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FREE QUASI-FREE STATES

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To a real Hilbert space and a one-parameter group of orthogonal transformations we associate a C^*-algebra which admits a free quasi-free state. This construction is a free-probability analog of the construction of quasi-free states on the CAR and CCR algebras. We show that under certain conditions, our C^*-algebras are simple, and the free quasi-free states are unique.

The corresponding von Neumann algebras obtained via the GNS construction are free analogs of the Araki-Woods factors. Such von Neumann algebras can be decomposed into free products of other von Neumann algebras. For non-trivial one-parameter groups, these von Neumann algebras are type III factors. In the case the one-parameter group is nontrivial and almost-periodic, we show that Connes' Sd invariant completely classifies these algebras.

1. Introduction.

We consider in this paper free analogs of the quasi-free states on the CAR and CCR algebras.

Quasi-free states are important in mathematics and physics, and a vast body of literature exists (a partial list includes [2, 1, 3, 7, 17, 22], see also the references in [7]). In particular, quasi-free states on the CAR algebra give rise, via the GNS construction, to the Araki-Woods factors. These factors are examples of hyperfinite type III factors.

The free probability theory of Voiculescu (see [29]) has parallels with the theory of hyperfinite algebras. For example, Voiculescu's Free Gaussian Functor ([27]) is a free analog of the CAR functor (see [14]). The principal idea of this paper, suggested to us by D.-V. Voiculescu, is to extend this parallel further, by constructing free analogs of quasi-free states. To this end, we in a functorial way associate to a real Hilbert space $\mathcal{H}_\mathbb{R}$ with a one-parameter group of orthogonal transformations U_t, a subalgebra $\Gamma(\mathcal{H}_\mathbb{R}, U_t)$ of the extension of the Cuntz algebra associated to the complexification of $\mathcal{H}_\mathbb{R}$. The restriction of the vacuum expectation then becomes the free quasi-free state φ_U.

One application of free quasi-free states is to consider the free analogs of Araki-Woods factors, i.e., the von Neumann algebras obtained in the GNS representations associated to free quasi-free states. We show that such algebras are always type III factors (unless the one-parameter group is trivial.)

Since free quasi-free states are constructed in a manner similar to the construction of the trace on the algebras of the Free Gaussian Functor, it becomes possible to model our algebras using "matricial models". These are generalizations of the random matrix techniques of [**28**, **29**]. Using these tools we prove that Connes' Sd invariant ([**9**]) is a complete invariant for the free analogs of Araki-Woods factors in the case of a nontrivial almost-periodic one-parameter group.

Examples of type III factors involved in free probability theory were obtained earlier by Rădulescu ([**21**, **20**]), Dykema ([**12**, **13**]) and Barnett ([**6**]). One of the results of this paper is that some of the factors considered in the above papers are isomorphic to free analogs of Araki-Woods factors. In particular, using our classification result, it becomes possible to find isomorphisms between such factors.

The rest of the paper is divided as follows. Section 2 is devoted to the definition and basic properties of the algebra $\Gamma(\mathcal{H}_{\mathbb{R}}, U_t)$ and of the free quasi-free state φ_U. Section 3 considers Tomita theory for the GNS representation associated to φ_U. In Section 4 we specialize to the case of two-dimensional real Hilbert spaces (these are "building blocks" out of which algebras corresponding to higher-dimensional Hilbert spaces with almost-periodic actions can be constructed). We introduce "generalized circular elements", and consider their polar decomposition.

Section 5 is devoted to matricial models and some applications. Lastly, in Section 6 we consider free analogs of Araki-Woods factors, and give their classification in the case of nontrivial almost-periodic actions.

Some of the results of this paper were announced in [**23**].

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2. Free Quasi-free states.

Given a separable real Hilbert space $\mathcal{H}_\mathbb{R}$, and a one-parameter group of orthogonal transformations U_t, consider the complex Hilbert space

$$\mathcal{H}_\mathbb{C} = \mathcal{H}_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}.$$

Denote by $\langle \cdot, \cdot \rangle$ the inner product of $\mathcal{H}_\mathbb{C}$ (this Hermitian inner product, as all Hermitian inner products in this paper, is assumed to be $\mathbb{C}$-linear in the *second* variable.) Embed $\mathcal{H}_\mathbb{R}$ into $\mathcal{H}_\mathbb{C}$ as $\mathcal{H}_\mathbb{R} \otimes 1$. As a real Hilbert space (with inner product $\mathrm{Re}\langle \cdot, \cdot \rangle$),

$$\mathcal{H}_\mathbb{C} \cong \mathcal{H}_\mathbb{R} \oplus i\mathcal{H}_\mathbb{R}.$$

The operator

$$* : x + iy \mapsto x - iy$$

for $x, y \in \mathcal{H}_\mathbb{R}$ is a well-defined bounded anti-linear operator on $\mathcal{H}_\mathbb{C}$. For $y \in \mathcal{H}_\mathbb{R}$, $x \in \mathcal{H}_\mathbb{C}$,

$$\langle x, y \rangle = \overline{\langle y, x \rangle} = \langle y, *x \rangle.$$

Also, $x \in \mathcal{H}_\mathbb{R}$ if and only if $*x = x$.

The one-parameter group U_t extends to a group of unitary transformations on $\mathcal{H}_\mathbb{C}$ by linearity. Let A be the closed (not necessarily bounded) operator such that $U_t = A^{it}$; let H be the closed (not necessarily bounded) operator such that $U_t = \exp(iHt)$. Thus $A = \exp(H)$. Since U_t is orthogonal for all t, its infinitesimal generator, iH, is an unbounded operator from $\mathcal{H}_\mathbb{R}$ to $\mathcal{H}_\mathbb{R}$. Thus H maps the intersection of its domain with $\mathcal{H}_\mathbb{R}$ (which is nonempty because the domains of H and iH are the same) into $i\mathcal{H}_\mathbb{R}$. Hence $*H = -H*$. Since $A = \exp(H)$, it follows that $*A = A^{-1}*$.

Define another inner product on $\mathcal{H}_\mathbb{C}$ by

$$\langle x, y \rangle_U = \left\langle \frac{2}{1 + A^{-1}} x, y \right\rangle.$$

This is an inner product because $A \geq 0$, so $2/(1 + A^{-1})$ is bounded and positive. Notice that $2/(1 + A^{-1})$ has an (unbounded) inverse; thus it has empty kernel. Hence this new inner product is non-degenerate. Notice that for $x, y \in \mathcal{H}_\mathbb{R}$,

$$\begin{aligned} \langle y, x \rangle_U &= \left\langle \frac{2}{1 + A^{-1}} y, x \right\rangle = \left\langle x, * \frac{2}{1 + A^{-1}} y \right\rangle \\ &= \left\langle x, \frac{2}{1 + A} y \right\rangle = \left\langle x, \frac{2A^{-1}}{1 + A^{-1}} y \right\rangle \\ &= \left\langle \frac{2}{1 + A^{-1}} x, A^{-1} y \right\rangle = \left\langle x, A^{-1} y \right\rangle_U . \end{aligned} \tag{1}$$

Let $\mathcal{H}$ be the complex Hilbert space obtained from $\mathcal{H}_{\mathbb{C}}$ by completing with respect to $\langle\cdot,\cdot\rangle_U$. Since U_t clearly preserves this inner product, it defines a one-parameter group of unitary transformations on $\mathcal{H}$; we denote this group once again by U_t. Notice that the norm induced on $\mathcal{H}_{\mathbb{R}}$ by $\langle\cdot,\cdot\rangle_U$ is the same as the original norm on $\mathcal{H}_{\mathbb{R}}$. Indeed, for $x \in \mathcal{H}_{\mathbb{R}}$,

$$\begin{aligned}\langle x, x\rangle_U &= \left\langle \frac{2}{1+A^{-1}} x, x\right\rangle \\ &= \langle x, x\rangle + \left\langle \frac{1-A^{-1}}{1+A^{-1}} x, x\right\rangle;\end{aligned}$$

but just as above,

$$\left\langle \frac{1-A^{-1}}{1+A^{-1}} x, x\right\rangle = \left\langle x, \frac{1-A}{1+A} x\right\rangle = -\left\langle x, \frac{1-A^{-1}}{1+A^{-1}} x\right\rangle = 0.$$

Thus we have constructed an isometric embedding of $\mathcal{H}_{\mathbb{R}}$ into a complex Hilbert space $\mathcal{H}$, which satisfies four properties:

(a) The restriction of the real part of the inner product on $\mathcal{H}$ is the inner product on $\mathcal{H}_{\mathbb{R}}$;

(b) $\mathcal{H}_{\mathbb{R}} + i\mathcal{H}_{\mathbb{R}}$ is dense in $\mathcal{H}$, and $\mathcal{H}_{\mathbb{R}} \cap i\mathcal{H}_{\mathbb{R}} = \{0\}$;

(c) U_t extends to a one-parameter group of unitaries on $\mathcal{H}$;

(d) The restriction of the imaginary part of the inner product on $\mathcal{H}$ to $\mathcal{H}_{\mathbb{R}}$ is given by

$$j = \left\langle i\frac{1-A^{-1}}{1+A^{-1}}\cdot,\cdot\right\rangle_{\mathcal{H}_{\mathbb{R}}},$$

where $U_t = A^{it}$.

In (d), to check that

$$i\frac{1-A^{-1}}{1+A^{-1}} x \in \mathcal{H}_{\mathbb{R}}$$

for $x \in \mathcal{H}_{\mathbb{R}}$, one observes that

$$*\left(i\frac{1-A^{-1}}{1+A^{-1}} x\right) = i\frac{1-A^{-1}}{1+A^{-1}}(*x)$$

and for $x \in \mathcal{H}_{\mathbb{R}}$, $*x = x$ and $Ax \in \mathcal{H}_{\mathbb{R}} + i\mathcal{H}_{\mathbb{R}}$. Moreover, for $y \in \mathcal{H}_{\mathbb{R}} + i\mathcal{H}_{\mathbb{R}}$, $y \in \mathcal{H}_{\mathbb{R}}$ if and only if $*y = y$.

Observe that these properties define the embedding of $\mathcal{H}_{\mathbb{R}}$ into $\mathcal{H}$. We record that for x, y in a certain dense subset of $\mathcal{H}_{\mathbb{R}}$, we have (by Equation (1))

$$\langle x, y\rangle_U = \langle y, A^{-1}x\rangle_U. \tag{2}$$

Remark 2.1. There is another way to construct the embedding $\mathcal{H}_\mathbb{R} \subset \mathcal{H}$. Let $\mathcal{K} = \mathcal{H}_\mathbb{R}$, and set $\mathcal{H} = \mathcal{K} \otimes_\mathbb{R} \mathbb{C}$; let U_t be the extension (by linearity) of U_t to $\mathcal{H}$, and let A be such that $A^{it} = U_t$. Define the embedding $\mathcal{H}_\mathbb{R} \hookrightarrow \mathcal{H}$ by

$$h \mapsto \frac{\sqrt{2}A^{1/4}}{\sqrt{A^{1/2} + A^{-1/2}}} h \otimes 1.$$

This embedding is isometric for the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}_\mathbb{R}}$ on $\mathcal{H}_\mathbb{R}$ and $\mathrm{Re}\langle \cdot, \cdot \rangle_\mathcal{H}$. Indeed,

$$\begin{aligned} &\mathrm{Re} \left\langle \frac{\sqrt{2}A^{1/4}}{\sqrt{A^{1/2} + A^{-1/2}}} h, \frac{\sqrt{2}A^{1/4}}{\sqrt{A^{1/2} + A^{-1/2}}} h \right\rangle_\mathcal{H} \\ &= \mathrm{Re} \left\langle \frac{2A^{1/2}}{A^{1/2} + A^{-1/2}} h, h \right\rangle_\mathcal{H} \\ &= \left\langle \frac{A^{1/2} + A^{-1/2}}{A^{1/2} + A^{-1/2}} h, h \right\rangle = \|h\|_{\mathcal{H}_\mathbb{R}}. \end{aligned}$$

The image of $\mathcal{H}_\mathbb{R}$ under this embedding is $\frac{A^{1/4}}{\sqrt{A^{1/2}+A^{-1/2}}} \mathcal{K} \otimes 1$. Notice that A has an (unbounded) inverse. If follows that since the complex span of $\mathcal{K} \otimes 1$ is dense in $\mathcal{H}$, so is the complex span of the image of $\mathcal{H}_\mathbb{R}$. Similarly, since $\mathcal{K} \otimes 1 \cap i(\mathcal{K} \otimes 1) = \emptyset$, the same property holds for the image of $\mathcal{H}_\mathbb{R}$. Since $A^{it} = U_t$, the restriction of U_t to the image of $\mathcal{H}_\mathbb{R}$ is the original one-parameter group on $\mathcal{H}_\mathbb{R}$. Lastly, it is easily seen that

$$\mathrm{Im} \left\langle \frac{\sqrt{2}A^{1/4}}{\sqrt{A^{1/2} + A^{-1/2}}} h, \frac{\sqrt{2}A^{1/4}}{\sqrt{A^{1/2} + A^{-1/2}}} g \right\rangle_\mathcal{H} = \left\langle i \frac{1 - A^{-1}}{1 + A^{-1}} h, g \right\rangle_{\mathcal{H}_\mathbb{R}}.$$

Thus this embedding satisfies properties (a)–(d) above.

Remark 2.2. In [**19**], Tomita theory was extended to the case of an arbitrary embedding of a real Hilbert space $\mathcal{H}_\mathbb{R}$ into a complex Hilbert space $\mathcal{H}$, satisfying (a) and (b) above. It is not hard to check that in our case U_{-t} satisfies the KMS condition with respect to $\mathcal{H}_\mathbb{R}$ (see [**19**] for definition). Thus U_{-t} is the modular automorphism group for the embedding $\mathcal{H}_\mathbb{R} \subset \mathcal{H}$.

Consider now the full Fock space

$$\mathcal{F}(\mathcal{H}) = \mathbb{C}\Omega \oplus \bigoplus_{n>0}^{\infty} \mathcal{H}^{\otimes n}$$

and for $h \in \mathcal{H}$, define left and right creation operators by

$$\ell(h) : \xi \mapsto h \otimes \xi, \quad r(h) : \xi \mapsto \xi \otimes h$$

where $\xi \in \mathcal{H}^{\otimes m}$ for some m. Let

$$s(h) = \operatorname{Re} \ell(h), \quad d(h) = \operatorname{Re} r(h).$$

Notice that U_t defines a one-parameter group $\mathcal{F}(U_t)$ of unitary transformations on the full Fock space by

$$\mathcal{F}(U_t)\xi_1 \otimes \cdots \otimes \xi_n = (U_t\xi_1) \otimes \cdots \otimes (U_t\xi_n).$$

We now define an analog of Voiculescu's Free Gaussian functor (see [27]), and define the free quasi-free states on the resulting algebras.

Definition 2.3. Given $\mathcal{H}_{\mathbb{R}}$ and U_t as above, define

(i) $\Gamma(\mathcal{H}_{\mathbb{R}}, U_t) = C^*(s(\mathcal{H}_{\mathbb{R}}))$;

(ii) π_U to be the obvious representation on the Fock space;

(iii) the **free quasi-free state** φ_U on this algebra to be the vacuum expectation $\langle \Omega, \Omega \cdot \rangle_U$.

In the case U_t is trivial, $\mathcal{H}$ is the same as $\mathcal{H}_{\mathbb{C}}$, and $\Gamma(\mathcal{H}_{\mathbb{R}}, U_t)$ coincides with the algebra $\Phi(\mathcal{H}_{\mathbb{R}})$ of the Free Gaussian functor of [27].

