PACIFIC JOURNAL OF MATHEMATICS
Vol. 177, No. 2, 1997

WIENER TAUBERIAN THEOREMS FOR $SL_2(\mathbb{R})$

Rudra P. Sarkar

In this article we prove a Wiener Tauberian theorem for $L^p(SL_2(\mathbb{R}))$, $1 \le p < 2$. Let G be the group $SL_2(\mathbb{R})$ and K its maximal compact subgroup $SO(2,\mathbb{R})$. Let M be $\{\pm I\}$. We show that if the Fourier transforms of a set of functions in $L^p(G)$ do not vanish simultaneously on any irreducible $L^{p-\epsilon}$-tempered representation for some $\epsilon > 0$, where they are assumed to be defined, and if for each M-type at least one of the matrix coefficients of any of those Fourier transforms does not 'decay too rapidly at ∞' in a certain sense, then this set of functions generate $L^p(G)$ as a $L^1(G)$-bimodule. As a key step towards this main theorem we prove a W-T Theorem for L^p-sections of certain line bundles over G/K. W-T theorems on $SL_2(\mathbb{R})$ have been proved so far, for biinvariant L^1 functions and for L^1 functions on the symmetric space $SL_2(\mathbb{R})/SO(2,\mathbb{R})$, where the generator is left K-finite. Our results are on the space of *all* L^p functions (resp. sections), $p \in [1,2)$ of $SL_2(\mathbb{R})$ (resp. of line bundles over $SL_2(\mathbb{R})/SO(2,\mathbb{R})$), without any restriction of K-finiteness on the generators.

1. Introduction.

Let G be the group $SL_2(\mathbb{R})$ and K its maximal compact subgroup $SO(2,\mathbb{R})$. We denote the characters of K by χ_n, $n \in \mathbb{Z}$. A complex valued function f on G is said to be of left (resp. right) K-type n if $f(kx) = \chi_n(k)f(x)$ (resp. $f(xk) = \chi_n(k)f(x)$), for all $k \in K$ and $x \in G$. For a class of functions $\mathcal{F}$ on G (e.g. $L^p(G)$), $\mathcal{F}_n$ will denote the corrsponding subclass of functions of right type n while $\mathcal{F}_{m,n}$ will contain functions which are also of left type m. The subclass of $\mathcal{F}$ consisting of functions with integral zero will be denoted by $\mathcal{F}^0$. Principal and discrete parts of the Fourier transform of a function f will be denoted by $\widehat{f}_H$ and $\widehat{f}_B$ respectively. Unless mentioned otherwise p will lie in $[1,2)$. Most of the time we follow the notation and terminogy of [**Ba**].

In this article we prove a Wiener Tauberian theorem (W-T theorem) for $L^p(SL_2(\mathbb{R}))$, $1 \le p < 2$. As a key step towards the main theorem we prove a W-T Theorem for L^p-sections of certain line bundles over G/K. The W-T Theorem for $SL_2(\mathbb{R})$ was first studied in [**E-M**] where two different versions of the theorem were obtained for biinvariant L^1 functions. More precisely, sufficient conditions for a single function $f \in L^1(G)_{0,0}$ to generate $L^1(G)_{0,0}$ as a closed ideal (under convolution) were established. The second version was extended in [**S**] to determine sufficient conditions on a K-finite function in $L^1(G/K)$ forcing it to generate $L^1(G/K)$ as a closed left $L^1(G)$ module. In a recent paper [**B-W**], sufficient conditions are obtained on a family of functions to generate $L^1(G)_{0,0}$.

Our point of departure in this article is the observation that both of the theorems of [**E-M**] can be extended to $L^p(G)_{n,n}$ for all $n \in \mathbb{Z}$. This is made possible by:

(1) The isomorphism between each L^p-Schwartz space $C^p(G)$ and its Fourier transform $C^p(\widehat{G})$ which respects the spherical types as well as the splitting in continuous and discrete parts ([**T**] and [**Ba**]) and hence allows the continuous and discrete components of a C^p-function to be treated separately.

(2) The fact that for any function of a fixed K-type, only finitely many discrete series representations are relevant.

The W-T theorem for $L^p(G)_{n,n}$ can be extended further to $L^p(G)_n$ to give an analogue of the $L^1(G/K)$ case treated in [**S**], using the techniques used there. However, these techniques do not work unless the generating functions are K-finite and are at most finitely many. But, coming to our problem, it is not difficult to see that one can not generate the whole of $L^p(G)$ by starting from a K-finite function or by finitely many of them. Besides, for generating $L^p(G)$ one has to generate $L^p(G)_n$ for every n. And every $L^p(G)_n$ should be generated by using the full strength of the generating function $f \in L^p(G)$, not simply by the right n type projection of f (which may be even zero!). All the projections of f in various right types should work together to generate $L^p(G)_n$, for a particular n. This rules out the finitely-many-generators model of [**S**] as a starting point of this extension. We bypass this obstacle by taking resort to the result in [**B-W**] for biinvariant L^1 functions. It enables us to prove a W-T Theorem for L^p-sections of certain line bundles over G/K without any K-finite restriction on the generator-sections. This is the intermediate step we mentioned above. Before stating our result we establish some more notation.

For each p let $\gamma = \frac{2}{p} - 1$ and define $\mathcal{S}^\gamma$ by

$$\mathcal{S}^\gamma = \{\lambda \in \mathbb{C} \mid |\Re\lambda| \leq 2/p - 1\}.$$

Let $\mathcal{S}^\gamma_\delta$ denote the augmented strip $\{\lambda \in \mathbb{C} \mid |\Re\lambda| \leq \gamma + \delta\}$ for $\delta > 0$. Let Γ_n denote the integers between 0 and n of parity opposite to n. Then for $f \in L^p(G)_n$ (equivalently for the L^p-sections of the bundle corresponding to

n) the natural domain of the continuous part of the Fourier transform, $\widehat{f}_H$ is $\mathcal{S}^\gamma$ while that of the discrete part $\widehat{f}_B$ is Γ_n.

Theorem 1.1. *Let $\{f^\alpha|\alpha \in \Lambda\}$ be a subset of $L^p(G)_n$, Λ being an index set, such that the Fourier transform $\widehat{f}^\alpha_H$ of each f^α has a holomorphic extension on $\overset{\circ}{\mathcal{S}}{}^\gamma_\delta$ for some $\delta > 0$ and all the matrix coefficients of $\widehat{f}^\alpha_H$ for all α vanish at infinity, that is, $\lim_{|\lambda|\to\infty} |(\widehat{f}^\alpha_H(\lambda))_{m,n}| = 0$ on $\mathcal{S}^\gamma_\delta$. Let there be an $\alpha_0 \in \Lambda$ such that one of its matrix coefficients, say $(\widehat{f}^{\alpha_0}_H)_{m_0,n}$ satisfies moreover the condition on decay at infinity:*

$$(*) \qquad \limsup_{|t|\to\infty} \left| \left(\widehat{f}^{\alpha_0}_H\right)_{m_0,n} (it) e^{K e^{|t|}} \right| > 0 \qquad \textit{for all } K > 0.$$

Also assume that the collections $\{\widehat{f}^\alpha_H|\alpha \in \Lambda\}$ and $\{\widehat{f}^\alpha_B|\alpha \in \Lambda\}$ do not have common zeros on $\mathcal{S}^\gamma_\delta$ and Γ_n respectively. Then the left $L^1(G)$ module generated by $\{f^\alpha|\alpha \in \Gamma\}$ is dense in $L^p(G)_n$.

