type III_λ factors. In case of type III_λ, $N \subset M$ is assumed to have a common discrete decomposition. Applying the results of Section 3 to the dual action θ on $\tilde{N} \subset \tilde{M}$, we obtain several assertions on the comparison between the type II and type III invariants of $N \subset M$. Also, some stability properties for a type III inclusion to have the same type II and type III invariants are given.

1. Preliminaries.

Throughout this paper, let $N \subset M$ be an inclusion of factors with finite index. We assume only that $N \subset M$ are σ-finite and not finite-dimensional. Let $E_0 \colon M \to N$ be the minimal conditional expectation with $\operatorname{Ind} E_0 = [M \colon N]_0$ $(= \lambda^{-1})$, the minimal index [**12**, **13**, **32**]. Let

$$\cdots \subset N_2 \subset N_1 \subset N_0 = N \subset M = M_0 \subset M_1 \subset M_2 \subset \cdots \tag{1.1}$$

be the Jones tower of tunnel and basic constructions with the Jones projections e_n $(\in M'_{n-2} \cap M_n)$ and the conditional expectations $E_n \colon M_n \to M_{n-1}$, $n \geq 1$, [**21**, **24**]. In the rest of this section, we collect definitions and preliminaries for later use.

1.1. Standard invariants. The faithful normalized trace (i.e. the so-called Markov trace) ϕ is defined on the tower of higher relative commutants $M' \cap M_n$ by

$$\phi|_{M'\cap M_n} = E_1 \circ \cdots \circ E_n|_{M'\cap M_n}, \qquad n \geq 1, \tag{1.2}$$

where the traciality of ϕ is due to [**28**]. In this paper, the trace considered on $\bigcup_n M' \cap M_n$ is always induced by the minimal conditional expectation, so that in case of a type II_1 inclusion we will assume that $N \subset M$ is *extremal* [**39**, 1.2.5], that is, the above ϕ comes from the trace on $\bigcup_n M_n$. Let R denote the finite von Neumann algebra generated by $\bigcup_n M' \cap M_n$ via the GNS representation with respect to ϕ. Also define the von Neumann subalgebras R_i of R by $R_i = \overline{\bigcup_n M_i' \cap M_n}$. Even when $N \subset M$ is not of type II_1, we define the *core* or the *standard part* of $N \subset M$ as $(N^{\mathrm{st}} \subset M^{\mathrm{st}}) = (\overline{\bigcup_n N_n' \cap N} \subset \overline{\bigcup_n N_n' \cap M})$ [**39**, 1.4.1], which is antiisomorphic to $R_1 \subset R$.

The *standard matrix* or *principal graph* $\Gamma_{N,M} = [a_{kl}]_{k\in K,\, l\in L}$ of $N \subset M$ is defined so that $[a_{kl}]_{k\in K_n,\, l\in L_n}$ is the inclusion matrix of $M' \cap M_{2n} \subset M' \cap M_{2n+1}$ and $[a_{kl}]^t_{k\in K_{n+1},\, l\in L_n}$ is that of $M' \cap M_{2n+1} \subset M' \cap M_{2n+2}$, where $K_0 = \{k_0 = *\} \subset K_1 \subset \cdots \subset K = \bigcup_n K_n$ and $L_0 \subset L_1 \subset \cdots \subset L = \bigcup_n L_n$. Let $(c_{n,k})_{k\in K_n}$ and $(q_{n,k})_{k\in K_n}$ be the dimension vector and the trace vector (i.e. the ϕ-values of minimal projections) of $M' \cap M_{2n}$. Since $q_{n+1,k} = \lambda q_{n,k}$ for all $k \in K_n$ and $n \geq 0$, the *standard eigenvector* $\vec{s} = (s_k)_{k\in K}$ is defined so that $s_{k_0} = 1$ and $(q_{n,k})_{k\in K_n} = (\lambda^n s_k)_{k\in K_n}$, $n \geq 0$, which satisfies $\Gamma_{N,M}\Gamma_{N,M}^t \vec{s} = \lambda^{-1}\vec{s}$. See [**37**, **39**] for details on the standard invariants of $N \subset M$. Following [**39**] we denote the standard invariants $(\Gamma_{N,M}, \vec{s}\,)$ of $N \subset M$ by $\mathcal{G}_{N,M}$.

The next proposition shown in [**16**] is useful to reduce problems on the standard invariants of general inclusions of factors to the type II_1 case.

Proposition 1.2. *For any inclusion $N \subset M$ of factors with finite index, there exists an extremal inclusion $B \subset A$ of type II_1 factors such that $[A\colon B] = [M\colon N]_0$ and $\mathcal{G}_{B,A} = \mathcal{G}_{N,M}$.*

1.3. Growth conditions. We can consider growth conditions of various type for the standard invariants of $N \subset M$. The strongest growth condition is the finite depth of $N \subset M$, i.e. $\#K < \infty$ where $\#K$ denotes the cardinal number of K. The principal graph $\Gamma_{N,M}$ of $N \subset M$ is said to have *polynomial growth* if $\lim_{n\to\infty} \frac{1}{n}(\#K_n)^{1/m} = 0$ for some $m \in \mathbf{N}$, and to have *subexponential growth* if $\lim_{n\to\infty} \frac{1}{n}\log(\#K_n) = 0$. Similarly, the standard eigenvector $\vec{s}$ of $N \subset M$ is said to have polynomial growth or subexponential growth if $\lim_{n\to\infty} \frac{1}{n}(\max_{k\in K_n} s_k)^{1/m} = 0$ for some $m \in \mathbf{N}$ or $\lim_{n\to\infty} \frac{1}{n}\log(\max_{k\in K_n} s_k) = 0$, respectively. It is said that $\mathcal{G}_{N,M}$ has polynomial growth or subexponential growth if both $\Gamma_{N,M}$ and $\vec{s}$ have the same growth, which is equivalent to $\lim_{n\to\infty} \frac{1}{n}(\sum_{k\in K_n} s_k)^{1/m} = 0$ for some $m \in \mathbf{N}$ or $\lim_{n\to\infty} \frac{1}{n}\log(\sum_{k\in K_n} s_k) = 0$, respectively.

It was shown in [**15**] that intermediate inclusions $Q \subset P$ of factors with $N \subset Q \subset P \subset M_n$ and descendant inclusions $Np \subset pM_np$ with projections $p \in N' \cap M_n$ satisfy the same growth conditions as $N \subset M$ does.

1.4. Amenability and strong amenability. Popa [**39**, 5.3.1] established a number of characterizations of the strong amenability for the standard invariant $\mathcal{G}_{N,M}$. Among other things, we state:

Proposition 1.5. *The following conditions are equivalent:*

(i) $\dim N^{\mathrm{st}'} \cap M^{\mathrm{st}} = \dim N' \cap M$;

(ii) $\|\Gamma_{N,M}\|^2 = |M\colon N|_0$ *and* $\Gamma_{N,M}$ *is ergodic, i.e.* M^{st} *(or* R*) is a factor;*

(iii) $H(M^{\mathrm{st}}|N^{\mathrm{st}}) = \log[M\colon N]_0$*, or equivalently* $H(R|R_1) = \log[M\colon N]_0$*, where* $H(R|R_1)$ *is the relative entropy* [**36**] *of* R *relative to* R_1 *with respect to* ϕ*.*

Here, it should be noted that although $N \subset M$ was assumed in [**39**, 5.3.1] to be an extremal inclusion of type II_1 factors, the above conditions are equivalent for general inclusions as well thanks to Proposition 1.2. Concerning the relative entropy $H(R|R_1)$, we have by [**36**, Proposition 3.4]

$$H(R|R_1) = \lim_{n\to\infty} H(M' \cap M_n | M_1' \cap M_n).$$

It is said that $\mathcal{G}_{N,M}$ is *amenable* if the equality $\|\Gamma_{N,M}\|^2 = |M\colon N|_0$ holds and is *strongly amenable* if the conditions of Proposition 1.5 hold. Note [**39**, 1.3.5] that

$$\|\Gamma_{N,M}\|^2 = \lim_{n\to\infty} (\dim M' \cap M_n)^{1/n}. \tag{1.3}$$

Now let B be a finite-dimensional algebra given a faithful normalized trace ϕ, and $f_1, \dots, f_m$ be the minimal central projections of B. Let us define a quantity $J(B)$ of entropy like by

$$J(B) = \sum_{j=1}^{m} \phi(f_j) \log(\dim Bf_j) = \sum_{j=1}^{m} b_j\beta_j \log b_j^2, \tag{1.4}$$

where $(b_1, \dots, b_m)$ and $(\beta_1, \dots, \beta_m)$ are the dimension vector and the trace vector (with respect to ϕ) of B. For instance, we have

$$J(M' \cap M_{2n}) = \sum_{k\in K_n} \phi(f_{n,k}) \log c_{n,k}^2 = \sum_{k\in K_n} c_{n,k}\lambda^n s_k \log c_{n,k}^2,$$

where $(f_{n,k})_{k\in K_n}$ is the set of minimal central projections of $M'\cap M_{2n}$. On the other hand, let $H(M'\cap M_n)$ denote the (von Neumann) entropy of $\phi|_{M'\cap M_n}$. Then it is known that $\lim_{n\to\infty}\frac{2}{n}H(M'\cap M_n)$ exists and is equal to the Connes-Størmer dynamical entropy of the canonical shift on (R,ϕ). (See [**3**, **5**] for definition and properties of the canonical shift.)

The next proposition was proved in [**16**, Theorem 4.5], which gives a convenient combinatorial characterization of the strong amenability for $\mathcal{G}_{N,M}$.

Proposition 1.6. *The limit* $\lim_{n\to\infty}\frac{1}{n}J(M'\cap M_n)$ *exists and*

$$\lim_{n\to\infty}\frac{1}{n}H(M'\cap M_n)+\lim_{n\to\infty}\frac{1}{n}J(M'\cap M_n)=\frac{1}{2}H(R|R_2).$$

Moreover, $\mathcal{G}_{N,M}$ *is strongly amenable if and only if*

$$\lim_{n\to\infty}\frac{2}{n}J(M'\cap M_n)=\log[M\colon N]_0.$$

Furthermore, it was shown in [**16**] that if $\mathcal{G}_{N,M}$ has subexponential growth, then it is strongly amenable.

Lemma 1.7. *Let* $C\subset B$ *be finite-dimensional algebras given a trace* ϕ *on* B, *and define* $J(C)$ *as* (1.4) *with respect to* $\phi|_C$. *Then* $J(C)\le J(B)$.

