PACIFIC JOURNAL OF MATHEMATICS
Vol. 178, No. 1, 1997

ON THE KAUFFMAN BRACKET SKEIN MODULE OF SURGERY ON A TREFOIL

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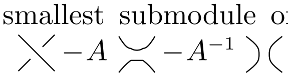
We estimate the rank of the Kauffman bracket skein module of each manifold obtained from integral surgery on a trefoil knot. It is well known that all but two of these manifolds contain no incompressible surfaces. We find that the two exceptions are exactly those whose skein module is not finitely generated, thereby extending a pattern that holds for all known compact orientable examples.

1. Introduction and basic definitions.

The Kauffman bracket skein module is an invariant of 3-manifolds, introduced by Przytycki [**5**], which has only been computed for a small collection of compact orientable examples: I-bundles over surfaces [**2**] [**5**], lens spaces [**3**] [**4**], and the exteriors of $(2, q)$-torus knots [**1**]. Although it has not been remarked upon in the previous literature, in all examples the module is finitely generated if and only if the manifold contains no essential surface. (We include reducing spheres and boundary reducing disks as essential surfaces.) The main result of this paper extends that pattern to include all integral surgeries on a right hand trefoil knot.

The result is obtained in several propositions spread over the last two sections of the paper. In this section we will introduce the basic definitions and some background material. In the next we will use Heegaard splittings to present the modules of the surgered manifolds. In Section 3 we will reduce these to finite presentations for all the manifolds without essential surfaces. Finally, in Section 4 we map the module onto a simpler specialization which turns out to be an algebra. For the two manifolds containing essential surfaces, it is easily seen to be infinite dimensional.

Let M be a 3-manifold. Its Kauffman bracket skein module is an algebraic invariant, $K(M)$, built from the set of all framed links in M. By a framed link we mean an embedded collection of annuli considered up to isotopy in M. The set of framed links is denoted $\mathcal{L}_M$ and it includes the empty link $\emptyset$.

Let R denote the ring of Laurent polynomials $\mathbb{Z}[A, A^{-1}]$ and $R(\mathcal{L}_M)$ the free R-module with basis $\mathcal{L}_M$. Let $S(M)$ be the smallest submodule of $R(\mathcal{L}_M)$ containing all possible expressions of the form ,

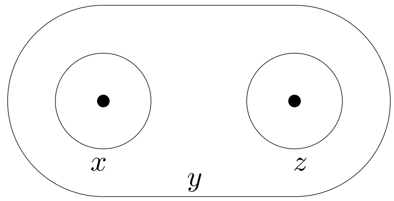
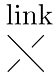
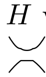
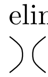


Figure 1. Knots x, y and z in H.

or $\bigcirc + A^2 + A^{-2}$. The first relation, called a skein relation, involves three links embedded identically except as the diagrams indicate, with framing annuli assumed to lie flat in the page. The second relation, called a framing relation, tells how to remove a trivial component from a link. We define $K(M)$ to be the quotient $R(\mathcal{L}_M)/S(M)$.

It will be necessary to understand the module for a genus two handlebody H. Any link L in H can be represented by a diagram in a twice punctured plane. The diagram determines a framing of the link, namely a set of annuli lying flat in the plane and parallel to the diagram. We will draw diagrams in the plane of the page using two dots to represent the punctures. For example, Figure 1 shows three framed knots in H called x, y and z.

Links in H can be formally multiplied by stacking their diagrams. The multiplication is not commutative; L_1L_2 means L_1 lies beneath L_2. Using this multiplication, we describe the set $\mathcal{B}_H$ of links in H whose diagrams have no crossings and no trivial components as $\mathcal{B}_H = \{x^iy^jz^k \mid (i,j,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}\}$. Here $\mathbb{N}$ denotes non-negative integers, and $x^0 = y^0 = z^0 = \emptyset$, but it will be more convenient to use 1 for the empty link.

Given any link L in H we may eliminate a crossing from its diagram via the relation  $= A$  $+ A^{-1}$ . This process can be repeated until there are only diagrams with no crossings. Then the trivial components of these are absorbed by the framing relations, expressing L as an R-linear combination of links in $\mathcal{B}_H$. This process is called *resolving* L and it shows that $\mathcal{B}_H$ generates $K(H)$. Przytycki [**5**] has shown that $\mathcal{B}_H$ is a free basis for $K(H)$. The formal multiplication in H makes $K(H)$ into a polynomial algebra, $R[x,y,z]$.

2. The setup.

The trefoil exterior, X, is obtained from H by attaching a 2-handle along the curve α in Figure 2. Adding a 2-handle to H affects the module by adding relations to the free presentation of $K(H)$. Every link in X can be isotoped into H and all skein and framing relations in X also hold in H. Hence, the only relations induced by the 2-handle are caused by links that are isotopic

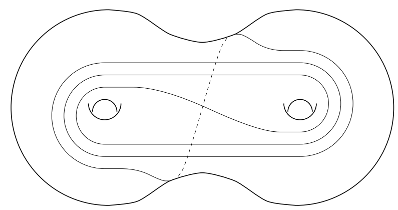


Figure 2. Attaching curve α in ∂H.

