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OPTIMAL REGULARITY OF HARMONIC MAPS FROM A RIEMANNIAN MANIFOLD INTO A STATIC LORENTZIAN MANIFOLD

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In this paper, we give an optimal regularity result for some class of weakly harmonic maps from a Riemannian manifold M into a static Lorentzian manifold. Our main result is the following: For such class of weakly harmonic map w, there exists closed set $\Sigma \subset M$ such that w is C^∞ in $M \setminus \Sigma$ and the Hausdorff dimension of Σ is less than or equal to $\dim M - 3$.

1. Introduction.

In this paper, we study regularity of harmonic maps from a Riemannian manifold into a static Lorentzian manifold.

By definition, N is a static Lorentzian manifold if and only if the following hold (see [10]):

(i) N is the form $N = N_0 \times \mathbf{R}$, where N_0 is a Riemannian manifold with a metric g_0.

(ii) The metric g of N is given by

$$g\left(\begin{pmatrix}\xi\\ \tau\end{pmatrix}, \begin{pmatrix}\xi'\\ \tau'\end{pmatrix}\right) = g_0(\xi, \xi') - \beta(x)\tau\tau'$$

for $\begin{pmatrix}\xi\\ \tau\end{pmatrix}, \begin{pmatrix}\xi'\\ \tau'\end{pmatrix} \in T_{(x,t)}N = T_xN_0 \times \mathbf{R}$, where $\beta : N_0 \to \mathbf{R}^+$ is a smooth positive function.

In such a case, we write $N = N_0 \times_\beta \mathbf{R}$.

In this paper we consider the case where N_0 is compact. We may assume, by Nash-Moser theorem, N_0 is a submanifold of $\mathbf{R}^k$ for some $k > 1$. By the compactness of N_0, there exist constants $\beta_{\min}, \beta_{\max} > 0$ such that $\beta_{\min} \le \beta(x) \le \beta_{\max}$ for all $x \in N_0$.

Let M be a Riemannian manifold with non-empty boundary ∂M. For a map $w = (u, t) : M \to N_0 \times_\beta \mathbf{R}$, we define the energy $\mathcal{E}(w)$ of w by:

$$\mathcal{E}(w) = \mathcal{E}(u, t) = \int_M |\nabla u|^2 dV - \int_M \beta(u)|\nabla t|^2 dV,$$

where dV is a volume measure on M.

By definition, $w = (u,t) \in H^1(M; N_0) \times H^1(M; \mathbf{R})$ is a (*weakly*) *harmonic map* if the following holds:

Since N_0 is compact, there exists a tubular neighborhood $\mathcal{O}$ of N_0 in $\mathbf{R}^k$ such that the nearest point projection $\Pi : \mathcal{O} \to N_0$ is a smooth fibration. For any $\phi \in C_0^\infty(M; \mathbf{R}^k)$ and $\zeta \in C_0^\infty(M; \mathbf{R})$, there holds:

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0} \mathcal{E}(\Pi \circ (u + \epsilon\phi), t + \epsilon\zeta) = 0.$$

This means that $w = (u,t)$ is a stationary point with respect to the variation of the target manifold.

The weakly harmonic map $w = (u,t)$ satisfies the following equations in the distributional sense (see [8]):

$$(1.1) \qquad \begin{cases} \Delta u + A(u)(\nabla u, \nabla u) + \frac{1}{2}\nabla\beta(u)|\nabla t|^2 = 0 & \text{in} \quad M \\ \operatorname{div}(\beta(u)\nabla t) = 0 & \text{in} \quad M \end{cases}.$$

Here, A is the 2nd fundamental form of the embedding $N_0 \hookrightarrow \mathbf{R}^k$.

We consider Equation (1.1) with the prescribed boundary condition on ∂M:

$$(1.2) \qquad w = (u,t) = (\varphi, \iota) \quad \text{on} \quad \partial M,$$

where $(\varphi, \iota) : \partial M \to N \times_\beta \mathbf{R}$ is a given smooth map.

When the target manifold is a Riemannian, there are many regularity theories. For example, Schoen-Uhlenbeck [13] studied the regularity of minimizing harmonic mappings. In [5], [6] and [7], Hélein studied the regularity of weakly harmonic mappings when the dimension of the base manifold is 2. In [1], Bethuel studied the regularity of stationary harmonic mappings. In general, weakly harmonic maps are very singular. In [11], Riviere constructed weakly harmonic mapping from B^m to S^n ($m \geq 3$, $n \geq 2$) whose singular set is B^m. So there is *no* regularity theory for general weakly harmonic maps.

On the other hand, if the target manifold is a Lorentzian manifold, in [4], Greco constructed a smooth harmonic map when target is static Lorentzian and domain manifold is two dimensional. In [8], the author proved that any weakly harmonic map from 2-dimensional Riemannian manifold into static Lorentzian manifold is smooth and regularity results for some classes of harmonic maps when target is a static Lorentzian and the dimension of domain manifold is greater than 2.

More precisely, in [8], we considered the following class of solutions when $\dim M \geq 3$:

For a given $u \in H^1_\varphi(M; N_0)$ ($H^1_\varphi(M; N_0)$ is the space of all $H^1(M; N_0)$-maps with boundary value φ on ∂M.), there exists a unique solution $t = t(u)$ for the 2nd equation of (1.1) with $t = \iota$ on ∂M.

Define the functional $\mathcal{F} : H^1_\varphi(M; N_0) \to \mathbf{R}$ by

$$\mathcal{F}(u) = \int_M |\nabla u|^2 dV - \int_M \beta(u)|\nabla t(u)|^2 dV.$$

It is shown in [8] that $\mathcal{F}$ is bounded from below in $H^1_\varphi(M; N_0)$ and $\mathcal{F}$ attains the $\inf_{u \in H^1_\varphi(M;N_0)} \mathcal{F}(u)$ for some $u \in H^1_\varphi(M; N_0)$ and $(u, t(u))$ solves the Equations (1.1) and (1.2) in a weak sense. In this paper, we call such solutions as *minimal type*.

The main result of [8] is the following:

Theorem 1.1 ([8]). *Let $(u, t) \in H^1(M; N_0) \times H^1(M; \mathbf{R})$ be a minimal type solution. Then there exists a closed set $\Sigma \subset M$ such that (u, t) is C^∞ in $M \setminus \Sigma$. Moreover $\mathcal{H}^{m-2}(\Sigma) = 0$, where $\mathcal{H}^{m-2}$ is the $(m-2)$-dimensional Hausdorff measure and $m = \dim M$.*

Minimal type solutions correspond to minimizing harmonic maps when target manifold is a Riemannian. (Note that there are no *energy minimizing harmonic mappings* when target manifold is a Lorentzian since $\inf \mathcal{E}_{(u,t) \in H^1_\varphi(M;N_0) \times H^1_\iota(M;\mathbf{R})}(u, t) = -\infty$.)