Moreover, in the case $p = 1$ and $n = 0$, if the collection $\{\widehat{f}^\alpha\}$ does not have any common zero on $\mathcal{S}^1_\delta$ except at ± 1 then the left ideal generated by $\{f^\alpha|\alpha \in \Lambda\}$ is dense in $L^1(G)^0_0$.

From now on, if a function satisfies the above decay condition $(*)$, we will simply write that it does not '*decay too rapidly at* ∞'. Let M be $\{\pm I\} \subset K$ and σ^+ and σ^- denote the trivial and the only nontrivial irreducible representations of M, *i.e.* $\widehat{M} = \{\sigma^+, \sigma^-\}$. Analogous to the K-types, we talk of functions f on G being of M-type σ^+ and σ^-.

We are now in a position to describe our final result where the hypothesis merely demands that the Fourier transforms of the generators do not vanish simultaneously on any irreducible $L^{p-\epsilon}$-tempered representation for some $\epsilon > 0$ and for each M-type at least one of the matrix coefficients of any of the Fourier transforms does not 'decay too rapidly at ∞'.

Theorem 1.2. *Let $\{f^\alpha|\alpha \in \Lambda\}$ be a subset of $L^p(G)$ such that for each $\alpha \in \Lambda$ the Fourier transform $\widehat{f}^\alpha_H$ has holomorphic extension on $\widehat{M} \times \overset{\circ}{\mathcal{S}}{}^\gamma_\delta$ for some $\delta > 0$ and all matrix coefficients $(\widehat{f}^\alpha_H(\sigma, .))_{m,n}$, $\sigma \in \widehat{M}$ and $m, n \in \mathbb{Z}$, satisfy $\lim_{|\lambda|\to\infty} |(\widehat{f}^\alpha_H(\sigma, \lambda))_{m,n}| = 0$ on $\mathcal{S}^\gamma_\delta$. Let two of the matrix coefficients, one from each party, not decay too rapidly at ∞. If $\{\widehat{f}^\alpha_H\}$ and $\{\widehat{f}^\alpha_B\}$ do not have common zero on $\widehat{M} \times \mathcal{S}^\gamma_\delta \cup \{D_+, D_-\}$ and $\mathbb{Z}^*$ respectively, where D_+ and D_- are mock discrete series, then for $p \in (1, 2)$ the $L^1(G)$-bimodule generated by $\{f^\alpha|\alpha \in \Lambda\}$ is dense in $L^p(G)$.*

Moreover, for the case $p = 1$, if there is at least one f^α with nonvanishing integral then the ideal generated by $\{f^\alpha|\alpha \in \Lambda\}$ is dense in $L^1(G)$. Otherwise, *the ideal is dense in* $L^1(G)^0$.

The $\epsilon > 0$ mentioned above the theorem is brought in by the need to choose a slightly larger domain for the Fourier transforms of the generating functions. It is a common feature in all W-T theorems proved so far; if a W-T theorem can be proved for $L^1(G)_{n,n}$ without imposing this condition then the corresponding stronger version of our result will immediately follow. We will come back to this in the concluding remarks.

2. Notation and Preliminaries.

We denote the L^p-Schwartz space of G by $C^p(G)$ [**Ba**, p. 13] and its image under Fourier transform by $C^p(\widehat{G})$. Similarly $L^p(\widehat{G})$ is the image under Fourier transform of $L^p(G)$. Functions of left K-type m and right K-type n are also mentioned as (m,n) type functions. By $f_{m,n}$ we denote the projection of f in left type m and right type n. Notation like f_m are always locally explained and may be consistent only locally. The element,

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

of K will be denoted by k_θ. For details of the parametrization of representations $\{\pi_{\sigma,\lambda} | (\sigma,\lambda) \in \widehat{M} \times \mathbb{C}\}$ of C and their realisation on $L^2(K)$ we refer to [**Ba**]. However there are a few minor variations. We will mention about them in this section. The standard orthonormal basis for $L^2(K)$ will be denoted by e_n, $n \in \mathbb{Z}$, where $e_n(k_\theta) = e^{in\theta}$, $k_\theta \in K$. The (m,n)-th matrix coefficient of the principle series representation $\pi_{\sigma,\lambda}$ will be denoted by $\Phi^{m,n}_{\sigma,\lambda}$. It will vanish when m or n does not belong to $\mathbb{Z}^\sigma$. The discrete series representations which occur as subrepresentations of $\pi_{\sigma,k}$, where k is positive integer, will be denoted by π_k and their matrix coefficients will be denoted by $\Phi^{m,n}_k$. Here the matrix coefficients are with respect to vectors e^k_m and e^k_n, which are suitable multiples of e_m and e_n, so as to have norm 1 in the representation space of π_k. Here we mention two estimates of the matrix coefficients:

(1) For every $x \in G$, $\sigma \in \widehat{M}$, $m,n \in \mathbb{Z}^\sigma$ and $\lambda \in \mathcal{S}^1$ $|\Phi^{m,n}_{\sigma,\lambda}(x)| \le 1$. ([**E-M**], 2.9.)

(2) For $p \in (1,2)$ and for arbitrary but fixed $\epsilon > 0$, there exists $C_\epsilon > 0$, such that $\int_G |\Phi^{m,n}_{s,\lambda}(x)|^q \le C_\epsilon$, for all $x \in G$, $\sigma \in \widehat{M}$, $m,n \in \mathbb{Z}^\sigma$ and $\lambda \in \mathcal{S}^{\gamma-\epsilon}$.

Here C_ϵ depends on ϵ and $\frac{1}{p} + \frac{1}{q} = 1$. This result can also be stated as: If an admissible representation is L^p tempered, then its matrix coefficients are in $L^{q+\delta}$ for any $\delta > 0$. (Follows from Theorem 4.1 of [**Ba**], which can be considered as the definition of L^p-temperedness and from the inequalities (3.3) and (3.4) of [**Ba**].)

