PACIFIC JOURNAL OF MATHEMATICS
Vol. 177, No. 2, 1997

RADON TRANSFORM ON FINITE SYMMETRIC SPACES

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The Radon transform belongs to the area of Inverse Problems. The reconstruction of a function from its projection or averages is a central point of study. Besides having direct applications in medical tomography, geophysics, there are also applications in signal processing, statistics and probability. Hence, it is useful to consider discretized versions of the Radon transform.

1. Introduction.

If G is a finite group, and S a fixed subset in G, then, given $f\colon G \to \mathbb{C}$, the Radon transform of f is defined by Diaconis and Graham [**3**] as

$$Rf(x) = \sum_{y \in xS} f(y). \tag{1}$$

See also Fill [**5**]. The goal of this paper is to describe the formulas which reconstruct f when the space of functions is a finite symmetric space, as well as to provide an example.

2. Radon transform on finite groups.

Assume that G is a finite group of Lie type and K is a subgroup of G such that (G, K) is a symmetric pair. Denote by $\widehat{G}$ the space of equivalence classes of unitary representations of G. If H_π is the representation space of $\pi \in \widehat{G}$, then $H_\pi^K \subset H_\pi$ is the space of K-fixed vectors in H_π. The dimension of H_π^K is $n(\pi)$. The set $\widehat{G}_K \subset \widehat{G}$ consists of all $\pi \in \widehat{G}$ for which $H_\pi^K \neq 0$. If $\pi \in \widehat{G}_K$ is irreducible, then $\omega_\pi\colon K \setminus G/K \to \mathbb{C}$ is a zonal spherical function for (G, K). The order of a set A is denoted by $|A|$.

The following theorem makes use of the Moore-Penrose inverse of an operator L which gives a least squares approximation to the original function in the case that the operator is not invertible. See Horn and Johnson [**6**]. This is necessary since in our application of this theorem to the Radon transform operator on finite upper half planes, we may have a non-invertible operator.

Theorem 1. *Fix $S \subset K\backslash G/K$, with (G,K) as above. For $f\colon K\backslash G/K \to \mathbb{C}$, with Radon transform Rf defined by (1), then $f(x)$ is least squares approximated by:*

$$|K|^2 \sum_{\pi\in\widehat{G}_K} n(\pi) \sum_{t\in K\backslash G/K} \lambda_\pi^{\div}\omega_\pi\left(t^{-1}\right) Rf(t)\omega_\pi(x),$$

with

$$\lambda_\pi^{\div} = \begin{cases} \left(\sum_{s\in S}\omega_\pi(s^{-1})\right)^{-1}, & \sum_{s\in S}\omega_\pi(s^{-1}) \neq 0, \\ zero, & otherwise. \end{cases}$$

Proof. The proof is a direct application of the Peter-Weyl theorem for compact, hence finite groups. The symmetric pair (G,K) is a Gelfand pair. Hence, the regular representation on $L^2(G/K)$ decomposes into a direct sum of representations of G. If $f \in L^2(G/K)$, then

$$f(xK) = f(x) = \frac{1}{|G|}\sum_{\pi\in\widehat{G}_K}\sum_{i=1}^{n(\pi)}\sum_{y\in G} n(\pi)\pi_{i1}(y^{-1})f(y)\pi_{i1}(x),$$

with π_{ij} the ij-th matrix entry of π, and $\pi_{11} = \omega_\pi$. Similarly, $f \in L^2(K\setminus G)$ has the decomposition:

$$f(Kx) = f(x) = \frac{1}{|G|}\sum_{\pi\in\widehat{G}_K}\sum_{j=1}^{n(\pi)}\sum_{y\in G} n(\pi)\pi_{1j}(y^{-1})f(y)\pi_{1j}(x).$$

If $f \in L^2(K\setminus G)\cap L^2(G/K) = L^2(K\setminus G/K)$, then

$$f(KxK) = f(x) = \frac{1}{|G|}\sum_{\pi\in\widehat{G}_K} n(\pi)\sum_{y\in G}\pi_{11}(y^{-1})f(y)\pi_{11}(x). \tag{2}$$