In fact, when $\beta \equiv$ const., (u, t) is a minimal type harmonic map if and only if $u \in H^1_\varphi(M; N_0)$ is a Dirichlet energy minimizing map and t is a harmonic function.

Also in such a case, by the regularity result of Schoen-Uhlenbeck [13], Theorem 1.1 may be improved.

However, there is a strong difference between $\mathcal{F}$-minimizing problem and Dirichlet energy minimization problem as in the Riemannian case. That is, the functional $\mathcal{F}$ is not a *local* functional. For example, $\mathcal{F}$-minimizer in $H^1_\varphi(M; N_0)$ does not have a local minimizing property. This non-localness comes from the fact that $t(u)$ is *implicitly* defined in M by u and ι as a solution of the equation $div(\beta(u)\nabla t(u)) = 0$ with $t(u) = \iota$ on ∂M. So we can not directly localize the problem. Since regularity problem is mainly local in the domain, this causes problems. This is the troublesome point in our problem.

However, in general case, as the above special case suggests, it is reasonable to conjecture that the size of the singular set Σ in Theorem 1.1 is $\dim \Sigma \leq m - 3$.

In this paper we prove such conjecture is true. Our main result is the following:

Theorem A. *Let $N = N_0 \times_\beta \mathbf{R}$ be a static Lorentzian manifold. Let $(u,t) \in H^1_\varphi(M;N_0) \times H^1_\iota(M;\mathbf{R})$ be a minimal type harmonic map. Then there exists a closed set $\Sigma \subset M$ such that (u,t) is C^∞ in $M \setminus \Sigma$. Moreover the Hausdorff dimension of Σ is less than or equal to $m-3$ when $m \geq 3$. When $m = 3$, Σ is a discrete set and when $m = 2$, Σ is empty.*

Remark 1.2. (a) The above singular set estimate is optimal. In fact, $u_0 = x/|x| : B^3 \to S^2$ is a Dirichlet energy minimizing map (see [**2**]) and $w = (u_0, t_0)$, where $t_0 \equiv$ *const*. defines a minimal type harmonic map which is singular at $0 \in B^3$. Here $B^3 = \{x \in \mathbf{R}^3 : |x| \leq 1\}$. The examples of higher dimensional cases are constructed in the same way.

(b) Since $t(u)$ satisfies the elliptic equation $\operatorname{div}(\beta(u)\nabla t(u)) = 0$ with $0 < \beta_{\min} \leq \beta(u(x)) \leq \beta_{\max} < +\infty$, $t(u)$ is always Hölder continuous in M by De Giorgi-Nash theorem [**9**].

(c) If $\dim M = 2$, in [**8**], we proved that *any* weakly harmonic map in $H^1(M;N_0) \times H^1(M;\mathbf{R})$ is smooth in M.

The crucial step in the proof of Theorem A is a derivation of the monotonicity inequality. Here new difficulties arise due to the fact that the functional $\mathcal{F}$ is not a *local* functional as stated above. Thus it turns out that we need to analyze the behavior of solutions of some elliptic equations under the deformations of the domain manifold. These are carried out in §2.

Next step consists of compactness result for the families of scaled maps. As stated above, there is non sense to consider "local minimizing" maps in order to study regularity properties of (global) minimizing maps (since global $\mathcal{F}$-minimizer is not in general local $\mathcal{F}$-minimizer), so we need to consider compactness properties of scaled maps of global minimizers (which has, in general, no minimizing properties) in order to study local properties of (global) minimizing maps.

Finally, using Federer's dimension reduction argument [**3**], [**12**], [**13**], we obtain Theorem A. These are carried out in §3.

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2. Monotonicity inequality.

In this section, we derive the monotonicity inequality for $\mathcal{F}$-minimizing map u in $H^1_\varphi(M;N_0)$. For simplicity, we consider the case $M = \Omega$ is flat, that is, M is the open bounded set Ω in $\mathbf{R}^m$. The general case is more involved, but essentially the same method is applicable with slight modifications.

Our main result is the following:

Proposition 2.1. *Let $u \in H^1_\varphi(\Omega; N_0)$ be a $\mathcal{F}$-minimizing map. For $a \in \Omega$ and $0 < R_1 \leq R_2 < 1/2 \operatorname{dist}(a, \partial\Omega)$, there exist constants $c, \alpha > 0$ (depending only on β and m) such that the following holds:*

$$R_1^{2-m} \int_{B_{R_1}(a)} |\nabla u|^2 dx + 2 \int_{R_1 \leq |x-a| \leq R_2} |x-a|^{2-m} \left| \frac{\partial u}{\partial r} \right|^2 dx$$
$$\leq R_2^{2-m} \int_{B_{R_2}(a)} |\nabla u|^2 dx + c(R_1^\alpha + R_2^\alpha).$$

Here $B_\rho(a) = \{x \in \mathbf{R}^m : |x-a| < \rho\}$.

Proof. Fix $\phi \in C_0^\infty(\Omega)$. For $|\epsilon|$ small, we set $\phi_\epsilon(x) = x + \epsilon\phi(x)$. For small $|\epsilon|$, $\phi_\epsilon : \Omega \to \Omega$ defines a diffeomorphism.

Let $u \in H^1_\varphi(M; N_0)$ be a $\mathcal{F}$-minimizer. We set $u_\epsilon = u \circ \phi_\epsilon^{-1}$. For small $|\epsilon|$, $u_\epsilon \in H^1_\varphi(\Omega; N_0)$.

We study the dependence of $\mathcal{F}(u_\epsilon)$ in ϵ.

By some calculation, we obtain:

(2.1)
$$\int_\Omega |\nabla u_\epsilon|^2 dx$$
$$= \int_\Omega |\nabla u|^2 dx + \epsilon \left(\int_\Omega |\nabla u|^2 \operatorname{div} \phi \ dx - 2 \sum_{\substack{1 \leq i,j \leq m \\ 1 \leq l \leq k}} \int_\Omega \frac{\partial u^l}{\partial x_i} \frac{\partial u^l}{\partial x_j} \frac{\partial \phi_i}{\partial x_j} dx \right) + O(\epsilon^2).$$

Next we study $\int_\Omega \beta(u_\epsilon) |\nabla t(u_\epsilon)|^2 \ dx$.