Proof. Let $(c_1,\dots,c_l)$ and $(\gamma_1,\dots,\gamma_l)$ be the dimension vector and the trace vector of C together with $(b_1,\dots,b_m)$ and $(\beta_1,\dots,\beta_m)$ of B. Let $[a_{ij}]_{1\le i\le l,\,1\le j\le m}$ be the inclusion matrix of $C\subset B$. Then we get

$$b_j=\sum_{i=1}^{l}c_ia_{ij}\quad(1\le j\le m),\qquad \gamma_i=\sum_{j=1}^{m}a_{ij}\beta_j\quad(1\le i\le l).$$

Hence the assertion follows from

$$\begin{aligned}J(C)&=2\sum_{i=1}^{l}c_i\gamma_i\log c_i\le 2\sum_{i,j}c_ia_{ij}\beta_j\log\left(\sum_{k=1}^{l}c_ka_{kj}\right)\\&=2\sum_{j=1}^{m}b_j\beta_j\log b_j=J(B).\end{aligned}$$

□

Proposition 1.8. *Let* $N\subset M$ *and* $\tilde N\subset\tilde M$ *be two inclusions of factors with* $[M\colon N]_0=[\tilde M\colon\tilde N]_0<\infty$ *such that* $M'\cap M_n\supset\tilde M'\cap\tilde M_n$ *and* $\phi|_{\tilde M'\cap\tilde M_n}=\tilde\phi|_{\tilde M'\cap\tilde M_n}$ *for all* $n\ge 0$, *where* $\tilde N\subset\tilde M_1\subset\tilde M_2\subset\cdots$ *is the Jones tower and* $\tilde\phi$ *is defined on* $\bigcup_n\tilde M'\cap\tilde M_n$ *as* (1.2). *If* $\mathcal{G}_{\tilde N,\tilde M}$ *is amenable or strongly amenable, then so is* $\mathcal{G}_{N,M}$, *respectively.*

Proof. The assertion concerning amenability is trivial by (1.3). The other follows from Proposition 1.6 and Lemma 1.7. □

1.9. Strongly outer automorphisms. Let $\mathrm{Aut}(M, N)$ denote the set of all automorphisms α of M such that $\alpha(N) = N$. For any $\alpha \in \mathrm{Aut}(M, N)$, since $\alpha \circ E_0 = E_0 \circ \alpha$ due to the uniqueness of minimal conditional expectation, we can (uniquely) extend α to the Jones tower $M_1 \subset M_2 \subset \cdots$ subject to $\alpha(e_n) = e_n$, $n \geq 1$, which are denoted by the same α. The extended α defines automorphisms of the higher relative commutants $M' \cap M_n$, $n \geq 0$. Note that $\alpha|_{\cup_n M' \cap M_n}$ is the opposite counterpart of the *standard part* or the *Loi part* α^{st} of α defined on $\bigcup_n N_n' \cap M$ [**29**, **40**].

Definition 1.10. An automorphism $\alpha \in \mathrm{Aut}(M, N)$ is said to be *strongly outer* [**6**] or *properly outer* [**40**] if the following equivalent conditions hold (see [**50**, Lemma 3.1] for the proof of equivalence):

(i) for every $n \geq 0$ and every $x \in M_n$, if $xy = \alpha(y)x$ for all $y \in M$ then $x = 0$;

(ii) for every $n \geq 0$ and every $x \in M_n$, if $xy = \alpha(y)x$ for all $y \in N$ then $x = 0$;

(iii) for every $n \geq 0$ and every $x \in M$, if $xy = \alpha(y)x$ for all $y \in N_n$ then $x = 0$.

Assume that M is an infinite factor. Let $\mathrm{End}(M)$ denote the endomorphisms of M and $\mathrm{Sect}(M) = \mathrm{End}(M)/\mathrm{Int}(M)$, the sectors. For any $\rho \in \mathrm{End}(M)$, the class of ρ in $\mathrm{Sect}(M)$ is denoted by $[\rho]$ and the conjugate sector $[\overline{\rho}]$ is defined by $\overline{\rho} = \rho^{-1} \circ \gamma$, where γ is the Longo canonical endomorphism [**31**] for $\rho(M) \subset M$. The sector theory [**33**, **17**] is quite important in theory of subfactors; for instance, a Jones tunnel of $N = \rho(M) \subset M$ (with finite index) is given as $M \supset \rho(M) \supset \rho\overline{\rho}(M) \supset \rho\overline{\rho}\rho(M) \supset \cdots$, and the standard invariants of $N \subset M$ are described by the irreducible decompositions of the sectors $[(\rho\overline{\rho})^n]$ and $[(\rho\overline{\rho})^n\rho]$. For instance, the irreducible decomposition of $[(\rho\overline{\rho})^n]$ is written as $[(\rho\overline{\rho})^n] = \oplus_{k \in K_n} c_{n,k}[\rho_k]$, where $[\rho_k]$, $k \in K$, are the sectors corresponding to the even vertices of the principal graph. The standard eigenvector $\vec{s}$ is the vector of statistical dimensions, i.e. $s_k = d(\rho_k)$ $(= [M\colon \rho_k(M)]_0^{1/2})$. For $\rho, \eta \in \mathrm{End}(M)$, we write $\eta \prec \rho$ if each irreducible component of $[\eta]$ is contained up to multiplicity in the irreducible decomposition of $[\rho]$.

The following result [**26**, Proposition 4] (also [**6**, Theorem 2]) characterizes strongly outer automorphisms in terms of sectors.

Proposition 1.11. *Assume that $N \subset M$ is an inclusion of infinite factors and $N = \rho(M)$ with $\rho \in \mathrm{End}(M)$. Then for every $\alpha \in \mathrm{Aut}(M, N)$ and $n \geq 0$, there exists a nonzero $x \in M_n$ such that $yx = x\alpha(y)$ for all $y \in M$ if and only if $\alpha \prec (\rho\overline{\rho})^n$, i.e. α appears as a sector in the irreducible decompositions of*

$(\rho\bar{\rho})^n$. *Hence* α *is strongly outer if and only if it does not appear in* $\bigsqcup_n (\rho\bar{\rho})^n$.

The above theorem shows, for example, the following (see [6], [25], [40, 1.6]):

1° The non-strongly outer automorphisms in $\mathrm{Aut}(M, N)$ form a group.

2° If $N \subset M$ has finite depth and $\alpha \in \mathrm{Aut}(M, N)$ is aperiodic in $\mathrm{Aut}(M)/\mathrm{Int}(M)$, then α is automatically strongly outer.

3° If $s_k > 1$ for all $k \in K \setminus \{k_0\}$, then any $\alpha \in \mathrm{Aut}(M, N) \setminus \mathrm{Int}(M)$ is strongly outer.

2. Actions of discrete groups.

In this section, let G be a discrete group and $\alpha \colon G \to \mathrm{Aut}(M, N)$ an action of G on $N \subset M$. Then α uniquely extends to actions on the Jones tower $M_1 \subset M_2 \subset \cdots$ subject to the conditions $\alpha_g(e_n) = e_n$, $g \in G$, which define actions on the higher relative commutants $M' \cap M_n$, $n \geq 0$. Assume that α is *strongly outer*, that is, α_g is strongly outer for any $g \in G \setminus \{e\}$. We set

$$\left(\tilde{N} \subset \tilde{M}\right) = (N \rtimes_\alpha G \subset M \rtimes_\alpha G).$$

Since the strong outerness of α implies that $\alpha \colon G \to \mathrm{Aut}(M)$ and $\alpha|_N \colon G \to \mathrm{Aut}(N)$ are outer in the usual sense, it follows that both $\tilde{M}$ and $\tilde{N}$ are factors (see [47, 22.3]). Noting [47, 19.13] that

$$\tilde{N} = (N \otimes \mathbf{B}(\ell^2(G)))^{\alpha\otimes \mathrm{Ad}\,\rho}, \qquad \tilde{M} = (M \otimes \mathbf{B}(\ell^2(G)))^{\alpha\otimes \mathrm{Ad}\,\rho},$$

we can canonically extend the minimal conditional expectation $E_0 \colon M \to N$ to $\tilde{E}_0 \colon \tilde{M} \to \tilde{N}$ by $\tilde{E}_0 = E_0 \otimes \mathrm{id}_{\mathbf{B}(\ell^2(G))}|_{\tilde{M}}$; equivalently

$$\tilde{E}_0\left(\sum x(g)\lambda(g)\right) = \sum E_0(x(g))\lambda(g), \qquad \sum x(g)\lambda(g) \in \tilde{M}, \tag{2.1}$$

where $\lambda(g) = 1 \otimes \lambda_g$ and λ_g (resp. ρ_g) is the left (resp. right) regular representation of G on $\ell^2(G)$. For brevity we let $M \subset \tilde{M}$ without the symbol π_α of embedding.

It is known [40, 1.5] (also implicitly in the proof of [6, Proposition 7]) that an action α of a discrete group G on $N \subset M$ is strongly outer if and only if the following equivalent conditions hold:

$$N_n' \cap (M \rtimes_\alpha G) = N_n' \cap M, \qquad n \geq 0, \tag{2.2}$$

$$M' \cap (M_n \rtimes_\alpha G) = M' \cap M_n, \qquad n \geq 0. \tag{2.3}$$

(The setting of $N \subset M$ being of type II_1 in [40] is irrelevant to these characterizations.)

Assume in the following lemmas that α is a strongly outer action of a discrete group G on $N \subset M$. Since $\operatorname{Ind} \tilde{E}_0 = \operatorname{Ind} E_0 < \infty$ (see the proof of [**14**, Theorem 2.8]), the inclusion $\tilde{N} \subset \tilde{M}$ has finite index. Indeed, we have:

Lemma 2.1. *The conditional expectation* $\tilde{E}_0 \colon \tilde{M} \to \tilde{N}$ *is minimal. Hence* $[M\colon N]_0 = [\tilde{M}\colon \tilde{N}]_0$.

Proof. Let $\mathcal{E}(M,N)$ denote the set of all faithful normal conditional expectations from M onto N. Any $E \in \mathcal{E}(M,N)$ extends to $\tilde{E} \in \mathcal{E}(\tilde{M},\tilde{N})$ as E_0 does to $\tilde{E}_0$, so that $\tilde{E}|_{\tilde{N}'\cap\tilde{M}} = E|_{\tilde{N}'\cap\tilde{M}}$ where $\tilde{N}' \cap \tilde{M} \subset N' \cap M$ by Popa's characterization (2.2). This means that any $F \in \mathcal{E}(\tilde{M},\tilde{N})$ is obtained as the extension of some $E \in \mathcal{E}(M,N)$, because [**7**, Théorème 5.3] says that $F \mapsto F|_{\tilde{N}'\cap\tilde{M}}$ is a bijection from $\mathcal{E}(\tilde{M},\tilde{N})$ onto the set of all faithful normal states on $\tilde{N}'\cap\tilde{M}$. Since $\operatorname{Ind} \tilde{E} = \operatorname{Ind} E$, it follows that $\tilde{E}_0$ is minimal. $\square$

In view of Lemma 2.1, we easily see that the Jones tower for $\tilde{N} \cap \tilde{M}$ is

$$\tilde{N} \subset \tilde{M} = \tilde{M}_0 \subset \tilde{M}_1 = M_1 \rtimes_\alpha G \subset \tilde{M}_2 = M_2 \rtimes_\alpha G \subset \cdots,$$

where the Jones projections are the same as e_n in (1.1) and the iterated conditional expectations $\tilde{E}_n \colon \tilde{M}_n \to \tilde{M}_{n-1}$ are the canonical extensions of E_n, $n \geq 0$, given as (2.1). Let $\tilde{\phi}$ be the trace on $\bigcup_n \tilde{M}' \cap \tilde{M}_n$ induced by $\{\tilde{E}_n\}$ as (1.2).

The following (1) was given in [**6**], and (2) is immediate from (1) and $E_n = \tilde{E}_n|_{M_n}$.

Lemma 2.2.

(1) $(M' \cap M_n)^\alpha = \tilde{M}' \cap \tilde{M}_n$, $n \geq 0$.

(2) $\phi|_{\tilde{M}'\cap\tilde{M}_n} = \tilde{\phi}|_{\tilde{M}'\cap\tilde{M}_n}$, $n \geq 0$.

Proposition 2.3. *Let α be a strongly outer action of a discrete group G on $N \subset M$. Then:*

(1) *If $\tilde{N} \subset \tilde{M}$ has finite depth, then so does $N \subset M$.*

(2) *If $\mathcal{G}_{\tilde{N},\tilde{M}}$ is amenable, then so is $\mathcal{G}_{N,M}$.*

(3) *If $\mathcal{G}_{\tilde{N},\tilde{M}}$ is strongly amenable, then so is $\mathcal{G}_{N,M}$.*

Proof. (1) (The argument below is found in [**40**].) If $\tilde{N} \subset \tilde{M}$ has finite depth, then for some n the central support of e_n in $\tilde{M}' \cap \tilde{M}_n = (M' \cap M_n)^\alpha$ is 1 (see [**9**, 4.6.3]), so that the support of e_n in $M' \cap M_n$ is also 1. Hence $N \subset M$ has finite depth.