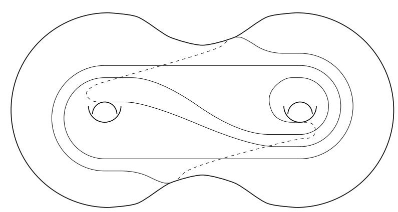


Figure 3. β_3.

in X but not in H. The difference comes from sliding a link across the 2-handle. Therefore, writing $sl(L)$ for any slide of L, the presentation of $K(X)$ has generators $\mathcal{B}_H$ and relations $\{L - sl(L) \mid L \in \mathcal{L}_H\}$ where $L - sl(L)$ is expanded in terms of $\mathcal{B}_H$. In [1] this is reduced to a free presentation with basis $\mathcal{B}_X = \{x^i y^j \mid j \leq 1\}$.

We will not include the proof here but there are a few internal details which will be relevant later. There are several relations among the links in $\mathcal{B}_H$ that hold in $K(X)$, two of which are particularly useful. Since x and z are both meridians of the knot and their framings agree, we have $x^m L = Lx^m = z^m L = Lz^m$ for any L. Also, for any $m \in \mathbb{N}$ there is an identity

$$A^2 x^m y^2 + x^{m+2} y - x^m y + A^{-2} x^{m+2} - (A^2 + A^{-2}) x^m = 0. \tag{1}$$

This follows by resolving the relation $(m, 2, 0) - f(m, 2, 0) = 0$ found in [1]. (The notation there is (i, j, k) for $x^i y^j z^k$.)

Although the algebra structure of $K(H)$ does not survive in $K(X)$ we will continue to use the multiplicative notation. It simply means that the links are stacked up in distinct horizontal slices of H. A nice consequence of this notation is that distributivity makes sense.

One obtains $X(r)$ from H by attaching a handle along α and another along a disjoint curve β_r. Figure 3 shows β_3. Any other β_r is obtained from β_3 by introducing $r - 3$ signed twists around the left hand hole. Figure 4 illustrates he behavior of several β_r's near the left hand hole.

We can present the module $K(X(r))$ using $\mathcal{B}_X$ as generators and all $L -$

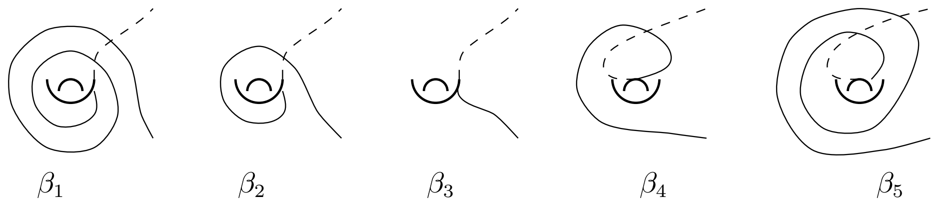


Figure 4. Behavior of β_r.

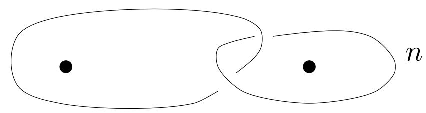


Figure 5. J_n.

$sl(L)$ as relations. This time $sl(L)$ is any slide of a link in X over the handle attached to β_r, and the relation is expanded in terms of $\mathcal{B}_X$. We do not intend to give a complete description of this presentation. Rather, we will expand a particular set of slides that suffice to eliminate all but a finite number of generators.

Before moving into the details of that process let us make one last observation about the relations. The curve β_r can be thought of as a framed link in X. The framing is given by an annulus parallel to β_r in ∂X. Any slide of L over the 2-handle corresponds to a band sum of L and β_r as framed links. To do calculations this way we need a diagram of β_r whose induced framing agrees with the one just defined. Figures 3 and 4 describe a picture of β_r which, projected into the page, has the correct framing.

3. Finiteness results.

First we define some links that will occur in the relations. Figure 5 depicts a family of links $\{J_n \mid n \in \mathbb{N}\}$. The integer n next to an arc signifies n copies of that component, all parallel in the page. Two other families, $\{L_r\}$ and $\{M_r\}$, are defined by example. In Figure 6 the pattern illustrated in L_2, L_1 and L_0 continues for $r \leq 2$, with L_r winding $2-r$ times around the dot. For $r \geq 3$ the pattern for L_3, L_4 and L_5 continues. Figure 7 defines M_r for all r except 2, 3 and 4. The diagrams are centered around the left hand hole of H and the pattern is obvious. When $r \leq 1$, M_r is a $(3-r, -1)$-torus knot with an extra -1 framing added. When $r \geq 5$, M_r is a $(r-3, 1)$-torus knot with an extra $+1$ framing.