Conjugation by the one-parameter group $\mathcal{F}(U_t)$ of unitaries on the Fock space sends $s(h)$ to $s(U_t(h))$ for $h \in \mathcal{H}_{\mathbb{R}}$, and thus leaves $\Gamma(\mathcal{H}_{\mathbb{R}}, U_t)$ invariant. The resulting one-parameter group of automorphisms on $\Gamma(\mathcal{H}_{\mathbb{R}}, U_t)$ is denoted by α.

Remark 2.4. In fact, any unitary transformation on $\mathcal{H}$ which leaves $\mathcal{H}_{\mathbb{R}}$ invariant gives rise in a similar way to an automorphism of $\Gamma(\mathcal{H}_{\mathbb{R}}, U_t)$. Thus any element of $O(\mathcal{H}_{\mathbb{R}}) \cap Sp(\mathcal{H}_{\mathbb{R}}, j)$ defines an automorphism of $\Gamma(\mathcal{H}_{\mathbb{R}}, U_t)$; here $O(\mathcal{H}_{\mathbb{R}})$ denotes the group of orthogonal transformations on $\mathcal{H}_{\mathbb{R}}$, j is as in property (d), and $Sp(\mathcal{H}_{\mathbb{R}}, j)$ denotes all invertible linear transformations on $\mathcal{H}_{\mathbb{R}}$ that preserve j.

Remark 2.5. The map from $\mathcal{H}_{\mathbb{R}}$ to $\Gamma(\mathcal{H}_{\mathbb{R}}, U_t)$ given by $h \mapsto s(h)$ is $\mathbb{R}$-linear. It has an inverse, which sends $s(h)$ to $s(h)\Omega$. Thus $\mathcal{H}_{\mathbb{R}}$ can be identified with the real subspace of $\Gamma(\mathcal{H}_{\mathbb{R}}, U_t)$ spanned by all $s(h)$, $h \in \mathcal{H}_{\mathbb{R}}$. In a similar way, we can define a $\mathbb{C}$-linear map $\hat{s} : \mathcal{H}_{\mathbb{R}} + i\mathcal{H}_{\mathbb{R}} \to \Gamma(\mathcal{H}_{\mathbb{R}}, U_t)$ by setting $\hat{s}(h + ig) = s(h) + is(g)$, $h, g \in \mathcal{H}_{\mathbb{R}}$, and identify $\mathcal{H}_{\mathbb{R}} + i\mathcal{H}_{\mathbb{R}}$ with an appropriate subspace of $\Gamma(\mathcal{H}_{\mathbb{R}}, U_t)$. Notice that both s and $\hat{s}$ are equivariant with respect to U_t acting on $\mathcal{H}_{\mathbb{R}}$ and $\mathcal{H}_{\mathbb{R}} + i\mathcal{H}_{\mathbb{R}}$ and α_t acting on $\Gamma(\mathcal{H}_{\mathbb{R}}, U_t)$. It follows that if $h \in \mathcal{H}_{\mathbb{R}}$ is an entire vector for U_t, then $s(h)$ is entire for α_t. Moreover, since $\hat{s}$ is $\mathbb{C}$-linear, we have that

$$\alpha_{ir}(s(h)) = \hat{s}(A^{-r}h)$$

for $r \in \mathbb{C}$. Notice also that s and $\hat{s}$ intertwine the restriction of $\langle\cdot,\cdot\rangle_U$ to $\mathcal{H}_\mathbb{R}$ and $\mathcal{H}_\mathbb{R} + i\mathcal{H}_\mathbb{R}$ and the restriction of $(x, y) \mapsto \varphi_U(x^*y)$ to the appropriate subspaces of $\Gamma(\mathcal{H}_\mathbb{R}, U_t)$. Finally, since $\|s(h)\| = \|h\|$, s is isometric for the Hilbert space norm on $\mathcal{H}_\mathbb{R}$ and the C^*-norm on $\Gamma(\mathcal{H}_\mathbb{R}, U_t)$.

Remark 2.6. We could also start with the inclusion $\mathcal{H}_\mathbb{R} \subset \mathcal{H}$, define U_t to be the modular group of the inclusion ([**19**]) and write $\Gamma(\mathcal{H}_\mathbb{R} \subset \mathcal{H})$. Alternatively, since the inclusion $\mathcal{H}_\mathbb{R} \subset \mathcal{K}$ is determined completely by $j = (\mathrm{Im}\langle\cdot,\cdot\rangle_\mathcal{K})|_{\mathcal{H}_\mathbb{R}}$, we could start with an appropriate anti-symmetric form j on $\mathcal{H}_\mathbb{R}$, and then construct $\mathcal{H}$ and U_t; then we could write $\Gamma(\mathcal{H}_\mathbb{R}, j)$.

Notice that $\Gamma(\mathcal{H}_\mathbb{R}, U_t)$ is generated by words in elements $s(h)$ for $h \in \mathcal{H}_\mathbb{R}$. Each $s(h)$ is distributed with respect to φ_U as a semicircular variable (see [**29**]). Explicitly, the value of φ_U on a word in $s(h_i)$ is given by

$$\varphi_U(s(h_1)\cdots s(h_n)) = 2^{-n} \sum_{\substack{(\{\gamma_1,\beta_1\},\dots,\{\gamma_{n/2},\beta_{n/2}\}) \\ \in NC(n),\ \gamma_i<\beta_i}} \prod_{k=1}^{n/2} \langle h_{\gamma_k}, h_{\beta_k}\rangle_U \tag{3}$$

for n even and is zero otherwise. This formula can be also taken as a definition of φ_U. Here $NC(k)$ stands for all non-crossing partitions of $\{1,\dots,k\}$, i.e., partitions for which whenever $a < b < c < d$, and a, c are in the same class, b, d are in the same class, then all a, b, c, d are in the same class.

Speicher in [**24**] defined, for an arbitrary functional ψ on an algebra generated by elements $x_1, x_2, \dots$, certain multilinear functionals ω_n, depending on ψ, which are called free cumulants of ψ. The following formula defines the free cumulants recursively; here $a_j \in \{x_1, x_2, \dots\}$:

$$\psi(a_1 \dots a_k) = \sum_{(A_1,\dots,A_s)\in NC(k)} \omega_{|A_1|}(a_{A_1})\omega_{|A_2|}(a_{A_2})\dots\omega_{|A_s|}(a_{A_s}), \quad \forall a_i,$$

where we use the following notation: By $(A_1,\dots,A_s) \in NC(k)$ we mean a non-crossing partition with classes $A_1,\dots,A_s$. Also, if S is a finite subset of $\mathbb{N}$, by $|S|$ we mean the number of elements of S, and a_S stands for the $|S|$-tuple of those a_i for which $i \in S$. (For example, if $S = \{1, 4, 6\}$, then by a_S we mean (a_1, a_4, a_6).)

In view of Formula (3), the free cumulants of φ_U are all zero, except second-order, given by

$$\omega_2(s(h), s(g)) = \varphi_U(s(h)s(g)).$$

This is analogous to the situation for the CAR and CCR cases; indeed, quasi-free states on those algebras can be characterized by requiring that all correlation functions (which are certain combinatorial multi-linear functionals defined in terms of values of states on words in generators) are all zero, except second-order (see e.g. [7]). Thus free cumulants of Speicher play the role of correlation functions in our case.

Remark 2.7. Suppose that A is some C^*-algebra generated by elements $x_1, x_2, \dots$, and φ is a faithful state on A, so that the cumulants ω_n of φ are all zero, except for $n = 2$. Take $\mathcal{H}_{\mathbb{R}}$ to be the closure of the real span of $x_1, x_2, \dots$ with respect to the norm $\|x\|_2 = \varphi(x^*x)$, and $\mathcal{H}$ to be the closure of the complex span of $x_1, x_2, \dots$ with respect to the same norm. Assume that $\mathcal{H}_{\mathbb{R}} \subset \mathcal{H}$ satisfies conditions (a) and (b) above. We see that $A \cong \Gamma(\mathcal{H}_{\mathbb{R}} \subset \mathcal{H})$ (in the notation of Remark 2.6) in a way that maps φ to the free quasi-free state on $\Gamma(\mathcal{H}_{\mathbb{R}} \subset \mathcal{H})$. The inner product on $\mathcal{H}$ is then $\langle x, y\rangle = \varphi(x^*y)$. In particular suppose $\mathcal{K}_{\mathbb{R}} \subset \mathcal{H}_{\mathbb{R}}$ is a real subspace, let $\mathcal{K}$ be the complex span (in $\mathcal{H}$) of $\mathcal{K}_{\mathbb{R}}$. Then by the above, we have $\Gamma(\mathcal{K}_{\mathbb{R}} \subset \mathcal{K}) \cong C^*(s(\mathcal{K}_{\mathbb{R}})) \subset \Gamma(\mathcal{H}_{\mathbb{R}} \subset \mathcal{H})$.

Suppose $\mathcal{K}_{\mathbb{R}} \subset \mathcal{H}_{\mathbb{R}}$ are real Hilbert spaces, and that P is the orthogonal projection onto $\mathcal{K}_{\mathbb{R}}$. We assume that P intertwines the modular groups of $\mathcal{H}_{\mathbb{R}} \subset \mathcal{H}$ and $\mathcal{K}_{\mathbb{R}} \subset \mathcal{K}$. This is of course equivalent to requiring that the inclusion map from $\mathcal{K}_{\mathbb{R}}$ to $\mathcal{H}_{\mathbb{R}}$ intertwine the modular groups, i.e., leave $\mathcal{K}_{\mathbb{R}}$ invariant. It follows that if we write $\mathcal{H}_{\mathbb{R}} = \mathcal{K}_{\mathbb{R}} \oplus_{\mathcal{H}_{\mathbb{R}}} \mathcal{K}_{\mathbb{R}}^{\perp_{\mathcal{H}_{\mathbb{R}}}}$, then the modular group of $\mathcal{H}_{\mathbb{R}} \subset \mathcal{H}$ preserves this decomposition. It follows that the $\mathcal{K}_{\mathbb{R}}$ and $\mathcal{K}_{\mathbb{R}}^{\perp_{\mathcal{H}_{\mathbb{R}}}}$ are perpendicular in the inner product of $\mathcal{H}$. Thus $\Gamma(\mathcal{H}_{\mathbb{R}} \subset \mathcal{H}) = \Gamma(\mathcal{K}_{\mathbb{R}} \subset \mathcal{K}) *_r \Gamma(\mathcal{K}_{\mathbb{R}}^{\perp_{\mathcal{H}_{\mathbb{R}}}} \subset \mathcal{V})$, where $\mathcal{V}$ is the complex span of $\mathcal{K}_{\mathbb{R}}^{\perp_{\mathcal{H}_{\mathbb{R}}}}$ in $\mathcal{H}$. Because of this free product decomposition, we see that there exists a completely positive map $\Phi : \Gamma(\mathcal{H}_{\mathbb{R}} \subset \mathcal{H}) \to \Gamma(\mathcal{K}_{\mathbb{R}} \subset \mathcal{K})$. Notice that this map is state-preserving.

Suppose now $\mathcal{H}_{\mathbb{R}}$, $\mathcal{K}_{\mathbb{R}}$ are two real Hilbert spaces, with one-parameter groups of orthogonal transformations $U_t = \exp(Ht)$ and $V_t = \exp(Kt)$, and suppose $A : \mathcal{H}_{\mathbb{R}} \to \mathcal{K}_{\mathbb{R}}$ is a contraction, such that A intertwines U_t and V_t, i.e., $V_t A = AU_t$ (equivalently, $KA = AH$). Let $\mathcal{L}_{\mathbb{R}} = \mathcal{H}_{\mathbb{R}} \oplus \mathcal{K}_{\mathbb{R}}$, and set

$$W_t = \exp\left(t\begin{pmatrix} H & 0 \\ 0 & K\end{pmatrix}\right),$$

$$B = \begin{pmatrix} (1 - A^*A)^{1/2} & A^* \\ A & -(1 - AA^*)^{1/2}\end{pmatrix}.$$

Then B is an orthogonal matrix, which commutes with W_t. Therefore the map

$$\Xi : \mathcal{H}_{\mathbb{R}} \ni \xi \mapsto B\begin{pmatrix}\xi \\ 0\end{pmatrix}$$

is an isometry from $\mathcal{H}_{\mathbb{R}}$ to $\mathcal{L}_{\mathbb{R}}$, which intertwines U_t and W_t. Also, the projection

$$P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

from $\mathcal{L}_{\mathbb{R}}$ onto $\mathcal{K}_{\mathbb{R}}$ intertwines W_t and V_t. The composition of the map Ξ and P is A.

Since Ξ induces an injection from $\Gamma(\mathcal{H}_{\mathbb{R}}, U_t)$ to $\Gamma(\mathcal{L}_{\mathbb{R}}, W_t)$, and P induces a completely positive map from $\Gamma(\mathcal{L}_{\mathbb{R}}, W_t)$ onto $\Gamma(\mathcal{K}_{\mathbb{R}}, V_t)$, we obtain the following:

Theorem 2.8. Γ *is a functor from the category of pairs of real Hilbert spaces with distinguished one-parameter automorphism groups and contractions that intertwine these groups, to the category of* C^**-algebras with distinguished states and state-preserving completely positive maps.*

This theorem is analogous to the properties of the Free Gaussian Functor ([**27**]) and the CAR functor ([**14**]).

Theorem 2.9. φ_U *is a KMS state for* α_t *at inverse temperature* 1*.*

This theorem actually follows from the results of Section 3, but we give a combinatorial proof nonetheless.

Proof. Recall that a state ψ satisfies the KMS condition (see e.g. [**7**]) for α_t at inverse temperature β if for all x, y which are entire for α_t, one has

$$\psi(xy) = \psi(y\alpha_{i\beta}(x)).$$

By [**7**], there is a dense subset of vectors in $\mathcal{H}_{\mathbb{R}}$ which are entire for U_t; since $\alpha_t s(g) = s(U_t(g))$, if $g \in \mathcal{H}_{\mathbb{R}}$ is entire for U_t, then also $s(g)$ is entire for α_t. Thus it is sufficient to check the KMS condition for words in $s(h)$, where h runs over a dense set of entire elements in $\mathcal{H}_{\mathbb{R}}$. It is clearly sufficient then to show that

$$\varphi_U(s(h_1)\dots s(h_n)) = \varphi_U(s(h_2)\dots s(h_n)\alpha_i(s(h_1)))$$

for h_i entire in $\mathcal{H}_{\mathbb{R}}$.

Remark that by Equation (3)

$$\varphi_U(s(g_1)\dots s(g_{n-1})\alpha_t(s(g_n))) \\ = \begin{cases} 2^{-n} \displaystyle\sum_{\substack{(\{\gamma_1,\beta_1\},\dots,\{\gamma_{n/2},\beta_{n/2}\}) \\ \in NC(n),\ \gamma_i<\beta_i}} \prod_{k=1}^{n/2} \langle f_{\gamma_k}, f_{\beta_k}\rangle_U, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

where $f_i = g_i$ for all $i \neq n$ and $f_n = U_t g_n$, $g_i \in \mathcal{H}_{\mathbb{R}}$, for all t. Since for a non-crossing partition $\{\{\gamma_1, \beta_1\}, \dots, \{\gamma_{n/2}, \beta_{n/2}\}\}$ with $\gamma_i < \beta_i$, it cannot happen that $\gamma_k = n$, it follows that the value of the above expression (which is clearly entire in t) at $t = i$ is given by

$$\varphi_U(s(g_1)\dots s(g_{n-1})\alpha_i(s(g_n)))$$
$$= \begin{cases} 2^{-n} \sum_{\substack{(\{\gamma_1,\beta_1\},\dots,\{\gamma_{n/2},\beta_{n/2}\}) \\ \in NC(n),\ \gamma_i<\beta_i}} \prod_{k=1}^{n/2} \langle f_{\gamma_k}, f_{\beta_k}\rangle_U, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

where $f_k = g_k$ for $k \neq n$ and $f_n = U_i g_n = A^{-1} g_n$, $g_k \in \mathcal{H}_{\mathbb{R}}$.