(2) and (3) follow from Lemmas 2.1, 2.2, and Proposition 1.8. $\square$

The above (3) was shown in [**51**] by a different method. Furthermore, we give another proof using the relative entropy in the following:

Remark 2.4. Since $\alpha|_{\bigcup_n M'\cap M_n}$ preserves ϕ, it can extend to a ϕ-preserving action of G on R (denoted by the same α). Note that for any $i \geq 0$

$$\begin{array}{ccc} M_i'\cap M_n & \subset & M'\cap M_n \\ \cup & & \cup \\ (M_i'\cap M_n)^\alpha & \subset & (M'\cap M_n)^\alpha \end{array}$$

is a commuting square with respect to ϕ. This implies that $\begin{array}{ccc} R_i & \subset & R \\ \cup & & \cup \\ R_i^\alpha & \subset & R^\alpha \end{array}$ is a commuting square with respect to ϕ and hence $H(R^\alpha|R_i^\alpha) \leq H(R|R_i)$ by [**3**, Lemma 13]. When α is strongly outer, since the extension of α to M_i is also strongly outer, Lemma 2.2 shows that R_i^α is the von Neumann algebra generated by $\bigcup_n \tilde{M}_i'\cap\tilde{M}_n$ with respect to $\tilde{\phi}$. Thus Proposition 1.5 proves Proposition 2.3(3) again.

In the rest of this section, let us prove the converse implications of Proposition 2.3 when G is a finite group. For this sake, it is important to look at the inclusion matrix (i.e. Bratteli diagram) of $(M'\cap M_n)^\alpha \subset M'\cap M_n$. First let us consider an action α of G on a finite-dimensional algebra B, where G is an arbitrary group. Let $f_1,\dots,f_m$ be the minimal central projections of B. Since α gives rise to permutations on $\{f_1,\dots,f_m\}$, we decompose $\{1,\dots,m\}$ into $J_1,\dots,J_r$ under the relation $j\sim j'$ when $\alpha_g(f_j)=f_{j'}$ for some $g\in G$. Set $\tilde{f}_k=\sum_{j\in J_k} f_j$ for $1\leq k\leq r$. Then $\tilde{f}_k$ are central projections in B^α and

$$(B^\alpha\subset B)=\bigoplus_{k=1}^r\left(B^\alpha\tilde{f}_k\subset B\tilde{f}_k\right).$$

So it suffices to assume that α is transitive on $\{f_1,\dots,f_m\}$. Then we have:

Lemma 2.5. *With the above notations and transitivity assumption, let $[a_{ij}]_{1\leq i\leq l,\,1\leq j\leq m}$ be the inclusion matrix of $B^\alpha\subset B$ and define a subgroup G_1 of G by $G_1=\{g\in G\colon \alpha_g(f_1)=f_1\}$. Then a_{ij} does not depend on $1\leq j\leq m$ and $[a_{ij}]_{1\leq i\leq l}$ is the inclusion matrix of $(Bf_1)^{\alpha_{G_1}}\subset Bf_1$ for any j.*

Proof. For each $1\leq j\leq m$, let $G_j=\{g\in G\colon \alpha_g(f_j)=f_j\}$ and choose $g_j\in G$ such that $\alpha_{g_j}(f_1)=f_j$. Then it is easy to check that $G_j=g_jG_1g_j^{-1}$ and

$$B^\alpha=\left\{\bigoplus_{j=1}^m\alpha_{g_j}(x)\colon x\in(Bf_1)^{\alpha_{G_1}}\right\}.$$

This means that $B^\alpha\cong(Bf_1)^{\alpha_{G_1}}$ and the inclusion $B^\alpha\subset B$ is written as $\bigoplus_{j=1}^m\alpha_{g_j}\colon (Bf_1)^{\alpha_{G_1}}\to\bigoplus_{j=1}^m Bf_j$. Since α_{g_j} transforms $(Bf_1)^{\alpha_{G_1}}\subset Bf_1$ to $(Bf_j)^{\alpha_{G_j}}\subset Bf_j$, we get the result. $\square$

Thus, the next lemma is enough for our purpose.

Lemma 2.6. *Let G be a finite group. Then there exists $W \in \mathbf{N}$ (depending only on G) such that if α is an action of G on $B = M_d(\mathbf{C})$ and $[a_i]_{1\le i\le l}$ is the inclusion matrix of $B^\alpha \subset B$ where $l = \dim Z(B^\alpha)$, then $l \le W$ and $a_i \le W$ for all $1 \le i \le l$.*

Proof. Let U be a projective unitary representation of G on $\mathbf{C}^d$ implementing α, i.e. $\alpha_g = \mathrm{Ad}\, U_g$, $g \in G$, so that $B^\alpha = (U_G)'$. For each $1 \le i \le l$ choose a minimal projection p_i in the ith summand of B^α. Then $a_i = \dim p_i\mathbf{C}^d$ and $U^{(i)} = U|_{p_i\mathbf{C}^d}$ is an irreducible projective unitary representation of G. Now let (H, Z) be a primitive central extension of G [48, §2.9]. So we write $H = \bigsqcup_{g\in G} Zt_g$ with a transversal $\{t_g\}$ of Z in H. Then for $1 \le i \le l$ there exists an irreducible unitary representation $V^{(i)}$ of H on $p_i\mathbf{C}^d$ such that $V^{(i)}_{st_g} = \lambda(s, g)U^{(i)}_g$ where $\lambda(s, g) \in \mathbf{C}$ for all $s \in Z$ and $g \in G$ (see [48, p. 263]). Since $V^{(i)}$ and $V^{(i')}$ are not equivalent when $i \ne i'$, we have the conclusion from usual theory on unitary representations of finite groups. □

Summarizing the above arguments, we obtain the following key lemma.

Lemma 2.7. *Let α be an action of a finite group G on $N \subset M$. Then there exists $W \in \mathbf{N}$ (depending only on G) such that for every $n \ge 0$ the inclusion matrix $[a_{ij}]$ of $(M' \cap M_n)^\alpha \subset M' \cap M_n$ satisfies the following:*

(1) $\#\{j : a_{ij} \ne 0\} \le W$ *for all* i,

(2) $\#\{i : a_{ij} \ne 0\} \le W$ *for all* j,

(3) $a_{ij} \le W$ *for all* i, j.

Theorem 2.8. *Let α be a strongly outer action of a finite group G on $N \subset M$. Define the standard invariants $\Gamma_{\tilde N,\tilde M}$ and $(\tilde s_k)_{k\in\tilde K}$ with $\tilde K = \bigcup_n \tilde K_n$ for $\tilde N \subset \tilde M$ as well as $\Gamma_{N,M}$ and $(s_k)_{k\in K}$ for $N \subset M$. With W given in Lemma 2.7, the following hold for every $n \ge 0$:*

(1) $W^{-1}(\#K_n) \le \#\tilde K_n \le W(\#K_n)$.

(2) $\max_{k\in K_n} s_k \le \max_{k\in\tilde K_n} \tilde s_k \le W^2 \max_{k\in K_n} s_k$.

(3) $\dim \tilde M' \cap \tilde M_n \le \dim M' \cap M_n \le W^4 \dim \tilde M' \cap \tilde M_n$. *Hence* $\|\Gamma_{\tilde N,\tilde M}\| = \|\Gamma_{N,M}\|$.

(4) $J(\tilde M' \cap \tilde M_n) \le J(M' \cap M_n) \le J(\tilde M' \cap \tilde M_n) + \log W^4$. *Hence* $\lim\limits_{n\to\infty} \frac{1}{n} J(\tilde M' \cap \tilde M_n) = \lim\limits_{n\to\infty} \frac{1}{n} J(M' \cap M_n)$.

Proof. (1) is trivial from Lemmas 2.2(1) and 2.7.

(2) The first inequality is obvious from Lemma 2.2. For each $n \ge 0$, let $(b_1, \dots, b_m)$ and $(\beta_1, \dots, \beta_m)$ be the dimension vector and the trace vector

of $M' \cap M_n$, and $(c_1, \dots, c_l)$ and $(\gamma_1, \dots, \gamma_l)$ be those of $(M' \cap M_n)^\alpha$, respectively. Also let $[a_{ij}]_{1\le i\le l,\,1\le j\le m}$ be the inclusion matrix of $(M' \cap M_n)^\alpha \subset M' \cap M_n$. Then Lemma 2.7 implies that

$$\gamma_i = \sum_{j=1}^m a_{ij}\beta_j \le W^2 \max_{1\le j\le m} \beta_j.$$

Since the trace vectors of $M' \cap M_{2n}$ and $\tilde{M}' \cap \tilde{M}_{2n} = (M' \cap M_{2n})^\alpha$ are $(\lambda^n s_k)_{k\in K_n}$ and $(\lambda^n \tilde{s}_k)_{k\in\tilde{K}_n}$ respectively, the above means that $\max_{k\in\tilde{K}_n} \tilde{s}_k \le W^2 \max_{k\in K_n} s_k$.

(3) With the notations in the proof of (2), we get

$$\begin{aligned}\dim(M' \cap M_n)^\alpha &\le \dim M' \cap M_n = \sum_{i=1}^m b_j^2 \\ &= \sum_{j=1}^m \left(\sum_{i=1}^l c_i a_{ij}\right)^2 \le \sum_{j=1}^m \left(\sum_{a_{ij}\ne 0} c_i^2\right)\left(\sum_{i=1}^l a_{ij}^2\right) \\ &\le W^3 \sum_{j=1}^m \sum_{a_{ij}\ne 0} c_i^2 = W^3 \sum_{i=1}^l \#\{j\colon a_{ij}\ne 0\} c_i^2 \\ &\le W^4 \dim(M' \cap M_n)^\alpha.\end{aligned}$$

(4) The first inequality follows from Lemmas 1.7 and 2.2. For the second, we get

$$\begin{aligned}&J(M' \cap M_n) - J((M' \cap M_n)^\alpha) \\ &= 2\sum_{i,j} c_i a_{ij}\beta_j \log\frac{b_j}{c_i} \le 2\log\left(\sum_{i,j} a_{ij} b_j \beta_j\right) \\ &\le 2\log\left(W \sum_{j=1}^m \#\{i\colon a_{ij}\ne 0\} b_j\beta_j\right) \le \log W^4.\end{aligned}$$

□

Proposition 1.6 and Theorem 2.8 give:

Corollary 2.9. *Let G and α be as in Theorem 2.8. Then:*

(1) *The growth of $\mathcal{G}_{\tilde{N},\tilde{M}}$ is the same as $\mathcal{G}_{N,M}$. Hence $\tilde{N} \subset \tilde{M}$ has finite depth or subexponential growth if and only if so does $N \subset M$, respectively.*

(2) *$\mathcal{G}_{\tilde{N},\tilde{M}}$ is amenable if and only if so is $\mathcal{G}_{N,M}$.*

(3) *$\mathcal{G}_{\tilde{N},\tilde{M}}$ is strongly amenable if and only if so is $\mathcal{G}_{N,M}$.*

The above (3) was shown in [**51**], while our proof is completely different from [**51**].