Next we develop a set of relations by sliding $x^n L_r$ and $x^n M_r$ over the surgery curve β_r. In what follows the x^n factor will always appear around

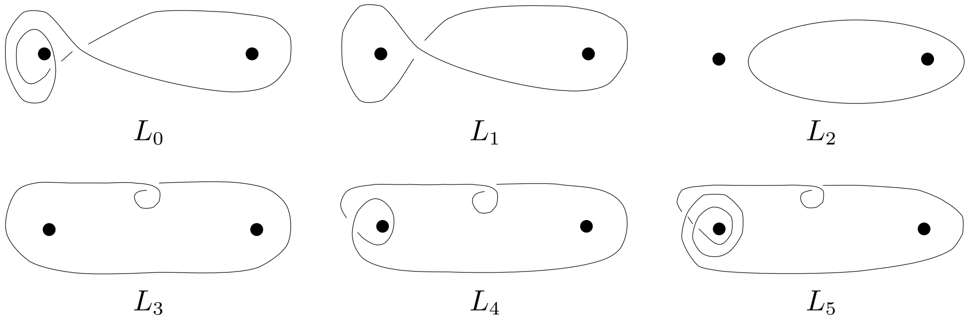


Figure 6. L_r.

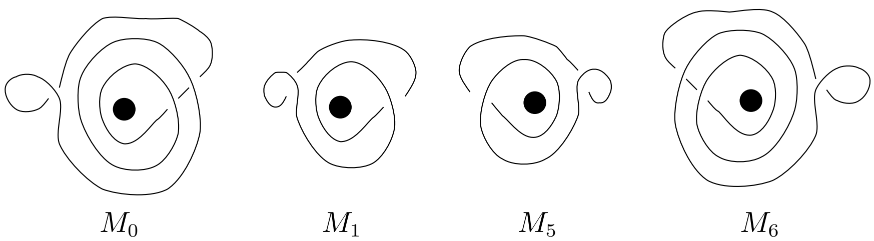


Figure 7. M_r.

the right hand dot in the diagrams.

Lemma 1. *For all r it is possible to slide $x^n L_r$ over β_r to get J_n.*

Proof. Figure 8 describes a diagram of a band sum of β_r with $x^n L_r$. The shaded rectangle is the band, which determines a slide, and the shaded disk is the region where the links depend on r. To fill in the shaded disk, refer to Figure 4 for β_r and Figure 6 for L_r. Within the shaded disk the twists of L_r are the same as those of β_r, so there is an isotopy that undoes all the winding. The result is always Figure 8 with the disk filled in as if $r = 3$. From there it is easy to see an isotopy to J_n. □

Lemma 2. *For $r \leq 1$ or $r \geq 5$ there is a slide of $x^n M_r$ over β_r yielding $\beta_3 \cup x^n$.*

Proof. This is just a matter of choosing a slide so that the winding of M_r undoes the winding of β_r. The correct choices for $r = 1$ and $r = 5$ are shown in Figure 9. The others are similar. □

Now we have to expand these links in terms of $\mathcal{B}_X$. We will do this by resolving all of them as links in H. If the resolution does not end up with

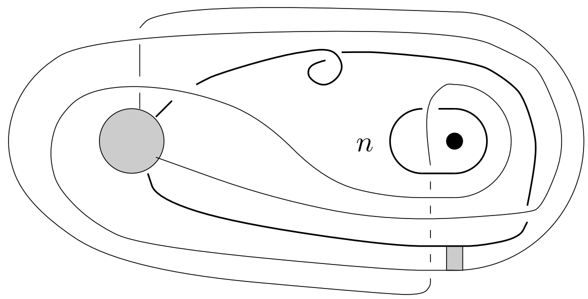
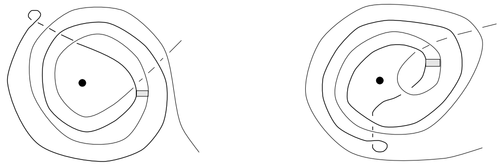


Figure 8. Band sum of $x^n L_r$ with β_r.

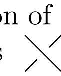
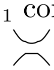
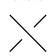
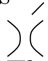


$r = 1$ $r = 5$

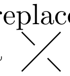
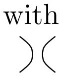
Figure 9. Slides of M_1 and M_2.

any powers of y larger than one then we have the right answer in $K(X)$. If y^2 occurs we will use Equation (1) to eliminate it. It will turn out that we do not need the exact resolutions, only a few higher degree terms. To speed calculations we let $\delta(m)$ denote any linear combination of terms with degree less than or equal to m.