The representation $\pi_{\sigma^-,0}$ has two irreducible subrepresentations, so called mock discrete series. We will denote them by D_+ and D_-. The representation spaces of D_+ and D_- contain $e_n \in L^2(K)$ respectively for positive odd n's and negative odd n's where $e_n(k_\theta) = e^{in\theta}$. For a function f, $\widehat{f}_H(D_+) \neq 0$ (resp. $\widehat{f}_H(D_-) \neq 0$) will mean that $\widehat{f}_H(\sigma^-, 0)$ has a nonzero matrix coefficient, say, $(\widehat{f}_H(\sigma^-, 0))_{m,n}$ where e_m and e_n are in D_+ (resp. D_-).

The space $C^p_H(\widehat{G})_{m,n}$ for $m, n \in \mathbb{Z}$, consists of the continuous maps $F\colon \mathcal{S}^\gamma \to \mathbb{C}$ satisfying the following properties: [**Ba**, p. 39]

(1) F is holomorphic on $\mathring{\mathcal{S}}^\gamma$, the interior of $\mathcal{S}^\gamma$,

(2) $F(\lambda) = \varphi^{m,n}_\lambda F(-\lambda)$ for all $\lambda \in \mathcal{S}^\gamma$,

(3) $\widehat{\rho}_{H,l,r}(F) < \infty$ for all $l \in \mathbb{N},\ r \in \mathbb{R}^+$,

(4) $F(k) = 0$ if $n, m < 0$, k is of parity opposite to that of m, n and $|k| \leq \min\{|m|, |n|, \gamma\}$,

where $\varphi^{m,n}_\lambda = \frac{P_{m,n}(\lambda)}{P_{m,n}(-\lambda)}$ for some polynomial $P_{m,n}$ in λ involving m and n (see [**Ba**], 7.1) and

$$\widehat{\rho}_{H,l,r}(F) = \sup_{\lambda \in \mathcal{S}^\gamma} \left| \left(\frac{d}{d\lambda}\right)^l F(\lambda) \right| (1 + |\lambda|)^r.$$

In particular $P_{n,n} = 1$. So in the definition of $C^p_H(\widehat{G})_{n,n}$ property 2 reduces to $F(\lambda) = F(-\lambda)$ and property 4 is not relevant.

Note that though $C^p_H(\widehat{G})_{n,n}$ is the image under Fourier transform of functions of $C^p_H(G)_{n,n}$ relative to the principal series representation, the definition is *independent* of n. Also note that this definition is in a sense independent of p. Only thing which changes with p is the width of the strip $\mathcal{S}^\gamma$ and so far we want to use results of complex analysis involving holomorphic functions on a vertical strip we are always in the same situation. For $p \in (1, 2)$, analysis is rather simpler for the fact that $\mathcal{S}^\gamma_\delta$ does not contain any integer point parametrizing the discrete series, *i.e.* no discrete series is embedded in any of the principal series parametrized by that strip.

Let

$$\mathbb{Z}(k) = \begin{cases} \{n \in \mathbb{Z} | n > k \ \text{ and of parity opposite to } \ k\} & \text{if } \ k > 0 \\ \{n \in \mathbb{Z} | n < k \ \text{ and of parity opposite to } \ k\} & \text{if } \ k < 0 \end{cases},$$

$$\mathbb{Z}^\gamma(n) = \begin{cases} \{k \in \mathbb{Z} | \gamma < k < n \ \text{ and of parity opposite to } \ n\} & \text{if } \ n > 0 \\ \{k \in \mathbb{Z} | n < k < -\gamma \ \text{ and of parity opposite to } \ n\} & \text{if } \ n < 0 \end{cases}$$

and

$$\Gamma_n = \mathbb{Z}^0(n).$$

Then Γ_n is the set of points parametrizing discrete series representations which are relevant to a function of right or left K-type n and $\mathbb{Z}^\gamma(n)$ consists of those elements in Γ_n which are outside the strip $\mathcal{S}^\gamma$.

Let $C^p_B(\widehat{G})_{n,n}$ be the set of all functions $F\colon \mathbb{Z}^\gamma(n) \to \mathbb{C}$. ([**Ba**, p. 37].)

Following [**B-W**], we define $A^p_\circ(\delta)$ to be the space of continuous functions $F\colon \mathcal{S}^\gamma_\delta \to \mathbb{C}$ satisfying the following properties:

(1) F is holomorphic on $\overset{\circ}{\mathcal{S}}{}^\gamma_\delta$,

(2) $F(\lambda) = F(-\lambda)$ for all $\lambda \in \mathcal{S}^\gamma_\delta$,

(3) $\lim_{|\lambda|\to\infty} F(\lambda) = 0$ for all $\lambda \in \mathcal{S}^\gamma_\delta$.

There is a conformal map ψ from the strip $\mathcal{S}^\gamma_\delta$ onto the unit disc $\mathbb{D}$ so that $\psi(-\lambda) = -\psi(\lambda)$ and it maps $i\mathbb{R}$ onto the line segment from i to $-i$, namely,

$$\psi(\lambda) = \frac{i(1 - e^{\pi i \lambda/2(\gamma+\delta)})}{(1 + e^{\pi i \lambda/2(\gamma+\delta)})}, \qquad \lambda \in \mathcal{S}^\gamma_\delta.$$

Let $A_0(\mathbb{D})$ be the algebra of all functions which are analytic on $\mathbb{D}$ and continuous in its closure such that $f(z) = f(-z)$ for all $z \in \mathbb{D}$ and $f(i) = f(-i) = 0$. So if $f \in A_0(\mathbb{D})$, then $f \circ \psi \in A^p_\circ(\delta)$.

Now one can generalize Lemma 1.3 and Lemma 1.4 of [**B-W**].

Lemma 2.1. *Fix $\delta > 0$. Then the set*

$$\mathcal{F} = \left\{ f \in C^p(G)_{n,n} \,\middle|\, \begin{array}{l} \widehat{f}_H \ \text{can be extended holomorphically to } \mathcal{S}^\gamma_\delta \\ \text{and } \widehat{f}_H(\lambda)e^{-K\lambda^2} \in A^p_\circ(\delta) \ \text{for some } K > 0 \end{array} \right\}$$

is dense in $C^p(G)_{n,n}$ (and hence in $L^p(G)_{n,n}$).

Proof. Take $g \in C^\infty_c(G)_{n,n} \subset C^p(G)_{n,n}$. Find F_m such that $F_m(\lambda) = \widehat{g}_H(\lambda)e^{\lambda^2/m} \in A^p_\circ(\delta)$ for all $m \in \mathbb{N}$. Now using the same argument as in Lemma 1.3 of [**B-W**] we show that $F_m \to \widehat{g}_H$ in $C^p_H(\widehat{G})_{n,n}$. Actually this proves that the set $\widehat{\mathcal{F}}_H$ of all $F \in C^p_H(\widehat{G})_{n,n}$ satisfying,

(i) F is holomorphically extendable to $\mathcal{S}^\gamma_\delta$ and

(ii) $F(\lambda)e^{-K\lambda^2} \in A^p_\circ(\delta)$ for some $K > 0$,

is dense in $C^p_H(\widehat{G})_{n,n}$.