Remark 2.10. It was proved in [**16**, Theorem 4.1] that $\lim_{n\to\infty} \frac{1}{n} H(Z(M' \cap M_n))$ exists and

$$\frac{1}{2} H(R|R_2) + \lim_{n\to\infty} \frac{1}{n} H(Z(M' \cap M_n)) = \lim_{n\to\infty} \frac{2}{n} H(M' \cap M_n).$$

In view of Lemma 2.2 and Remark 2.4, this formula applied to $\tilde{N} \subset \tilde{M}$ reads as

$$\frac{1}{2} H(R^\alpha|R_2^\alpha) + \lim_{n\to\infty} \frac{1}{n} H(Z((M' \cap M_n)^\alpha)) = \lim_{n\to\infty} \frac{2}{n} H((M' \cap M_n)^\alpha).$$

Under the assumption of Theorem 2.8, it is not difficult to show by Lemma 2.7 that

$$\lim_{n\to\infty} \frac{1}{n} H(Z(M' \cap M_n)) = \lim_{n\to\infty} \frac{1}{n} H(Z((M' \cap M_n)^\alpha)),$$
$$\lim_{n\to\infty} \frac{1}{n} H(M' \cap M_n) = \lim_{n\to\infty} \frac{1}{n} H((M' \cap M_n)^\alpha).$$

The last equality means that the dynamical entropies of the canonical shifts for $N \subset M$ and for $\tilde{N} \subset \tilde{M}$ are identical. Combining the above estimates yields $H(R^\alpha|R_2^\alpha) = H(R|R_2)$, which implies Corollary 2.9(3) again.

Indeed, the assertion for finite depth in Corollary 2.9(1) holds true without the strong outerness assumption. To show this, we mention the following lemma due to Wierzbicki [**49**]. We say that a square $\begin{matrix} Q & \subset & P \\ \cup & & \cup \\ N & \subset & M \end{matrix}$ of general factors with $[P\colon N]_0 < \infty$ is a *commuting square* if the commuting square condition is satisfied for the minimal conditional expectations: For instance, $E_Q(M) \subset N$ for the minimal conditional expectation $E_Q\colon P \to Q$ (see [**9**, 4.2.1] for other equivalent conditions). Furthermore, such a commuting square is said to be *nondegenerate* [**39**, 1.1.5] if $\overline{\operatorname{span}}\, MQ = P$, which is equivalent to the co-commuting square condition in [**44**].

Lemma 2.11. *Let $\begin{matrix} Q & \subset & P \\ \cup & & \cup \\ N & \subset & M \end{matrix}$ is a nondegenerate commuting square of factors with $[P\colon N]_0 < \infty$ as above. If both $N \subset M$ and $M \subset P$ have finite depth, then so does $N \subset P$.*

Proof. This was proved in [**49**] for type II_1 factors, so that we only indicate the reduction to the type II_1 case. This can be done by taking tensor products with a type III_1 factor and then by taking crossed products by the modular automorphism group (see the proof [**16**] of Proposition 1.2 for details). Note that the nondegeneracy condition is preserved under these procedures. □

Proposition 2.12. *Let G be a finite group. Let α be an action of G on $N \subset M$, which is outer on both N and M. Then $N \subset M$ has finite depth if and only if so does $\tilde{N} \subset \tilde{M}$.*

Proof. Since $\mathrm{span} M\tilde{N} = \mathrm{span}\, M\lambda(G) = \tilde{M}$, the commuting square $\begin{matrix} \tilde{N} & \subset & \tilde{M} \\ \cup & & \cup \\ N & \subset & M \end{matrix}$ is nondegenerate. Since $N \subset \tilde{N}$ and $M \subset \tilde{M}$ have depth 2, the above lemma shows that the finite depth of $N \subset M$ (resp. $\tilde{N} \subset \tilde{M}$) implies that of $N \subset \tilde{M}$. The latter condition implies the finite depth of $\tilde{N} \subset \tilde{M}$ (resp. $N \subset M$) by [**2**] (also [**15**, Theorem 2.2]). □

3. Actions of locally compact abelian groups.

In this section, we assume that G is a locally compact abelian group. Let $\alpha\colon G \to \mathrm{Aut}(M,N)$ be a continuous action of G on $N \subset M$, which extends to continuous actions to $M_1 \subset M_2 \subset \cdots$ as in Section 2. Here, the continuity of the extensions of α is immediate from $M_n = \mathrm{span}\, M_{n-1}e_nM_{n-1}$, $n \ge 1$. Define

$$\tilde{N} \subset \tilde{M} = \tilde{M}_0 \subset \tilde{M}_1 \subset \tilde{M}_2 \subset \cdots \tag{3.1}$$

by $\tilde{M}_n = M_n \rtimes_\alpha G$, $n \ge 0$. Then for every $n \ge 0$, $E_n\colon M_n \to M_{n-1}$ canonically extends to $\tilde{E}_n\colon \tilde{M}_n \to \tilde{M}_{n-1}$ by $\tilde{E}_n = E_n \otimes \mathrm{id}_{\mathbf{B}(L^2(G))}|_{\tilde{M}_n}$. Let $\hat{\alpha}\colon \hat{G} \to \mathrm{Aut}(\tilde{M})$ be the dual action of α. Since $\hat{\alpha}_t(\tilde{N}) = \tilde{N}$, i.e. $\hat{\alpha}_t \in \mathrm{Aut}(\tilde{M},\tilde{N})$ for all $t \in \hat{G}$, the Takesaki duality says [**47**, 19.5] that

$$\left(\tilde{N} \rtimes_{\hat{\alpha}} \hat{G} \subset \tilde{M} \rtimes_{\hat{\alpha}} \hat{G}\right) \cong (N \otimes \mathbf{B}(L^2(G)) \subset M \otimes \mathbf{B}(L^2(G))). \tag{3.2}$$

We consider the following assumptions:

(A) Both $\tilde{N} \subset \tilde{M}$ are factors (see [**47**, 21.6] concerning the factorness of crossed products).

(B) α_g is strongly outer for any $g \in G \setminus \{e\}$.

(C) $\hat{\alpha}_t$ is strongly outer for any $t \in \hat{G} \setminus \{\hat{e}\}$.

As was noted in Section 2, if G is discrete, then assumption (A) automatically follows from (B). Throughout this section, (A) will be assumed. Then we have:

Lemma 3.1.

(1) *$\tilde{E}_0 \colon \tilde{M} \to \tilde{N}$ is minimal and hence $[M \colon N]_0 = [\tilde{M} \colon \tilde{N}]_0$.*

(2) *(3.1) is the Jones tower for $\tilde{N} \subset \tilde{M}$ iterated from $\tilde{E}_0$, where the Jones projections are the same as e_n in (1.1).*

(3) *The extensions of $\hat{\alpha}$ to $\tilde{M}_1 \subset \tilde{M}_2 \subset \cdots$ subject to $\hat{\alpha}_t(e_n) = e_n$, $t \in \hat{G}$, are the dual actions of $\alpha|_{M_n}$, $n \geq 1$.*

Proof. (1) was shown in [**14**, Theorem 2.8] by using the Takesaki duality (3.2). Then (2) and (3) are readily checked. □

Lemma 3.2.

(1) $(M' \cap M_n)^\alpha = (\tilde{M}' \cap \tilde{M}_n)^{\hat{\alpha}}$, $n \geq 0$,

(2) $\phi|_{(M' \cap M_n)^\alpha} = \tilde{\phi}|_{(\tilde{M}' \cap \tilde{M}_n)^{\hat{\alpha}}}$, $n \geq 0$.

Proof. We have (1) because:

$$\begin{aligned}(M' \cap M_n)^\alpha &= M' \cap M_n \cap \lambda(G)' = \tilde{M}' \cap M_n \\ &= \tilde{M}' \cap \left(\tilde{M}_n\right)^{\hat{\alpha}} = \left(\tilde{M}' \cap \tilde{M}_n\right)^{\hat{\alpha}}\end{aligned}$$

thanks to $M_n = (\tilde{M}_n)^{\hat{\alpha}}$, while (2) is obvious as Lemma 2.2(2). □

Note that if $\sup_n \dim Z((M' \cap M_n)^\alpha) < \infty$, then both $N \subset M$ and $\tilde{N} \subset \tilde{M}$ have finite depth by Lemma 3.2(1) (see the proof of Proposition 2.3(1)). Although it will not be needed in this paper, it is worth noting that $\{(M_i' \cap M_j)^\alpha\}_{0 \leq i \leq j}$ is a λ-sublattice of $\{M_i' \cap M_j\}_{0 \leq i \leq j}$ in the sense of [**43**] and thus by [**43**, Theorem 3.1] there exists an extremal inclusion $N^0 \subset M^0$ of type II_1 factors such that $(M_i' \cap M_j)^\alpha = M_i^{0'} \cap M_j^0$, $0 \leq i \leq j$. Here, Theorems 1.5 and 1.6 show that $\mathcal{G}_{N^0, M^0}$ is strongly amenable if and only if

$$\lim_{n \to \infty} H\left((M' \cap M_n)^\alpha | (M_1' \cap M_n)^\alpha\right) = \log[M \colon N]_0,$$

or equivalently $\lim_{n \to \infty} \frac{2}{n} J((M' \cap M_n)^\alpha) = [M \colon N]_0$.

Lemma 3.2 says that it should be necessary to look at the inclusions $(M' \cap M_n)^\alpha \subset M' \cap M_n$ and $(\tilde{M}' \cap \tilde{M}_n)^{\hat{\alpha}} \subset \tilde{M}' \cap \tilde{M}_n$ if we want to compare $\mathcal{G}_{N,M}$ and $\mathcal{G}_{\tilde{N},\tilde{M}}$. It is crucial for this sake to analyze the behavior of $\alpha|_{\bigcup_n M' \cap M_n}$ and $\hat{\alpha}|_{\bigcup_n \tilde{M}' \cap \tilde{M}_n}$. On the other hand, Kosaki [**26**] studied the "eigenvalue problem" for the dual action $\theta|_{\bigcup_n \tilde{M}' \cap \tilde{M}_n}$ for an inclusion $N \subset M$ of type III factors to obtain some structure results for type III inclusions. From these considerations, we are led to deal with the eigenvalues of $\alpha|_{M' \cap M_n}$ and $\hat{\alpha}|_{\tilde{M}' \cap \tilde{M}_n}$.

Namely, we define

(3.3)
$$\begin{aligned}\mathrm{Eig}\,(\alpha|_{M'\cap M_n}) &= \Big\{t\in\hat G\colon \alpha_g(x)=\langle g,t\rangle x,\ g\in G,\\ &\qquad\qquad \text{for some nonzero }\ x\in M'\cap M_n\Big\},\\ \mathrm{Eig}\,(\hat\alpha|_{\tilde M'\cap\tilde M_n}) &= \Big\{g\in G\colon \hat\alpha_t(x)=\langle g,t\rangle x,\ t\in\hat G,\\ &\qquad\qquad \text{for some nonzero }\ x\in \tilde M'\cap\tilde M_n\Big\}.\end{aligned}$$

Then we can consider growth conditions of $\alpha|_{\bigcup_n M'\cap M_n}$ as those of $\#\,\mathrm{Eig}(\alpha|_{M'\cap M_n})$, $n\ge 0$. For instance, we say that $\alpha|_{\bigcup_n M'\cap M_n}$ has *subexponential growth* if

$$\lim_{n\to\infty}\frac{1}{n}\log\#\,\mathrm{Eig}(\alpha|_{M'\cap M_n})=0,$$

and it has *polynomial growth* if there exists $m\in\mathbf{N}$ such that

$$\#\,\mathrm{Eig}(\alpha|_{M'\cap M_n})\le(n+1)^m,\qquad n\ge 0.$$

When we replace $(N\subset M,\alpha)$ by $(N\otimes P\subset M\otimes P,\ \alpha\otimes 1)$ with an infinite factor P, the Jones tower for $N\otimes P\subset M\otimes P$ is $\{M_n\otimes P\}$ and that for $((N\otimes P)\rtimes_{\alpha\otimes 1}G\subset(M\otimes P)\rtimes_{\alpha\otimes 1}G)=(\tilde N\otimes P\subset\tilde M\otimes P)$ is $\{\tilde M_n\otimes P\}$. The dual action of $\alpha\otimes 1$ is $\tilde\alpha\otimes 1$. It is a simple fact that condition (B) is equivalent to that for $\alpha\otimes 1$ and (C) is equivalent to that for $\hat\alpha\otimes 1$. Thus, to compare $\mathcal{G}_{N,M}$ and $\mathcal{G}_{\tilde N,\tilde M}$, it may be assumed without loss of generality that both $N\subset M$ and $\tilde N\subset\tilde M$ are infinite factors. Furthermore, we may assume by [**34**, Lemma 2.3] that $N=\rho(M)$ for some $\rho\in\mathrm{End}(M)$ and $\tilde N=\eta(\tilde M)$ for some $\eta\in\mathrm{End}(\tilde M)$. In this setting, Proposition 1.11 shows that (B) and (C) are respectively equivalent to the following:

(B′) α_g does not appear in $\bigsqcup_n(\rho\bar\rho)^n$ for any $g\in G\setminus\{e\}$,
(C′) $\hat\alpha_t$ does not appear in $\bigsqcup_n(\eta\bar\eta)^n$ for any $t\in\hat G\setminus\{\hat e\}$.