Lemma 3. *If $n \geq 1$ then $J_n = A^{-2n}x^{n+1} + (A^{-2n+2} - A^{2n+2})x^{n-1}y + \delta(n-1)$.*

Proof. The relation  $= A^{-2}$  $+ (A - A^{-3})$  is easily derived from the standard one. Using it, we can unlink one copy of x from J_n. The result is a recursive formula, $J_n = A^{-2}xJ_{n-1} + (1 - A^4)K_{n-1}$, where K_{n-1} is the link in Figure 10. The resolution of K_{n-1} contains a term $A^{2n-2}x^{n-1}y$ given by smoothing every crossing as  $\rightsquigarrow$ . Let D be a diagram resulting from any smoothing  $\rightsquigarrow$  in K_{n-1}, and let l be a horizontal line through the dots in Figure 10. There is an isotopy of D so that it meets l in no more than $2n - 2$ points, insuring that $D = \delta(n-1)$. Now the recursion formula becomes $J_n = A^{-2}xJ_{n-1} + (A^{2n-2} - A^{2n+2})x^{n-1}y + \delta(n-1)$. The result follows easily by induction, noting that $J_0 = x$. $\square$

Lemma 4. *For $r \geq 3$ we have $L_r = -A^r x^{r-3}y + \delta(r-3)$.*

Proof. The framing relation implies that the kink in L_r can be replaced with the coefficient $-A^3$. Smoothing all the remaining crossings via  $\rightsquigarrow$ 

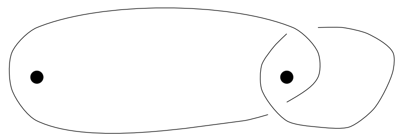


Figure 10. K_{n-1}.

gives $-A^r x^{r-3} y$. By an intersection argument like the one above, the other terms in the resolution contribute $\delta(r-3)$. □

Lemma 5. *For $r \leq 2$ we have $L_r = A^r x^{1-r} y + A^{r-2} x^{3-r} + \delta(1-r)$.*

Proof. First resolve the rightmost crossing in the diagram of L_r to get $AL' + A^{-1} x L''$. Here L'' is a knot that winds $2-r$ times about the left hand dot. A simple intersection argument shows that L'' resolves into $A^{r-1} x^{2-r} + \delta(-r)$. For L' we use an argument similar to the proof of Lemma 4 to show $L' = A^{r-1} x^{1-r} y + \delta(1-r)$. Combining the resolutions of L' and L'' proves the lemma. □

Lemma 6. *For $r \leq 1$ we have $M_r = -A^{r-5} x^{3-r} + \delta(1-r)$.*

Proof. This an intersection argument identical to the one used to resolve L'' above, except that there is a kink in the diagram contributing $-A^{-3}$. □

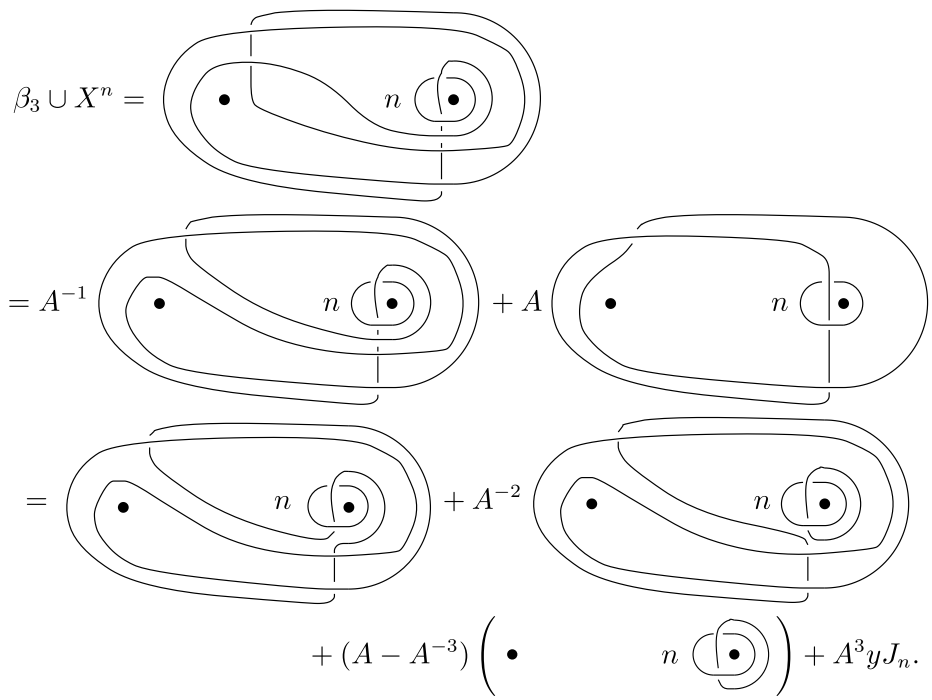
Lemma 7. *For $r \geq 5$ we have $M_r = -A^{r-1} x^{r-3} + \delta(r-5)$.*

Proof. This is the last proof with all the crossings reversed. □

Lemma 8. *In $K(X)$ the link $\beta_3 \cup x^n$ resolves as*

$$A^{-2n-3} x^{n+3} + (A^{2n+3} + A^{-2n-1}) x^{n+1} y + \delta(n+1).$$

Proof. We begin with the partial resolution



Write this as $D_1 + A^{-2}D_2 + (A - A^{-3})D_3 + A^3 y J_n$. Since resolving D_1 and D_2 would be overly cumbersome, we continue by sliding them over the curve α (Figure 2). The slide of D_1 is shown in Figure 11. We leave it to the reader to check that the result is the link in Figure 12, after which the usual intersection argument gives $D_1 = \delta(n+1)$. A similar slide of D_2 gives Figure 13, a link which resolves into $A^{-2n-1}x^{n+3} + A^{-2n+1}x^{n+1}y + \delta(n+1)$. Since D_3 is clearly just $\delta(n+1)$, we now have only yJ_n to resolve.