So $\widehat{\mathcal{F}}_H \oplus C^p_B(\widehat{G})_{n,n}$ is dense in $C^p(\widehat{G})_{n,n}$. But $\{\widehat{f} | f \in \mathcal{F}\} = \widehat{\mathcal{F}}_H \oplus C^p_B(\widehat{G})_{n,n}$ as for $f \in \mathcal{F}$ there is no restriction on its Fourier transform with respect to discrete series representations □

Lemma 2.2. *Let $\delta > 0$. Let f_i, f be functions of type (n,n) such that $\widehat{f}_{iH}, \widehat{f}_H \in A^p_\circ(\delta)$. If there is a $K > 0$ such that*

$$\widehat{f}_H(\lambda)e^{-K\lambda^2} \in A^p_\circ(\delta)$$

and the sequence $\widehat{f}_{iH}(\lambda)$ *converges to* $\widehat{f}_H(\lambda)e^{-K\lambda^2}$ *in the topology of* $A^p_\circ(\delta)$*, then* $\widehat{f}_{iH}(\lambda)e^{K\lambda^2}$ *converges to* $\widehat{f}_H$ *in* $C^p_H(\widehat{G})_{n,n}$ *topology.*

Proof. Same as the proof of Lemma 1.4 of [**B-W**]. □

Let $f \in L^1(G)$, then the operator valued Fourier transform $\widehat{f}_H(\pi_{\sigma,\lambda}) = \int f(x)(\pi_{\sigma,\lambda})(x)\,dx$ exists for $\Re\lambda = 0$. Now consider the matrix coefficient of this operator valued Fourier transform, $(\widehat{f}_H(\pi_{\sigma,\lambda})e_m, e_n) = \int f(x)\Phi^{n,m}_{\sigma,\lambda}(x)\,dx$, for $\sigma \in \widehat{M}$, $\Re\lambda = 0$, $m, n \in \mathbb{Z}^\sigma$. This last expression makes sense for all $\lambda \in \mathcal{S}^1$ and defines a holomorphic function on that strip for each m, n.

For $f \in L^p(G)$, $1 \le p \le 2$, we proceed to define the operator Fourier transform in the following way. Write $f = f_1 + f_2$, where $f_1 \in L^2(G)$ and $f_2 \in L^1(G)$. For instance, $f_1 = f.\chi_{\{|f|<1\}}$ and $f_2 = f.\chi_{\{|f|\ge 1\}}$. Then $(\widehat{f}_{1H}, \widehat{f}_{1B})$ is defined by the Plancherel theorem *a.e.* on $\Re\lambda = 0$ and on $k \in \mathbb{Z}$. The corresponding transforms for f_2 are defined above and we write $\widehat{f}_H(\pi_{\sigma,\lambda}) = \widehat{f}_{1H}(\pi_{\sigma,\lambda}) + \widehat{f}_{2H}(\pi_{\sigma,\lambda})$ and $\widehat{f}_B(\pi_k) = \widehat{f}_{1B}(\pi_k) + \widehat{f}_{2B}(\pi_k)$, for $\sigma \in \widehat{M}$, *a.e.* on $\Re\lambda = 0$ and $k \in \mathbb{Z}$. It can be easily checked that $\widehat{f}_H$ and $\widehat{f}_B$ are independent of the expression $f = f_1 + f_2$. For $\Re\lambda = 0$, we also have $(\widehat{f}_H)_{m,n}(\sigma, \lambda) = \int f_{m,n}(x)\Phi^{m,n}_{\sigma,\lambda}(x)\,dx = \widehat{f}_{m,n}(\lambda)$, where $f_{m,n}$ is the projection of f on (m, n) type ([**Ba**, 2.4]) and σ is determined by the parity of m and n. Moreover by the estimate (2) above in this section, $\widehat{f}_{m,n}$ extends to a holomorphic function on the strip $\mathcal{S}^\gamma$.

At this point we make an observation on the hypotheses of Theorem 1.1 and Theorem 1.2. We assume in both of them that the operator Fourier transforms $\widehat{f}_H(\pi_{\sigma,\lambda})$ are defined for $\lambda \in \mathcal{S}^{\gamma+\delta}$ for some $\delta > 0$, to keep the statement relatively simple. However, what we really need and make use of is only that the matrix coefficients of the transforms $(\widehat{f}_H)_{m,n}(\lambda)$ have analytic extensions on $\mathcal{S}^{\gamma+\delta}$ beyond their natural domain $\mathcal{S}^\gamma$. The extension of the operator transform is infact a mere notational convenience.

3. Main Results.

We now extend Theorem 1.1 of [**B-W**] from $L^1(G)_{0,0}$ to $L^p(G)_{n,n}$. As always, $\gamma = \frac{2}{p} - 1$.

Theorem 3.1. *Let* $\{f^\alpha | \alpha \in \Lambda\}$ *be a subset of* $L^p(G)_{n,n}$*, where* Λ *is an index set. Suppose, for some* $\delta > 0$*, each* $\widehat{f}^\alpha_H$ *can be extended holomorphically to* $\mathring{\mathcal{S}}^\gamma_\delta$ *and satisfy* $\lim_{|\lambda|\to\infty} \widehat{f}^\alpha_H(\lambda) = 0$ *in* $\mathring{\mathcal{S}}^\gamma_\delta$*. Let there exist an* $\alpha_0 \in \Lambda$ *such that* $f^{\alpha_0}_H$ *does not 'decay too rapidly at* ∞*'. Moreover, if the collections* $\{\widehat{f}^\alpha_H\}$ *and* $\{\widehat{f}^\alpha_B\}$ *do not have common zeros on* $\mathcal{S}^\gamma_\delta$ *and* Γ_n *respectively then the*

$L^1(G)_{n,n}$-module generated by $\{f^\alpha | \alpha \in \Lambda\}$ is dense in $L^p(G)_{n,n}$.

Proof. Since $\widehat{f}^\alpha_H \in A^p_0(\delta)$ for all $\alpha \in \Lambda$ and f^{α_0} does not 'decay too rapidly at ∞', by Beurling-Rudin theorem (see [**H**]) and Lemma 1.2 of [**B-W**], the algebraic ideal $\widehat{\mathcal{I}}$ generated by $\widehat{f}^\alpha_H$'s is dense in $A^p_0(\delta)$.