Proposition 3.3.
(1) *Assume that $\tilde N\subset\tilde M$ are infinite and $\tilde N=\eta(\tilde M)$ for some $\eta\in\mathrm{End}(\tilde M)$. Then for every $n\ge 0$*

$$\mathrm{Eig}(\alpha|_{M'\cap M_n})\subset\Big\{t\in\hat G\colon \hat\alpha_t\prec(\eta\bar\eta)^n\Big\},$$

and the equality holds if α satisfies (B).

(2) *Assume that $N \subset M$ are infinite and $N = \rho(M)$ for some $\rho \in \mathrm{End}(M)$. Then for every $n \geq 0$*

$$\mathrm{Eig}(\hat{\alpha}|_{\tilde{M}' \cap \tilde{M}_n}) \subset \{g \in G \colon \alpha_g \prec (\rho\bar{\rho})^n\},$$

and the equality holds if $\hat{\alpha}$ satisfies (C).

Proof. (1) The proof below is on the same lines as in [**26**], while we give it for completeness. It suffices by Proposition 1.11 to show that for any fixed $t \in \hat{G}$

$$\begin{aligned}\{x \in M' \cap M_n \colon \alpha_g(x) = \langle g, t\rangle x,\ g \in G\}\\ \subset \left\{x \in \tilde{M}_n \colon yx = x\hat{\alpha}_t(y),\ y \in \tilde{M}\right\},\end{aligned} \tag{3.4}$$

and the equality holds if (B) is satisfied. Let $\mathcal{H}$ denote the right-hand side of (3.4). If x belongs to the left-hand side of (3.4), then since $M = \tilde{M}^{\hat{\alpha}}$, we get $yx = xy = x\hat{\alpha}_t(y)$, $y \in M$. Also we get

$$\lambda(g)x = \alpha_g(x)\lambda(g) = \langle g, t\rangle x\lambda(g) = x\hat{\alpha}_t(\lambda(g)), \qquad g \in G.$$

Hence $x \in \mathcal{H}$ and so (3.4) is shown.

Next assume (B). Note that $\mathcal{H}$ is a finite-dimensional $\hat{\alpha}$-invariant subspace of $\tilde{M}_n$, because it is the space of intertwiners between two sectors of finite index. Since $x_1x_2^* \in \tilde{M}' \cap \tilde{M}_n$ when $x_1, x_2 \in \mathcal{H}$, we can define an inner product on $\mathcal{H}$ by $\langle x_1, x_2\rangle = \tilde{\phi}(x_1x_2^*)$. Then $\hat{\alpha}$ acts on $\mathcal{H}$ as a unitary group of $\hat{G}$. Hence by the spectral decomposition, there exist a basis $\{x_j\}_{j=1}^m$ of $\mathcal{H}$ and $\{g_j\}_{j=1}^m \subset G$ such that $\hat{\alpha}_s(x_j) = \langle g_j, s\rangle x_j$, $s \in \hat{G}$. Set $z_j = x_j\lambda(g_j^{-1})$ in $\tilde{M}_n$. Then $z_j \in M_n$ follows from

$$\hat{\alpha}_s(z_j) = \langle g_j, s\rangle x_j\hat{\alpha}_s\left(\lambda(g_j^{-1})\right) = z_j, \qquad s \in \hat{G}.$$

We get for every $y \in M$

$$yz_j\lambda(g_j) = z_j\lambda(g_j)\hat{\alpha}_t(y) = z_j\lambda(g_j)(y) = z_j\alpha_{g_j}(y)\lambda(g_j),$$

so that $yz_j = z_j\alpha_{g_j}(y)$. Since $z_j \neq 0$, (B) gives $g_j = e$ and so $x_j \in \tilde{M}_n^{\hat{\alpha}} = M_n$. This shows that $\mathcal{H} \subset M' \cap M_n$. Moreover, if $x \in \mathcal{H}$ then

$$\alpha_g(x)\lambda(g) = \lambda(g)x = x\hat{\alpha}_t(\lambda(g)) = \langle g, t\rangle x\lambda(g)$$

and hence $\alpha_g(x) = \langle g, t\rangle x$, $g \in G$. Therefore the equality holds in (3.4).

(2) Via the Takesaki duality (3.2) with $\hat{\hat{\alpha}} \cong \alpha \otimes \mathrm{Ad}\,\lambda^*$, applying the above (1) to $(\tilde{N} \subset \tilde{M}, \hat{\alpha})$ instead of $(N \subset M, \alpha)$, we have

$$\mathrm{Eig}\left(\hat{\alpha}|_{\tilde{M}' \cap \tilde{M}_n}\right) \subset \{g \in G \colon \alpha_g \otimes \mathrm{Ad}\,\lambda_g^* \prec (\rho\bar{\rho})^n \otimes 1\}$$

with the equality in case of (C). Since $\alpha_g \otimes \mathrm{Ad}\,\lambda_g^* \prec (\rho\bar\rho)^n \otimes 1$ if and only if $\alpha_g \prec (\rho\bar\rho)^n$, we get the result. $\square$

Lemmas 3.2(1) and Proposition 3.3 give:

Corollary 3.4.
(1) *If $\hat\alpha$ satisfies* (C), *then $(\tilde M' \cap \tilde M_n)^{\hat\alpha} = M' \cap M_n$ for all $n \ge 0$.*
(2) *If α satisfies* (B), *then $(M' \cap M_n)^{\alpha} = \tilde M' \cap \tilde M_n$ for all $n \ge 0$.*

Consequently, we obtain the assertions (1)-(3) of Proposition 2.3 under (B) and the reverse assertions under (C).

In the rest of the section, we assume that G is of the form $\mathbf{R}^d \times \mathbf{T}^{d'} \times G_0$ where d, d' are nonnegative integers and G_0 is a finitely generated abelian group. In other words, G is written as $G = G_1 \times \cdots \times G_N$ where each G_n is $\mathbf{R}$ or $\mathbf{T}$ or a cyclic group. Let us show that the growth of $\mathcal{G}_{\tilde N, \tilde M}$ can be controlled by those of $\mathcal{G}_{N,M}$ and $\alpha|_{\bigcup_n M' \cap M_n}$. The following is a key lemma.

Lemma 3.5. *Let G be as above, B be a finite-dimensional algebra given a faithful normalized trace ϕ, and α be an action of G on B. Let $[a_{ij}]_{1\le i\le l,\, 1\le j\le m}$ be the inclusion matrix of $B^\alpha \subset B$, and $(\beta_1, \dots, \beta_m)$ and $(\gamma_1, \dots, \gamma_l)$ be the trace vectors of B and B^α, respectively, where $m = \dim Z(B)$ and $l =$ $\dim Z(B^\alpha)$. Let $\mathrm{Eig}(\alpha)$ be the set of eigenvalues of α (see (3.3)) and put $W = \#\,\mathrm{Eig}(\alpha)$. Then the following hold with N given above:*
(1) $\sum_j a_{ij} \le W^N$ *for all i,*
(2) $\sum_i a_{ij} \le W^N$ *for all j,*
(3) $\max_i \gamma_i \le W^N \max_j \beta_j$.
Furthermore, if G is $\mathbf{R}$ or $\mathbf{T}$ or a cyclic group, then
(4) $W \le l^2 \max_i \gamma_i / \min_j \beta_j$.

Proof. First suppose $G = \mathbf{R}$. Since α fixes the central projections of B by continuity, we may assume that $B = M_d(\mathbf{C})$ (i.e. $m = 1$) and hence there exists a selfadjoint element H in $M_d(\mathbf{C})$ such that $\alpha_g = \mathrm{Ad}\, e^{\sqrt{-1}gH}$, $g \in \mathbf{R}$. Take the spectral decomposition $H = \sum_{i=1}^l t_i p_i$. Then $B^\alpha = \bigoplus_{i=1}^l p_i B p_i$ and $\mathrm{Eig}(\alpha) = \{t_i - t_{i'} : 1 \le i,\ i' \le l\}$. So we have $a_{i1} = 1$ for all i, $\sum_i a_{i1} = l \le W$, and $W \le l^2$. Since $\gamma_i = 1/d = \beta_1$, the desired assertions hold in this case. The case $G = \mathbf{T}$ is similarly shown.

Second suppose $G = \langle g \rangle$, a cyclic group. Set $\alpha = \alpha_g$. By the argument preceding Lemma 2.5, we may assume that α is transitive on the minimal central projections $f_1, \dots, f_m$ of B; more precisely $\alpha^j(f_1) = f_{j+1}$, $1 \le j < m$, and $\alpha^m(f_1) = f_1$. By the proof of Lemma 2.5 we have

$$B^\alpha = \left\{ \bigoplus_{j=0}^{m-1} \alpha^j(x) \colon x \in (Bf_1)^{\alpha^m} \right\} \cong (Bf_1)^{\alpha^m}$$

and $\alpha^m|_{Bf_1} = \mathrm{Ad}\, U$ with a unitary U in Bf_1. Take the spectral decomposition $U = \sum_{i=1}^{l} \lambda_i p_i$, so that $\mathrm{Eig}(\alpha^m|_{Bf_1}) = \{\lambda_i\lambda_{i'}^{-1} \colon 1 \le i,\ i' \le l\}$. Since $a_{ij} = 1$ for all i, j, we have $\sum_j a_{ij} = m$ and $\sum_i a_{ij} = l$. But $m \le W$ is seen because α acts on $\bigoplus_{j=1}^{m} \mathbf{C} f_j \cong \mathbf{C}^m$ as a cyclic permutation matrix. If $\alpha^m(x) = \lambda x$ for $0 \ne x \in Bf_1$, then

$$\alpha\left(\sum_{j=0}^{m-1} \lambda^{-j/m}\alpha^j(x)\right) = \lambda^{1/m}\sum_{j=0}^{m-1} \lambda^{-j/m}\alpha^j(x),$$

showing $\lambda^{1/m} \in \mathrm{Eig}(\alpha)$. This implies that $l \le \#\,\mathrm{Eig}(\alpha^m|_{Bf_1}) \le W$. Also $\gamma_i/\beta_j = m$ for all i, j. Moreover, if $\alpha(x) = \mu x$ for $0 \ne x \in B$, then $\alpha(xf_j) = \mu x f_{j+1}$ for $1 \le j < m$ and $\alpha(xf_m) = \mu x f_1$, implying $\alpha^m(xf_1) = \mu^m x f_1$ with $xf_1 \ne 0$. This shows that $W \le \#\,\mathrm{Eig}(\alpha^m|_{Bf_1}) m \le l^2 m$. Hence the desired assertions hold in case of a cyclic group.