If $n = 0$ then $yJ_n = xy$, and combining with the above resolutions we obtain $\beta_3 = A^{-3}x^3 + (A^3 + A^{-1})xy + \delta(1)$. If $n \geq 1$ then Lemma 3 provides

$$A^3 y J_n = A^{-2n+3}x^{n+1}y + (A^{-2n+5} - A^{2n+5})x^{n-1}y^2 + y\delta(n-1). \tag{2}$$

Unfortunately, $x^{n-1}y^2$ is not in $\mathcal{B}_X$. Also, the expression $y\delta(n-1)$ may contain terms of the form $x^m y^2$ in which $m \leq n-2$. For these we invoke Equation (1) in a slightly revised form:

$$x^m y^2 = -A^{-2}x^{m+2}y + \delta(m+2). \tag{3}$$

If $m \leq n-2$ then Equation (3) becomes $x^m y^2 = \delta(n+1)$. Hence $y\delta(n-1) = \delta(n+1)$. For the $x^{n-1}y^2$ term we use Equation (3) to rewrite Equation (2) as

$$A^3 y J_n = A^{2n+3}x^{n+1}y + \delta(n+1).$$

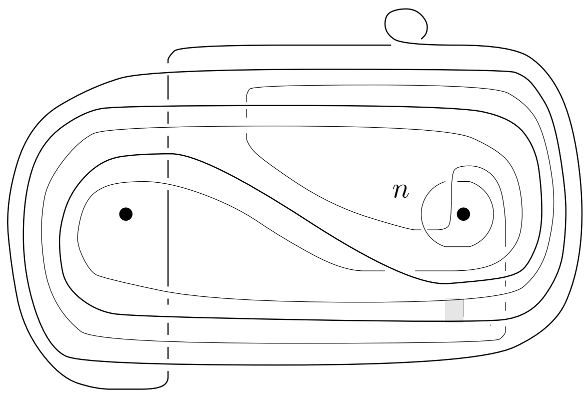


Figure 11. Band sum of D_1 with α.

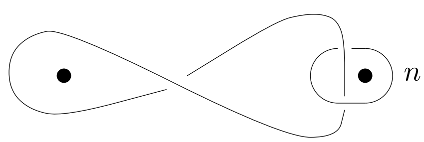


Figure 12. Slide of D_1 over α.

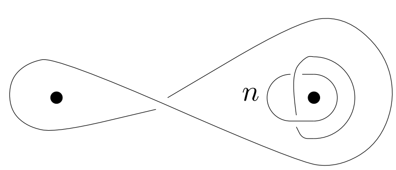


Figure 13. Slide of D_2 over α.

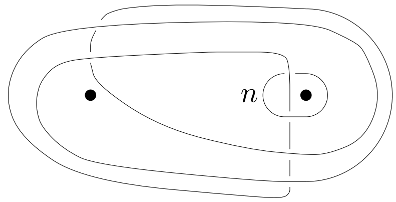
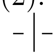
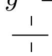
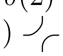


Figure 14. $\beta_4 \cup x^n$.

This, together with the resolutions of D_1, D_2 and D_3, gives the desired result. □

We will also need resolutions of the links $\beta_4 \cup x^n$ and $\beta_2 \cup x^n$. The diagram we have been using for β_4 is the diagram in Figure 3 with an extra positive twist around the left hand hole of H. However, both holes of H are meridians of the trefoil, so there is an isotopy of X restricted to a neighborhood of ∂X which moves the positive twist to the other hole. The result is shown in Figure 14.

Lemma 9. *In $K(X)$ we have $\beta_4 \cup x^n = A^{2n+4} x^{n+2} y + \delta(n+2)$.*

Proof. The proof is by induction on n. First resolve Figure 14 with $n = 0$ to get $\beta_4 = (A^2 - A^6)y^2 + x^2 y + \delta(2)$. Since $y^2 = -A^{-2}x^2 y + \delta(2)$ we have the $n = 0$ case. For $n \geq 1$, we use  $= A^2$  $+ (A^{-1} - A^3)$  to unlink a copy of x in Figure 14. The result is

$$\beta_4 \cup x^n = A^2 x(\beta_4 \cup x^{n-1}) + (A^{-1} - A^3)(\beta_3 \cup x^{n-1}).$$

Induction and Lemma 8 finish the proof. □

Lemma 10. *In $K(X)$ we have $\beta_2 \cup x^n = A^{-2n-4} x^{n+4} + \delta(n+3)$.*

Proof. We know that Figure 14 is $\beta_4 \cup x^n$. From it we can obtain diagrams of $\beta_3 \cup x^n$ and $\beta_2 \cup x^n$ by introducing (respectively) one and two negative twists around the left hand dot. Near that dot the new diagram of $\beta_2 \cup x^n$ will look like β_1 in Figure 4. Resolving the innermost crossing in that figure yields

$$\beta_2 \cup x^n = A^{-1} x(\beta_3 \cup x^n) - A^{-2}(\beta_4 \cup x^n).$$

The result follows from Lemmas 8 and 9. □

We are now ready to prove the finiteness theorems. All of them are obtained with essentially the same induction argument, which we formalize in the following lemma.