Note that if $\lambda_\pi = \sum_{s\in S}\omega_\pi(s^{-1})$, and $\mathbf{1}_{S^{-1}}(z) = 1$, for $z \in S^{-1}$ and 0 otherwise, then

$$\begin{aligned}\frac{\lambda_\pi}{|G|}\sum_{y\in G}\omega_\pi(y^{-1})f(y) &= \frac{1}{|G|}\sum_{z\in G}\sum_{y\in G}\mathbf{1}_{S^{-1}}(z)\omega_\pi(z^{-1})\omega_\pi(y^{-1})f(y)\\ &= \frac{|K|}{|G|}\sum_{z\in G/K}\sum_{y\in G}\mathbf{1}_{S^{-1}}(z)\omega_\pi(z^{-1})\omega_\pi(y^{-1})f(y)\\ &= \frac{1}{|G|}\sum_{z\in G/K}\sum_{y\in G}\mathbf{1}_{S^{-1}}(z)\sum_{k\in K}\omega_\pi(z^{-1}ky^{-1})f(y)\end{aligned}$$

$$= \frac{1}{|G|} \sum_{z \in G} \sum_{y \in G} \mathbf{1}_{S^{-1}}(z) \omega_\pi(z^{-1} y^{-1}) f(y)$$

$$= \frac{1}{|G|} \sum_{t \in G} \sum_{y \in G} \mathbf{1}_{S^{-1}}(y^{-1} t) \omega_\pi(t^{-1}) f(y)$$

$$= \frac{1}{|G|} \sum_{t \in G} \omega_\pi(t^{-1}) \sum_{y \in tS} f(y)$$

$$= \frac{1}{|G|} \sum_{t \in G} \omega_\pi(t^{-1}) Rf(t)$$

$$= \frac{|K|^2}{|G|} \sum_{t \in K \backslash G/K} \omega_\pi(t^{-1}) Rf(t). \tag{3}$$

Substituting this result into Formula (2) gives the intended result. □

Theorem 2. *Fix $S_0 \subset G/K$, with (G, K) as before. If, given $f\colon G/K \to \mathbf{C}$, $Rf\colon G/K \to \mathbf{C}$ is its Radon transform, then*

$$f(x) = |K| \sum_{\pi \in \widehat{G}_K} \sum_{i=1}^{n(\pi)} n(\pi) \lambda_\pi^{\div} \sum_{t \in G/K} \pi_{i1}(t^{-1}) Rf(t) \pi_{i1}(x),$$

with $\lambda_\pi^{\div}$ as before. Moreover, fix $S_1 \subset G/K$. If, given $g\colon K \setminus G \to \mathbf{C}$, $Rg\colon K \setminus G \to \mathbf{C}$ is its Radon transform, then

$$g(x) = |K| \sum_{\pi \in \widehat{G}_K} \sum_{j=1}^{n(\pi)} n(\pi) \lambda_\pi^{\div} \sum_{t \in G/K} \pi_{1j}(t^{-1}) Rf(t) \pi_{1j}(x).$$

Proof. We have

$$f(xK) = f(x) = \frac{1}{|G|} \sum_{\pi \in \widehat{G}_K} \sum_{y \in G} \sum_{i=1}^{n(\pi)} n(\pi) \pi_{i1}(y^{-1}) \pi_{i1}(x) f(y).$$

Note that if $\lambda_\pi = \sum_{s \in S_0} \pi_{11}(s^{-1})$, then

$$\lambda_\pi \sum_{y \in G} \pi_{i1}(y^{-1}) f(y) = \frac{|K|}{|G|} \sum_{z \in G/K} \sum_{y \in G} 1_{S_0^{-1}}(z) \pi_{11}(z^{-1}) \pi_{i1}(y^{-1}) f(y)$$

$$= \frac{1}{|G|} \sum_{z \in G/K} \sum_{y \in G} 1_{S_0^{-1}}(z) \sum_{k \in K} \pi_{i1}(z^{-1} k y^{-1}) f(y)$$

$$= \frac{1}{|G|} \sum_{z \in G/K} \sum_{y \in G} 1_{S_0^{-1}}(z) \pi_{i1}(z^{-1}y^{-1}) \sum_{k \in K} f(yk)$$

$$= \frac{1}{|G|} \sum_{t \in G} \sum_{y \in G} 1_{S_0^{-1}}(y^{-1}t) \pi_{i1}(t^{-1}) f(y)$$

$$= \frac{1}{|G|} \sum_{t \in G} \pi_{i1}(t^{-1}) Rf(t). \tag{4}$$

Hence,

$$f(x) = \sum_{\pi \in \widehat{G}_K} \sum_{i=1}^{n(\pi)} n(\pi) \lambda_\pi^{\div} \sum_{t \in G} \pi_{i1}(t^{-1}) Rf(t) \pi_{i1}(x).$$