Finally suppose $G = G_1 \times \cdots \times G_N$ and so $\hat{G} = \hat{G}_1 \times \cdots \times \hat{G}_N$, where each G_n is $\mathbf{R}$ or $\mathbf{T}$ or a cyclic group. Let $B_0 = B$ and for $1 \le n \le N$, B_n be the fixed point algebra of $\alpha_{G_1\times\cdots\times G_n}$. Then $B^\alpha = B_N \subset \cdots \subset B_1 \subset B$. Let $\Gamma^{(n)} = [a_{ij}^{(n)}]$ be the inclusion matrix of $B_n \subset B_{n-1}$ for $1 \le n \le N$. Since $B_n = (B_{n-1})^{\alpha_{G_n}}$, the above cases show that $\sum_j a_{ij}^{(n)} \le \#\,\mathrm{Eig}(\alpha_{G_n}|_{B_{n-1}})$ and $\sum_i a_{ij}^{(n)} \le \#\,\mathrm{Eig}(\alpha_{G_n}|_{B_{n-1}})$ for all n, i, j. We further get $\#\,\mathrm{Eig}(\alpha_{G_n}|_{B_{n-1}}) \le W$ for all n, because the eigenspaces of $\alpha_{G_n}|_{B_{n-1}}$ is invariant for any $\alpha_{G_{n'}}$, $n' \ne n$. Since $[a_{ij}] = \Gamma^{(N)}\Gamma^{(N-1)}\cdots\Gamma^{(1)}$, these estimates yields (1) and (2). Also (3) is easily checked. The last assertion is already shown in the above cases of $N = 1$. $\square$

Theorem 3.6. *Let G be of the form $\mathbf{R}^d \times \mathbf{T}^{d'} \times G_0$ with nonnegative integers d, d' and a finitely generated abelian group G_0.*

(1) *Assume that $\alpha|_{\bigcup_n M'\cap M_n}$ has subexponential growth. Then $\|\Gamma_{N,M}\| \le \|\Gamma_{\tilde{N},\tilde{M}}\|$ and $\lim_{n\to\infty} \frac{1}{n} J(M' \cap M_n) \le \lim_{n\to\infty} \frac{1}{n} J(\tilde{M}' \cap \tilde{M}_n)$. Hence, if $\mathcal{G}_{N,M}$ is amenable or strongly amenable, then so is $\mathcal{G}_{\tilde{N},\tilde{M}}$, respectively. The converse holds true as well if α satisfies* (B).

(2) *Assume that α satisfies* (B). *If both $\mathcal{G}_{N,M}$ and $\alpha|_{\bigcup_n M'\cap M_n}$ have polynomial growth or subexponential growth, then so does $\mathcal{G}_{\tilde{N},\tilde{M}}$, respectively. Moreover, the converse holds true if G is $\mathbf{R}$ or $\mathbf{T}$ or a cyclic group.*

Proof. (1) Put $W_n = \#\,\mathrm{Eig}(\alpha|_{M'\cap M_n})$, $n \ge 0$. Then for each $n \ge 0$, (1) and (2) of Lemma 3.5 imply (1)-(3) of Lemma 2.7 for the inclusion matrix of $(M' \cap M_n)^\alpha \subset M' \cap M_n$, where W is replaced by W_n^N. Hence the proof of Theorem 2.8(3) gives

$$\dim M' \cap M_n \le W_n^{4N} \dim (M' \cap M_n)^\alpha \le W_n^{4N} \dim \tilde{M}' \cap \tilde{M}_n$$

by Lemma 3.2(1). Also the proof of Theorem 2.8(4) gives

$$J(M'\cap M_n)\le J\left((M'\cap M_n)^\alpha\right)+\log W_n^{4N}\le J\left(\tilde{M}'\cap\tilde{M}_n\right)+\log W_n^{4N}$$

thanks to Lemma 1.7. Since $\lim_{n\to\infty}W_n^{1/n}=1$ by assumption, we have $\|\Gamma_{N,M}\|\le\|\Gamma_{\tilde{N},\tilde{M}}\|$ and $\lim_{n\to\infty}\frac{1}{n}J(M'\cap M_n)\le\lim_{n\to\infty}\frac{1}{n}J(\tilde{M}'\cap\tilde{M}_n)$, showing the first part. The converse assertion under (B) is immediate from Corollary 3.4(2).

(2) By (1)-(3) of Lemma 3.5 and Corollary 3.4(2), we have (1) and (2) of Theorem 2.8 with W_n^N in place of W; namely

$$W_n^{-N}(\#K_n)\le\#\tilde{K}_n\le W_n^N(\#K_n), \tag{3.5}$$

$$\max_{k\in K_n}s_k\le\max_{k\in\tilde{K}_n}\tilde{s}_k\le W_n^{2N}\max_{k\in K_n}s_k. \tag{3.6}$$

These imply the first assertion. Conversely, assume that G is $\mathbf{R}$ or $\mathbf{T}$ or a cyclic group and that $\mathcal{G}_{\tilde{N},\tilde{M}}$ has subexponential growth (resp. polynomial growth). Then Lemma 3.5(4) yields

$$W_{2n}\le\left(\#\tilde{K}_n\right)^2\max_{k\in\tilde{K}_n}\tilde{s}_k\Big/\min_{k\in K_n}s_k=\left(\#\tilde{K}_n\right)^2\max_{k\in\tilde{K}_n}\tilde{s}_k$$

thanks to $\min_{k\in K}s_k=1$. This implies that $\alpha|_{\bigcup_n M'\cap M_n}$ has subexponential growth (resp. polynomial growth). Hence so does $\mathcal{G}_{N,M}$ too by (3.5) and (3.6). □

In what follows, we assume that $N\subset M$ has finite depth. Then it is known [9, 4.6.3] that there exists $n_0\in\mathbf{N}$ such that

$$M'\cap M_{n+1}=\mathrm{span}((M'\cap M_n)e_{n+1}(M'\cap M_n)),\quad n\ge n_0. \tag{3.7}$$

Set $A=M'\cap M_{n_0}$ and $p_n=e_{n_0+n}$, $n\ge1$, for brevity. Our aim below is to prove that $\alpha|_{\bigcup_n M'\cap M_n}$ automatically polynomial growth in this case. We need the following lemmas.

Lemma 3.7. *With the assumption and the notations above,*

(1) $[A,p_n]=0$, $n\ge2$,

(2) $p_1Ap_1\subset p_1A$,

(3) $p_2p_1Ap_2=p_2A$,

(4) $p_np_{n-1}\cdots p_1Ap_n=p_np_{n-2}\cdots p_1A$, $n\ge3$.

Proof. (1) is trivial from $e_n\in M'_{n-2}\cap M_n$, and (2) follows from

$$e_{n_0+1}Ae_{n_0+1}=e_{n_0+1}E_{n_0}(A)\subset e_{n_0+1}A.$$

We have (3) and (4) because

$$\begin{aligned}
&e_{n_0+n}(e_{n_0+n-1}\cdots e_{n_0+1}A)e_{n_0+n}\\
&= e_{n_0+n}E_{n_0+n-1}(e_{n_0+n-1}\cdots e_{n_0+1}A)\\
&= e_{n_0+n}E_{n_0+n-1}(e_{n_0+n-1})e_{n_0+n-2}\cdots e_{n_0+1}A\\
&= e_{n_0+n}e_{n_0+n-2}\cdots e_{n_0+1}A.
\end{aligned}$$

□

Lemma 3.8. *For every $n \geq 1$,*

$$M' \cap M_{n_0+n} = \operatorname{span}(Ap_1Ap_2p_1A\cdots Ap_np_{n-1}\cdots p_1A). \tag{3.8}$$

Proof. By induction on n. The case $n=1$ is (3.7). Suppose that (3.8) holds for $n-1$. Then by (3.7) and repeated use of Lemma 3.7, we compute as follows:

$$\begin{aligned}
&M' \cap M_{n_0+n}\\
&= \operatorname{span}((Ap_1Ap_2p_1A\cdots Ap_{n-1}\cdots p_1A)p_n(Ap_1Ap_2p_1A\cdots Ap_{n-1}\cdots p_1A))\\
&= \operatorname{span}(Ap_1A\cdots Ap_{n-1}\cdots p_3(p_2p_1Ap_1Ap_2)p_1Ap_3p_2p_1A\cdots A\\
&\qquad\qquad p_{n-2}\cdots p_1Ap_n\cdots p_1A)\\
&= \operatorname{span}(Ap_1A\cdots Ap_{n-1}\cdots p_3p_2(A)p_1Ap_3p_2p_1A\cdots A\\
&\qquad\qquad p_{n-2}\cdots p_1Ap_n\cdots p_1A)\\
&= \operatorname{span}(Ap_1A\cdots Ap_{n-1}\cdots p_4(p_3p_2p_1Ap_3)p_2p_1Ap_4\cdots p_1A\cdots A\\
&\qquad\qquad p_{n-2}\cdots p_1Ap_n\cdots p_1A)\\
&= \operatorname{span}(Ap_1A\cdots Ap_{n-1}\cdots p_3(p_1A)p_2p_1Ap_4\cdots p_1A\cdots Ap_n\cdots p_1A)\\
&= \operatorname{span}(Ap_1A\cdots Ap_{n-2}\cdots p_2(p_1Ap_1A)p_{n-1}\cdots p_5(p_4\cdots p_1Ap_4)p_3p_2p_1A\\
&\qquad\qquad p_5\cdots p_1A\cdots Ap_{n-2}\cdots p_1Ap_n\cdots p_1A)\\
&= \operatorname{span}(Ap_1A\cdots Ap_{n-2}\cdots p_1Ap_{n-1}\cdots p_4(p_2p_1A)p_3p_2p_1Ap_5\cdots p_1A\\
&\qquad\qquad \cdots Ap_{n-2}\cdots p_1Ap_n\cdots p_1A)\\
&= \operatorname{span}(Ap_1A\cdots Ap_{n-2}\cdots p_3(p_2p_1Ap_2p_1A)p_{n-1}\cdots p_1Ap_5\cdots p_1A\\
&\qquad\qquad \cdots Ap_{n-2}\cdots p_1Ap_n\cdots p_1A)\\
&= \operatorname{span}(Ap_1A\cdots Ap_{n-2}\cdots p_1Ap_{n-1}\cdots p_6(p_5\cdots p_1Ap_5)p_4\cdots p_1A\\
&\qquad\qquad \cdots Ap_{n-2}\cdots p_1Ap_n\cdots p_1A).
\end{aligned}$$

Continuing the above process, we arrive at

$$M' \cap M_{n_0+n}$$

$$
\begin{aligned}
&= \operatorname{span}(Ap_1A\cdots Ap_{n-2}\cdots p_1Ap_{n-1}(p_{n-2}\cdots p_1Ap_{n-2})p_{n-3}\cdots p_1A \\
&\qquad\qquad p_n\cdots p_1A) \\
&= \operatorname{span}(Ap_1A\cdots Ap_{n-2}\cdots p_1Ap_{n-1}p_{n-2}(p_{n-4}\cdots p_1A)p_{n-3}\cdots p_1A \\
&\qquad\qquad p_n\cdots p_1A) \\
&= \operatorname{span}(Ap_1A\cdots Ap_{n-2}p_{n-3}(p_{n-4}\cdots p_1Ap_{n-4})p_{n-5}\cdots p_1A \\
&\qquad\qquad p_{n-1}\cdots p_1Ap_n\cdots p_1A) \\
&= \operatorname{span}(Ap_1A\cdots Ap_{n-3}p_{n-4}p_{n-5}(p_{n-6}\cdots p_1Ap_{n-6})p_{n-7}\cdots p_1A \\
&\qquad\qquad p_{n-2}\cdots p_1Ap_{n-1}\cdots p_1Ap_n\cdots p_1A).
\end{aligned}
$$

Repeat the final step in the above for n even or odd separately. Then (3.8) for n is obtained. ∎

Theorem 3.9. *If $N \subset M$ has finite depth, then $\alpha|_{\bigcup_n M'\cap M_n}$ has polynomial growth.*