Lemma 11. *Let $q \in R$ and $N \in \mathbb{N}$. If, for all $m \geq N$, there are relations $x^m = \delta(m-1)$ and $x^m y = qx^{m+1} + \delta(m)$ in $K(X(r))$, then the module is finitely generated.*

Proof. Clearly $K(X(r))$ is generated by all expressions of the form $\delta(n)$. By inducting on n we can show that every $\delta(n)$ is equivalent to some $\delta(N)$. If $n \leq N$ this is obvious. Choose $n > N$ and assume it is true for all $\delta(n-1)$. Using the given relations we have $x^n = \delta(n-1) = \delta(N)$ and $x^{n-1}y = qx^n + \delta(n-1) = \delta(N)$. Hence every $\delta(n) = \delta(N)$. Finally, every $\delta(N)$ can be achieved using a finite set of generators. □

Proposition 1. *If $r \geq 7$ then $K(X(r))$ is finitely generated.*

Proof. For all n the relations $x^n M_r = \beta_3 \cup x^n$ and $x^n L_r = J_n$ hold in $K(X(r))$. Following Lemmas 3, 4, 7 and 8 these resolve into $-A^{r-1}x^{n+r-3} + \delta(n+r-5) = A^{-2n-3}x^{n+3} + \delta(n+2)$ and $-A^r x^{n+r-3}y + \delta(n+r-3) = \delta(n+1)$. Since $r \geq 7$ each relation contains a distinct highest degree term with a unit coefficient. Isolating the highest degree terms creates relations $x^{n+r-3} = \delta(n+r-4)$ and $x^{n+r-3}y = \delta(n+r-3)$. These satisfy the hypotheses of Lemma 11 with $q = 0$ and $N = r-3$. □

Proposition 2. *$K(X(5))$ is finitely generated.*

Proof. For $r = 5$ the relations from the preceding proof specialize to $-A^4x^{n+2} + \delta(n) = A^{-2n-3}x^{n+3} + \delta(n+2)$ and $-A^5x^{n+2}y + \delta(n+2) = \delta(n+1)$. Isolating the highest degree terms produces $x^{n+3} = \delta(n+2)$ and $x^{n+2}y = \delta(n+2)$, after which Lemma 11 applies with $N = 3$. □

Proposition 3. *$K(X(4))$ is finitely generated.*

Proof. This time we use relations coming from a slide of an unknot times x^n and from the slide of x^{n+1} shown in Figure 15. These are $\beta_4 \cup x^n = (-A^2 - A^{-2})x^n$ and $-A^{-3}\beta_3 \cup x^n = x^{n+1}$. Lemmas 8 and 9 resolve these into $A^{2n+4}x^{n+2}y + \delta(n+2) = \delta(n)$ and $-A^{-2n-6}x^{n+3} + \delta(n+2) = x^{n+1}$. Solving for highest degree terms gives the familiar relations $x^{n+2}y = \delta(n+2)$ and $x^{n+3} = \delta(n+2)$. □

Proposition 4. *$K(X(3))$ is finitely generated.*

Proof. Using the unknot slide and the usual slide of L_3 we have $\beta_3 \cup x^n = (-A^2 - A^{-2})x^n$ and $x^n L_3 = J_n$. We can rewrite these as $x^{n+3} = \delta(n+2)$ and $x^n y = -A^{-2n-3}x^{n+1} + \delta(n)$. We now apply Lemma 11 with $N = 3$ and $q = -A^{-2n-3}$. □

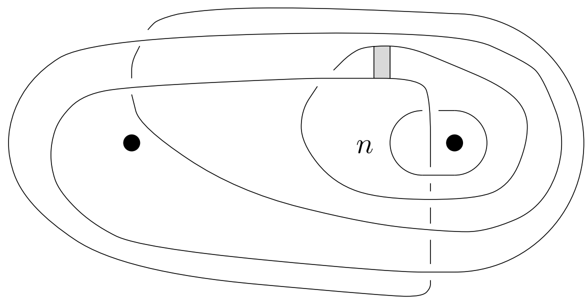


Figure 15. Band sum of x^{n+1} with β_4.