Here some difficulties arise due to the non-localness of $t(u)$ with respect to u. That is, $t(u_\epsilon)$ is defined implicitly in Ω by u_ϵ and ι as a solution of the equations

$$\begin{cases} \operatorname{div}(\beta(u_\epsilon) \nabla t(u_\epsilon)) = 0 & \text{in} \quad \Omega, \\ t(u_\epsilon) = \iota & \text{on} \quad \partial\Omega. \end{cases}$$

So the equality $t(u_\epsilon) \circ \phi_\epsilon = t(u)$ does not hold in general. Thus we need the analysis of the behavior of $t(u_\epsilon) \circ \phi_\epsilon$ with respect to ϵ.

We compute, by the change of variable $x = \phi_\epsilon(y)$,

(2.2)
$$\int_\Omega \beta(u_\epsilon(x)) |\nabla t(u_\epsilon)(x)|^2 \ dx$$

$$= \int_\Omega \beta(u)|\nabla t(u_\epsilon)(\phi_\epsilon(y))|^2(1+\epsilon \operatorname{div}\phi)dy + O(\epsilon^2)\int_\Omega |\nabla t(u_\epsilon)(\phi_\epsilon(y))|^2 dy.$$

We set $\hat{t}_\epsilon := t(u_\epsilon)\circ\phi_\epsilon$.

Since

$$\nabla t(u_\epsilon)(\phi_\epsilon(y)) = J^{-1}(\phi_\epsilon(y))\nabla\hat{t}_\epsilon(y),$$

where $\nabla t(u_\epsilon) = \left(\frac{\partial t(u_\epsilon)}{\partial x_1},\dots,\frac{\partial t(u_\epsilon)}{\partial x_m}\right)^t$, $\nabla\hat{t}_\epsilon(y) = \left(\frac{\partial \hat{t}_\epsilon}{\partial y_1},\dots,\frac{\partial \hat{t}_\epsilon}{\partial y_m}\right)^t$ and $J(\phi_\epsilon) =$ jacobian of ϕ_ϵ, and

$$J^{-1}(\phi_\epsilon(y)) = \left(\delta_{ij} - \epsilon\frac{\partial\phi_j}{\partial y_i} + O(\epsilon^2)\right)_{ij},$$

we obtain

(2.3)
$$|\nabla t(u_\epsilon)(\phi_\epsilon(y))|^2 = |\nabla\hat{t}_\epsilon(y)|^2 - 2\epsilon\sum_{1\le i,j\le m}\frac{\partial\hat{t}_\epsilon}{\partial y_i}\frac{\partial\hat{t}_\epsilon}{\partial y_j}\frac{\partial\phi_j}{\partial y_i} + O(\epsilon^2)R_1(\nabla\hat{t}_\epsilon,\phi).$$

Here $R_1(\nabla\hat{t}_\epsilon,\phi)$ is a quadratic form in $\nabla\hat{t}_\epsilon$.

Combining (2.2) and (2.3), we obtain:

$$\begin{aligned}
&\int_\Omega \beta(u_\epsilon)|\nabla t(u_\epsilon)|^2\,dx\\
&= \int_\Omega \beta(u)\left\{\left|\nabla\hat{t}_\epsilon\right|^2 - 2\epsilon\sum_{1\le i,j\le m}\frac{\partial\hat{t}_\epsilon}{\partial y_i}\frac{\partial\hat{t}_\epsilon}{\partial y_j}\frac{\partial\phi_j}{\partial y_i}\right\}(1+\epsilon\operatorname{div}\phi)\,dy\\
&\quad + O(\epsilon^2)R_2(\nabla\hat{t}_\epsilon,\phi)\\
&= \int_\Omega\beta(u)|\nabla\hat{t}_\epsilon|^2dy + \epsilon\left(\int_\Omega\beta(u)\left|\nabla\hat{t}_\epsilon\right|^2\operatorname{div}\phi - 2\beta(u)\sum_{1\le i,j\le m}\frac{\partial\hat{t}_\epsilon}{\partial y_i}\frac{\partial\hat{t}_\epsilon}{\partial y_j}\frac{\partial\phi_j}{\partial y_i}dy\right)\\
&\quad + O(\epsilon^2)R_3\left(\nabla\hat{t}_\epsilon,\phi\right).
\end{aligned} \tag{2.4}$$

Here $R_2(\nabla\hat{t}_\epsilon,\phi)$ and $R_3(\nabla\hat{t}_\epsilon,\phi)$ are quadratic functionals in $\nabla\hat{t}_\epsilon$.

Since $t(u_\epsilon)$ is a critical point of the functional $t\mapsto\int_\Omega\beta(u_\epsilon)|\nabla t|^2\,dx$, by (2.4), $\hat{t}_\epsilon$ satisfies the following for any $\Phi\in C_0^\infty(\Omega;\mathbf{R})$:

$$\begin{aligned}
&\int_\Omega\beta(u)\nabla\hat{t}_\epsilon\cdot\nabla\Phi\,dy + \epsilon\left(\int_\Omega\beta(u)\nabla\hat{t}_\epsilon\cdot\nabla\Phi\operatorname{div}\phi\,dy\right.\\
&\qquad\left. - \int_\Omega\beta(u)\sum_{1\le i,j\le m}\frac{\partial\Phi}{\partial y_i}\frac{\partial\hat{t}_\epsilon}{\partial y_j}\frac{\partial\phi_i}{\partial y_j}\,dy - \int_\Omega\beta(u)\sum_{1\le i,j\le m}\frac{\partial\hat{t}_\epsilon}{\partial y_i}\frac{\partial\Phi}{\partial y_j}\frac{\partial\phi_i}{\partial y_j}\,dy\right)
\end{aligned}$$

$$+ O(\epsilon^2) R_3' \left(\nabla \Phi, \nabla \hat{t}_\epsilon, \phi \right). \tag{2.5}$$

Here $R_3'(\nabla\Phi, \nabla \hat{t}_\epsilon, \phi)$ is a linear form in $\nabla\Phi$ and $\nabla \hat{t}_\epsilon$.