Let $h \in \mathcal{F}$, where $\mathcal{F}$ is as in Lemma 2.1. Then $\widehat{h}_H(\lambda)e^{-K\lambda^2} \in A^p_0(\delta)$. So there is a sequence $F_n \in \widehat{\mathcal{I}}$ such that $F_n \to \widehat{h}_H(\lambda)e^{-K\lambda^2}$. Since $F_n \in \widehat{\mathcal{I}}$ and $e^{K\lambda^2} \in A^p_0(\delta)$, $F_n e^{K\lambda^2} \in \widehat{\mathcal{I}}$ and by Lemma 2.2 $F_n e^{K\lambda^2}$ converges to $\widehat{h}_H(\lambda)$ in the topology of $C^p_H(\widehat{G})_{n,n}$. Thus, we get a sequence in the ideal generated by $\{\widehat{f}^\alpha e^{K'\lambda^2}\}_{\alpha\in\Lambda}$, for some $K' < K$, in $C^p_H(\widehat{G})_{n,n}$ which converges to $\widehat{h}_H$. For $p = 1$ we will choose this sequence. For $p > 1$, using the facts that $C^1_H(\widehat{G})_{n,n}$ is dense in $C^p_H(\widehat{G})_{n,n}$ and $C^p_H(\widehat{G})_{n,n}$ is a Frechét algebra, we will choose the sequence to be in the module generated by $\{\widehat{f}^\alpha e^{K'\lambda^2}\}_{\alpha\in\Lambda}$ over $C^1_H(\widehat{G})_{n,n}$.

Let $\mathbb{Z}^\gamma(n) = \{k_i | 1 \le i \le r\}$. By hypothesis, for each i, $1 \le i \le r$, there exists an $s_i \in \Lambda$ such that $f^{s_i}_B(k_i) \neq 0$. Let $\Lambda' = \{s_i | 1 \le i \le r\} \subset \Lambda$. Let

$$g^{s_i}(k_j) = \delta_{i,j}\frac{\widehat{h}_B(k_j)}{\widehat{f}^{s_j}_B(k_j)} \qquad \text{for } \ 1 \le i, \ j \le r$$

and

$$g^\alpha(k) = 0 \qquad \text{for all } \ k \in \Gamma_n, \ \text{ for all } \ \alpha \in \Lambda - \Lambda'.$$

Now if we define $\mathcal{G}_{nB}(k) = \sum_\alpha \widehat{f}^\alpha_B(k) g^\alpha(k)$ for $k \in \Gamma_n$, then $\mathcal{G}_{nB}$ converges to $\widehat{h}_B$ in the topology of $C^p_B(\widehat{G})_{n,n}$. This proves the theorem in view of the existence of an isomorphism between $C^p(\widehat{G})_{n,n}$ and $C^p(G)_{n,n}$ and the injectiveness of the Fourier transform on $L^p(G)_{n,n}$. □

Working towards a proof of Theorem 1.1 stated in the introduction, we recall that if $\widehat{f} = (\widehat{f}_H, \widehat{f}_B) \in L^p(\widehat{G})$ then $(\widehat{f}_B(k))_{m,n} = \eta^{m,n}(k)(\widehat{f}_H(k))_{m,n}$ for $k \in \mathcal{S}^\gamma$ where $\eta^{m,n}(k)$ is a positive number described in [**Ba**, p. 30]. Therefore $(\widehat{f}_B(k))_{m,n} \neq 0 \Leftrightarrow (\widehat{f}_H(k))_{m,n} \neq 0$.

Suppose that $\widehat{f}_B(k) \neq 0$ for all $k \in \Gamma_n$. Then it implies the following:

(a) If n is a positive then f has at least one (non-zero component of) left type m such that $m \ge n$, because for every $m < n$, $(\widehat{f}_B(n-1))_{m,n} = 0$. Similarly, when n is negative f has at least one left type m for some $m \le n$.

(b) Let $f \in L^1(G)_n$. If n is even then there is exactly one point in the strip $\mathcal{S}^1$ namely $+1$ or -1, which parametrizes a relevant discrete series representation depending on whether n is greater or less than zero and δ can be chosen carefully to avoid any other such point in $\mathcal{S}^1_\delta$. So when $n > 0$ by above hypothesis $\widehat{f}_B(1) \neq 0$. In fact there is a K-type

m such that $m \in \mathbb{Z}(1)$ and $(\widehat{f}_B(1))_{m,n} \neq 0$. This is equivalent to saying that $(\widehat{f}_H(1))_{m,n} \neq 0$. For $n < 0$ one can have a similar statement. When n is odd neither the original strip $\mathcal{S}^1$ nor the (carefully chosen) augmented strip $\mathcal{S}^1_\delta$ has any point-parameter of discrete series representation relevant to this K-type n. For $p \in (1, 2)$ the strip $\mathcal{S}^\gamma$ and its δ augmentation $\mathcal{S}^\gamma_\delta$ can avoid points which parametrize discrete series representation.

Proof of Theorem 1.1. We will first consider the case when $p = 1$ and the collection indexed by Λ contains exactly on function, $f \in L^p(G)_n$. Let $f_m(x) = \int_0^{2\pi} e^{-im\theta} f(k_\theta x)\, d\theta$ for all $m \in \mathbb{Z}$. Then f_m is an (m, n) type function and (m, n)-th matrix coefficient of $\widehat{f}_H$, $(\widehat{f}_H)_{m,n} = \widehat{f}_{mH}$. In particular $(\widehat{f}_H)_{m_0,n} = \widehat{f}_{m_0 H}$. (Please note that the use of the notation f_m for left projection of f to m type is absolutely local and has no connection with the rest of the paper.)

Let $\mathcal{G}_m(\lambda) = e^{-\lambda^4} P_m(\lambda)$ where P_m is to be chosen a polynomial of λ involving m, n. When $m.n > 0$ (*i.e.* when m is of same sign as n), then P_m is in fact the numerator of the rational function $\phi_\lambda^{m,n}$ (see [**Ba**, 7.1]), *i.e.*, $\phi_\lambda^{m,n} = P_m(\lambda)/P_m(-\lambda)$. Hence $e^{-\lambda^4} P_m(\lambda) = \phi_\lambda^{m,n} e^{-\lambda^4} P_m(-\lambda)$ which shows that $\mathcal{G}_m(\lambda) = e^{\lambda^4} P_m(\lambda) \in C^1_H(\widehat{G})_{n,m}$.

If $m.n < 0$ then we will have to choose the polynomial in a slightly different way:

Case 1. Let n be odd. Then take the polynomial $P'_m(\lambda) = P_m(\lambda).\lambda^2$. Now $P'_m(0) = 0$ and $e^{-\lambda^4} P'_m(\lambda) = \varphi_\lambda^{n,m} e^{-\lambda^4} P'_m(\lambda)$. Therefore, in this case $\mathcal{G}_m(\lambda) = e^{-\lambda^4} P'_m(\lambda) \in C^1_H(\widehat{G})_{n,m}$.

Case 2. Let n be even (hence $|n|, |m| \geq 2$). Then the required polynomial is given by $P''_m(\lambda) = P_m(\lambda).(1 - \lambda^2)$. So $P''_m(\pm 1) = 0$ and $e^{-\lambda^4} P''_m(\lambda) = \varphi_\lambda^{n,m} e^{-\lambda^4} P''_m(\lambda)$. Hence here $\mathcal{G}_m = e^{-\lambda^4} P''_m(\lambda) \in C^1_H(\widehat{G})_{n,m}$.