Proposition 5. *$K(X(2))$ is finitely generated.*

Proof. We construct relations from an unknot slide and by sliding M_1 over β_2. We saw in Lemma 2 that M_1 slides over β_1 to form β_3. We can apply this trick to the new diagram of β_2 introduced in the proof of Lemma 10. Doing this we find that M_1 slides over β_2 yielding β_4. Therefore our relations are $x^n M_1 = \beta_4 \cup x^n$ and $\beta_2 \cup x^n = (-A^2 - A^{-2})x^n$. We can resolve these using Lemmas 6, 9 and 10, giving relations $-A^{-4}x^{n+2} + \delta(n) = A^{2n+4}x^{n+2}y + \delta(n+2)$ and $A^{-2n-4}x^{n+4} + \delta(n+3) = \delta(n)$. These are equivalent to $x^{n+2}y = \delta(n+2)$ and $x^{n+4} = \delta(n+3)$. □

Proposition 6. *If $r \le -2$ then $K(X(r))$ is finitely generated.*

Proof. Here we resolve $x^n M_r = \beta_3 \cup x^n$ and $x^n L_r = J_n$ using Lemmas 3, 5, 6 and 8. The resulting relations are $-A^{r-5}x^{n+3-r} + \delta(n+1-r) = A^{-2n-3}x^{n+3} + \delta(n+2)$ and $A^r x^{n+1-r}y + A^{r-2}x^{n+3-r} + \delta(n+1-r) = \delta(n+1)$. Since $r \le -2$ these can be rewritten as $x^{n+3-r} = \delta(n+1-r)$ and $x^{n+1-r}y = -A^{-2}x^{n+3-r} + \delta(n+1-r)$. For each n substitute the first relation into the second to obtain new relations $x^{n+3-r} = \delta(n+1-r)$ and $x^{n+1-r}y = \delta(n+1-r)$. □

Proposition 7. *$K(X(-1))$ is finitely generated.*

Proof. In this case we use the above relations at $r = -1$. They are $x^{n+4} = -A^{-2n+3}x^{n+3} + \delta(n+2)$ and $x^{n+2}y = -A^{-2}x^{n+4} + \delta(n+2)$. Substituting the first into the second gives $x^{n+4} = \delta(n+3)$ and $x^{n+2}y = A^{-2n+1}x^{n+3} + \delta(n+2)$, satisfying the hypotheses of Lemma 11 with $N = 4$ and $q = A^{-2n+1}$. □

Proposition 8. *$K(X(1))$ is finitely generated.*

Proof. We use the relations of Proposition 6 specialized at $r = 1$ and expanded in more detail. They are $-A^{-4}x^{n+2} + \delta(n) = A^{-2n-3}x^{n+3} + (A^{2n+3} +$

$A^{-2n-1})x^{n+1}y + \delta(n+1)$ and $Ax^ny + A^{-1}x^{n+2} + \delta(n) = A^{-2n}x^{n+1} + \delta(n)$. First we rewrite these as $x^{n+3} = -A^{2n-1}x^{n+2} - (A^{4n+6} + A^2)x^{n+1}y + \delta(n+1)$ and $x^ny = -A^{-2}x^{n+2} + A^{-2n-1}x^{n+1} + \delta(n)$. Then, for each n, we substitute the first relation evaluated at n into the second evaluated at $n+1$. The new relations can be written as $x^{n+3} = \delta(n+2)$ and $x^{n+1}y = -(A^{-2n-7} + A^{-6n-7})x^{n+2} + \delta(n+1)$. Once again, Lemma 11 applies. □

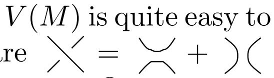
Since it is well known that only $X(6)$ and $X(0)$ admit essential surfaces, Propositions 1–8 prove half of the main theorem.

Theorem 1. *If $X(r)$ does not contain an essential surface then $K(X(r))$ is finitely generated.*

In the next section we will complete the picture by showing that $K(X(0))$ and $K(X(6))$ are not finitely generated.

4. Infiniteness results.

In this section we will work with a specialization of $K(M)$ given by setting $A = 1$ and mapping $\mathbb{Z}$ onto $\mathbb{Z}/2\mathbb{Z}$. The result is a $\mathbb{Z}/2\mathbb{Z}$-vector space, $V(M)$, which is finitely generated whenever $K(M)$ is. Therefore, our goal will be to show that $V(X(0))$ and $V(X(6))$ are infinite dimensional.

It turns out that $V(M)$ is quite easy to study. This is because the skein and framing relations are  and $\bigcirc = 0$. Hence, $V(M)$ does not see crossings or framings. One consequence is that $V(M)$ is a commutative algebra generated by the set of free homotopy classes of loops in M. The multiplication is disjoint union and the unit is $\emptyset$. To avoid confusion we will use the notation $\mathcal{A}(M)$ to refer to the algebra.

By specializing the proof in [**1**] we see that $\mathcal{B}_X$ is a basis for $V(X)$. We can then present $\mathcal{A}(X(r))$ as a quotient of $\mathcal{A}(X)$ by a finitely generated ideal. For a given r fix slides of y and x over β_r, denoting them $f(y)$ and $g(x)$ respectively.