By density, (2.5) holds for any $\Phi \in H_0^1(\Omega; \mathbf{R})$. We take $\Phi = \hat{t}_\epsilon - t(u) \in H_0^1(\Omega; \mathbf{R})$. Then we have

$$\begin{aligned}
&\int_\Omega \nabla \hat{t}_\epsilon \cdot \nabla(\hat{t}_\epsilon - t(u)) \, dy + \epsilon \left(\int_\Omega \beta(u) \nabla \hat{t}_\epsilon \cdot \nabla(\hat{t}_\epsilon - t(u)) \operatorname{div} \phi \, dy \right.\\
&\quad - \int_\Omega \beta(u) \sum_{1 \le i,j \le m} \frac{\partial}{\partial y_j}(\hat{t}_\epsilon - t(u)) \frac{\partial \hat{t}_\epsilon}{\partial y_j} \frac{\partial \phi_i}{\partial y_j} \, dy \\
&\quad \left. - \int_\Omega \beta(u) \sum_{1 \le i,j \le m} \frac{\partial \hat{t}_\epsilon}{\partial y_j} \frac{\partial}{\partial y_i}(\hat{t}_\epsilon - t(u)) \frac{\partial \phi_i}{\partial y_j} \, dy \right) \\
&\quad + O(\epsilon^2) R_3'(\nabla(\hat{t}_\epsilon - t(u)), \nabla \hat{t}_\epsilon, \phi) = 0.
\end{aligned} \tag{2.6}$$

On the other hand, since $\operatorname{div}(\beta(u)\nabla t(u)) = 0$, we have:

$$\int_\Omega \beta(u) \nabla t(u) \cdot \nabla(\hat{t}_\epsilon - t(u)) \, dy = 0. \tag{2.7}$$

Subtracting (2.7) form (2.6), we obtain:

$$\int_\Omega |\nabla \hat{t}_\epsilon - \nabla t(u)|^2 \, dy \le C_1(|\epsilon| + \epsilon^2) \|\nabla \hat{t}_\epsilon\|_{L^2(\Omega)} \|\nabla \hat{t}_\epsilon - \nabla t(u)\|_{L^2(\Omega)}, \tag{2.8}$$

where $c_1 > 0$ is a constant which may depend on ϕ but does not depend on ϵ.

We claim that $\hat{t}_\epsilon$ is bounded in H^1.

To prove this, first observe that by (2.4) there exists $c_2 > 0$ independent of ϵ with $|\epsilon|$ small such that

$$\int_\Omega |\nabla \hat{t}_\epsilon|^2 \, dy \le c_2 \int_\Omega |\nabla t(u_\epsilon)|^2 \, dy.$$

Let h be the harmonic extension of ι to Ω, i.e.,

$$\begin{cases} \Delta h = 0 & \text{in} \quad \Omega, \\ h = \iota & \text{on} \quad \partial\Omega. \end{cases}$$

Then by the minimizing property of $t(u_\epsilon)$, we obtain

$$\int_\Omega \beta(u_\epsilon) |\nabla t(u_\epsilon)|^2 \, dy \le \int_\Omega \beta(u_\epsilon) |\nabla h|^2 \, dy \le \beta_{\max} \int_\Omega |\nabla h|^2 \, dy.$$

Therefore we have

$$\int_\Omega |\nabla t(u_\epsilon)|^2\ dy \le \frac{\beta_{\max}}{\beta_{\min}} \int_\Omega |\nabla h|^2\ dy = \text{independent of } \ \epsilon.$$

This completes the proof of the claim.

Combining this claim with (2.8), we get

$$\int_\Omega |\nabla \hat{t}_\epsilon - \nabla t(u)|^2\ dy \le c_3\epsilon^2, \tag{2.9}$$

where c_3 is a constant independent of ϵ.

We set $\hat{t}_\epsilon = t(u) + \alpha_\epsilon$. Then

$$\alpha_\epsilon\big|_{\partial\Omega} = 0 \tag{2.10}$$

and

$$\int_\Omega |\nabla \alpha_\epsilon|^2\ dy = O(\epsilon^2). \tag{2.11}$$

By (2.4), we obtain

$$\begin{aligned}
&\int_\Omega \beta(u_\epsilon)|\nabla t(u_\epsilon)|^2\ dx \\
&= \int_\Omega \beta(u)|\nabla t(u) + \nabla\alpha_\epsilon|^2\ dy + \epsilon\Big(\int_\Omega \beta(u)|\nabla t(u) + \nabla\alpha_\epsilon|^2 \operatorname{div}\phi\ dy \\
&\quad - 2\int_\Omega \beta(u) \sum_{1\le i,j\le m} \left(\frac{\partial t(u)}{\partial y_i} + \frac{\partial \alpha_\epsilon}{\partial y_i}\right)\left(\frac{\partial t(u)}{\partial y_j} + \frac{\partial \alpha_\epsilon}{\partial y_j}\right)\frac{\partial \phi_j}{\partial y_i}\ dy\Big) + O(\epsilon^2) \\
&= \int_\Omega \beta(u)|\nabla t(u)|^2\ dy + 2\int_\Omega \beta(u)\nabla t(u)\cdot\nabla\alpha_\epsilon\ dy + \int_\Omega \beta(u)|\nabla\alpha_\epsilon|^2\ dy \\
&\quad + \epsilon\Big(\int_\Omega \beta(u)|\nabla t(u)|^2 \operatorname{div}\phi\ dy + 2\int_\Omega \beta(u)\nabla t(u)\cdot\nabla\alpha_\epsilon \operatorname{div}\phi\ dy \\
&\quad + \int_\Omega \beta(u)|\nabla\alpha_\epsilon|^2 \operatorname{div}\phi\ dy \\
&\quad - 2\sum_{1\le i,j\le m}\Big\{\int_\Omega \beta(u)\frac{\partial t(u)}{\partial y_i}\frac{\partial t(u)}{\partial y_j}\frac{\partial \phi_i}{\partial y_j}\ dy + \int_\Omega \beta(u)\frac{\partial \alpha_\epsilon}{\partial y_i}\frac{\partial t(u)}{\partial y_j}\frac{\partial \phi_i}{\partial y_j}\ dy \\
&\quad + \int_\Omega \beta(u)\frac{\partial t(u)}{\partial y_i}\frac{\partial \alpha_\epsilon}{\partial y_i}\frac{\partial \phi_j}{\partial y_i}\ dy + \int_\Omega \beta(u)\frac{\partial \alpha_\epsilon}{\partial y_i}\frac{\partial \alpha_\epsilon}{\partial y_j}\frac{\partial \phi_i}{\partial y_j}\ dy\Big\}\Big) + O(\epsilon^2).
\end{aligned} \tag{2.12}$$

Here, $\int_\Omega \beta(u)\nabla t(u)\cdot\nabla\alpha_\epsilon\ dy = 0$ by $\operatorname{div}(\beta(u)\nabla t(u)) = 0$ and (2.10).