Now we compare the expressions for

$$\varphi_U(s(h_1)\dots s(h_n))$$

and

$$\varphi_U(s(h_2)\dots s(h_n)\alpha_i(s(h_1)))$$

using the above formulae.

For n odd, both expressions are zero. For n even, given a non-crossing partition

$$\{\{\gamma_1, \beta_1\}, \dots, \{\gamma_{n/2}, \beta_{n/2}\}\},$$

$\gamma_k < \beta_k$, the term corresponding to this partition in the expression for

$$\varphi_U(s(h_1)\dots s(h_n))$$

is

$$\prod_k \langle h_{\gamma_k}, h_{\beta_k}\rangle_U.$$

We associate to this term the term in the sum for

$$\varphi_U(s(h_2)\dots s(h_n)\alpha_i(s(h_1)))$$

associated to the partition

$$\{\{\gamma'_1, \beta'_1\}, \dots, \{\gamma'_{n/2}, \beta'_{n/2}\}\},$$

where

$$\gamma'_k = \gamma_k - 1, \quad \beta'_k = \beta_k - 1,$$

and the arithmetic is performed modulo n. The resulting term in the sum for

$$\varphi_U(s(h_2)\dots s(h_n)\alpha_i(s(h_1)))$$

is then equal to

$$\prod_k \langle f_{\gamma_k}, f_{\beta_k}\rangle_U,$$

where $f_{\gamma_i} = h_{\gamma_i}$, $f_{\beta_i} = h_{\beta_i}$ for $\gamma_i \neq 1$, and for $\gamma_j = 1$, $f_{\beta_j} = A^{-1}h_1$, $f_{\gamma_j} = h_{\beta_j}$. The equality of the two terms follows since

$$\langle h, g\rangle_U = \langle g, A^{-1}h\rangle_U$$

(this was established in Equation (2)). □

Remark 2.10. The proof above can be easily adapted to prove the following statement: Suppose φ is a functional on an algebra A, and $x_1, \ldots, x_n$ are some generators of A. Suppose α_t is a one-parameter group of automorphisms on A, such that $\alpha_t(x_k)$ is a linear combination of $x_1, \ldots, x_n$. Then ψ is KMS for α_t if and only if each cumulant ω_k of ψ satisfies:

$$\omega_k(a_1, \ldots, a_k) = \omega_k(a_2, \ldots, a_k, \alpha_i(a_1)),$$

where $a_j \in \{x_1, \ldots, x_n\}$. Of course, when α_t is trivial, this reduces to the well-known characterization of when a functional is tracial in terms of its cumulants.

Suppose $\mathcal{H}_{\mathbb{R}}^{(k)}$ is a family of real Hilbert spaces, with one-parameter groups of orthogonal transformations $U_t^{(k)}$. Consider on $\bigoplus_k \mathcal{H}_{\mathbb{R}}^{(k)}$ the one-parameter group $\bigoplus_k U_t^{(k)}$. Then it is easy to see that if $\mathcal{H}_{\mathbb{R}}^{(k)} \subset \mathcal{H}^{(k)}$, satisfy for each k (a)–(d) above, then so does $(\bigoplus_k \mathcal{H}_{\mathbb{R}}^{(k)}) \subset (\bigoplus_k \mathcal{H}^{(k)})$. This, combined with the fact that cumulants of the free product of two states are sums of cumulants of those states (see [**24**, **15**]) proves

Theorem 2.11. *Let $\mathcal{H}_{\mathbb{R}}^{(k)}$, $U_t^{(k)}$ be as above. Then*

$$\mathop{\ast}_k \left(\Gamma(\mathcal{H}_{\mathbb{R}}^{(k)}, U_t^{(k)}), \varphi_{U^{(k)}}\right) = \left(\Gamma(\bigoplus_k \mathcal{H}_{\mathbb{R}}^{(k)}, \bigoplus_k U_t^{(k)}), \varphi_{\bigoplus_k U^{(k)}}\right).$$

Remark 2.12. Notice that if $\mathcal{H}_{\mathbb{R}}$ is a finite-dimensional vector space (or if in general the eigenvectors of A densely span $\mathcal{H}$, i.e., U_t is almost periodic), $(\mathcal{H}_{\mathbb{R}}, U_t)$ can be written as a direct sum of two-dimensional real Hilbert spaces with nontrivial actions, and one-dimensional real Hilbert spaces with trivial actions. Remark also that $\Gamma(\mathbb{R}, \mathrm{id}_t)$ is a commutative C^*-algebra, isomorphic to $C[-1,1]$; the free quasi-free state defines the semicircular measure on it (see [**27**]). Thus $\Gamma(\mathcal{H}_{\mathbb{R}}, U_t)$ in this case is the free product of algebras of the form $\Gamma(\mathbb{R}^2, V_t)$ with V_t nontrivial, and/or $\Gamma(\mathbb{R}, \mathrm{id}_t)$.

3. The Fock space representation π_U.

The vacuum vector Ω is clearly cyclic for the representation π_U. Given the inclusion $\mathcal{H}_\mathbb{R} \subset \mathcal{H}$, consider the real subspace

$$\mathcal{H}'_\mathbb{R} = i\mathcal{H}_\mathbb{R}^{\perp_{\mathrm{Re}\langle\cdot,\cdot\rangle_U}} = \{g \in \mathcal{H} : \langle g, h\rangle_U \in \mathbb{R}, \forall h \in \mathcal{H}_\mathbb{R}\}.$$

This subspace is the "commutant" (in the sense of [**19**, **18**]) of $\mathcal{H}_\mathbb{R}$. The results of this section, and Remark 2.2 allow one to view our construction as a "functor" that translates Tomita theory for $\mathcal{H}_\mathbb{R} \subset \mathcal{H}$ into Tomita theory of the representation of $\Gamma(\mathcal{H}_\mathbb{R}, U_t)''$ on the Fock space.

Lemma 3.1. *Let $B = d(\mathcal{H}'_\mathbb{R})''$. Then $B \subset \Gamma(\mathcal{H}_\mathbb{R}, U_t)'$.*

Proof. In general, if $\xi \perp \Omega$, then $[d(h), s(g)]\xi = 0$, for all h, g. Thus we only need to show that $[d(h), s(g)]\Omega = 0$ for $h \in \mathcal{H}'_\mathbb{R}, g \in \mathcal{H}_\mathbb{R}$. We have

$$d(h)s(g)\Omega = \langle h, g\rangle_U \Omega + g \otimes h$$

while

$$s(g)d(h)\Omega = \langle g, h\rangle_U \Omega + g \otimes h.$$

But $\langle g, h\rangle_U \in \mathbb{R}$ so $\langle g, h\rangle_U = \langle h, g\rangle_U$. □

Remark that $\mathcal{H}'_\mathbb{R} + i\mathcal{H}'_\mathbb{R}$ is dense in $\mathcal{H}$, so Ω is clearly cyclic for B; it follows that it is separating for $\Gamma(\mathcal{H}_\mathbb{R}, U_t)''$. Thus Tomita theory (see e.g. [**26**]) applies. Recall that the operator S is defined on a dense set of vectors of the form $x\Omega$, $x \in \Gamma(\mathcal{H}_\mathbb{R}, U_t)''$ by

$$S : x\Omega \mapsto x^*\Omega.$$

Clearly it is enough to specify the values of S on tensors of the form $h_1 \otimes \cdots \otimes h_n$ for $h_i \in \mathcal{H}_\mathbb{R}$.

Lemma 3.2. *Consider $\xi = h_1 \otimes \cdots \otimes h_n$, where $h_i \in \mathcal{H}_\mathbb{R}$. Then $S\xi = h_n \otimes \cdots \otimes h_1$.*

Proof. The proof proceeds by induction on the degree of the tensor ξ. If $\xi = h$, $h \in \mathcal{H}_\mathbb{R}$, then $\xi = s(h)\Omega$, and since $s(h)$ is self-adjoint, $S\xi = \xi$. Suppose the formula holds for all $n < k$. Then

$$h_1 \otimes \cdots \otimes h_k = s(h_1)\dots s(h_k)\Omega - w,$$

where w is a sum of vectors of the form

$$h_1 \otimes \cdots \otimes h_r \langle h_{r+1}, h_{r+2}\rangle_U h_{r+3} \otimes \cdots \otimes h_s \langle h_{s+1}, h_{s+2}\rangle_U \dots,$$

where there is at least one term $\langle h_k, h_{k+1}\rangle_U$. Applying S to w, by anti-linearity of S, amounts to switching the order of all h_i, just as applying it to $s(h_1)\dots s(h_k)\Omega$ does. □

The adjoint of S is defined by the formula

$$\langle Sx, y\rangle_U = \langle S^* y, x\rangle_U.$$

Thus for $h_i, g_i \in \mathcal{H}_{\mathbb{R}}$,

$$\langle S^* h_1 \otimes \dots \otimes h_n, g_m \otimes \dots \otimes g_1\rangle_U = \delta_{nm} \prod_{j=1}^{n} \langle g_j, h_j\rangle_U$$

which by Equation (2) and because $A = A^*$, is the same as

$$\delta_{nm} \prod_{j=1}^{n} \langle A^{-1} h_j, g_j\rangle_U = \left\langle (A^{-1})^{\otimes n} h_n \otimes \dots \otimes h_1, g_m \otimes \dots \otimes g_1\right\rangle_U .$$

Thus

$$S^*(h_1 \otimes \dots \otimes h_n) = (A^{-1})^{\otimes n} h_n \otimes \dots \otimes h_1. \tag{4}$$

It follows that the modular operator $\Delta = S^* S$ acts by

$$\Delta : h_1 \otimes \dots \otimes h_n \mapsto (A^{-1})^{\otimes n}(h_1 \otimes \dots \otimes h_n). \tag{5}$$

Notice that $\Delta^{it} = U_{-t}$, thus α_{-t} is the modular group of φ_U; compare Remark 2.2 and Theorem 2.9.

Lastly, let $S = J\Delta^{1/2}$ be the polar decomposition of S, where J is an anti-linear isometry, $J^2 = 1$. Thus $\Delta^{1/2} = JS$, so

$$J(h_1 \otimes \dots \otimes h_n) = \Delta^{1/2}(h_n \otimes \dots \otimes h_1)$$

for $h_i \in \mathcal{H}_{\mathbb{R}}$. Thus the modular conjugation J is defined by anti-linearity and

$$J(h_1 \otimes \dots \otimes h_n) = (A^{-1/2})^{\otimes n} h_n \otimes \dots \otimes h_1. \tag{6}$$

With the help of J, we can now identify the commutant of $\Gamma(\mathcal{H}_{\mathbb{R}}, U_t)$.

Theorem 3.3. $\Gamma(\mathcal{H}_{\mathbb{R}}, U_t)' = d(\mathcal{H}'_{\mathbb{R}})''$.

Proof. By Lemma 3.1, it is sufficient to show that

$$J\Gamma(\mathcal{H}_{\mathbb{R}}, U_t)J \subset d(\mathcal{H}'_{\mathbb{R}})''.$$

Since conjugation by J is an anti-homomorphism, it is sufficient to show that

$$Js(h)J \in d(\mathcal{H}'_{\mathbb{R}})''$$

for all $h \in \mathcal{H}_{\mathbb{R}}$. We first claim that

$$Js(h) = d(A^{-1/2}h)J.$$

For $h_i \in \mathcal{H}_{\mathbb{R}}$,

$$Js(h)h_1 \otimes \cdots \otimes h_n = J\langle h, h_1\rangle_U h_2 \otimes \cdots \otimes h_n + Jh \otimes h_1 \otimes \cdots \otimes h_n,$$

which since J is anti-linear, is equal to

$$\langle h_1, h\rangle_U (A^{-1/2})^{\otimes(n-1)} h_n \otimes \cdots \otimes h_2 + (A^{-1/2})^{\otimes(n+1)} h_n \otimes \cdots \otimes h_1 \otimes h.$$

On the other hand,

$$d(A^{-1/2}h)Jh_1 \otimes \cdots \otimes h_n$$

is equal to

$$\langle A^{-1/2}h, A^{-1/2}h_1\rangle_U (A^{-1/2})^{\otimes(n-1)} h_n \otimes \cdots \otimes h_2 + (A^{-1/2})^{\otimes(n+1)} h_n \otimes \cdots \otimes h_1 \otimes h.$$

Thus all we need to show is that

$$\langle h_1, h\rangle_U = \langle A^{-1/2}h, A^{-1/2}h_1\rangle_U.$$

But the right hand side is the same (since $A = A^*$) as $\langle h, A^{-1}h_1\rangle_U$, which by Equation (2) is equal to $\langle h_1, h\rangle_U$.

To conclude the proof it is sufficient to show that for any h in a certain dense subset of $\mathcal{H}_{\mathbb{R}}$,

$$A^{-1/2}h \in \mathcal{H}'_{\mathbb{R}}.$$

To check this, all we need to show is that

$$\langle A^{-1/2}h, g\rangle_U \in \mathbb{R}$$

for all $g \in \mathcal{H}_{\mathbb{R}}$. But

$$\begin{aligned}\langle A^{-1/2}h, g\rangle_U &= \left\langle \frac{2A^{-1/2}}{1+A^{-1}}h, g \right\rangle \\ &= \left\langle g, \frac{2A^{1/2}}{1+A}h \right\rangle = \left\langle \frac{2}{1+A^{-1}}g, A^{-1/2}h \right\rangle \\ &= \left\langle g, A^{-1/2}h \right\rangle_U,\end{aligned}$$

and so is real. ∎

4. The two-dimensional case.

Consider $\mathcal{H}_{\mathbb{R}} = \mathbb{R}^2$ with a nontrivial action U_t. In a suitable orthonormal basis e_1, e_2, U_t acts as the matrix

$$\begin{pmatrix} \cos\log(\lambda)t & -\sin\log(\lambda)t \\ \sin\log(\lambda)t & \cos\log(\lambda)t \end{pmatrix},$$

where $0 < \lambda < 1$. In this case the inner product $\langle\cdot,\cdot\rangle_U$ is defined by the requirements that the norms of e_i are 1, and $\langle e_1, e_2\rangle_U = -i(\lambda-1)/(\lambda+1)$. Therefore the desired inclusion $\mathcal{H}_{\mathbb{R}} \subset \mathcal{H}$ can be obtained, for example, if one picks an orthonormal basis g, h for $\mathbb{C}^2 = \mathcal{H}$ and lets $\mathcal{H}_{\mathbb{R}}$ be the real span of

$$e_1 = \frac{1}{\sqrt{1+\alpha^2}}(g + \alpha h)$$

and

$$e_2 = \frac{1}{\sqrt{1+\alpha^2}}(ig - i\alpha h),$$

where $\alpha = 1/\sqrt{\lambda}$. Thus $\Gamma(\mathcal{H}_{\mathbb{R}}, U_t)$ can be viewed as generated by $s(e_1)$ and $s(e_2)$.