In every case

$$\begin{aligned}\mathcal{G}_m(\lambda)\widehat{f}_{mH}(\lambda) &= e^{-\lambda^4} P_m(\lambda)\widehat{f}_{mH}(\lambda)\\ &= e^{-\lambda^4} P_m(-\lambda)\phi_\lambda^{n,m}\phi_\lambda^{m,n}\widehat{f}_m(-\lambda)\\ &= \mathcal{G}_m(-\lambda)\widehat{f}_{mH}(-\lambda)\end{aligned}$$

since $\widehat{f}_m(\lambda) = \phi^{m,n}(\lambda)\widehat{f}_m(-\lambda)$, f_m being an (m, n) type function and $\phi_\lambda^{m,n} = \frac{1}{\phi_\lambda^{n,m}}$) (see [**Ba**, Proposition 7.2 and Equation 9.8]). This shows that for all m, $\mathcal{G}_m \widehat{f}_{mH}(\lambda)$ is the Fourier transform of an (n, n) type function with respect to principal series representation π_λ. It is obvious that they can be holomorphically extended to the strip $\overset{\circ}{\mathcal{S}}{}^1_\delta$ and that $\lim_{|\lambda| \to \infty} |\mathcal{G}_m(\lambda)\widehat{f}_{mH}(\lambda)| = 0$ as the claims hold for f_m's.

We will now show that $\mathcal{G}_{m_0}.\widehat{f}_{m_0H}$ does not 'decay too rapidly at ∞'. Let $K > 0$ be fixed. Take a K' such that $0 < K' < K$. Then

$$\begin{aligned}\left|\mathcal{G}_{m_0}(it)\widehat{f}_{m_0H}(it)e^{Ke^{|t|}}\right| &= \left|\widehat{f}_{m_0H}(it)P_{m_0}(it)e^{Ke^{|t|}-\lambda^4}\right| \\ &= \left|\widehat{f}_{m_0H}(it)P_{m_0}(it)e^{(K-K')e^{|t|}-t^4}e^{K'e^{|t|}}\right|.\end{aligned}$$

So

$$\limsup_{|t|\to\infty}\left|\mathcal{G}_{m_0}(it)\widehat{f}_{m_0H}(it)e^{Ke^{|t|}}\right| > 0$$

as

$$\limsup_{|t|\to\infty}\left|\widehat{f}_{m_0H}(it)e^{(K-K')e^{|t|}-t^4}\right| > 0, \quad \left|e^{K'e^{|t|}}\right| > 0.$$

Next, we want to show that for each $\lambda \in \mathcal{S}_\delta^\gamma$ there is an m such that $\mathcal{G}_m(\lambda)\widehat{f}_{mH}(\lambda) \neq 0$. The only possible zeros of the polynomials P_m, P'_m and P''_m in the strip $\mathcal{S}_\delta^1$ are ± 1 and 0 (where δ is chosen carefully). Let us investigate this more closely:

(i) $P_{n,m}(-1) = 0$ if and only if $n = 0$ and $m \neq 0$,
$P_{n,m}(+1) \neq 0$ for all $m \neq 0$,
if $n \neq 0$, $P_{n,0}(+1) = 0$,
therefore, $P'_{n,m}(\pm 1) \neq 0$ for all $m \neq 0$;

(ii) $P_{n,m}(0) \neq 0$ so, $P''_{n,m}(0) \neq 0$;

(iii) $P'_{n,m}(0) = 0$ and $P''_{n,m}(\pm 1) = 0$.

By hypothesis there is an m such that $\widehat{f}_{mH}(0) \neq 0$. So, if n is odd, then $n.m > 0$. Because otherwise, $\Phi^{m,n}_{\sigma^+,0} \equiv 0$ ([**Ba**, Proposition 7.1]) which implies that $\widehat{f}_{mH}(0) = 0$. So the zeros of the polynomials P'_m at 0 will never be relevant.

If n is even then $\widehat{f}_{mH}$ is nonzero either at $+1$ or at -1 and so n, m can not be of opposite sign. (If $n.m < 0$ then both $|n|, |m| \geq 2$ being even integers. Then $\Phi_1^{m,n} \equiv 0$ and hence $\widehat{f}_{mH}(\pm 1) = 0$.) So we can forget about the zeros of $P''_{m,n}$ at ± 1.

Thus the only polynomials we are concerned about are $P_{n,0}$ with $n \neq 0$ which have zero at $+1$ and $P_{0,m}$ with $m \neq 0$ which have zero at -1. For this we have the following remedies:

When $n > 0$, by the discussion preceeding this proof, there exists an f_r (as one of the projections of f of type (r, n) with $r \in \mathbb{Z}(1)$) such that $\widehat{f}_{rH}(1) \neq 0$ and so $\mathcal{G}_r\widehat{f}_{rH}(1) \neq 0$.

When $n < 0$, by discussion (b) above, there exists an $s \in \mathbb{Z}(-1)$ such that $\widehat{f}_{sH}(-1) \neq 0$. But $\widehat{f}_{sH}(1)\phi_1^{n,s} = \widehat{f}_{sH}(-1)$ (see [**Ba**, 9.8]), and $\phi_\lambda^{n,s}$ has no pole at $\lambda = 1$ (see [**Ba**, Proposition 7.2]). This implies that $\widehat{f}_{sH}(1) \neq 0$ (see [**Ba**, Proposition 7.2(v)]) and as $s \in \mathbb{Z}(-1)$, $s \neq 0$. Hence $\mathcal{G}_s\widehat{f}_{sH}(1) \neq 0$.

If $n = 0$ then $\widehat{f}_{mH} = 0$ for all $m \neq 0$ as $\Phi_1^{m,0} \equiv 0$ (see [**Ba**] Proposition 7.1). But, then $\widehat{f}_H(1) \neq 0$ forces $\widehat{f}_{0H}(-1) = \widehat{f}_{0H}(1)$ to be non-zero. Therefore, zeros of the polynomial $P_{0,m}$ with $m \neq 0$ will not concern us.

Let $\mathcal{G}_m(k) = e^{-k^4} P_m(k)$ for all $k \in \Gamma_n$. Now let for a $k_0 \in \Gamma_n$, $\widehat{f}_{m_0 B}(k_0) \neq 0$. Then $m_0 \in \mathbb{Z}(k_0)$. Therefore $P_{m_0}(k_0) \neq 0$ as all the zeros of the polynomial are either between m_0 and n or between $-m_0$ and $-n$ (see [**Ba**] Proposition 7.1). Now by isomorphism of $C^1(G)_{n,m}$ and its Fourier transform $C^1(\widehat{G})_{n,m}$, for every m, there exists $g_m \in C^1(G)_{n,m}$ such that $\widehat{g}_{mH}(\lambda) = \mathcal{G}_m(\lambda)$ for $\lambda \in \mathcal{S}_\delta^1$ and $\widehat{g}_{mB}(k) = \mathcal{G}_m(k)$ for $k \in \Gamma_n$. So we have established that the set of $L^1(G)_{n,n}$ functions $\{g_m * f_m | m \in \mathbb{Z}^\sigma\}$ satisfies all the conditions of Theorem 3.1 and hence the ideal generated by them is dense in $L^1(G)_{n,n}$. But $g_m * f_m = g_m * f$; so the result follows for the fact that the left $L^1(G)$ module generated by $L^1(G)_{n,n}$ is all of $L^1(G)_n$.