Thus, by (2.11) and (2.12) we obtain

$$\int_\Omega \beta(u_\epsilon)|\nabla t(u_\epsilon)|^2\ dx$$

$$
\begin{aligned}
&= \int_\Omega \beta(u)|\nabla t(u)|^2\ dy + \epsilon\Big(\int_\Omega \beta(u)|\nabla t(u)|^2 \operatorname{div}\phi\ dy \\
&\quad - 2\int_\Omega \beta(u) \sum_{1\le i,j\le m} \frac{\partial t(u)}{\partial y_i}\frac{\partial t(u)}{\partial y_j}\frac{\partial \phi_j}{\partial y_i}\ dy\Big) + O(\epsilon^2).
\end{aligned} \tag{2.13}
$$

Combining (2.1) and (2.13) we get

$$
\begin{aligned}
\mathcal{F}(u_\epsilon) &= \int_\Omega |\nabla u|^2\ dx - \int_\Omega \beta(u)|\nabla t(u)|^2\ dx + \epsilon\Big(\int_\Omega |\nabla u|^2 \operatorname{div}\phi\ dx \\
&\quad - 2\sum_{\substack{1\le ij\le m\\ 1\le l\le k}} \int_\Omega \frac{\partial u^l}{\partial x_i}\frac{\partial u^l}{\partial x_j}\frac{\partial \phi_j}{\partial x_i}\ dx - \int_\Omega \beta(u)|\nabla t(u)|^2 \operatorname{div}\phi\ dx \\
&\quad + 2\sum_{1\le i,j\le m} \int_\Omega \beta(u)\frac{\partial t(u)}{\partial x_i}\frac{\partial t(u)}{\partial x_j}\frac{\partial \phi_j}{\partial x_i}\ dx\Big) + O(\epsilon^2) \\
&= \mathcal{F}(u) + \epsilon\Big(\int_\Omega |\nabla u|^2 \operatorname{div}\phi\ dx - 2\sum_{\substack{1\le i,j\le m\\ 1\le l\le k}} \int_\Omega \frac{\partial u^l}{\partial x_i}\frac{\partial u^l}{\partial x_j}\frac{\partial \phi_j}{\partial x_i}\ dx \\
&\quad - \int_\Omega \beta(u)|\nabla t(u)|^2 \operatorname{div}\phi\ dx \\
&\quad + 2\sum_{1\le i,j\le m} \int_\Omega \beta(u)\frac{\partial t(u)}{\partial x_i}\frac{\partial t(u)}{\partial x_j}\frac{\partial \phi_j}{\partial x_i}\ dx\Big) + O(\epsilon^2).
\end{aligned} \tag{2.14}
$$

Since $u_\epsilon \in H^1_\varphi(\Omega; N_0)$, by the minimality of u we have $\mathcal{F}(u) \le \mathcal{F}(u_\epsilon)$ for small $|\epsilon|$.

Therefore by (2.14) we obtain

$$
\begin{aligned}
&\int_\Omega |\nabla u|^2 \operatorname{div}\phi\ dx - 2\sum_{\substack{1\le i,j\le m\\ 1\le l\le k}} \int_\Omega \frac{\partial u^l}{\partial x_i}\frac{\partial u^l}{\partial x_j}\frac{\partial \phi_i}{\partial x_j}\ dx \\
&- \int_\Omega \beta(u)|\nabla t(u)|^2 \operatorname{div}\phi\ dx + 2\sum_{1\le i,j\le m} \int_\Omega \beta(u)\frac{\partial t(u)}{\partial x_i}\frac{\partial t(u)}{\partial x_j}\frac{\partial \phi_j}{\partial x_i}\ dx = 0
\end{aligned}
$$

for all $\phi \in C_0^\infty(\Omega; \mathbf{R}^n)$.

Here we take, in particular, $\phi(x) = \zeta(r/\rho)(x-a)$, where $\zeta \in C^\infty(\mathbf{R})$ is such that $\zeta(r) \equiv 1$ for $r \le 1/2$, $\zeta(r) \equiv 0$ for $r \ge 1$, $\zeta' \le 0$ and $\rho > 0$ satisfies $B_\rho(a) \subset\subset \Omega$.

Then by some computations, we obtain

$$
-\rho\int_\Omega |\nabla u|^2 \frac{d}{d\rho}\left\{\phi\left(\frac{r}{\rho}\right)\right\}\ dx + (m-2)\int_\Omega |\nabla u|^2 \phi\left(\frac{r}{\rho}\right)\ dx
$$

$$
\begin{aligned}
&+ 2\rho \int_\Omega \left|\frac{\partial u}{\partial r}\right|^2 \frac{d}{d\rho}\left\{\phi\left(\frac{r}{\rho}\right)\right\}\ dx + \rho \int_\Omega \beta(u)|\nabla t(u)|^2 \frac{d}{d\rho}\left\{\phi\left(\frac{r}{\rho}\right)\right\}\ dx \\
&- (m-2)\int_\Omega \beta(u)|\nabla t(u)|^2 \phi\left(\frac{r}{\rho}\right)\ dx \\
&- 2\rho \int_\Omega \beta(u)\left|\frac{\partial t(u)}{\partial r}\right|^2 \frac{d}{d\rho}\left\{\phi\left(\frac{r}{\rho}\right)\right\}\ dx = 0.
\end{aligned}
\tag{2.15}
$$

Letting ϕ approach to the characteristic function of $(-\infty, 1]$, we obtain

$$
\begin{aligned}
&- \rho\frac{d}{d\rho}\int_{B_\rho(a)} |\nabla u|^2\ dx + (m-2)\int_{B_\rho(a)} |\nabla u|^2\ dx \\
&+ 2\rho\frac{d}{d\rho}\int_{B_\rho(a)} \left|\frac{\partial u}{\partial r}\right|^2\ dx + \rho\frac{d}{d\rho}\int_{B_\rho(a)} \beta(u)|\nabla t(u)|^2\ dx \\
&- (m-2)\int_{B_\rho(a)} \beta(u)|\nabla t(u)|^2\ dx - 2\rho\frac{d}{d\rho}\int_\Omega \beta(u)\left|\frac{\partial t(u)}{\partial r}\right|^2\ dx = 0.
\end{aligned}
\tag{2.16}
$$

(2.16) is equivalent to the following:

$$
\begin{aligned}
&\frac{d}{d\rho}\left\{\rho^{2-m}\int_{B_\rho(a)} |\nabla u|^2\ dx - \rho^{2-m}\int_{B_\rho(a)} \beta(u)|\nabla t(u)|^2\ dx\right\} \\
&\quad = 2\rho^{2-m}\int_{\partial B_\rho(a)} \left|\frac{\partial u}{\partial r}\right|^2\ dx - 2\rho^{2-m}\int_{\partial B_\rho(a)} \beta(u)\left|\frac{\partial t(u)}{\partial r}\right|^2\ dx.
\end{aligned}
\tag{2.17}
$$