Proof. Let A and p_n be as above. Since α acts as a unitary group on the Hilbert space A equipped with the inner product induced by ϕ, we can choose $\{t_j\}_{j=1}^m$ in $\hat{G}$ and an orthonormal basis $\{x_j\}_{j=1}^m$ of A such that $\alpha_g(x_j) = \langle g, t_j\rangle x_j$ for all $g \in G$ and $1 \le j \le m$. Since $\alpha_g(p_n) = p_n$, we get

$$
\begin{aligned}
&\alpha_g(x_{j_0}p_1x_{j_1}p_2p_1x_{j_2}\cdots x_{j_{n-1}}p_n\cdots p_1x_{j_n}) \\
&\quad = \langle g, t_{j_0}t_{j_1}\cdots t_{j_n}\rangle x_{j_0}p_1x_{j_1}p_2p_1x_{j_2}\cdots x_{j_{n-1}}p_n\cdots p_1x_{j_n}
\end{aligned}
$$

for any $j_0, \dots, j_n \in \{1, \dots, m\}$. Lemma 3.8 means that

$$
\begin{aligned}
&M'\cap M_{n_0+n} \\
&\quad = \operatorname{span}\{x_{j_0}p_1x_{j_1}p_2p_1x_{j_2}\cdots x_{j_{n-1}}p_n\cdots p_1x_{j_n} : j_0, \dots, j_n \in \{1, \dots, m\}\},
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\operatorname{Eig}\left(\alpha|_{M'\cap M_{n_0+n}}\right) &\subset \left\{\prod_{k=0}^n t_{j_k} : j_0, \dots, j_n \in \{1, \dots, m\}\right\} \\
&= \left\{\prod_{j=0}^m t_j^{n_j} : n_1, \dots, n_m \ge 0,\ \sum_{j=1}^m n_j = n+1\right\}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\#\operatorname{Eig}\left(\alpha|_{M'\cap M_{n_0+n}}\right) \\
&\quad \le \#\left\{(n_1, \dots, n_m) : n_1, \dots, n_m \ge 0,\ \sum_{j=1}^m n_j = n+1\right\}
\end{aligned}
$$

$$\leq (n+2)^m \leq (n_0+n+1)^m,$$

completing the proof. □

By Theorems 3.6 and 3.9 we obtain:

Corollary 3.10. *Let G be as in Theorem 3.6. Assume that $N \subset M$ has finite depth. Then $\mathcal{G}_{\tilde{N},\tilde{M}}$ is strongly amenable. Furthermore, $\mathcal{G}_{\tilde{N},\tilde{M}}$ has polynomial growth whenever α satisfies* (B).

The results stated in Theorem 3.6 and Corollary 3.10 can be reversed, where $(N \subset M, \alpha)$ and $(\tilde{N} \subset \tilde{M}, \hat{\alpha})$ are interchanged with (C) instead of (B).

4. Type II and type III invariants.

In this section, we apply the results of Section 3 to compare the type II and type III standard invariants for inclusions of type III factors. Let us consider either of the following two cases:

Case 1. Let $N \subset M$ be an inclusion of type III_1 factors with finite index and set $\sigma = \sigma^{\psi \circ E_0}$, the modular automorphism group, where ψ is a faithful normal state on N and $E_0\colon M \to N$ is the minimal conditional expectation. Since $\sigma|_N = \sigma^\psi$, the inclusion $\tilde{N} \subset \tilde{M}$ of type II_∞ factors is defined by

$$\left(\tilde{N} \subset \tilde{M}\right) = (N \rtimes_\sigma \mathbf{R} \subset M \rtimes_\sigma \mathbf{R}).$$

Then the Takesaki duality says that

$$(N \subset M) \cong \left(\tilde{N} \rtimes_\theta \mathbf{R} \subset \tilde{M} \rtimes_\theta \mathbf{R}\right),$$

where θ is the dual action of σ.

Case 2. Let $N \subset M$ be an inclusion of type III_λ factors $(0 < \lambda < 1)$ with finite index. Assume that $N \subset M$ admits a common discrete decomposition:

$$(N \subset M) = \left(\tilde{N} \rtimes_\theta \mathbf{Z} \subset \tilde{M} \rtimes_\theta \mathbf{Z}\right).$$

Here $\tilde{N} \subset \tilde{M}$ is an inclusion of type II_∞ factors and θ is the dual automorphism of the modular action σ with the period $T = -2\pi/\log\lambda$. Hence

$$\left(\tilde{N} \subset \tilde{M}\right) \cong (N \rtimes_\sigma (\mathbf{R}/T\mathbf{Z}) \subset M \rtimes_\sigma (\mathbf{R}/T\mathbf{Z})).$$

Note [**29**, **30**] that an irreducible inclusion $N \subset M$ of type III_λ automatically has a common discrete decomposition, but not in general.

In the above cases, the canonical extension $\tilde{E}_0\colon \tilde{M} \to \tilde{N}$ of the minimal conditional expectation E_0 is the conditional expectation with respect to the canonical trace tr on $\tilde{M}$ ($\mathrm{tr}\,|_{\tilde{N}}$ is the canonical trace on $\tilde{N}$). Thus we can write

$$\left(\tilde{N} \subset \tilde{M}\right) = (B \otimes \mathbf{B}(\mathcal{H}) \subset A \otimes \mathbf{B}(\mathcal{H}))$$

with an extremal inclusion $B \subset A$ of type II_1 factors. The Jones tower $\{M_n\}$ for $N \subset M$ is identified with $\{\tilde{M}_n \rtimes_\theta \mathbf{R}\}$ (or $\{\tilde{M}_n \rtimes_\theta \mathbf{Z}\}$), where $\{\tilde{M}_n\}$ is the Jones tower for $\tilde{N} \subset \tilde{M}$.

Since θ is trace-scaling, i.e. $\mathrm{tr}\circ\theta = e^{-s}\,\mathrm{tr}$, $s \in \mathbf{R}$ (or $\mathrm{tr}\circ\theta = \lambda\,\mathrm{tr}$), it is seen [**40**, 1.6] that θ_t, $t \neq 0$, are strongly outer (or θ is a strongly outer action of $\mathbf{Z}$) on $\tilde{N} \subset \tilde{M}$. It was observed in [**28**, **29**] (see also Corollary 3.4) that

$$\left(\tilde{M}' \cap \tilde{M}_n\right)^\theta = M' \cap M_n, \qquad n \geq 0.$$

When $N = \rho(M)$ for some $\rho \in \mathrm{End}(M)$, Proposition 3.3 shows that the growth of $\theta|_{\bigcup_n \tilde{M}' \cap \tilde{M}_n}$ is determined by that of $\#\{t \in [0,T)\colon \sigma_t \prec (\rho\bar{\rho})^n\}$, $n \geq 0$, where $T = \infty$ in Case 1 and $T = -2\pi/\log\lambda$ in Case 2.

In this way, we are in the situation supposed in Section 3. We write $\mathcal{G}_{\mathrm{III}} = \mathcal{G}_{N,M}$ and $\mathcal{G}_{\mathrm{II}} = \mathcal{G}_{\tilde{N},\tilde{M}}$, and refer to them as the type III and the type II standard invariants of $N \subset M$, respectively. Before stating the theorem, we recall some known results concerning the difference of type II and type III standard invariants.

1° Let $N \subset M$ be as in Case 1 or Case 2 above and assume that $N = \rho(M)$ with $\rho \in \mathrm{End}(M)$. The type II and type III principal graphs of $N \subset M$ are different if and only if a modular automorphism σ_t ($t \notin T(M)$) appears in $\bigsqcup_n (\rho\bar{\rho})^n$. Thus $\mathcal{G}_{\mathrm{II}} = \mathcal{G}_{\mathrm{III}}$ if $N \subset M$ is a type III_1 inclusion with finite depth. These were proved in [**18**] (and seen from Proposition 3.3). Moreover, Izumi [**19**] announced a corresponding result in the type III_0 case in terms of "modular endomorphisms".

2° If $N \subset M$ is a type III_1 inclusion whose type II principal graph (or dual principal graph) include no circle, then $\mathcal{G}_{\mathrm{II}} = \mathcal{G}_{\mathrm{III}}$ (see the proof of [**28**, Corollary 7]). Thus, concerning $N \subset M$ of type III_1 with $[M\colon N]_0 \leq 4$, the difference of type II and type III graphs occurs only when $N \subset M$ is a locally trivial inclusion having the type II graph $A_n^{(1)}$ and the type III graph $A_{\infty,\infty}$ (see [**30**, **45**]).

3° Let $N \subset M$ be of type III_λ with a common discrete decomposition. In case of index less than 4, a graph change occurs only when the type II graph is D_{2n} and the type III graph is A_{4n-3} (see [**28**]). In case of index 4, there are a variety of graph changes as was listed in [**30**, Theorem 4.4].

4° The coincidence of $\mathcal{G}_{\mathrm{II}}$ and $\mathcal{G}_{\mathrm{III}}$ is obviously necessary for a type III inclusion to split as a type II_1 inclusion tensored with a common type III factor. According to [**29**, **22**], if $N \subset M$ is an AFD type III_λ inclusion of finite depth with a common discrete decomposition and $\mathcal{G}_{\mathrm{II}} = \mathcal{G}_{\mathrm{III}}$, then there exists an AFD type II_1 inclusion $B \subset A$ such that $(N \subset M) \cong (B \otimes R_\lambda \subset A \otimes R_\lambda)$, R_λ being the AFD type III_λ factor. The splitting theorem in the final form was recently presented by Popa [**42**] under the strong amenability condition including the type III_1 case.

5° Note [**36**] that if $N \subset M$ is of type II_1 such that $4 < [M\colon N] < 3+2\sqrt{2}$ and $N' \cap M \neq \mathbf{C}1$, then $N \subset M$ is locally trivial and $[M\colon N]_0 = 4$. According to [**11**], there is a small possibility for (dual) principal graphs of irreducible type II_1 inclusion $N \subset M$ with $4 < [M\colon N] < 3+\sqrt{3}$. In fact, the graphs of such inclusions are restricted to A_∞ or a few series of finite graphs. Furthermore, in the finite depth case, the graphs for each possible index value are just a pair of principal and dual principal ones. These results show by Proposition 1.2 that if $N \subset M$ is an inclusion of arbitrary factors with $4 < [M\colon N]_0 < 3 + \sqrt{3}$, then $N \subset M$ is irreducible and its graph receives the same restriction as in [**11**]. Let $N \subset M$ be as in Case 1 or Case 2. It is clear that when the type II or type III graph of $N \subset M$ is A_∞, a graph change is impossible. Also, it is impossible that the type II and type III graphs of $N \subset M$ are a dual pair of different ones, as is obvious from $\dim N' \cap M_{n-1} = \dim M' \cap M_n$. So we see that $\mathcal{G}_{\mathrm{II}} = \mathcal{G}_{\mathrm{III}}$ whenever $4 < [M\colon N]_0 < 3 + \sqrt{3}$.

Now we state the next theorem, which is immediate from the results of Section 3.

Theorem 4.1. *Let $N \subset M$ be an inclusion of type* III *factors in Case* 1 *or Case* 2*. Then:*

(1) *If $\mathcal{G}_{\mathrm{III}}$ of $N \subset M$ is of finite depth, amenable, or strongly amenable, then so is $\mathcal{G}_{\mathrm{II}}$, respectively.*

(2) *If $\theta|_{\bigcup_n \tilde{M}' \cap \tilde{M}_n}$ has subexponential growth, then $\mathcal{G}_{\mathrm{III}}$ of $N \subset M$ is amenable or strongly amenable if and only if so is $\mathcal{G}_{\mathrm{II}}$, respectively.*

(3) *$\mathcal{G}_{\mathrm{III}}$ of $N \subset M$ has polynomial growth or subexponential growth if and only if so do both $\mathcal{G}_{\mathrm{II}}$ and $\theta|_{\bigcup_n \tilde{M}' \cap \tilde{M}_n}$, respectively.*

(4) *If the type* II *inclusion $\tilde{N} \subset \tilde{M}$ has finite depth, then $\mathcal{G}_{\mathrm{III}}$ of $N \subset M$ has polynomial growth* (*hence it is strongly amenable*).