Let

$$y = \frac{s(e_1) + is(e_2)}{2}.$$

Then $C^*(y) = \Gamma(\mathcal{H}_{\mathbb{R}}, U_t)$; moreover,

$$\alpha_t(y) = \exp(-i\log(\lambda)t)y;$$

notice that y is entire for α. If we let $\ell_1 = \ell(g)$, $\ell_2 = \ell(h)$, then y is up to a constant multiple equal to

$$\ell_2 + \sqrt{\lambda}\ell_1^*.$$

Definition 4.1. An element y is a $*$-probability space (A, φ), whose $*$-distribution is equal to the $*$-distribution of the element $\ell_2 + \sqrt{\lambda}\ell_1^*$ with respect to the vacuum expectation, for some $0 \le \lambda \le 1$, will be called a **generalized circular element.**

For trivial U_t (formally $\lambda = 1$), such a y is (up to a constant multiple) the circular element of Voiculescu (see [**28**, **29**]), which justifies the term "generalized circular element". In that case in the polar decomposition $y = ub$, u is a Haar unitary (i.e., all moments of u are zero, except zeroth), and b is quarter-circular; in particular, b has no atoms in its distribution; also, the distribution of b^2 is free Poisson. Moreover, u and b are free. We aim to prove a similar result, for $0 < \lambda < 1$. The proof is very similar to one in [28].

Lemma 4.2. *Suppose (A, φ) is a C^*-probability space, $1 \in B \subset A$ is a subalgebra, $v \in A$ and $u \in A$, such that u is a unitary and*
(1) $\varphi(v) = \varphi(v^*) = \varphi(u) = \varphi(u^*) = 0$,
(2) *u is $*$-free from $B \cup \{v, v^*\}$,*
(3) *if $b \in B$, then $\varphi(b) = 0$ implies that $\varphi(vb) = \varphi(vbv^*) = \varphi(bv^*) = 0$.*
(4) $v^*v = 1$.
Then uv and B are $*$-free.

Proof. Let $c = \varphi(vv^*)$. Notice that

$$\varphi\left((uv)^k((uv)^*)^l\right) = \delta_{kl}c^k. \tag{7}$$

Indeed, replacing in $uv \dots uvv^*u^* \dots v^*u^*$ the term vv^* by $vv^* - c + c$, we have

$$\varphi\left((uv)^k((uv)^*)^l\right) = \varphi\left((uv)^{k-1}u(vv^* - c)u^*(uv)^{l-1}\right) + c\varphi\left((uv)^{k-1}(uv)^{l-1}\right) \tag{8}$$

and $\varphi((uv)^{k-1}u(vv^* - c)u^*(uv)^{l-1}) = 0$, since it can be written as $u^{r_1}x_1u^{r_2}x_2 \dots u^{r_s}$, where $r_i = \pm 1$, and x_i is either v, v^* or $vv^* - c$. In any case, $\varphi(x_i) = 0$, $x_i \in C^*(v)$, and moreover $\varphi(u^{r_i}) = 0$, so the freeness condition applies. If in Equation (7) $k \neq l$, then applying Equation (8) several times we eventually get $\varphi(uvuvuv \dots uv)$, or $\varphi((uv)^*(uv)^* \dots (uv)^*)$, both of which are zero by freeness of u and v. If $k = l$, then we eventually get $c^k\varphi(1) = c^k$. Thus Equation (7) holds.

Since uv is an isometry, $(uv)^*(uv) = v^*u^*uv = 1$, $C^*(uv)$ is (densely) linearly spanned by the *irreducible nontrivial words* in (uv), i.e., words of the form

$$(uv)^k((uv)^*)^l, \quad k, l \geq 0, \quad k + l > 0,$$

and also 1. Thus, using Equation (7), the linear subspace of $C^*(uv)$ consisting of elements the value of φ on which is zero, is (densely) linearly spanned by elements of the form $(uv)^k((uv)^*)^l - \delta_{kl}c^k$, $k, l \geq 0$, $k + l \neq 0$. It follows that to check freeness of uv and B, we must show that for $b_k \in B$, $\varphi(b_k) = 0$ (except possibly b_0 and/or b_n are equal to 1),

$$\varphi(\underbrace{b_0w_1b_1w_2 \dots w_nb_n}_{W}) = 0 \tag{9}$$

where

$$w_j = (uv)^{k_j}((uv)^*)^{l_j} - \delta_{k_jl_j}c^{k_j}, \tag{10}$$

$k_j, l_j \geq 0$, $k_j + l_j > 0$, for all j. We shall prove Equation (9) under a weaker assumption, which is that w_j is also allowed to be

$$(uv)^{s_j} u(vv^* - c)u^*((uv)^*)^{t_j}, \tag{11}$$

$s_j, t_j \geq 0$.

Given such a W, for each i such that w_i is as in Equation (10) with both k_j and l_j nonzero, but not as in Equation (11), i.e.,

$$w_i = \underbrace{(uv)(uv)\dots(uv)}_{k_i-1\ uv\text{'s}} u(vv^*)u^* \underbrace{(v^*u^*)(v^*u^*)\dots(v^*u^*)}_{l_i-1\ (uv)^*\text{'s}} - \delta_{k_il_i}c^{k_i}$$

replace this w_i by

$$\begin{aligned} w_i &= \left((uv)^{k_i-1}u(vv^* - c)u^*((uv)^*)^{l_i-1}\right) \\ &\quad + \left((uv)^{k_i-1}u(c)u^*((uv)^*)^{l_i-1} - \delta_{k_il_i}c^{k_i}\right) \\ &= A_i + B_i. \end{aligned} \tag{12}$$

In other words, we replace vv^* in the middle of the word w_i by $(vv^* - c) + c$ and redistribute.

After such a replacement is done, W can be rewritten as a sum of terms, in which some w_i are replaced by A_i's and some by B_i's. Consider the term where all replacements are replacements by A_i's. This term can be written as

(13)
$$b_0 u^{r_1} a_1 u^{r_2} a_2 \dots u^{r_m} b_n, \quad r_i = \pm 1, \quad a_i \in \{v, v^*, vv^* - c, b_k, vb_k, b_kv^*, vb_kv^*\}.$$

In any case, $a_i \in C^*(B \cup \{v\})$ and $\varphi(a_i) = 0$ by the hypothesis. Thus by freeness of u and $C^*(B \cup \{v\})$, φ is zero on such a term.

In the rest of the terms at least one w_i is replaced by B_i. Then, since

$$B_i = c\left((uv)^{k_i-1}((uv)^*)^{l_i-1} - \delta_{k_i-1,l_i-1}c^{k_i-1}\right),$$

we see that such a term is once again

$$b_0 w_1' b_1 w_2' b_2 \dots b_n,$$

so of the same form as W in Equation (9), but now with the total number of symbols v and u strictly smaller than the total number of such symbols in W. Thus applying our replacement procedure to each of these terms repeatedly, we finally get $\varphi(W) = \varphi(\sum W_i)$, where each W_i has the same form as W in Equation (9), but for which the substrings w_i are either as in Equation (10) with k_i or l_i equal to zero, or w_i is as in Equation (11) (so that no further replacements can be performed). But then each W_i can be rewritten as in Equation (13), so as before $\varphi(W_i) = 0$. Thus $\varphi(W) = 0$. □

Lemma 4.3. *For $0 < \lambda < 1$, $y = \ell_1 + \sqrt{\lambda}\ell_2{}^*$, the distribution of y^*y with respect to φ_U has no atoms; moreover, y^*y is invertible. If $y = v(y^*y)^{1/2}$ is the polar decomposition of y, then $v^*v = 1$, $vv^* \neq 1$.*

Proof. Notice that φ_U is the vacuum expectation. Let

$$R = y^*y = 1 + \lambda\ell_2\ell_2^* + \sqrt{\lambda}(\ell_2\ell_1 + (\ell_2\ell_1)^*),$$

and let Q be the projection

$$Q = \ell_2\ell_2^* - \ell_2\ell_1\ell_1^*\ell_2^*.$$

Finally, let $\mathcal{X} \subset \mathcal{F}(\mathbb{C}^2)$ be the subspace spanned by the vacuum vector Ω and all tensors of the form $(\ell_2\ell_1\Omega)^{\otimes n}$. Then for all $\xi \in \mathcal{X}$, we have

$$R\xi = (R - \lambda Q)\xi,$$

since $Q\mathcal{X} = 0$. Moreover, $\mathcal{X}$ is invariant under $\ell_2\ell_1$, $(\ell_2\ell_1)^*$ and $\ell_2\ell_2^*$, so under R; also, $\Omega \in \mathcal{X}$. It follows that the distribution of R is the same as the distribution of $(R - \lambda Q)$. But the latter is (up to a constant multiple) $(1 + \sqrt{\lambda}\ell)(1 + \sqrt{\lambda}\ell^*)$, where $\ell = \ell_2\ell_1$. But ℓ in itself is $*$-distributed as, e.g., ℓ_1. So the distribution of y^*y is the same as the distribution of

$$\left(1 + \sqrt{\lambda}\ell_1\right)\left(1 + \sqrt{\lambda}\ell_1^*\right).$$

We note that when $0 < \lambda < 1$, $1 + \sqrt{\lambda}\ell_1$ is invertible. It follows that the distribution of y^*y has support bounded away from zero. This coupled with the earlier results showing that the vacuum expectation is a faithful state, shows $v^*v = 1$, and that y^*y is invertible. Notice that the modular group acts on v by scaling it by $\exp(-i\log(\lambda)t)$. Thus by the KMS-condition applied to $1 = \varphi_U(v^*v)$ we have that $\varphi_U(vv^*) = \lambda \neq 1$. Thus $vv^* \neq 1$.

It remains to compute the distribution of y^*y, i.e., of

$$\left(1 + \sqrt{\lambda}\ell\right)\left(1 + \sqrt{\lambda}\ell^*\right) = A$$

(here ℓ is a creation operator). Let φ denote the vacuum expectation. Then

$$\varphi(A^k) = \varphi\left(\left(1 + \sqrt{\lambda}\ell\right)B^{k-1}\left(1 + \sqrt{\lambda}\ell^*\right)\right),$$

where

$$B = \left(1 + \sqrt{\lambda}\ell^*\right)\left(1 + \sqrt{\lambda}\ell\right) = (1 + \lambda) + \left(2\sqrt{\lambda}\right)s,$$

s denoting the semicircularly distributed variable $(\ell+\ell^*)/2$. Observe that

$$\varphi(A^k)=\varphi(B^{k-1});$$

this is because $\varphi(\ell x)=\varphi(x\ell^*)=0$, for all x.

Notice that the distribution of B is supported away from zero for $0<\lambda<1$. Let $f\cdot dx$ be the distribution of B, where f is a continuous function supported away from zero, and dx is the Lebesgue measure on the real line; let ν be the distribution of A. Then for $k>0$,

$$\varphi(A^k)=\int x^k d\nu=\varphi(B^{k-1})=\int x^{k-1}f(x)\cdot dx=\int x^k\frac{f(x)}{x}\cdot dx.$$

It follows that the distribution of A is $\frac{f(x)}{x}\cdot dx$ perhaps plus a point mass at zero. But we have seen above that it is supported away from zero. So the distribution of A is $f(x)/x\cdot dx$, and in particular has no atoms. $\square$

Remark 4.4. The distribution of y^*y is

$$\frac{\sqrt{4\lambda-(t-(1+\lambda))^2}}{2\pi\lambda t}dt$$

and is supported on the interval $((1-\sqrt{\lambda})^2,(1+\sqrt{\lambda})^2)$. The free Poisson distribution with R-transform $\lambda/(1-z)$ is (see [**29**])

$$\frac{\sqrt{4\lambda-(t-(1+\lambda))^2}}{2\pi t}dt+(1-\lambda)\delta_0$$

and is supported on the interval $((1-\sqrt{\lambda})^2,(1+\sqrt{\lambda})^2)$. It follows that all moments (except zeroth) of this free Poisson distribution are equal to λ times the corresponding moments of y^*y. By the KMS condition, we find that $\varphi_U((yy^*)^k)=\lambda\varphi_U((y^*y)^k)$, $k>0$. It follows that the distribution of yy^* is free Poisson. This is analogous to the statement that cc^* is free Poisson, if c is a circular variable.

Lemma 4.5. *Let $y=\ell_1+\sqrt{\lambda}\ell_2^*$ for $0<\lambda<1$, and let $y=vb$ be its polar decomposition, $b=(y^*y)^{1/2}$. Let u be a Haar unitary which is $*$-free from y. Then uv and b are $*$-free with respect to the vacuum expectation.*

Proof. Let φ stand for the vacuum expectation. We want to apply Lemma 4.2 to u, v and $B=C^*(b)$.

Condition (1) is immediate, since the modular group action preserves the state, so $\varphi(v)=\varphi(-v)$.

Condition (2) is fulfilled because u is $*$-free from y, and $W^*(y)$ contains both v and b.

Next, $\varphi(vd) = \varphi(dv^*) = 0$ for all $d \in B$, because the generator of B, b, is fixed under the modular group action, and since φ is KMS, it is invariant with respect to the modular automorphism group, so we get $\varphi(vd) = -\varphi(vd)$, etc. Also,

$$\varphi(vdv^*) = \varphi(dv^*v)\lambda$$

by the KMS-condition. But $v^*v = 1$ by Lemma 4.3. So (3) and (4) follow. □

Lemma 4.6. *Let $y = \ell_1 + \sqrt{\lambda}\ell_2^*$. Let u be a Haar unitary which is $*$-free from y. Then the $*$-distribution of y is the same as the $*$-distribution of uy.*

Proof. Since the joint distribution of free random variables only depends on their individual distributions, we are free to choose u as we like, as long as u and y are free. In particular, we may assume that u and ℓ_1, ℓ_2 are all $*$-free with respect to some state φ, and that the restriction of φ to $C^*(\ell_1, \ell_2)$ is the vacuum expectation ψ. Now, uy is obtained from y by replacing ℓ_1 by $u\ell_1$ and ℓ_2 by $\ell_2 u^*$.

It thus suffices to show that (ℓ_1, ℓ_2) have the same joint distribution as $(u\ell_1, \ell_2 u^*)$. The value of ψ on a word in ℓ_1, ℓ_2 and their adjoints is 1, if and only if this word reduces to 1 using the relations

$$\ell_i^*\ell_j = \delta_{ij}1,$$

and is zero otherwise. Similarly, the value of φ on a word in $u\ell_1, \ell_2 u^*$ and their adjoints is 1 if and only if this word reduces to 1 using the above relations and the fact that u is a unitary, and is zero otherwise. Indeed, if the word does not reduce, it means that it is of the form

$$u^{k_0} L_1 u^{k_1} L_2 \dots u^{k_{r-1}} L_r u^{k_r},$$

where $k_t \neq 0$ for $0 < t < r$, and each L_i is a nontrivial irreducible word in ℓ_1, ℓ_2 and their adjoints. Thus $\psi(L_i) = 0$, and the value of φ on the whole word is zero by freeness.