The reconstruction formula for g follows similarly. □

3. Radon transform on finite upper half planes.

Let $q = p^r$, with p an odd prime, and $\mathbb{F}_q$ be a finite field of characteristic q. Fix $\delta \in \mathbb{F}_q^\times$ to be a nonsquare. If $\theta^2 = \delta$, then $\mathbb{F}_q(\theta)$ is the unique quadratic extension. Consider the group $G = GL(2, \mathbb{F}_q)$, and take K to be the subgroup

$$K = \left\{ \begin{pmatrix} a & b\delta \\ b & a \end{pmatrix} : a^2 - b^2\delta \neq 0 \right\}. \tag{5}$$

Note that K is isomorphic to the multiplicative group of $\mathbb{F}_q(\theta)$ and thus has order $q^2 - 1$.

$$G/K \cong \left\{ \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} : y \in \mathbb{F}_q^\times, \quad x \in \mathbb{F}_q \right\}. \tag{6}$$

In this instance, when G and K are as indicated, G/K is a finite analogue of the Poincaré upper half plane discussed by Angel et al [1], Terras [8], [9] and Velasquez [10]. Define a function $k \colon G/K \times G/K \to \mathbb{R}$ by

$$k(z, w) = \frac{N(z - w)}{\operatorname{Im}(z) \operatorname{Im}(w)}, \tag{7}$$

with

$$N\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = x^2 - y^2\theta^2, \qquad \operatorname{Im}\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = y. \tag{8}$$

The elements of $K \setminus G/K$ are the sets

$$S_r = \left\{ \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} : k\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) = r \right\} \tag{9}$$

with $r \in \mathbb{F}_q$.

According to Soto-Andrade [7], Terras [8], the zonal spherical functions associated to the principal series representations of G are functions $\omega_a \colon K \setminus G/K \to \mathbb{C}$, with $a \in F_q^\times$. Given $t \in S_b \subset K \setminus G/K$, some $b \in F_q$, then

$$\omega_a(t) = \omega_a(b) = \frac{1}{q+1} \sum_{z \in S_b} \nu_a(\log_h(\operatorname{Im}(z))), \tag{10}$$

with $a \in \mathbb{F}_q^\times$, ν_a, a multiplicative character of $\mathbb{F}_q^\times$, and h, a generator of $\mathbb{F}_q^\times$, $\omega_a = 1$.

By results of [7], Terras [8], the zonal spherical functions associated to the cuspidal representations of G are the functions $\omega_l \colon K \setminus G/K \to \mathbb{C}$, for $l \in \mathbb{F}_q^\times$. Let ϵ be the sign character of $\mathbb{F}_q$ defined as

$$\epsilon(d) = \begin{cases} 1, & d \text{ square} \\ -1, & d \text{ nonsquare} \end{cases}. \tag{11}$$

Fix $l \in \mathbb{F}_q^\times$, τ_l, multiplicative character of $(\mathbb{F}_q(\theta))^\times$ such that $\tau_l \neq \tau_l^q$. If $z \in \mathbb{F}_q(\theta)$, write $z = \operatorname{Re}(z) + \theta \operatorname{Im}(z)$. Let

$$u = \{z \in \mathbb{F}_q(\theta) \colon Nz = \operatorname{Re}(z) - \delta \operatorname{Im}(z) = 1\}.$$

Given $t \in S_c \subset K \setminus G/K$, with $c = \frac{a}{\delta} - 2$, $a \in \mathbb{F}_q$,

$$\omega_l(t) = \omega_l(c) = \sum_{z \in U} \epsilon(c + 2\operatorname{Re}(z))\tau_l(z). \tag{12}$$

The rest of the spherical functions are calculated by Evans [4] to be

$$\omega_{\kappa,l}(t) = \sum_{x \in \mathbb{F}_q} \omega_l \left(\begin{pmatrix} 0 & -1 \\ 1 & x \end{pmatrix} t \right) \kappa(x), \tag{13}$$

where $l = 1, \dots, \frac{q-1}{2}$ and κ is a fixed additive character of $\mathbb{F}_q$.

Define for $x \in K \setminus G/K$,

$$b(x) = k\left(x\sqrt{\delta},\ \sqrt{\delta}\right), \tag{14}$$

where $k(z, w)$ is defined in Formula (7).