The following example shows that the above (4) is best possible.

Example 4.2. Consider locally trivial inclusions determined by modular automorphisms. Let P be a type III_1 factor and σ the modular automorphism group with respect to a faithful normal state φ_0 on P. Choose $r_0 = 0$, $r_1, \dots, r_m \in \mathbf{R}$ and define

$$N = \left\{ \sum_{i=0}^{m} \sigma_{r_i}(x) \otimes e_{ii} \colon x \in P \right\} \subset M = P \otimes M_{m+1}(\mathbf{C}),$$

where $\{e_{ij}\}_{0 \le i,j \le m}$ is the matrix units of $M_{m+1}(\mathbf{C})$. Then $[M : N]_0 = (m+1)^2$ and the minimal conditional expectation $E_0 \colon M \to N$ is given by

$$E_0 \left(\sum_{i,j=0}^{m} x_{ij} \otimes e_{ij} \right) = \sum_{i=0}^{m} \sigma_{r_i} \left(\frac{1}{m+1} \sum_{j=0}^{m} \sigma_{r_j}^{-1}(x_{jj}) \right) \otimes e_{ii}.$$

Set $\psi(\sum_{i=0}^{m} \sigma_{r_i}(x) \otimes e_{ii}) = \varphi_0(x)$, $x \in P$, and $\varphi = \psi \circ E_0$. Since $\varphi = \varphi_0 \otimes \tau$ with the normalized trace τ on $M_{m+1}(\mathbf{C})$ and so $\sigma^{\varphi} = \sigma \otimes \mathrm{id}$, it is easy to see that the type II inclusion $(\tilde{N} \subset \tilde{M}) = (N \rtimes_{\sigma^{\psi}} \mathbf{R} \subset M \rtimes_{\sigma^{\varphi}} \mathbf{R})$ is given as follows:

$$\tilde{N} = \left\{ \sum_{i=0}^{m} \tilde{\sigma}_{r_i}(\tilde{x}) \otimes e_{ii} \colon \tilde{x} \in \tilde{P} \right\} \subset \tilde{M} = \tilde{P} \otimes M_{m+1}(\mathbf{C}),$$

where $\tilde{P} = P \rtimes_{\sigma} \mathbf{R}$ and $\tilde{\sigma}_r$ is the canonical extension of σ_r, i.e. $\tilde{\sigma}_r(x) = \sigma_r(x)$, $x \in P$, and $\tilde{\sigma}_r(\lambda(t)) = \lambda(t)$, $t \in \mathbf{R}$. Since $\tilde{\sigma}_r \in \mathrm{Int}(\tilde{P})$, it follows that

$$\left(\tilde{N} \subset \tilde{M} \right) \cong \left(\tilde{P} \otimes \mathbf{C}1 \subset \tilde{P} \otimes M_{m+1}(\mathbf{C}) \right)$$

and hence $\tilde{N} \subset \tilde{M}$ has depth 1. On the other hand, the standard invariants of locally trivial inclusions were computed in [**1**], [**39**, 5.1.5], and [**45**, **46**]. In our setting, $N \subset M$ has infinite depth. Indeed, choose isometries $v_0, \dots, v_m$ in P with $\sum_{i=0}^{m} v_i v_i^* = 1$ and define $\rho \in \mathrm{End}(P)$ by $\rho(x) = \sum_{i=0}^{m} v_i \sigma_{r_i}(x) v_i^*$. Then we can readily see that $(N \subset M) \cong (\rho(P) \subset P)$ and $[\rho] = [\mathrm{id}] \oplus [\sigma_{r_i}] \oplus \cdots \oplus [\sigma_{r_m}]$. Hence the irreducible components of $[(\rho\bar{\rho})^n]$ are

$$\left\{ [\sigma_r] \colon r = \sum_{i,j=1}^{m} n_{ij}(r_i - r_j),\ n_{ij} = 0, 1, 2, \dots, \sum_{i,j=1}^{m} n_{ij} = n \right\}.$$

In particular, if $r_1, \dots, r_m$ are linearly independent over the rationals, then $\theta|_{\bigcup \tilde{M}' \cap \tilde{M}_n}$ has the polynomial growth of exactly order n^m. Also, let P be a type III_λ factor $(0 < \lambda < 1)$ with φ_0 a λ-trace. Then $N \subset M$ is an example of Case 2, and we have the same conclusion when $r_1, \dots, r_m$, $-2\pi/\log\lambda$ are linearly independent over the rationals.

We end with stability properties for a type III inclusion to have the same $\mathcal{G}_{\mathrm{II}}$ and $\mathcal{G}_{\mathrm{III}}$.

Proposition 4.3. *Let $N \subset M$ be an inclusion of AFD type* III_1 *factors with finite index. Assume that $N \subset M$ has the same type* II *and type* III *principal graphs. Then:*

(1) *Any intermediate inclusion $Q \subset P$ of factors with $N \subset Q \subset P \subset M_n$ has the same type* II *and type* III *graphs.*

(2) *Any descendant inclusion $Np \subset pM_np$ with a projection $p \in N' \cap M_n$ has the same type* II *and type* III *graphs.*

Proof. By assumption of AFD, all factors in question are isomorphic, so that we are free to use the sector technique. For $\rho \in \mathrm{End}(M)$, we say that ρ (or $\rho(M) \subset M$) satisfies (#) if $\mathcal{G}_{\mathrm{II}} = \mathcal{G}_{\mathrm{III}}$ for $\rho(M) \subset M$, that is, any modular automorphism σ_t ($t \neq 0$) does not appear in $\bigsqcup_n (\rho\overline{\rho})^n$. In the following, let $\rho, \eta \in \mathrm{End}(M)$ with $d(\rho), d(\eta) < \infty$. First let us show:

(a) If ρ satisfies (#), then so does $\overline{\rho}$.

(b) If ρ satisfies (#) and if $\eta \prec (\rho\overline{\rho})^n$ or $\eta \prec (\rho\overline{\rho})^n\rho$ for some $n \geq 0$, then η satisfies (#).

(c) If $\rho\eta$ satisfies (#), then so does ρ.

Since $(M \subset M_1) \cong (\overline{\rho}(M) \subset M)$ where M_1 is the basic construction for $N = \rho(M) \subset M$ (see [**33**, p. 296]), it suffices for (a) to show that the strong outerness of σ_t on $N \subset M$ is equivalent to that of σ_t on $M \subset M_1$. But this is immediate from the equivalence of conditions in Definition 1.10. If $\eta \prec (\rho\overline{\rho})^n$ or $\eta \prec (\rho\overline{\rho})^n\rho$, then the irreducible components of $\bigsqcup_k (\eta\overline{\eta})^k$ are contained in $\bigsqcup_k (\rho\overline{\rho})^k$. Hence (b) holds. Since $\rho\overline{\rho} \prec (\rho\eta)(\overline{\rho\eta})$, (c) follows.

(1) Let $N = \rho(M) \subset M$ with $\rho \in \mathrm{End}(M)$ and assume that ρ satisfies (#). Since

$$(M \subset M_{2n}) \cong ((\rho\overline{\rho})^n(M) \subset M), \quad (M \subset M_{2n+1}) \cong ((\overline{\rho}\rho)^n\overline{\rho}(M) \subset M),$$

it follows from (a) and (b) that $M \subset M_n$ satisfies (#) for all $n \geq 0$. Since $(N_1 \subset N) \cong (\overline{\rho}(M) \subset M)$, $N \subset M_n$ also satisfies (#) for all $n \geq 0$. Hence it is enough to show that if $N \subset P \subset M$ then $P \subset M$ and $N \subset P$ satisfy (#). Write $P = \rho(M)$ and $N = \rho_1(M)$ with $\rho, \rho_1 \in \mathrm{End}(M)$, and set $\eta = \rho^{-1}\rho_1$. Then $\eta \in \mathrm{End}(M)$ and $\rho\eta = \rho_1$ satisfies (#) by assumption. Hence (c) implies that ρ (i.e. $P \subset M$) satisfies (#). Moreover, it is easy to see that $N \subset M$ satisfies (#) if and only if so does $M' \subset N'$. So the above case can be applied to $M' \subset P' \subset N'$, so that $N \subset P$ satisfies (#).

(2) The case $n = 1$ is enough by (1). For any projection $p \in N' \subset M$, there exists $\eta \in \mathrm{End}(M)$ such that $\eta \prec \rho$ and $(Np \subset pMp) \cong (\eta(M) \subset M)$. Hence (b) shows the result. $\square$

The above proposition holds true also when $N \subset M$ is an inclusion of AFD type III_λ factors with a common discrete decomposition. But we assume for (1) that $Q \subset P$ is of type III_λ and has a common discrete decomposition too. When $N \subset M$ actually splits into the form $B \otimes L \subset A \otimes L$ with a type II_1 inclusion $B \subset A$, it is rather trivial that $Q \subset P$ and $Np \subset pM_n p$ in Proposition 4.3 split by the same L.

For an inclusion $N \subset M$ of type III factors such that $Z(M \rtimes_\sigma \mathbf{R}) = Z(N \rtimes_\sigma \mathbf{R})$ and $N = \rho(M)$ for some $\rho \in \mathrm{End}(M)$, Kosaki [**27**] recently introduced the *relative T-set* $T(M,N)$ by

$$T(M,N) = \left\{ t \in \mathbf{R} \colon \sigma_t \prec \bigsqcup_n (\rho\bar{\rho})^n \right\}.$$

When $N \subset M$ is as in Case 1 or Case 2, the above 1° mean that $\mathcal{G}_{\mathrm{II}} = \mathcal{G}_{\mathrm{III}}$ if and only if $T(M,N) = T(M)$. Furthermore, the proof of Proposition 4.3 shows that $T(P,Q) \subset T(M,N) = T(M_n, M_i)$ for any $i < n$ and $T(pM_np, Np) \subset T(M,N)$.

Let $N^1 \subset M^1$ and $N^2 \subset M^2$ be type III inclusions in Case 1 or Case 2. Then the tensor product inclusion $N^1 \otimes N^2 \subset M^1 \otimes M^2$ is in Case 1 or Case 2 as well. For example, if $N^i \subset M^i$ is of type III_{λ_i} $(0 < \lambda_i < 1,\ i = 1,2)$, then $N^1 \otimes N^2 \subset M^1 \otimes M^2$ is in Case 1 or Case 2 accordingly as $\log \lambda_1 / \log \lambda_2$ irrational or not. Assume that $N^i = \rho_i(M^i)$ for some $\rho_i \in \mathrm{End}(M^i)$, $i = 1,2$. Then the following was shown in [**27**] by using the sector technique:

$$T(M^1 \otimes M^2,\ N^1 \otimes N^2) = T(M^1, N^1) \cap T(M^2, N^2).$$

In particular, this implies the following:

Proposition 4.4. *With the above assumption, if both $N^1 \subset M^1$ and $N^2 \subset M^2$ have the same type* II *and type* III *graphs, then so does $N^1 \otimes N^2 \subset M^1 \otimes M^2$.*

Remark 4.5. The notion of the central freeness for subfactors was introduced in [**41**, **42**], and it was shown in [**42**, Chapters 3, 4] that a type III inclusion $N \subset M$ is centrally free if and only if $\mathcal{G}_{\mathrm{II}} = \mathcal{G}_{\mathrm{III}}$ for $N \subset M$. So Propositions 4.3 and 4.4 are considered as stability properties of the central freeness. Indeed, similar results concerning the central freeness were given in [**41**, **42**].

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