So all we need to show now is that a word in ℓ_1, ℓ_2 and their adjoints reduces to 1 if and only if the corresponding word with ℓ_1 replaced by $u\ell_1$ and ℓ_2 replaced by $\ell_2 u^*$, reduces to 1. If the former reduces to 1, then the latter clearly reduces to 1. If the former reduces to 0, then it must contain a string of the form

$$\ell_1^*\ell_2,$$

(or $\ell_2^*\ell_1$) which becomes

$$\ell_1^* u^* \ell_2 u^*$$

(or $u\ell_2^* u\ell_1$) and it is easy to see that a word containing such a string cannot be reducible. Finally, if the word in ℓ_i's does not reduce to either 1 or 0, it is seen easily that it cannot be reduced to 1 after the replacement is done. □

Remark 4.7. The proof of the above lemma is the only place where our proof differs in an essential way from [**28**], where a random matrix model was used to prove a similar lemma. Our proof can be used to give a completely combinatorial proof of Voiculescu's result on polar decomposition of a circular element. We also would like to mention a combinatorial proof of Voiculescu's result given recently by Banica ([**5**]).

Theorem 4.8. *Assume $0 < \lambda < 1$, and let $y = (s(e_1) + is(e_2))/2$, with e_i as in the beginning of Section 4. Then in the polar decomposition $y = vb$, v is a non-unitary isometry with $v^*v = 1$, and b is invertible. Moreover, with respect to the free quasi-free state φ_U, b has a distribution with no atoms, $\varphi_U(v^k(v^*)^l) = \delta_{kl}\lambda^k$ and furthermore v and b are $*$-free.*

Proof. Since the distribution of uy and y are the same (Lemma 4.6) it follows that the joint distribution of v and b is the same as the joint distribution of uv and b (since $uy = (uv)b$ is the polar decomposition of uy). But uv and b are free by Lemma 4.5. Thus v and b are $*$-free. The results on distribution of b are contained in Lemma 4.3. Lastly, by the KMS-condition, $\varphi_U(v^k(v^*)^l) = \lambda^k\varphi_U((v^*)^l v^k)$. If $k = l$, this is 1, since $v^*v = 1$; if $k \neq l$, this is either $\varphi_U(v^{k-l})$ or $\varphi_U((v^*)^{l-k})$, so zero, since φ_U is invariant under the modular action, which scales v by nontrivial constants. □

Corollary 4.9. *For nontrivial U_t and $\mathcal{H}_\mathbb{R}$ two-dimensional, with the above notation, $\Gamma(\mathcal{H}_\mathbb{R}, U_t) = C^*(y) = C^*(v) *_r C^*(b)$, the reduced free product with respect to the restriction of φ_U to $C^*(v)$ and $C^*(b)$.*

Proof. If U_t is nontrivial, b is invertible, and so $v \in C^*(y)$. □

Notice $C^*(b)$ is isomorphic to $C[-1, 1]$. Thus $W^*(b)$ is a diffuse commutative von Neumann algebra, and can be viewed as generated by, for example, a semicircular element.

The algebra $C^*(v)$ is isomorphic to the algebra $\mathcal{T}$ of Toeplitz operators: by universality of the Toeplitz operators, there is a map from $\mathcal{T}$ into $C^*(v)$, sending the generating isometry S of $\mathcal{T}$ to v; but this map is equivariant with respect to the circle action on $\mathcal{T}$, scaling S by complex number of modulus 1, and α_t, so the map is an isomorphism. Notice then that $W^*(v)$ is isomorphic to bounded operators on a Hilbert space, and thus is generated by matrix units $\{e_{ij}\}_{i,j=0}^{\infty}$. The restriction of φ_U to this algebra is given by

$$\varphi_U(e_{ij}) = \delta_{ij}\lambda^j(1-\lambda).$$

Remark 4.10. Notice that for $\lambda \neq 1$, $C^*(y)$ contains a nontrivial projection, namely vv^*. But for any nontrivial U_t, the algebra $\Gamma(\mathcal{H}_{\mathbb{R}}, U_t)$ contains a unital subalgebra isomorphic to $\Gamma(\mathcal{H}'_{\mathbb{R}}, U'_t)$, where $\mathcal{H}'_{\mathbb{R}}$ is two-dimensional and U'_t is nontrivial. This is because $\mathcal{H}_{\mathbb{R}}$ obviously contains a two-dimensional real Hilbert space $\mathcal{H}'_{\mathbb{R}}$ with the property that the restriction of the inner product of $\mathcal{H}$ to $\mathcal{H}'_{\mathbb{R}}$ is not real. Thus by the above, $\Gamma(\mathcal{H}_{\mathbb{R}}, U_t)$ contains a projection. Since the algebras of the Free Gaussian Functor of Voiculescu do not contain nontrivial projections ([**16**]) it follows that $\Gamma(\mathcal{H}_{\mathbb{R}}, U_t)$ is not isomorphic to any of those algebras.

Suppose U_t is almost-periodic, i.e., its eigenvectors densely span $\mathcal{H}$, and assume that $\dim \mathcal{H}_{\mathbb{R}} \geq 3$. Recall that if ψ is a state on an algebra B, then the centralizer of ψ is the subalgebra

$$B^{\psi} = \{x \in B : \psi(xy) = \psi(yx), \quad \forall y \in B\}.$$

Remark that if y is as in Theorem 4.8, then y^*y is in the centralizer of the free quasi-free state, because of the KMS condition.

We can write, using Corollary 4.9, $\Gamma(\mathcal{H}_{\mathbb{R}}, U_t) = C_1 *_r C_2$, where the centralizer of the restriction of φ_U to C_i contains a copy of $C[-1,1]$. Moreover, we may assume that the restriction of φ_U to this $C[-1,1]$ is the Lebesgue measure. Since $C[-1,1]$ contains continuous functions on the circle, we can find in C_1 a unitary a, such that $\varphi_U(a) = 0$, and we can find in C_2 unitaries b, c such that $\varphi_U(b) = \varphi_U(c) = \varphi_U(b^*c) = 0$. Notice that then the free product state is invariant under the adjoint actions of a, b, c, since they are in the centralizer. Avitzour proved (see Proposition 3.1 in [**4**]) that under such an assumption, for all $x \in C_1 *_r C_2 = \Gamma(\mathcal{H}_{\mathbb{R}}, U_t)$, $\varphi_U(x)1$ is in the closure of the convex hull of the set $\{uxu^*\}$, where u runs through the group generated by a, b, c above.

Theorem 4.11. *Suppose* $\dim \mathcal{H}_{\mathbb{R}} \geq 3$ *and* U_t *is almost-periodic. Then*

(i) $\Gamma(\mathcal{H}_{\mathbb{R}}, U_t)$ *is simple;*

(ii) φ_U *is the unique KMS-state for* α *at inverse temperature* 1*;*

(iii) *If* U_t *is nontrivial, there are no states on* $\Gamma(\mathcal{H}_{\mathbb{R}}, U_t)$ *which are KMS for* α *at inverse temperature* $\beta \neq 1$.

Proof. (i) Suppose J is a closed ideal in $\Gamma(\mathcal{H}_{\mathbb{R}}, U_t)$, and $0 \neq x \in J$. Since φ_U is faithful, $\varphi_U(x^*x) \neq 0$. But $\varphi_U(x^*x)1$ is in the closure of the convex hull of $\{uxu^*\}$, where u is in the group generated by a, b, c, which is a subset of J. So $1 \in J$.

(ii,iii) If ψ is another KMS-state for α at inverse temperature β, then a, b, c are still in the centralizer of ψ. This follows from the KMS condition, and the fact that α fixes a, b, c. Thus for any x, ψ is constant on the convex hull of $\{uxu^* : u \text{ in the group generated by } a, b, c\}$. Since $\varphi_U(x)1$ is in the closure of this convex hull, we see that $\psi(x) = \varphi_U(x)$. If U_t is nontrivial, since U_t is almost-periodic, it is easily seen that there is an element $y \in \Gamma(\mathcal{H}_{\mathbb{R}}, U_t)$, for which $\alpha_t(y) = \exp(-it \log \lambda)y$, and $\varphi_U(yy^*) = \lambda\varphi_U(y^*y) \neq 0$. It follows that ψ cannot be a KMS state at inverse temperature $\beta \neq 1$. □

Remark that statement (3) of the above theorem shows that it is impossible, for U_t almost-periodic and nontrivial, and $\dim \mathcal{H}_{\mathbb{R}} > 2$, to find an isomorphism between $\Gamma(\mathcal{H}_{\mathbb{R}}, U_t)$ and $\Gamma(\mathcal{H}_{\mathbb{R}}, V_t)$, where $V_t = U_{\kappa t}$, $\kappa \neq 1$, which is "covariant" with respect to V_t and U_t (i.e., one that would transfer $t \mapsto U_t$ into $t \mapsto V_{t/\kappa}$). While Properties (1) and (2) are very similar to the situation one has for quasi-free states on the CAR algebra, Property (3) is quite different.

The above theorem, with the same proof, also holds in the case when $\dim \mathcal{H}_{\mathbb{R}} = 2$, and U_t has period $2\pi/\log\lambda$, with $1 \geq \lambda \geq 1/2$. In this case, to find a, b, c, one uses Corollary 4.9 to rewrite $\Gamma(\mathcal{H}_{\mathbb{R}}, U_t)$ as the free product of $C[-1, 1]$ (wherein one finds b, c) and the Toeplitz operators. It is easily seen that when $1 \geq \lambda \geq 1/2$, there exists a unitary c in the centralizer of φ_U in the Toeplitz operators with $\varphi_U(c) = 0$.

In general, by Remark 2.7, if $\dim \mathcal{H}_{\mathbb{R}} \geq 3$, we can write $\mathcal{H}_{\mathbb{R}} = \overline{\bigcup \mathcal{H}_{\mathbb{R}}^{(k)}}$ and thus $\Gamma(\mathcal{H}_{\mathbb{R}}, U_t) = \overline{\bigcup \Gamma(\mathcal{H}_{\mathbb{R}}^{(k)} \subset \mathcal{H}^{(k)})}$, where $3 \leq \dim \mathcal{H}_{\mathbb{R}}^{(k)} < \infty$, $\mathcal{H}^{(k)}$ is the complex span (in $\mathcal{H}$) of $\mathcal{H}_{\mathbb{R}}^{(k)}$, and we are using the notation of Remark 2.6. Since by the above theorem, each $\Gamma(\mathcal{H}_{\mathbb{R}}^{(k)} \subset \mathcal{H}^{(k)})$ is simple, we have

Corollary 4.12. *Suppose* $\dim \mathcal{H}_{\mathbb{R}} \geq 3$. *Then* $\Gamma(\mathcal{H}_{\mathbb{R}}, U_t)$ *is simple.*

In fact, this argument shows that for each $x \in \Gamma(\mathcal{H}_{\mathbb{R}}, U_t)$, $\varphi_U(x)1$ is in the closure of the convex hull of $\{uxu^* : u \in \Gamma(\mathcal{H}_{\mathbb{R}}, U_t) \text{ unitary}\}$.

5. Matricial models.

In this section we develop several "matricial models," which have applications as technical tools. The models are to a certain extend generalizations of random matrix models (see [**29**, **28**, **11**]).

In what follows, let $\mathcal{E}_\infty$ be the Cuntz algebra on an infinite number of generators (see [**10**]), which can be viewed as generated by creation operators $\ell(h)$ where h are in some Hilbert space. The vacuum expectation $\langle \Omega, \cdot \Omega \rangle$

defines a state ψ on this algebra. It is known (e.g., [29]) that the operators $\ell(h)$ and $\ell(g)$ are $*$-free with respect to ψ if $h \perp g$. We shall say that a family F of isometries $\{L_k\}$ is $*$-distributed (with respect to some state) as a family of free creation operators, if the joint $*$-distribution of F is the same as the joint $*$-distribution of $\{\ell(g_k)\}$ with respect to the vacuum state for some orthonormal family $\{g_k\}$. Notice that then L_k are all $*$-free.

Let $M = M_N$ be either $N \times N$ matrices, or bounded operators on a separable Hilbert space (in which case we write $N = \infty$). Notice that M is generated by matrix units $\{e_{ij}\}_{i,j=0}^{N-1}$. Let ω be a state on M, given by

$$\omega = \mathrm{Tr}(\mathrm{diag}(c_0, c_1, \dots)\cdot),$$

where c_i are some nonnegative constants, $\sum c_i = 1$. For example, if N is finite, we can take $c_i = 1/N$, so ω is the normalized trace; another case of interest will be when $c_k = \lambda^k/\kappa$, where $0 < \lambda < 1$ and κ is chosen so that $\sum c_k = 1$. Let $C = \mathcal{E}_\infty \otimes M$, and consider the state $\theta = \psi \otimes \omega$ on C.

Theorem 5.1. *With the above notation, given a set of unit vectors $\{h_{ij}^k\}$ such that $h_{ij}^k \perp h_{ij'}^{k'}$ for all i, j, j', k, k' with $(j,k) \neq (j',k')$, consider in C the operators*

$$L_k = \sum_{i,j=0}^{N-1} \ell(h_{ij}^k) \otimes e_{ij}\sqrt{c_i}$$

(*interpreted as a weak limit when $N = \infty$*). *Then*

(i) *$L_s^* L_t = \delta_{st} 1$, and $\{L_k\}_k$ are jointly $*$-distributed with respect to θ as free creation operators, in particular are $*$-free;*

(ii) *$C^*(\{L_k\}_k)$ is free from the algebra generated by the projections $\{1 \otimes e_{ii}\}_{i=0}^{N-1}$.*

Proof. (i) First, we check that L_s is an isometry:

$$L_s^* L_s = \sum_{ijkl} \left(\ell(h_{ji}^s)^* \otimes e_{ij}\sqrt{c_j}\right)\left(\ell(h_{kl}^s) \otimes e_{kl}\sqrt{c_k}\right).$$

For $e_{ij}e_{kl}$ to be nonzero, we must have $j = k$; since $h_{ji}^s \perp h_{jl}^s$ for $i \neq l$, we also must have $i = l$. Thus the sum is equal to

$$\sum_{ij} c_j 1 \otimes e_{ii}$$

which is the identity, since $\sum_i c_i = 1$. In exactly the same way we get $L_r^*(L_s) = 0$ for $r \neq s$. To check that $\{L_k\}_k$ are jointly $*$-distributed as a family of free creation operators, in view of $L_r^* L_s = \delta_{rs} 1$, it is sufficient to check that θ is zero on any nontrivial irreducible word w in L_k's and L_k^*'s, i.e., a word of the form

$$w = L_{i_1} \dots L_{i_n} L_{j_1}^* \dots L_{j_m}^*,$$

where $n, m \geq 0$, $n + m > 0$. But viewed as a matrix, w has as entries sums of terms like

$$\ell(h_{a_1 b_1}^{i_1}) \dots \ell(h_{a_n b_n}^{i_n}) \ell(h_{c_1 d_1}^{j_1})^* \dots \ell(h_{c_m d_m}^{j_m})^*.$$

Each such term has zero expectation with respect to the state ψ; it follows that $\theta(w) = 0$, as desired.

(ii) Since $C^*(\{L_k\})$ is densely linearly spanned by words in L_k's and L_k^*'s, and 1; by (i), the linear subspace of this algebra of all elements that have zero expectation with respect to θ is densely linearly spanned by nontrivial irreducible words of the form $L_{i_1} \dots L_{i_n} L_{j_1}^* \dots L_{j_m}^*$, $n, m \geq 0$, $n + m > 0$. Similarly, the linear subspace of the algebra generated by the projections $1 \otimes e_{ii}$ consisting of elements that have expectation zero with respect to θ is linearly spanned by elements of the form $1 \otimes e_{ii} - c_i 1$. Thus to check freeness in (ii) it is sufficient to show that $\theta(W) = 0$, where

$$W = f_0 w_1 f_1 \dots w_n f_n,$$

w_i is of the form $L_{i_1} \dots L_{i_n} L_{j_1}^* \dots L_{j_m}^*$, $m, n \geq 0$, $m + n \neq 0$, and f_i is of the form $1 \otimes e_{jj} - c_j$, except possibly f_0 and/or f_n are 1.