Integrating (2.17) from R_1 to R_2 we obtain

$$
\begin{aligned}
&R_2^{2-m}\int_{B_{R_2}(a)} |\nabla u|^2\ dx - R_1^{2-m}\int_{B_{R_1}(a)} |\nabla u|^2\ dx \\
&\quad - R_2^{2-m}\int_{B_{R_2}(a)} \beta(u)|\nabla t(u)|^2\ dx + R_1^{2-m}\int_{B_{R_1}(a)} \beta(u)|\nabla t(u)|^2\ dx \\
&= 2\int_{R_1\le|x-a|\le R_2} |x-a|^{2-m}\left|\frac{\partial u}{\partial r}\right|^2\ dx \\
&\quad - 2\int_{R_1\le|x-a|\le R_2} \beta(u)|x-a|^{2-m}\left|\frac{\partial t(u)}{\partial r}\right|^2\ dx.
\end{aligned}
\tag{2.18}
$$

Since $t(u)$ is a solution of the equation $\operatorname{div}(\beta(u)\nabla t(u)) = 0$ in Ω and $0 < \beta_{\min} \le \beta(u) \le \beta_{\max} < +\infty$, by De Giorgi-Nash theorem and p. 59, Lemma

4.3 in [9], there exist constants $\alpha > 0$ and $c > 0$ depending only on $\beta_{\max}$, $\beta_{\min}$ and m such that the following hold for $0 < r < \operatorname{dist}(a, \partial\Omega)$:

$$r^{2-m} \int_{B_r(a)} |\nabla t(u)|^2 \, dx \le c r^{\alpha}, \tag{2.19}$$

$$\int_{B_r(a)} |x-a|^{2-m} |\nabla t(u)|^2 \, dx \le c r^{\alpha}. \tag{2.20}$$

Combining (2.18), (2.19) and (2.20), there exists constants $\alpha > 0$ and $c > 0$ depending only on β and m such that

$$\begin{aligned} & R_1^{2-m} \int_{B_{R_1}(a)} |\nabla u|^2 \, dx + 2 \int_{R_1 \le |x-a| \le R_2} |x-a|^{2-m} \left| \frac{\partial u}{\partial r} \right|^2 \, dx \\ & \le R_2^{2-m} \int_{B_{R_2}(a)} |\nabla u|^2 \, dx + c(R_1^{\alpha} + R_2^{\alpha}). \end{aligned}$$

This completes the proof of Proposition 2.1. □

3. Proof of Theorem A.

In this section, we prove our main theorem A. We recall the following two facts from [8].

Lemma 3.1 ([8, Lemma 3.4]). *Let $u \in H^1_{\varphi}(M; N_0)$ be a $\mathcal{F}$-minimizer. Let $B_r(a)$ be a ball such that $B_r(a) \subset\subset \Omega$ and let $v \in H^1(B_r(a); N_0)$ with $v = u$ on $\partial B_r(a)$. Then there exists $\eta \in H^1_{\varphi}(M; N_0)$ (depending on u, v, a and r) such that the following holds:*

$$\begin{aligned} & \int_{B_r(a)} |\nabla u|^2 \, dx - \int_{B_r(a)} \beta(u) |\nabla t(u)|^2 \, dx \\ & \le \int_{B_r(a)} |\nabla v|^2 \, dx + \beta_{\max} \int_{B_r(a)} |\nabla t(\eta)|^2 \, dx. \end{aligned}$$

The following small energy regularity theorem is proved in [8]:

Lemma 3.2 ([8, Lemma 3.10]). *There exist $R_0 >$, $\bar\epsilon > 0$, $\gamma > 0$ and $c > 0$ such that if a $\mathcal{F}$-minimizing map $u \in H^1_{\varphi}(M; N_0)$ satisfies $R^{2-m} \int_{B_{2R}(a)} |\nabla u|^2 \, dx < \bar\epsilon$ for some $R < \min\{R_0, \frac{1}{2}\operatorname{dist}(a, \partial\Omega)\}$, then u is γ-Hölder continuous in $B_R(a)$ and the following holds for any $x, y \in B_R(a)$:*

$$|u(x) - u(y)| \le c|x-y|^{\gamma}.$$

We begin the following:

Proposition 3.3. *Let $u \in H^1_\varphi(M; N_0)$ be a $\mathcal{F}$-minimizing map. For a given $B > 0$, there exists $\epsilon_0 >$, $c > 0$, $\delta > 0$ and $\rho_0 > 0$ such that if $\rho < \rho_0$, $B_\rho(b) \subset \subset \Omega$ $(b \in \Omega)$, $\rho^{2-n} \int_{B_\rho(b)} |\nabla u|^2 \, dx \le B$ and $\rho^{-n} \int_{B_\rho(b)} |u - (u)_{\rho,b}|^2 \, dx \le \epsilon_0$ $((u)_{\rho,b} = 1/|B_\rho(b)| \int_{B_\rho(b)} u \, dx)$, then u is δ-Hölder continuous in $B_{\frac{\rho}{2}}(b)$ and $|u(x) - u(y)| \le c|x - y|^\delta$ holds for any $x, y \in B_{\rho/2}(b)$.*

Proof. By Fubini's theorem there exists $\sigma \in [3/4\rho, \rho]$ such that

$$\int_{\partial B_\sigma(b)} |u - (u)_{\rho,b}|^2 \, ds \le \frac{8}{\rho} \int_{B_\rho(b)} |u - (u)_{\rho,b}|^2 \, dx \le 8\epsilon_0 \rho^{m-1} \tag{3.1}$$

$$\int_{\partial B_\sigma(b)} |\nabla u|^2 \, ds \le \frac{8}{\rho} \int_{B_\rho(b)} |\nabla u|^2 \, dx \le 8B\rho^{m-3}. \tag{3.2}$$

By (3.1) and (3.2) we have

$$\begin{aligned} &\sigma^{4-2n} \left(\int_{\partial B_\sigma(b)} |u - (u)_{\rho,b}|^2 \, ds \right) \left(\int_{\partial B_\sigma(b)} |\nabla u|^2 \, ds \right) \\ &\le 64B\epsilon_0 \rho^{2n-4} \sigma^{4-2n} \le \frac{245}{3} B\epsilon_0. \end{aligned} \tag{3.3}$$