Note. The Radon transform (1) for S given in (9) need not be invertible. See [**2**] where it can be seen that for finite upper half planes G/K the adjacency operators (which are the Radon transforms) for $q = 5, 7, 11$ are only invertible in 2 out of $q-2$ cases.

Theorem 3. *Let (G, K) be as described in formulas (5) to (6) and $f\colon K \setminus G/K \to \mathbb{C}$ with Radon transform defined by (1). The reconstruction of f is as follows. For $x \in K \setminus G/K$, $x = S_r$, $r \in \mathbb{F}_q$, with $b(x) = k(x\sqrt{\delta}, \sqrt{\delta}) = r$ as in (14), then $f(x)$ is least squares approximated by*

(15)

$$\frac{(q^2-1)^2}{(q+1)^2}\Bigg\{\sum_{a\in\mathbb{F}_q^\times}\sum_{b\in\mathbb{F}_q}\sum_{s\in S_b}\nu_a(\log_h(\operatorname{Im}(s^{-1})))\lambda_a^{\div}\sum_{y\in S_{b(x)}}\nu_a(\log_h(\operatorname{Im}(y)))Rf(y)$$
$$+(q+1)\sum_{l\in\mathbb{F}_{q^2}^\times}\sum_{m\in\mathbb{F}_q}\sum_{z,u\in U}\epsilon\left(\frac{m}{\delta}-2+2\operatorname{Re}(u^{-1})\right)$$
$$\cdot\,\tau_l(u^{-1})\lambda_l^{\div}\epsilon\left(\frac{b(x)}{\delta}-2+2\operatorname{Re}(z)\right)\tau_l(z)Rf(z)\Bigg\},$$

with

$$\lambda_a^{\div} = \begin{cases}(\lambda_a)^{-1}, & \lambda_a \neq 0,\\ \textit{zero}, & \textit{otherwise},\end{cases} \quad \textit{if}$$
$$\lambda_a = \sum_{\substack{b(p)\in\mathbb{F}_q\\ p\in S\subset K\setminus G/K}}\ \sum_{t\in S_{b(p)}}\nu_a(\log_h(\operatorname{Im}(t^{-1})))$$

and

$$\lambda_l^{\div} = \begin{cases}(\lambda_l)^{-1}, & \lambda_l \neq 0,\\ \textit{zero}, & \textit{otherwise},\end{cases} \quad \textit{if}$$
$$\lambda_l = \sum_{\substack{b(p)\in\mathbb{F}_q\\ p\in S\subset K\setminus G/K}}\ \sum_{v\in U}\epsilon\left(\frac{b(p)}{\delta}-2+2\operatorname{Re}(v^{-1})\right)\tau_l(v^{-1}).$$

Proof. This is proved by specializing Theorem 1 with $G = GL(2, \mathbb{F}_q)$ and K defined by Equation (5). □

Theorem 4. *Let (G, K) be as described above. If $Rf\colon G/K \to \mathbb{C}$ is the Radon transform, the least squares reconstruction of $f(t)$ is as follows. Set*

$$c(x,t) = b\left(\begin{pmatrix}0 & -1\\ 1 & x\end{pmatrix}t\right) = r,$$

where b is defined above in Formula (14). Then f is least squares approximated by

(16)

$$\frac{(q^2-1)^2}{(q+1)^2}\Bigg\{\sum_{a\in\mathbb{F}_q^\times}\sum_{b\in\mathbb{F}_q}\sum_{s\in S_b}\nu_a(\log_h(\operatorname{Im}(s^{-1})))\lambda_a^{\div}\sum_{y\in S_{b(x)}}\nu_a(\log_h(\operatorname{Im}(y)))Rf(y)$$
$$+(q+1)\sum_{l\in\mathbb{F}_{q^2}^\times}\sum_{x,m\in\mathbb{F}_q}\sum_{z,u\in U}\epsilon\left(\frac{m}{\delta}-2+2\operatorname{Re}(u^{-1})\right)$$
$$\cdot\,\tau_l(u^{-1})\lambda_l^{\div}\epsilon\left(\frac{c(x,t)}{\delta}-2+2\operatorname{Re}(z)\right)\tau_l(z)\kappa(x)Rf(z)\Bigg\},$$

where $\lambda_a^{\div}$ and $\lambda_l^{\div}$ are as defined in Theorem 3.

Proof. This is proved by specializing Theorem 2 with $G = GL(2,\mathbb{F}_q)$ and K defined by Equation (5). □

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Received September 2, 1993 and revised July 5, 1996.

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