Now, if W does not contain a substring of the form $L_r^* f_s L_t$, it is easily seen that as a matrix it has entries that have expectation zero with respect to ψ, so $\theta(W) = 0$. If W contains such a substring, $W = 0$. Indeed,

$$L_r^* 1 \otimes e_{ss} L_t = \sum_{ijkl} \left(\ell(h_{ji}^r)^* \otimes e_{ij} \sqrt{c_j} \right) (1 \otimes e_{ss}) \left(\ell(h_{kl}^t) \otimes e_{kl} \sqrt{c_k} \right). \tag{14}$$

For a nonzero term, we must have $j = s$, $s = k$, so $j = k$. If $r \neq t$, then $h_{ji}^r \perp h_{kl}^t$ for all i, l, by the hypothesis of the theorem, so all terms in the above sum vanish. Thus if $r \neq t$, $L_r^*(1 \otimes e_{ss} - c_s) L_t = L_r^*(1 \otimes e_{ss}) L_t - c_s L_r^* L_t = 0$. In the case $r = t$, we get a nonzero term in Equation (14) if and only if $i = l$, since otherwise h_{ji}^r and h_{kl}^t are orthogonal. Thus the sum in the equation is equal to $\sum_i 1 \otimes e_{ii} c_s = c_s 1$. Thus $L_r^*(1 \otimes e_{ss} - c_s) L_t = L_r^*(1 \otimes e_{ss}) L_t - c_s L_r^* L_t = 0$. □

The next model assumes more freeness among entries of L, but shows that then L is free from the off-diagonal matrix units e_{ij}.

Theorem 5.2. *Given an orthonormal set of vectors $\{h_{ij}^k\}$, consider operators L_k in C given by*

$$L_k = \sum_{i,j=0}^{N-1} \ell(h_{ij}^k) \otimes e_{ij} \sqrt{c_i}$$

(interpreted as a weak limit when $N = \infty$). Then

(i) $L_s^* L_t = \delta_{st} 1$, *and* $\{L_k\}_k$ *are jointly* $*$*-distributed with respect to* θ *as free creation operators, in particular are* $*$*-free;*

(ii) $C^*(\{L_k\}_k)$ *is free from the algebra generated by the matrix units* $\{1 \otimes e_{ij}\}_{i,j=0}^{N-1}$.

Proof. (i) Follows from (i) of Theorem 5.1. (ii) Since elements $1 \otimes e_{ij} - \delta_{ij} c_i$ span the linear subspace of the algebra generated by $\{1 \otimes e_{ij}\}$ consisting of elements that have zero expectation with respect to θ, just as in the proof of (ii) of Theorem 5.1, it is sufficient to check that $\theta(W) = 0$, where

$$W = f_0 w_1 f_1 \dots w_n f_n.$$

Here w_i is of the form $L_{i_1} \dots L_{i_p} L_{j_1}^* \dots L_{j_q}^*$, $p, q \geq 0$, $p + q > 0$, and f_i is of the form $1 \otimes e_{ij} - \delta_{ij} c_i$, except possibly f_0 and/or f_n are 1. Once again, if W does not contain a substring of the form $L_r^* f_s L_t$, then when viewed as a matrix it has entries that have zero expectation with respect to ψ, and so $\theta(W) = 0$. If W contains such a substring, it is zero. Indeed, if f_s is of the form $1 \otimes e_{kk} - c_k$, $L_r^* f_s L_t = 0$ by the computation in the proof of (ii) of Theorem 5.1. If $f_s = e_{pq}$, $p \neq q$, then

$$L_r^* f_s L_t = \sum_{ijkl} \left(\ell(h_{ji}^r)^* \otimes e_{ij} \sqrt{c_j} \right) (1 \otimes e_{pq}) \left(\ell(h_{kl}^t) \otimes e_{kl} \sqrt{c_k} \right).$$

For a nonzero term we must have $j = p$, $q = k$, so $j \neq k$. But then $h_{ji}^r \perp h_{kl}^t$. So the above expression is zero. □

Remark 5.3. Theorem 5.2 implies results of Voiculescu and Dykema (e.g., Proposition 5.1.7 of [**29**]) but does not rely on random results models (like in [**11**]), and is more general, allowing an arbitrary state on the matrix units.

As an application of the techniques we elaborated, we shall consider the algebra $\Gamma(\mathcal{H}_{\mathbb{R}}, U_t)''$ for $\mathcal{H}_{\mathbb{R}} \cong \mathbb{R}^2$ and U_t nontrivial with period $2\pi/\log(\lambda)$. By Remark 2.12, this is a "building block" out of which all $\Gamma(\mathcal{H}_{\mathbb{R}}, U_t)''$ for almost periodic U_t are constructed. We shall denote such an algebra by T_λ; we'll abuse notation and write φ_λ for φ_U. Notice that T_λ (as well as free products of countably many copies thereof) comes faithfully represented on a separable Hilbert space, thus has a separable predual. In what follows, we shall write $(M, \varphi) \cong (N, \psi)$, or say that these pairs are isomorphic, if there exists an isomorphism of M with N, which is state-preserving.

Theorem 5.4. $(T_\lambda, \varphi_\lambda) * (L^\infty[-1,1], \mu) \cong (T_\lambda, \varphi_\lambda)$, *where* μ *is the semicircular measure on* $L^\infty[-1,1]$.

Proof. By Theorem 4.8, T_λ is the free product of a diffuse commutative von Neumann algebra (generated, e.g., by a semicircular element a), and an algebra generated by matrix units $\{f_{ij}\}_{i,j=0}^{\infty}$ with the state whose value on f_{ij} is $\delta_{ij}\lambda^j(1-\lambda)$.

For an orthonormal family $\{h_{ij}\}$, take in Theorem 5.2

$$N=\infty, \quad c_k=(1-\lambda)\lambda^k,$$

and put

$$s=L+L^*, \quad L=\sum_{ij}\ell(h_{ij})\otimes e_{ij}\sqrt{c_i};$$

thus s and the matrix units $1\otimes e_{ij}$ are free. Then T_λ can be viewed as generated by s and the matrix units $1\otimes e_{ij}$, and the state φ_λ can be identified with the restriction of the state θ of Theorem 5.2 to this algebra.

Similarly, $(T_\lambda,\varphi_\lambda)*(L^\infty[-1,1],\mu)$ can be viewed as generated by matrix units $1\otimes e_{ij}$ as well as elements $L'+(L')^*$, $L''+(L'')^*$, where $L'=\sum\ell(h'_{ij})\otimes e_{ij}\sqrt{c_i}$, $L''=\sum\ell(h''_{ij})\otimes e_{ij}\sqrt{c_i}$, and $\{h'_{ij}\}\cup\{h''_{ij}\}$ form an orthonormal family; the free product state can be identified with the restriction of θ.

It is obvious from these considerations, that if we set

$$C=(1\otimes e_{00})T_\lambda(1\otimes e_{00}),$$
$$D=(1\otimes e_{00})((T_\lambda,\varphi_\lambda)*(L^\infty[-1,1],\mu))(1\otimes e_{00}),$$

then

$$(T_\lambda,\varphi_\lambda)\cong(C,\theta(1\otimes e_{00}\cdot 1\otimes e_{00}))\otimes(B(H),\mathrm{Tr}(\mathrm{diag}(c_0,c_1,\dots)\cdot))$$

and

$$(T_\lambda,\varphi_\lambda)*(L^\infty[-1,1],\mu)$$
$$\cong(D,\theta(1\otimes e_{00}\cdot 1\otimes e_{00}))\otimes(B(H),\mathrm{Tr}(\mathrm{diag}(c_0,c_1,\dots)\cdot)).$$

Thus to prove the statement of the theorem, it is sufficient to prove that

$$(C,\theta(1\otimes e_{00}\cdot 1\otimes e_{00}))\cong(D,\theta(1\otimes e_{00}\cdot 1\otimes e_{00})).$$

By Lemma 5.2.1 of [**29**], C is generated by elements of the form

$$c_{ij}=(1\otimes e_{0i})(L+L^*)(1\otimes e_{j0})=(\ell(h_{ij})\sqrt{c_i}+\ell(h_{ji})^*\sqrt{c_j})\otimes e_{00}.$$

Similarly, D is generated by elements

$$d'_{ij}=(1\otimes e_{0i})(L'+(L')^*)(1\otimes e_{j0})=(\ell(h'_{ij})\sqrt{c_i}+\ell(h'_{ji})\sqrt{c_j})\otimes e_{00}$$

and

$$d''_{ij}=(1\otimes e_{0i})(L''+(L'')^*)(1\otimes e_{j0})=(\ell(h''_{ij})\sqrt{c_i}+\ell(h''_{ji})\sqrt{c_j})\otimes e_{00}.$$

With respect to $\frac{1}{\theta(1\otimes e_{00})}\theta(1\otimes e_{00}\cdot 1\otimes e_{00})$, the elements c_{ij} are free for different $i \leq j$ (notice that $c_{ij}^* = c_{ji}$). Moreover, c_{ij}, up to a constant multiple, is distributed as $\ell + \ell^*$ for $i = j$, and $\ell + \sqrt{\lambda^{i-j}}\ell'$ for $i < j$, where ℓ, ℓ' are two free creation operators. By results of Section 4, we have that c_{ij} generates either $T_{\lambda^{j-i}}$, $i < j$, or $L^\infty[-1,1]$ if $i = j$. Thus the cutdown C, taken with the state $\frac{1}{\theta(1\otimes e_{00})}\theta(1\otimes e_{00}\cdot 1\otimes e_{00})$, is isomorphic in a state-preserving way to

$$\left(\mathop{\ast}_{k\in\mathbb{Z}}(L^\infty[-1,1],\mu)\right) * \left(\mathop{\ast}_{k\in\mathbb{Z}}(T_\lambda,\varphi_\lambda)\right) * \left(\mathop{\ast}_{k\in\mathbb{Z}}(T_{\lambda^2},\varphi_{\lambda^2})\right) * \dots$$

Similarly, with respect to $\frac{1}{\theta(1\otimes e_{00})}\theta(1\otimes e_{00}\cdot 1\otimes e_{00})$, the family $\{d'_{ij} : i \leq j\} \cup \{d''_{kl} : k \leq l\}$ is free and generates the cutdown algebra D. Just as above, we get that D, taken with the state $\frac{1}{\theta(1\otimes e_{00})}\theta(1\otimes e_{00}\cdot 1\otimes e_{00})$, is isomorphic in a state-preserving way to

$$\left(\mathop{\ast}_{k\in\mathbb{Z}}(L^\infty[-1,1],\mu)\right) * \left(\mathop{\ast}_{k\in\mathbb{Z}}(T_\lambda,\varphi_\lambda)\right) * \left(\mathop{\ast}_{k\in\mathbb{Z}}(T_{\lambda^2},\varphi_{\lambda^2})\right) * \dots$$

Thus C and D are isomorphic in a state-preserving way. □

Corollary 5.5. $(T_\lambda,\varphi_\lambda) * (L(\mathbb{F}_n),\mathrm{trace}) \cong (T_\lambda,\varphi_\lambda)$.

6. Associated von Neumann algebras.

In this section we consider a general $\mathcal{H}_\mathbb{R}$ with an action U_t. First, we consider the case when U_t is almost periodic; in this case the eigenvalues for A densely span $\mathcal{H}$. By Remark 2.12, T_λ is a "building block" for these algebras: by Theorems 2.11, 5.4, for nontrivial U_t, the algebras are just (finite or infinite) free products of T_λ's.

Connes in [8] defined the T invariant of a factor M to be the subgroup of $\mathbb{R}$

$$T(M) = \{t \in \mathbb{R} : \sigma_t^\varphi \text{ is inner}\}$$

where φ is some faithful normal weight, and σ^φ denotes the corresponding modular group. (See [8, 25] for further discussion of the T invariant.)

The S invariant of a factor M was defined in [8] to be the intersection over all faithful normal weights φ of the spectra of the modular operators Δ^φ. M is a type III factor if an only if $0 \in S(M)$; in that case Connes' III_λ classification of M in terms of its S invariant is as follows:

$$S(M) = \begin{cases} \{\lambda^n : n \in \mathbb{Z}\} \cup \{0\}, & \text{if } M \text{ is type III}_\lambda,\ 0<\lambda<1 \\ [0,+\infty), & \text{if } M \text{ is type III}_1 \\ \{0,1\}, & \text{if } M \text{ is type III}_0 \end{cases}.$$

Recall that the centralizer of a faithful normal state φ on M is defined to be

$$M^{\varphi} = \{x \in M : \varphi(xy) = \varphi(yx), \quad \forall y \in M\}.$$

If M^{φ} is a factor, then $S(M)$ is equal to the spectrum of the modular operator corresponding to φ.

Barnett in [6] proved that if M, N are two von Neumann algebras with states φ and ψ, and if M^{φ} contains a discrete finite group of at least 3 orthogonal unitaries, while N^{ψ} contains a discrete finite group of at least 2 orthogonal unitaries, then $M * N$ is a full type III_λ factor with $\lambda \neq 0$, (see [9] for definition of full factors), the centralizer of the free product state in $M * N$ is a factor, and moreover the T invariant $T(M * N)$ is given by

$$\{t \in \mathbb{R} : \sigma_t^{\psi} = \mathrm{id},\ \sigma_t^{\varphi} = \mathrm{id}\}$$

where σ^{θ} denotes the modular group of θ. Since $(T_\lambda, \varphi_\lambda) \cong (T_\lambda, \varphi_\lambda) * (L^{\infty}[-1,1], \mu)$, ($\mu$ is the semicircular measure) the centralizer of φ_λ in T_λ contains a diffuse commutative von Neumann algebra, and so a subgroup of four orthogonal unitaries; moreover for U_t almost-periodic, $\Gamma(\mathcal{H}_{\mathbb{R}}, U_t)''$ is a free product of T_λ's and a diffuse commutative von Neumann algebra. Using this, the fact that for a factor of type III_λ with separable predual, $\lambda \neq 0$,

$$T(M) = \begin{cases} \{0\}, & \text{if } \lambda = 1 \\ \dfrac{2\pi}{\log(\lambda)}\mathbb{Z}, & \text{if } 0 < \lambda < 1 \end{cases}$$

and that the modular action for the free quasi-free state on T_λ has, by definition, period $2\pi/\log(\lambda)$, we obtain, using the notation $\mathbb{R}_+^{\times}$ for the multiplicative group of positive real numbers:

Theorem 6.1. *Suppose U_t is almost-periodic, and let G be the closed subgroup of $\mathbb{R}_+^{\times}$ generated by the spectrum of A ($U_t = A^{it}$). Then*

$$\Gamma(\mathcal{H}_{\mathbb{R}}, U_t)'' \text{ is } \begin{cases} \text{type } \mathrm{III}_1, & \text{if } G = \mathbb{R}_+ \\ \text{type } \mathrm{III}_\lambda, & \text{if } G = \lambda^{\mathbb{Z}},\ 0 < \lambda < 1. \\ \text{type } \mathrm{II}_1, & \text{if } G = \{1\} \end{cases}$$

Moreover, $\Gamma(\mathcal{H}_{\mathbb{R}}, U_t)''$ is full.

Of course, the type II_1 case corresponds to trivial U_t, and $\Gamma(\mathcal{H}_{\mathbb{R}}, \mathrm{id}_t)'' \cong L(\mathbb{F}_{\dim \mathcal{H}_{\mathbb{R}}})$ by results of [27].