The case when Λ is an arbitrary index set hardly needs a separate proof. In fact, out of each f^α by projections we get f_j^α for all $j \in \mathbb{Z}$ which are functions of type (j, n). Now we apply previous arguments to the collection $\{\widehat{f}_j^\alpha | \alpha \in \Lambda,\ j \in \mathbb{Z}\}$ of functions in $L^1(\widehat{G})_n$. This completes the proof for L^1 case.

The proof for $p > 1$ will almost follow the above word for word. In fact, the case $p > 1$ is simpler as the troublesome points ± 1 are not in the (carefully chosen) strip $\mathcal{S}_\delta^\gamma$. Note that, whatever p we are working with, we will always get a $C_H^1(\widehat{G})$-function, namely $P(\lambda)e^{-\lambda^4}$, to change the K-type of the Fourier transforms. So arguments similar to that of the previous theorem will take care of this function. □

Now we are in a position to consider the final result, Theorem 1.2, stated in the introduction. Before proving it let us note that trivial representation is an irreducible L^1-tempered representation. It is a subrepresentation of the principal series representation $\pi_{\sigma^+,-1}$ [**Ba**, p. 16]. Fourier transform of $f(x)$ with respect to the trivial representation is $\int_G f(x)\,dx$. In fact,

$$\int_G f(x)\,dx = \int_G f(x)\Phi_{-1}^{0,0}(x)\,dx = \left(\widehat{f}(-1)\right)_{0,0} = \left(\widehat{f}(1)\right)_{0,0}.$$

So the hypothesis $\int_G f(x)\,dx \neq 0$ actually means that Fourier transform of f with respect to trivial representation is nonzero.

Proof of Theorem 1.2. As we have seen in the proof of previous theorem, it is enough to consider the case when $p = 1$ and the collection contains a single function, namely f. Let f_i be the projection of f to $L^1(G)$, for every $i \in \mathbb{Z}$. For each i and m in $\mathbb{Z}$ we choose a polynomial $P_{i,m}$ in λ involving i and m as explained below.

When $i.m > 0$, $P_{i,m}$ is simply the numerator of a rational function $\varphi_\lambda^{i,m}$ (see [**Ba**, 7.1]). Then $P_{i.m}(\lambda)e^{-\lambda^4} \in C^1_H(\widehat{G})_{i,m}$. So there exists a $g_{i,m} \in C^1(G)_{i,m}$ such that $\widehat{g}_{i,m\,H}(\lambda) = P_{i,m}(\lambda)e^{-\lambda^4}$ and $\widehat{g}_{i,m\,B}(k) = P_{i,m}(k)e^{-k^4} \in C^1_B(G)_{i,m}$ for $k \in \Gamma_i$.

When $i.m < 0$ and i, m are odd integers, we will have to use polynomial $P'_{i,m}(\lambda) = \lambda^2 P_{i,m}$ where $P_{i,m}$ is as above. By L^1-Schwartz space isomorphism between $C^1(G)_{i,m}$ and $C^1(\widehat{G})_{i,m}$ we can find $g_{i,m}$ so that $\widehat{g}_{i,m\,H}(\lambda) = P'_{i,m}(\lambda)e^{-\lambda^4}$ and $\widehat{g}_{i,m\,B}(k) = P'_{i,m}.e^{-k^4}$. When $i.m < 0$ and i, m are even integers, the required polynomial will be $P''_{i,m} = (1-\lambda^2).P_{i,m}$ and as above we can find a $g_{i,m}$. So for all $m \in \mathbb{Z}$ we can construct a collection of functions

$$\mathcal{F}_m = \{f_i * g_{i,m} | i \in \mathbb{Z}\}$$

contained in $L^1(G)_m$.

First let us deal with the case $m \neq 0$. We will show that the collection $\mathcal{F}_m$ satisfies the conditions of Theorem 1.1 and hence generates $L^1(G)_m$.

We will find a function $f_i * g_{i,m}$ in this collection so that its Fourier transform at $(\sigma^+, 0)$ or at $(\sigma^-, 0)$ is nonzero according as it is of even or odd parity. If m is even neither $P_{i,m}$ nor $P''_{i,m}$ has any zero at 0. So if the (r,s)-th matrix coefficient of $\widehat{f}$ is nonzero at $(\sigma^+, 0)$, $f_s * g_{s,m}$ will serve the purpose. If m is odd and positive, we will have to consider the nonzero matrix coefficient $\widehat{f}_{u_1,v_1\,H}(\sigma^-, 0)$ where both u_1 and v_1 are positive. Such a matrix coefficient exists by the hypothesis $\widehat{f}_H(D_+) \neq 0$. Then $f_{v_1} * g_{v_1,m}(\sigma^-, 0)$ will be nonzero. We are using the fact that v_1 and m are both being positive, $\widehat{g}_{v_1,m}(\sigma^-, 0) \neq 0$. Similarly for the case $m < 0$ we will consider a nonzero matrix coefficient corresponding to two K-finite vectors e_{u_2} and e_{v_2}, where u_2 and v_2 are both negative. Next we note that

$$\text{if } |i| > |m| \text{ then } P_{i,m}(1) \neq 0 \text{ and } P_{i,m}(-1) \neq 0,$$
$$\text{and if } |i| < |m| \text{ then } P_{i,m}(1) \neq 0 \text{ and } P_{i,m}(-1) \neq 0, \text{ only when } i = 0.$$

We will come across the polynomial $P_{0,m}$ only when we are dealing with an even m.

As $\widehat{f}_B(-1) \neq 0$ there is a matrix coefficient say $(\widehat{f}_B(-1))_{r,s}$ which is nonzero. Then obviously $r, s < -1$ and r, s are even integers. So, in particular, $s \neq 0$ and $\widehat{f}_{s,H}(\sigma^+, -1) \neq 0$. Therefore when $m < 0$, $f_s * g_{s,m}$ is the function in the collection $\mathcal{F}_m$ which has nonzero Fourier transform at $(\sigma^+, -1)$. Again as $\widehat{f}_B(+1) \neq 0$ there is a non-zero matrix coefficient, say $(\widehat{f}_B(1))_{r_2,s_2}$, where $r_2, s_2 > 1$. Now as $(\widehat{f}_H(-1))_{r_2,s_2} = \varphi^{r_2,s_2}_{\sigma^+,-1}(\widehat{f}_H(1))_{r_2,s_2}$ and as $\varphi^{r_2,s_2}_{\sigma^+,-1}$ has no zero at -1 (see [**Ba**] Proposition 7.2), $(\widehat{f}_H(-1))_{r_2,s_2} \neq 0$. Therefore when $m > 0$, $f_{s_2} * g_{s_2,m}$ is a function in $\mathcal{F}_m$ such that its Fourier transform is nonzero at $(\sigma^+, -1)$. Any other point of $\mathcal{S}^1_\delta$ including $+1$ is not a zero of the polynomials $P_{i,m}$ when $m \neq 0$ and as we are appealing to the nonvanishing of the Fourier transform at disgrete series the polynomials P'_m and P''_m are not relevant at all.