Here we take $\epsilon_0 > 0$ such that $254/3B\epsilon_0 < \delta'^2 \epsilon^q$, where $\epsilon > 0$ is a constant to be determined later and $\delta' > 0$, $q > 0$ are constants appearing in the following extension lemma due to Schoen -Uhlenbeck (Lemma 4.3 in [13]):

Lemma 3.4 ([13, Lemma 4.3])**.** *There exist $\delta' > 0$ and $q > 0$ such that if $\epsilon \in (0, 1)$ and $u \in H^1(\partial B_\sigma(b); N_0)$ satisfies*

$$\sigma^{4-2m} \left(\int_{\partial B_\sigma(b)} |\nabla^T u|^2 \, ds \right) \left(\int_{\partial B_\sigma(b)} |u - \xi|^2 \, ds \right) \le \delta'^2 \epsilon^q$$

for some $\xi \in \mathbf{R}^k$, then there exists $\bar{u} \in H^1(B_\sigma(b); N_0)$ such that $\bar{u} = u$ on $\partial B_\sigma(b)$ and the following holds:

$$\int_{B_\sigma(b)} |\nabla \bar{u}|^2 \, dx \le c \left(\epsilon\sigma \int_{\partial B_\sigma(b)} |\nabla^T u|^2 \, ds + \epsilon^{-q} \sigma^{-1} \int_{\partial B_\sigma(b)} |u - \xi|^2 \, ds \right).$$

By this lemma, there exists $\bar{u} \in H^1(B_\sigma(b); N_0)$ such that $\bar{u} = u$ on $\partial B_\sigma(b)$ and, by (3.1), (3.2),

$$\int_{B_\sigma(b)} |\nabla \bar{u}|^2 \, dx \le c(8\epsilon\sigma\rho^{m-3}B + 8\epsilon^{-q}\sigma^{-1}\epsilon_0\rho^{m-1})$$

$$\leq c\left(8B\epsilon\rho^{m-2} + \frac{32}{3}\epsilon_0\epsilon^{-q}\rho^{m-2}\right)$$
$$\leq c\left(8B\epsilon + \frac{32}{3}\epsilon_0\epsilon^{-q}\right)\rho^{m-2}. \tag{3.4}$$

On the other hand, by Lemma 3.1, there exists $\eta \in H^1_\varphi(\Omega; N_0)$ (depending on σ, b, u) such that

$$\int_{B_\sigma(b)} |\nabla u|^2\, dx \leq \int_{B_\sigma(b)} |\nabla \bar{u}|^2\, dx + \beta_{\max}\left(\int_{B_\sigma(b)} |\nabla t(u)|^2 + |\nabla t(\eta)|^2\, dx\right). \tag{3.5}$$

Combining (3.4) with (3.5) we obtain

$$\int_{B_\sigma(b)} |\nabla u|^2\, dx \leq c\left(8B\epsilon + \frac{32}{3}\epsilon_0\epsilon^{-q}\right)\rho^{m-2}$$
$$+ \beta_{\max}\left(\int_{B_\sigma(b)} |\nabla t(u)|^2 + |\nabla t(\eta)|^2\, dx\right). \tag{3.6}$$

By De Giorgi-Nash theorem, there exist $c > 0$, $\alpha > 0$ (as in (2.19)) depending only on β such that

$$\sigma^{2-m}\int_{B_\sigma(b)} |\nabla t(u)|^2\, dx \leq c\sigma^\alpha, \qquad \sigma^{2-m}\int_{B_\sigma(b)} |\nabla t(\eta)|^2\, dx \leq c\sigma^\alpha. \tag{3.7}$$

Combining (3.6) and (3.7) we obtain

$$\int_{B_\sigma(b)} |\nabla u|^2\, dx \leq c\left(8B\epsilon + \frac{32}{3}\epsilon_0\epsilon^{-q} + \rho_0^\alpha\right)\rho^{m-2}. \tag{3.8}$$

First we take $\epsilon >$ such that $c \cdot 8B\epsilon < \bar{\epsilon}/3$. Then choose $\epsilon_0 > 0$ such that $c \cdot 32/3\epsilon_0\epsilon^{-q} < \bar{\epsilon}/3$ (we may assume $254/3B\epsilon_0 < \delta'^2\epsilon^q$ also holds). Finally choose $\rho_0 > 0$ such that $c \cdot \rho_0^\alpha < \bar{\epsilon}/3$. Then for $\rho < \rho_0$, we have, by (3.8),

$$\rho^{2-m}\int_{B_\sigma(b)} |\nabla u|^2\, dx \leq \bar{\epsilon}.$$

Thus by Lemma 3.2, u is Hölder continuous in $B_{\rho/2}(b)$ and $|u(x) - u(y)| \leq c|x-y|^\delta$ holds for some $c > 0$ and $\delta > 0$. □

For $a \in \Omega$ and $\lambda > 0$, define the scaled map $u_{\lambda,a} := u(\lambda x + a)$.

Our next subject is to study the behavior of $u_{\lambda,a}$ for $\mathcal{F}$-minimizer u as $\lambda \downarrow 0$. For later purpose (see the proof of Corollary 3.6), we consider somewhat more general case.

Let $a \in \Omega$ and $R < 1/2\,\mathrm{dist}(a, \partial\Omega)$. Let u be a $\mathcal{F}$-minimizing map in $H^1_\varphi(M; N_0)$. For $0 < \lambda_i \leq 1$, $a_i \in B_R(a)$ with $\lambda_i \downarrow 0$ and $a_i \to a$, we consider the sequence $\{u_{\lambda_i,a_i}\}$.

Lemma 3.5. *Let u, λ_i and a_i be as above. Then there exists a subsequence of $\{u_{\lambda_i,a_i}\}$ (which we also write $\{u_{\lambda_i,a_i}\}$) such that for any $r > 0$ the following hold:*

(a) $u_{\lambda_i,a_i} \to u_\infty$ *(for some $u_\infty \in H^1_{\mathrm{loc}}(\mathbf{R}^m; N_0)$) locally uniformly in $B_r(0) \setminus \Sigma_\infty$, where $\Sigma_\infty \subset B_r(0)$ is a closed subset with $\mathcal{H}^{m-2}(\Sigma_\infty) = 0$. Moreover u_∞ is continuous in $B_r(0) \setminus \Sigma_\infty$.*

(b) $u_{\lambda_i,a_i} \to u_\infty$ *strongly in $H^1(B_r(0))$.*

(c) *In particular if $a_i \equiv a$, then $\frac{\partial u_\infty}{\partial r} = 0$.*

Proof. (a) We fix $r > 0$. For i large we have $B_{r\lambda_i}(a_i) \subset \Omega$. We may assume without loss of generality that this hold for all i.