Notice that $S(\Gamma(\mathcal{H}_{\mathbb{R}}, U_t)'')$ is the spectrum of the modular operator since the centralizer of the free quasi-free state is a factor.

The above theorem can also be obtained using the results of Dykema ([**13**]); those results also imply that the centralizer of the free quasi-free state on $\Gamma(\mathcal{H}_{\mathbb{R}}, U_t)''$ for nontrivial almost-periodic U_t is isomorphic to $L(\mathbb{F}_\infty)$, and the discrete core (see [**13**] for details) is isomorphic to $L(\mathbb{F}_\infty) \otimes B(H)$.

Remark 6.2. In the proof of Theorem 6.1 we have only used the fact that $(T_\lambda, \varphi_\lambda)$ is stable under taking free products with a diffuse commutative von Neumann algebra. Thus a similar theorem holds for any von Neumann algebra A with a non-tracial faithful normal state φ, for which $(A, \varphi) \cong (A, \varphi) * (L^\infty[-1, 1], \mu)$ (μ is the semicircular measure): A is then necessarily a full type III factor, not of type III_0, and its T invariant is given by $\{t \in \mathbb{R} : \sigma_t^\varphi = \mathrm{id}\}$. Such a property may be called free absorption.

In what follows it will be notationally convenient to write $(T_\lambda, \varphi_\lambda)$ for $(T_{1/\lambda}, \varphi_{1/\lambda})$ when $\lambda > 1$, and (T_1, φ_1) for $(L(\mathbb{F}_2), \mathrm{tr})$.

Proposition 6.3. *For $\lambda \neq 1$,*

$$(T_\lambda, \varphi_\lambda) * (T_\mu, \varphi_\mu) \cong (T_\lambda, \varphi_\lambda) * \left(\mathop{\ast}_{j,k \in \mathbb{Z}} (T_{\mu\lambda^k}, \varphi_{\mu\lambda^k}) \right).$$

Proof. Using Theorem 4.8, we can rewrite $(T_\lambda, \varphi_\lambda) * (T_\mu, \varphi_\mu)$ as a free product of:

(a) An algebra generated by matrix units f_{ij} with a state whose value on f_{ij} is $(1 - \lambda)\lambda^j \delta_{ij}$ (these come from the polar part of the polar decomposition of the generalized circular element generating T_λ, see Section 4);

(b) A diffuse commutative von Neumann algebra, generated by a semicircular element (this comes from the positive part of the polar decomposition);

(c) T_μ, which can be viewed as generated by an appropriate generalized circular element.

Thus if in Theorem 5.2 we take

$$N = \infty, \quad c_k = (1 - \lambda)\lambda^k,$$

and some orthonormal family $\{h_{ij}^1\} \cup \{h_{ij}^2\} \cup \{h_{ij}^3\}$, we can view $(T_\lambda, \varphi_\lambda) * (T_\mu, \varphi_\mu)$ as the algebra generated by

(a) Matrix units $1 \otimes e_{ij}$

(b) $L_1 + L_1^*$

(c) $L_2 + \sqrt{\mu} L_3^*$

where

$$L_k = \sum_{i,j} \ell(h_{ij}^k) \otimes e_{ij} \sqrt{c_i}.$$

In this picture, the free product state on $(T_\lambda,\varphi_\lambda)*(T_\mu,\varphi_\mu)$ is identified with the state θ of Theorem 5.2.

Consider now the algebra

$$(1\otimes e_{00})\left((T_\lambda,\varphi_\lambda)*(T_\mu,\varphi_\mu)\right)(1\otimes e_{00})$$

with the state given by the restriction of the free product state. By Lemma 5.2.1 of [**29**], this cutdown is generated by elements of the form

$$c_{kl}=1\otimes e_{0k}(L_1+L_1^*)1\otimes e_{l0}$$

and

$$d_{kl}=1\otimes e_{0k}(L_2+\sqrt{\mu}L_3^*)1\otimes e_{l0},$$

which are easily seen to be

$$c_{kl}=(\ell(h^1_{kl})\sqrt{c_k}+\ell(h^1_{lk})^*\sqrt{c_l})\otimes e_{00}$$

and

$$d_{kl}=(\ell(h^2_{kl})\sqrt{c_k}+\ell(h^3_{lk})^*\sqrt{c_l\mu})\otimes e_{00}.$$

Since h^i_{jk} are orthonormal, we see that the family

$$\{c_{ij}:i\leq j\}\cup\{d_{ij}\}$$

in $(1\otimes e_{00})((T_\lambda,\varphi_\lambda)*(T_\mu,\varphi_\mu))(1\otimes e_{00})$, is $*$-free (with respect to $\frac{1}{\theta(1\otimes e_{00})}\theta(1\otimes e_{00}\cdot 1\otimes e_{00})$). Moreover, by results of Section 4, c_{ij} generates $L^\infty[-1,1]$ (the restriction of $\frac{1}{\theta(1\otimes e_{00})}\theta$ to which is a semicircular measure) for $i=j$ and $T_{\lambda^{j-i}}$ otherwise (the restriction of $\frac{1}{\theta(1\otimes e_{00})}\theta$ to this is $\varphi_{\lambda^{j-i}}$). Similarly, d_{ij} generates $T_{\mu\lambda^{j-i}}$. It follows that

$$(1\otimes e_{00})((T_\lambda,\varphi_\lambda)*(T_\mu,\varphi_\mu))(1\otimes e_{00}) \cong\left(\mathop{\ast}_{i,j\in\mathbb{Z}}(T_{\lambda^i},\varphi_{\lambda^i})\right)*\left(\mathop{\ast}_{i,j\in\mathbb{Z}}(T_{\lambda^i\mu},\varphi_{\lambda^i\mu})\right) \tag{15}$$

in a state-preserving way.

Consider now the right hand side of the statement of our proposition,

$$(T_\lambda,\varphi_\lambda)*\left(\mathop{\ast}_{j,k\in\mathbb{Z}}(T_{\mu\lambda^k},\varphi_{\mu\lambda^k})\right).$$

Just as above, for a certain orthonormal family of vectors $\{g^1_{ij}\}\cup\{g^{kl,2}_{ij}\}\cup\{g^{kl,3}_{ij}\}$, we can view this algebra as being generated by

(a) matrix units $e_{ij}\otimes 1$

(b) the semicircular element $L_1 + L_1^*$
(c) generalized circular elements $L_2^{(kl)} + \sqrt{\mu\lambda^k} L_3^{(kl)}$
where

$$L_1 = \sum_{ij} \ell(g_{ij}^1) \otimes e_{ij} \sqrt{c_i}$$

and

$$L_s^{(kl)} = \sum_{ij} \ell(g_{ij}^{kl,s}) \otimes e_{ij} \sqrt{c_i}, \quad s = 1, 2.$$

Once again, the free product state is identified with θ. Considering the cutdown of

$$(T_\lambda, \varphi_\lambda) * \left(\mathop{\ast}_{j,k \in \mathbb{Z}} (T_{\mu\lambda^k}, \varphi_{\mu\lambda^k}) \right)$$

by $1 \otimes e_{00}$, we get an algebra generated by the entries of the matrices in (b) and (c), i.e., elements of the form

$$r_{ij} = (\ell(g_{ij}^1)\sqrt{c_i} + \ell(g_{ji}^1)^* \sqrt{c_j}) \otimes e_{00}$$

and

$$t_{ij}^{kl} = \left(\ell(g_{ij}^{kl,2})\sqrt{c_i} + \ell(g_{ij}^{kl,3})^* \sqrt{c_j \mu\lambda^k} \right) \otimes e_{00}.$$

We see that the family

$$\{r_{ij} : i \leq j\} \cup \{t_{ij}^{kl}\}$$

is $*$-free and that moreover, r_{ii} generates $L^\infty[-1,1]$, r_{ij} generates $T_{\lambda^{j-i}}$ for $j \neq i$, and t_{ij}^{kl} generates $T_{\lambda^{j-i+k}\mu}$. Thus the cutdown algebra, considered with the state $\frac{1}{\theta(1\otimes e_{00})}\theta(1 \otimes e_{00} \cdot 1 \otimes e_{00})$ is isomorphic, in a state-preserving way, to

$$\left(\mathop{\ast}_{k,l \in \mathbb{Z}} (T_{\lambda^k}, \varphi_{\lambda^k}) \right) * \left(\mathop{\ast}_{i,j,k,l \in \mathbb{Z}} (T_{\lambda^{j-i+k}\mu}, \varphi_{\lambda^{j-i+k}\mu}) \right).$$

It is trivial to see that this is isomorphic (in a state-preserving way) to the algebra of Equation (15). The statement of the theorem follows, since it is obvious from our constructions that the algebras on the right and left sides in the theorem are isomorphic to their cutdowns by $1 \otimes e_{00}$ (taken with restriction of θ), tensor $(B(H), \mathrm{Tr}(\mathrm{diag}(c_0, c_1, \dots)\cdot)).$ □

Theorem 6.4. *Suppose U_t is almost-periodic and nontrivial, and let H be the (not necessarily closed) subgroup of $\mathbb{R}_+$ generated by the point spectrum* *of A. Then $(\Gamma(\mathcal{H}_{\mathbb{R}}, U_t)'', \varphi_U)$ depends up to state-preserving isomorphisms only on H.*

Proof. Write $\Gamma(\mathcal{H}_{\mathbb{R}}, U_t)''$ as the free product

$$\mathop{\ast}_k (T_{\lambda_k}, \varphi_{\lambda_k}),$$

$\lambda_k \neq 1$. Then H is the (not necessarily closed) subgroup of $\mathbb{R}_+^\times$ generated by the λ_k's. By Theorem 5.4,

$$(T_\lambda, \varphi_\lambda) \cong (T_\lambda, \varphi_\lambda) * (T_1, \varphi_1).$$

Applying Proposition 6.3 to $(T_\lambda, \varphi_\lambda) * (T_1, \varphi_1)$ with $1 \neq \lambda$, we find that

$$(T_\lambda, \varphi_\lambda) \cong (T_\lambda, \varphi_\lambda) * \left(\mathop{\ast}_{j,k \in \mathbb{Z}} (T_{\lambda^k}, \varphi_{\lambda^k}) \right).$$

Thus

$$(T_\lambda, \varphi_\lambda) \cong (T_\lambda, \varphi_\lambda) * (T_\lambda, \varphi_\lambda).$$

By Proposition 6.3, for $\lambda, \mu \neq 1$,

$$\begin{aligned}(T_\lambda, \varphi_\lambda) * (T_\mu, \varphi_\mu) &\cong (T_\lambda, \varphi_\lambda) * \left(\mathop{\ast}_{i,j \in \mathbb{Z}} (T_{\lambda^i \mu}, \varphi_{\lambda^i \mu}) \right) \\ &\cong (T_\lambda, \varphi_\lambda) * \left(\mathop{\ast}_{i,j \in \mathbb{Z}} (T_{\lambda^i \mu}, \varphi_{\lambda^i \mu}) * (T_\mu, \varphi_\mu) \right)\end{aligned}$$

which by the above is isomorphic to

$$\mathop{\ast}_{i,j \in \mathbb{Z}} (T_{\lambda^i \mu^j}, \varphi_{\lambda^i \mu^j}).$$

Now rewrite $\ast_k (T_{\lambda_k}, \varphi_{\lambda_k})$, using the above isomorphisms, as

$$\mathop{\ast}_{r \geq 0} \mathop{\ast}_{i_1, \ldots, i_r} (T_{\lambda_{i_1}}, \varphi_{\lambda_{i_1}}) * \cdots * (T_{\lambda_{i_r}}, \varphi_{\lambda_{i_r}}) \cong \mathop{\ast}_{r \geq 0} \mathop{\ast}_{i_1, \ldots, i_r} \mathop{\ast}_{\nu \in H_{i_1, \ldots, i_r}} (T_\nu, \varphi_\nu),$$

where $H_{i_1, \ldots, i_r}$ is the subgroup generated by $\lambda_{i_1}, \ldots, \lambda_{i_r}$. By the above equations, since

$$H = \bigcup_r \bigcup_{i_1, \ldots, i_r} H_{i_1, \ldots, i_r},$$

we get finally that

$$\mathop{\ast}_k (T_{\lambda_k}, \varphi_{\lambda_k}) \cong \mathop{\ast}_{\nu \in H} (T_\nu, \varphi_\nu),$$

which clearly only depends on H. □

Corollary 6.5. *For a given $0 < \lambda < 1$, all type* III_λ *factors of the form* $\Gamma(\mathcal{H}_\mathbb{R}, U_t)''$ *are isomorphic.*

For M a type III_1 factor, Connes defined in [9] the Sd invariant of M as the intersection over all faithful normal almost periodic weights (i.e., weights for which eigenvectors of the associated modular operators densely span the representation space) of the point spectra of the corresponding modular operators.

Notice that for U_t almost-periodic but not periodic (so $\Gamma(\mathcal{H}_\mathbb{R}, U_t)''$ is type III_1), the state φ_U is almost periodic, as then the eigenvectors of the modular operator Δ densely span $\mathcal{F}(\mathcal{H})$. The point spectrum of Δ is precisely the subgroup H in Theorem 6.4; thus the Sd invariant is nontrivial and is not all of $\mathbb{R}_+$. Moreover, since the centralizer of the free quasi-free state is a factor, and $\Gamma(\mathcal{H}_\mathbb{R}, U_t)''$ is full, by Lemma 4.8 of [9], the Sd invariant is the point spectrum of Δ. It follows that the group H of Theorem 6.4 is an invariant of the factor $\Gamma(\mathcal{H}_\mathbb{R}, U_t)''$. In the case that $\Gamma(\mathcal{H}_\mathbb{R}, U_t)''$ is type III_λ, $0 < \lambda < 1$, the group H is the S invariant. Thus in any case

Theorem 6.6. *For U_t, U_t' almost-periodic and nontrivial, $\Gamma(\mathcal{H}_\mathbb{R}, U_t)''$ is isomorphic to $\Gamma(\mathcal{H}_\mathbb{R}', U_t')''$ if and only if the point spectra of the modular operators corresponding to φ_U and $\varphi_{U'}$ coincide.*

If U_t is almost-periodic and nontrivial, then $\Gamma(\mathcal{H}_\mathbb{R}, U_t)'' * \Gamma(\mathcal{H}_\mathbb{R}, U_t)'' \cong \Gamma(\mathcal{H}_\mathbb{R}, U_t)''$ (in fact, in a way that maps the state $\varphi_U * \varphi_U$ to φ_U).

In [21], Rădulescu showed that if ψ_λ for $0 < \lambda < 1$ is a state on $M_{2\times 2}$, defined on matrix units $\{f_{ij}\}_{i,j=0}^1$ by

$$\psi_\lambda(f_{ij}) = \delta_{ij}\lambda^j/(1+\lambda),$$

and $L^\infty[-1,1]$ is endowed with the semicircular measure μ, then

$$D = (M_{2\times 2}, \psi_\lambda) * (L^\infty[-1,1], \mu)$$

is a type III_λ factor with core isomorphic to $L(\mathbb{F}_\infty) \otimes B(H)$ (see [8, 26, 25] for definitions).

Theorem 6.7. *The factor $(D, \psi_\lambda * \mu)$ of Rădulescu is isomorphic to $(T_\lambda, \varphi_\lambda)$.*

Proof. Setting in Theorem 5.2

$$N = 2, \quad c_i = \frac{\lambda^i}{1+\lambda}$$

we can view D as generated by matrix units $\{1\otimes e_{ij}\}_{i,j=0}^{1}$ and a semicircular element

$$L' + (L')^*,$$

where we set

$$L' = \sum_{ij} \ell(h_{ij}) \otimes e_{ij}\sqrt{c_j},$$

and $\{h_{ij}\}_{i,j=0}^{1}$ is an orthonormal family. Under such an identification, the state $\psi_\lambda * \mu$ is identified with the restriction of the state θ of Theorem 5.2.