Lemma 12. $\mathcal{A}(X(r)) = \mathcal{A}(X)/(y + f(y), g + g(x), \beta_r)$.

Proof. We know that $V(X(r)) = V(X)/W(r)$ where $W(r)$ is the span of all $L + sl(L)$. Links can be homotoped through each other, so $sl(L_1L_2)$ is either $sl(L_1)L_2$ or $sl(L_2)L_1$, depending on which link meets the band. This implies that $W(r)$ is an ideal. Since $\beta_r = \bigcirc + sl(\bigcirc)$, it is clear that $(y+f(y), g+g(x), \beta_r) \subset W(r)$. Hence it suffices to show the reverse inclusion. To this end let L be a link in X and $sl(L)$ any band sum with β_r.

We choose a resolution $L = \sum r_iL_i$ in $V(X)$ satisfying the following conditions. First, every skein relation involved in the resolution must be chosen so that the original intersection of L and the band remains unchanged. Second, although the framing relation in $V(X)$ allows any trivial component to be absorbed, we will not do so if that component meets the band. Finally, either $L_i \in \mathcal{B}_X$ or $L_i = L_i' \cup \bigcirc$ with $L_i' \in \mathcal{B}_X$. Furthermore, the latter occurs only when the band is attached to $\bigcirc$. Because the band is undisturbed by this resolution we have $L + sl(L) = \sum r_i(L_i + sl(L_i))$.

Now we look at the three possible behaviors of $sl(L_i)$. If the band meets a trivial component then $sl(L_i) = x^j y^k \beta_r$ and $L_i = 0$. If the band meets a copy of x then, after homotopy, it is the same as the band creating $g(x)$ or it is that band with a half twist. If there is no twist then $L_i + sl(L_i) = x^j y^k(x + g(x))$. If there is a twist it can be thought of as a single crossing. Resolving it gives $L_i + sl(L_i) = x^j y^k(x + g(x)) + x^j y^{k+1}\beta_r$. An identical argument works when the band meets a copy of y. $\square$

Lemma 13. *Let $\mathcal{I}$ be the ideal of $\mathcal{A}(X)$ generated by $x^2 + y$. As a vector space, $\mathcal{A}(X)/\mathcal{I}$ is infinite dimensional.*

Proof. This vector space is presented with generators $\mathcal{B}_X$ and any spanning set of $\mathcal{I}$ as relations. Since every element of $\mathcal{I}$ is of the form $(\sum_i x^i + \sum_j x^j y)(x^2 + y)$, the set $\{x^{i+2} + x^i y \mid i \in \mathbb{N}\} \cup \{x^{j+2}y + x^j y^2 \mid j \in \mathbb{N}\}$ spans it. However, Equation (1) gives $x^{j+2}y + x^j y^2 = x^{j+2} + x^j y$, so we need only $\{x^{i+2} + x^i y \mid i \in \mathbb{N}\}$ as relations. There are exactly enough relations here to eliminate every $x^i y$ from $\mathcal{B}_X$, leaving a vector space generated by $\{x^i \mid i \in \mathbb{N}\}$. $\square$

Proposition 9. *$V(X(0))$ is infinite dimensional.*[1]

Proof. There is an obvious slide of x over β_0 so that $g(x) = \beta_1$. For $f(y)$ we use the slide shown in Figure 8, but with L_3 instead of L_0. It is a simple matter to resolve these into $y + f(y) = x^4 + x^2 y$, $x + g(x) = x^5 + x^3 y$, and $\beta_0 = x^6 + x^4 y + x^4 + x^2 y$. Thus $(y + f(y), x + g(x), \beta_0) \subset \mathcal{I}$, which means $V(X(0))$ contains the vector space underlying $\mathcal{A}(X)/\mathcal{I}$. $\square$

Proposition 10. *$V(X(6))$ is infinite dimensional.*[2]

Proof. As above, there is an obvious slide of x so the $g(x) = \beta_5$, and we use Figure 8 for $f(y)$. This time the relations resolve as $y - f(y) = x^2 + x^2 y$, $x + g(x) = x^3 + x^3 y$, and $\beta_6 = x^4 + x^4 y + x^2 + x^2 y$. This time $(y + f(y), x +$

[1]Since the surgery is along the boundary of a Seifert surface, $H_1(X(0))$ is infinite. At the time this article was first written, the author was unaware that Przytycki had shown infinite $H_1(M)$ implies infinitely generated $K(M)$.

[2]Here the surgery is along the cabling slope. Hence, $X(0) = L(2,1)\#L(3,1)$, a manifold with finite homology groups.

$g(x), \beta_6)$ lies in the ideal generated by $1 + y$. However, as in the proof of Lemma 13, $\mathcal{A}(X)/(1 + y)$ is infinite dimensional. □

Propositions 9 and 10 prove the other half of the main theorem.

Theorem 2. *If $X(r)$ contains an essential surface then $K(X(r))$ is not finitely generated.*

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Received June 12, 1995 and revised October 15, 1996.

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