To find a function in $\mathcal{F}_m$ whose Fourier transform does not 'decay too rapidly at ∞' we will get the matrix coefficient of the parity of m which has that property. Let that matrix coefficient be the (α, β)-th one. Then the Fourier transform of $f_\beta * g_{\beta m}$ also will not 'decay too rapidly at ∞'.

The collection $\mathcal{F}_m$ for $m \neq 0$ satisfies the conditions of Theorem 1.1 and hence generates $L^1(G)_m$ under left convolution.

We will now treat the case $m = 0$. Consider $\mathcal{F}_0$. For any $i \neq 0$, $P_{i,0}(1) = 0$ and $P_{i,0}(-1) \neq 0$. $P_{0,0}(\pm 1) = 1$.

By hypothesis $\int_G f(x)\,dx = (\widehat{f}(1))_{0,0} = \widehat{f}_0(1) \neq 0$ (see discussion preceeding this proof). So $f_0 * g_{0,0}$ is the function in the collection $\mathcal{F}_0$ which has nonzero Fourier transform at $+1$. So $\mathcal{F}_0$ generates $L^1(G)_0$ under left convolution.

Now as $f_i * g_{i,m} = f * g_{i,m}$, for every m, elements of $\mathcal{F}_m$ are right convolutions of a single function f. So the two sided (closed) ideal generated by f contains $L^1(G)_m$ for all m. The smallest closed right G-invariant subspace of $L^1(G)$ containing $L^1(G)_m$ for all $m \in \mathbb{Z}$ is $L^1(G)$ itself. Hence the first part of the theorem follows.

If we omit the condition $\int_G f(x)\,dx \neq 0$, there is no effect on the collection $\mathcal{F}_m$ for $m \neq 0$. But in this case $\mathcal{F}_0$ will generate $L^1(G)^0_0$, the space of $L^1(G)_0$ functions with integral zero. Note that $L^1(G)_m$ for any $m \neq 0$ is contained in $L^1(G)^0$, $L^1(G)_m = L^1(G)^0_m$. So in this case the function f under left and right convolution generates an ideal which contains $L^1(G)^0_m$ for all $m \in \mathbb{Z}$. The smallest closed right G-invariant subspace of $L^1(G)^0$ containing all $L^1(G)^0_m$ is $L^1(G)^0$. Hence the second part of the theorem. □

4. Concluding Remarks.

As for any group G of *real rank* one $C^p(\widehat{G})_{0,0}$ is identical with $C^p(\widehat{SL_2(\mathbb{R})})_{0,0}$, the proofs of W-T theorem for biinvariant L^1 functions on $SL_2(\mathbb{R})$ given in [**B-W**] and its extension to all $p \in [1, 2)$ in this article, will actually go through for biinvariant L^p functions on any such G.

We came to know about a research announcement [**B-B-H-W**] which has proposed a proof for the conjecture 1.1 of [**B-W**] (*i.e.* for removing the restriction of a wider strip), a slightly different 'not decay too rapidly' condition: $\inf_{f \in M} \delta_\infty(\widehat{f}) = 0$. Here $M \subset L^1(G)_{0,0}$ is the set of generators and $\delta_\infty(f) = -\limsup_{t \to +\infty} e^{\pi t} \log |\widehat{f}(it)|$. The techniques of Theorem 1.1 will give an immediate extension of this to a W-T theorem for the symmetric space $SL_2(\mathbb{R})/SO(2, \mathbb{R})$. This verifies our remarks in the introduction following Theorem 1.2 for the case of $L^p(G)_0$. More precisely assuming the results in [**B-B-H-W**], we can prove:

Let $\{f^\alpha | \alpha \in \Lambda\}$ *be a subset of* $L^p(G)_0$ *and*

$$\inf_{\alpha\in\Lambda,\, n\in\mathbb{Z}} \delta_\infty \left(\left(\widehat{f}^{\alpha}_{H} \right)_{n,0} \right) = 0.$$

Moreover if $\{\widehat{f}^{\alpha}_{H}\}$ *do not have common zeros on* $\mathcal{S}^\gamma$, *then for* $p \in [1,2)$ *the left* $L^1(G)$*-module generated by* $\{f^\alpha | \alpha \in \Lambda\}$ *is dense in* $L^p(G)_0$.

Acknowledgements. A. Sitaram made me interested in the W-T type problems. He also made me aware of the papers [**B-W**] and [**B-B-H-W**]. S.C. Bagchi is responsible for introducing me to this subject and for providing an exposure. Numerous conversations with them and with J. Sengupta were of valuable assistance. The author is grateful to all of them.

References

[Ba] W.H. Barker, *L^p harmonic analysis on $SL(2,\mathbb{R})$*, Memoir of the American Mathematical Society, **393** (1988).

[B-B-H-W] Y. Ben Natan, Y. Benyamini, H. Hedenmalm and Y. Weit, *Wiener's Tauberian Theorem in $L^1(G/K)$, and harmonic functions in the unit disc*, Bulletin AMS, **32(1)** (1995).

[B-W] Y. Benyamini and Y. Weit, *Harmonic analysis of spherical functions on $SU(1,1)$*, Ann. Inst. Fourier, Grenoble, **42(3)** (1992), 671-694.

[E-M] L. Ehrenpreis and F.I. Mauther, *Some properties of the Fourier transform on semi simple Lie groups* I, Annals of Math., **61** (1955), 406-439.

[H] K. Hoffman, *Banach spaces of analytic functions*, Prentice-Hall, Inc., N.J., 1962.

[S] A. Sitaram, *On an analogue of Wiener Tauberian Theorem for symmetric spaces of the non-compact type*, Pacific Journal of Math., **133** (1988), 197-208.

[T] P.C. Trombi, *Harmonic analysis of $C^p(G\colon F)(1 \leq p < 2)$*, Journal of Functional Analysis, **40** (1981), 84-125.

Received January 30, 1995 and revised September 7, 1995. The author is supported by a research award of national board for Higher Mathematics, India.

Indian Statistical Institute
203, B. T. Road
Calcutta 700035, India
E-mail address: res9423@isical.ernet.in