The cutdown by $1 \otimes e_{00}$ is generated by elements

$$(\ell(h_{00}) + \ell(h_{00})^*) \otimes e_{00}, \quad (\ell(h_{11}) + \ell(h_{11})^*) \otimes e_{00}$$

and

$$(\ell(h_{01}) + \sqrt{\lambda}\ell(h_{10})^*) \otimes e_{00}.$$

It follows that the cutdown algebra, considered with the state $\frac{1}{\theta(1\otimes e_{00})}\theta(1 \otimes e_{00} \cdot 1 \otimes e_{00})$, is isomorphic to

$$(T_\lambda, \varphi_\lambda) * (L^\infty[-1,1], \mu) * (L^\infty[-1,1], \mu) \cong (T_\lambda, \varphi_\lambda) \tag{16}$$

where all isomorphisms are state-preserving (the last isomorphism is by Theorem 5.4). Notice that since D is type III, this implies that D and T_λ are isomorphic, though perhaps in a way that does not preserve states.

Similarly, $(T_\lambda, \varphi_\lambda)$ can be viewed as generated by matrix units

$$\{1 \otimes f_{ij}\}_{i,j=0}^{\infty},$$

as well as the semicircular element

$$L + L^*,$$

where

$$L = \sum_{ij} \ell(g_{ij}) \otimes f_{ij}\sqrt{c_i'}, \quad c_j' = (1-\lambda)\lambda^j;$$

here $\{g_{ij}\}$ form an orthonormal family. The state φ_λ can be identified with the restriction of the state θ' given by the tensor product of the state $\mathrm{Tr}(\mathrm{diag}(c_0', c_1', \dots))$ and the vacuum expectation.

Consider

$$p_{00} = \sum_i 1 \otimes f_{2i2i}, \quad p_{11} = 1 - p_{00}$$

and

$$p_{01} = \sum_i 1 \otimes f_{2i2i+1}, \quad p_{10} = p_{01}^*.$$

Then $\{p_{ij}\}_{i,j=0}^1$ is a system of matrix units; moreover, the restriction of θ' to $C^*(\{p_{ij}\})$ is the same as the restriction of θ to $C^*(\{1 \otimes e_{ij}\})$. Consider elements

$$\begin{aligned} L_{ee} &= \sum_{ij} \ell(h_{2i}) \otimes f_{2i,2j} \sqrt{c'_{2i}} & L_{eo} &= \sum_{ij} \ell(h_{2i}) \otimes f_{2i,2j+1} \sqrt{c'_{2i}} \\ L_{oe} &= \sum_{ij} \ell(h_{2i+1}) \otimes f_{2i+1,2j} \sqrt{c'_{2i+1}} & L_{oo} &= \sum_{ij} \ell(h_{2i+1}) \otimes f_{2i+1,2j+1} \sqrt{c'_{2i+1}}. \end{aligned}$$

Then

$$L + L^* = L_{ee} + L_{ee}^* + L_{oe} + L_{oe}^* + L_{eo} + L_{eo}^* + L_{oo} + L_{oo}^*.$$

The cutdown $p_{00} T_\lambda p_{00}$ is generated by the matrix units $\{1 \otimes f_{2i2j}\}$ as well as

$$p_{00}(L + L^*)p_{00}, \quad p_{01}(L + L^*)p_{10}, \quad p_{01}(L + L^*)p_{00}.$$

These are respectively equal to

$$L_{ee} + L_{ee}^*, \quad p_{01}(L_{oo} + L_{oo}^*)p_{10}, \quad p_{01}(L_{oe} + L_{eo}^*).$$

Let

$$\begin{aligned} L_1 &= \sqrt{1+\lambda} L_{ee}, & L_2 &= \sqrt{1 + \frac{1}{\lambda}} p_{01} L_{oo} p_{10} \\ L_3 &= \sqrt{1 + \frac{1}{\lambda}} p_{01} L_{oe}, & L_4 &= \sqrt{1+\lambda} L_{eo} p_{10}. \end{aligned}$$

Then by Theorem 5.2, L_i's are $*$-free from each other and from the matrix units $\{1 \otimes f_{2i2j}\}$ with respect to the state $\frac{1}{\theta'(p_{00})}\theta'(p_{00} \cdot p_{00})$. Moreover, each L_i is distributed (with respect to that state) as a creation operator. Notice that $p_{00} T_\lambda p_{00}$ is generated by

$$\{1 \otimes f_{2i2j}\}, \quad L_1 + L_1^*, \quad L_2 + L_2^*, \quad L_3 \sqrt{\lambda} + L_4^*.$$

Thus this cutdown, taken with the state $\frac{1}{\theta'(p_{00})}\theta'(p_{00} \cdot p_{00})$, is isomorphic in a state-preserving way to the free product of

(a) $(L^\infty[-1, 1], \mu)$ (generated by $L_1 + L_1^*$);

(b) $(T_{\lambda^2}, \varphi_{\lambda^2})$ (generated by $L_2 + L_2^*$ and the matrix units $\{1 \otimes f_{2i2j}\}$), and

(c) $(T_\lambda, \varphi_\lambda)$ (generated by $L_3 \sqrt{\lambda} + L_4^*$).

By Theorem 6.4, this is isomorphic in a state-preserving way to T_λ, and thus to the algebra in Equation (16). The statement of the theorem now follows, since T_λ is isomorphic to its cutdown by p_{00} (taken with the restriction of θ'), tensor $(M_{2\times 2}, \psi_\lambda)$, and similarly D is isomorphic to its cutdown by $1 \otimes e_{00}$, tensored with $(M_{2\times 2}, \psi_\lambda)$. □

Corollary 6.8. *T_λ is a type III_λ factor with core isomorphic to $L(\mathbb{F}_\infty) \otimes B(H)$.*

This is of course because the III_λ factor of Rădulescu has this core.

By [**12**], if one takes a state ψ_λ on $N \times N$ matrices, given by

$$\mathrm{Tr}(\mathrm{diag}(c_0, c_1, \dots)\cdot),$$

where $c_i/c_{i+1} = \lambda \in (0,1)$, then

$$D = (M_{N\times N}, \psi_\lambda) * L^\infty[-1,1]$$

is a type III factor (here $L^\infty[-1,1]$ is taken with the semicircular measure). Proceeding as in the proof of Theorem 6.7, we see that D can be modeled by matrix units $1 \otimes e_{ij}$, and a semicircular element $L + L^*$. Cutting down by $1 \otimes e_{11}$ gives us an algebra generated by the entries of $L + L^*$, i.e., elements of the form

$$\ell(h_{ij})\sqrt{c_i} + \ell^*(h_{ji})\sqrt{c_j}.$$

But such an element generates T_{c_i/c_j}; moreover, for different (i,j) (with $i \le j$), such elements are free. By an argument just as in the proofs of Theorem 6.7, and Theorem 6.6, we find that

Proposition 6.9. *Let D_N be $M_{N\times N} *_r L^\infty[-1,1]$, where the state on $M_{N\times N}$ is ψ_λ, and the state on $L^\infty[-1,1]$ is the semicircular measure. Then $D_N \cong T_\lambda$ in a state-preserving way. In particular, for different N, the algebras D_N are isomorphic.*

In a similar way one can express in terms of algebras of Theorem 6.6 algebras of the form $M_{N\times N} *_r L^\infty[-1,1]$, where the state on the matrices is arbitrary (so for a suitable choice of matrix units is given by $\mathrm{Tr}(\mathrm{diag}(c_0,\dots)\cdot)$), and the state on $L^\infty[-1,1]$ is the semicircular measure. Using Theorem 6.6, one can determine precisely when such algebras are isomorphic, in terms of the constants c_i.

This is of interest in relation to the work of Dykema ([**13**]). For example, if one could show that reduced free products of $M_{N\times N}$ and $M_{K\times K}$ (with nontracial states) are stable under taking free products with diffuse commutative von Neumann algebras, it would be possible to use Theorem 6.6 to give a classification of such algebras; indeed, $M_{N\times N} *_r M_{K\times K} *_r L^\infty[-1,1]$ is isomorphic, by the above theorems, to $\Gamma(\mathcal{H}_\mathbb{R}, U_t)''$ for some finite-dimensional $\mathcal{H}_\mathbb{R}$. It seems an interesting question in general to ask whether all "free type" type III factors should be stable under taking free products with diffuse commutative von Neumann algeras (where the free product is with respect to some class of states on the type III factor).

Theorem 6.10. *If the action U_t is has no eigenvectors on $\mathcal{H}$, $\Gamma(\mathcal{H}_\mathbb{R}, U_t)''$ is a type* III_1 *factor.*

Proof. The action $\mathcal{F}(U_t)$ on the full Fock space will have no eigenvectors either, so the centralizer of φ_U in $\Gamma(\mathcal{H}_\mathbb{R}, U_t)''$ is trivial. Thus $\Gamma(\mathcal{H}_\mathbb{R}, U_t)''$ is a factor. Since in particular the centralizer is a factor, the S invariant is the spectrum of Δ, so all of $[0, +\infty)$. So $\Gamma(\mathcal{H}_\mathbb{R}, U_t)''$ is a type III_1 factor. □

Since the centralizer of φ_U in this algebra is trivial there cannot exist a state-preserving isomorphism of $(\Gamma(\mathcal{H}_\mathbb{R}, U_t)'', \varphi_U)$ with $(\Gamma(\mathcal{H}_\mathbb{R}, U_t)'', \varphi_U) * (L^\infty[-1,1], \mu)$, where μ is a semicircular measure, since the centralizer of the free product state in the algebra on the right hand side contains $L^\infty[-1,1]$. It is not known whether $\Gamma(\mathcal{H}_\mathbb{R}, U_t)''$ has free absorption for some different state.

For a general $(\mathcal{H}_\mathbb{R}, U_t)$, we can rewrite this pair as a direct sum of $(\mathcal{H}'_\mathbb{R}, U'_t)$, with U'_t almost-periodic, and $(\mathcal{H}''_\mathbb{R}, U''_t)$, with U''_t ergodic. Thus

$$\Gamma(\mathcal{H}_\mathbb{R}, U_t)'' = \Gamma(\mathcal{H}'_\mathbb{R}, U'_t)'' * \Gamma(\mathcal{H}''_\mathbb{R}, U''_t)''.$$

By [12, 6], we get the following corollary (notice that when U_t is nontrivial, at least one of terms in the free product above is type III; moreover, all of the algebras involved are diffuse, i.e., contain no minimal projections):

Corollary 6.11. *If U_t is nontrivial, $\Gamma(\mathcal{H}_\mathbb{R}, U_t)''$ is a type* III *factor.*

References

[1] H. Araki, *A lattice of von Neumann algebras associated with the quantum theory of a free Bose field*, J. Math. Phys., **4** (1963), 1343-1362.

[2] ———, *On quasifree states of CAR and Bogoliubov automorphisms*, Publ. RIMS Kyoto Univ., **6** (1970), 385-442.

[3] H. Araki and E.J. Woods, *A classification of factors*, Publ. RIMS Kyoto Univ. series A, **4** (1968), 51-130.

[4] D. Avitzour, *Free products of C^*-algebras*, Trans. AMS, **271**(**2**) (1982), 423-435.

[5] T. Banica, *On the polar decomposition of circular variables*, Preprint, Paris, 1995.

[6] L. Barnett, *Free products of von Neumann algebras of type* III, Proc. Amer. Math. Soc., **123** (1995), 543-553.

[7] O. Bratteli and D.W. Robinson, *Operator algebras and quantum statistical mechanics*, Texts and monographs in physics, vol. 1,2, Springer-Verlag, 1981.

[8] A. Connes, *Une classification des facteurs de type* III, Ann. Scient. Éc. Norm. Sup., **6** (1973), 133-252.

[9] ______, *Almost periodic states and factors of type* III_1, J. Funct. Anal., **16** (1974), 415-455.

[10] J. Cuntz, *Simple C^*-algebras generated by isometries*, Commun. Math. Phys., **57** (1977), 173-185.

[11] K. Dykema, *On certain free product factors via an extended matrix model*, J. Funct. Anal., **112** (1993), 31-60.

[12] ______, *Factoriality and Connes' invariant $\mathcal{T}(\mathcal{M})$ for free products of von Neumann algebras*, J. reine angew. Math., **450** (1994), 159-180.

[13] ______, *Free products of finite dimensional and other von Neumann algebras with respect to non-tracial states*, Free Probability (D.-V. Voiculescu , ed.) Fields Institute Communications, AMS, 1977, pp. 41-88.

[14] R. Kadison, *Notes on Fermi gas*, Symposia math., **XX** (1976), 425-431.

[15] A. Nica, *R-transforms of free joint distributions and non-crossing partitions*, J. Funct. Anal., **135** (1996), 271-296.

[16] M. Pimsner and D.-V. Voiculescu, *K-groups of reduced crossed products by free groups*, J. Operator Theory, **8** (1982), 131-156.

[17] R.T. Powers and E. Størmer, *Free states of the canonical anticommutation relations*, Commun. Math. Phys., **16** (1970), 1-31.

[18] M.A. Rieffel, *A commutation theorem and duality for free Bose fields*, Commun. Math. Phys., **39** (1974), 153-164.

[19] M.A. Rieffel and A. Van Daele, *A bounded operator approach to Tomita-Takesaki theory*, Pacific J. of Math., **69**(**1**) (1977), 187-221.

[20] F. Rădulescu, *A one parameter group of automorphisms of $L(\mathbb{F}_\infty) \otimes B(\mathcal{H})$ scaling the trace*, C.R. Acad. Sci. Paris, **314**(**1**) (1992), 1027-1032.

[21] ______, *A type* III_λ *factor with core isomorphic to the von Neumann algebra of a free group, tensor $B(H)$*, Recent advances in operator algebras (Orléans, 1992), Astérique, 1995, pp. 203-209.

[22] D. Shale and W. Stinespring, *States of the Clifford algebra*, Ann. Math., **80** (1964), 365-381.

[23] D. Shlyakhtenko, *Free quasi-free states*, C. R. Acad. Sci. Paris, **322**, Série I (1996), 645-649.

[24] R. Speicher, *Multiplicative functions on the lattice of non-crossing partitions and free convolution*, Math. Annalen, **298** (1994), 193-206.

[25] Ş. Strătilă, *Modular theory in operator algebras*, Editura Academiei, Bucharest and Abacus Press, England, 1981.

[26] Ş. Strătilă and L. Zsidó, *Lectures on von Neumann algebras*, Editura Academiei, Bucharest and Abacus Press, England, 1975, 1979.

[27] D.-V. Voiculescu, *Symmetries of some reduced free product C^*-algebras*, Operator Algebras and Their Connections with Topology and Ergodic Theory, Lecture Notes in Mathematics, vol. 1132, Springer Verlag, (1985), 556-588.

[28] ———, *Circular and semicircular systems and free product factors*, Operator Algebras, Unitary Representations, Enveloping Algebras, and Invariant Theory, Progress in Mathematics, vol. 92, Birkhäuser, Boston, (1990), 45-60.

[29] D.-V. Voiculescu, K. Dykema and A. Nica, *Free random variables*, CRM monograph series, vol. 1, American Mathematical Society, 1992.

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