By Proposition 2.1, there exists $c > 0$ independent of i such that

$$\int_{B_r(0)} |\nabla u_{\lambda_i,a_i}|^2\, dx = \lambda_i^{2-m} \int_{B_{r\lambda_i}(a_i)} |\nabla u|^2\, dx < c.$$

Therefore there exists a subsequence of $\{u_{\lambda_i,a_i}\}$ (we also denote it as $\{u_{\lambda_i,a_i}\}$) such that

$$u_{\lambda_i,a_i} \rightharpoonup u_\infty \quad \text{weakly in} \quad H^1(B_r(0)),$$
$$u_{\lambda_i,a_i} \to u_\infty \quad \text{strongly in} \quad L^2(B_r(0)).$$

For $b \in B_r(0)$ and $\rho < \mathrm{dist}(b, \partial B_r(0))$ small we assume

$$\rho^{-m} \int_{B_\rho(b)} |u_\infty - (u_\infty)_{\rho,b}|^2\, dx < \epsilon_0, \tag{3.10}$$

where $\epsilon_0 > 0$ is as in Proposition 3.3 for $B = c$.

Since $u_{\lambda_i,a_i} \to u_\infty$ in $L^2(B_r(0))$, for i large enough we have

$$\rho^{-m} \int_{B_\rho(b)} |u_{\lambda_i,a_i} - (u_\infty)_{\rho,b}|^2\, dx < \epsilon_0. \tag{3.11}$$

By the change of variable, we obtain from (3.11)

$$(\lambda_i\rho)^{-m} \int_{B_{\lambda_i\rho}(a_i+\lambda_i b)} |u - (u_\infty)_{\rho,b}|^2\, dx < \epsilon_0. \tag{3.12}$$

On the other hand, for i large enough,

$$\rho^{2-m} \int_{B_\rho(b)} |\nabla u_{\lambda_i,a_i}|^2\, dx = (\lambda_i\rho)^{2-m} \int_{B_{\lambda_i\rho}(a_i+\lambda_i b)} |\nabla u|^2\, dx \leq c.$$

By Proposition 3.3, u is δ-Hölder continuous in $B_{\lambda_i\rho/2}(a_i+\lambda_i b)$ and $|u(x)-u(y)|\le c|x-y|^\delta$ for $x,y\in B_{\lambda_i\rho/2}(a_i+\lambda_i b)$. By rescaling, u_{λ_i,a_i} is δ-Hölder continuous in $B_{\rho/2}(b)$ and $|u_{\lambda_i,a_i}(x)-u_{\lambda_i,a_i}|\le c\lambda_i^\delta|x-y|^\delta\le c|x-y|^\delta$ for $x,y\in B_{\rho/2}(b)$. Thus $\{u_{\lambda_i,a_i}\}$ is equi-continuous and equi-bounded in $B_\rho(b)$ and by Arzela-Ascoli's theorem, for some subsequence (we also denote it $\{u_{\lambda_i,a_i}\}$) such that

$$u_{\lambda_i,a_i}\to u_\infty \quad \text{uniformly in} \quad B_{\frac{\rho}{2}}(b).$$

By Poincaré inequality, if $\rho^{2-m}\int_{B_\rho(b)}|\nabla u_\infty|^2\,dx$ is small enough, then (3.10) holds. So by the standard covering argument, there exists closed subset Σ_∞ with $\mathcal{H}^{m-2}(\Sigma_\infty)=0$ such that

$$u_{\lambda_i,a_i}\to u_\infty \quad \text{locally uniformly in} \quad B_r(0)\setminus\Sigma_\infty.$$

Since $r>0$ is arbitrary, by diagonal sequence argument, we obtain (a).

(b) We fix $r>0$ as in (a). We prove $u_{\lambda_i,a_i}\to u_\infty$ strongly in $H^1(B_r(0))$.

Let Σ_∞ be as in part (a). Since $\mathcal{H}^{m-2}(\Sigma_\infty\cap B_r(0))=0$, for any $\epsilon>0$, there exists balls $\{B_{r_i}(x_i)\}$ such that $\sum_i r_i^{m-2}<\epsilon$, $\Sigma_\infty\cap B_r(0)\subset\bigcup_i B_{r_i}(x_i)$, $x_i\in B_r(0)$.

By Proposition 2.1, there exists $c>0$ independent of k and i such that

$$r_i^{2-m}\int_{B_{r_i}(x_i)}|\nabla u_{\lambda_k,a_k}|^2\,dx=(\lambda_k r_i)^{2-m}\int_{B_{\lambda_k r_i}(a_k+\lambda_k x_i)}|\nabla u|^2\,dx\le c.$$

Therefore we have

$$\sum_i\int_{B_{r_i}(x_i)}|\nabla u_{\lambda_k,a_k}|^2\,dx\le c\sum_i r_i^{m-2}<c\epsilon \tag{3.13}$$

for all large k.

On the other hand, by the first equation of (1.1), we obtain

$$\Delta u_{\lambda_k,a_k}+A(u_{\lambda_k,a_k})(\nabla u_{\lambda_k,a_k},\nabla u_{\lambda_k,a_k}) + \frac{1}{2}\nabla\beta(u_{\lambda_k,a_k})|\nabla t(u)_{\lambda_k,a_k}|^2=0 \quad \text{in} \quad B_r(0).$$

Here $t(u)_{\lambda_k,a_k}(x)=t(u)(\lambda_k x+a_k)$.

Thus we have

$$\begin{aligned}\Delta&(u_{\lambda_k,a_k}-u_{\lambda_j,a_j})\\ &=-A(u_{\lambda_k,a_k})(\nabla u_{\lambda_k,a_k},\nabla u_{\lambda_k,a_k})+A(u_{\lambda_j,a_j})(\nabla u_{\lambda_j,a_j},\nabla u_{\lambda_j,a_j})\\ &\quad-\frac{1}{2}\nabla\beta(u_{\lambda_k,a_k})|\nabla t(u)_{\lambda_k,a_k}|^2+\frac{1}{2}\nabla\beta(u_{\lambda_j,a_j})|\nabla t(u)_{\lambda_j,a_j}|^2.\end{aligned} \tag{3.14}$$

Take $\varphi \in C_0^\infty(B_r(0) \setminus \bigcup_i B_{r_i}(x_i))$ arbitrary. Multiplying $\varphi(u_{\lambda_k,a_k} - u_{\lambda_j,a_j})$ by (3.14) and integrating over $B_r(0)$, we obtain

$$\begin{aligned}
&\int_{B_r(0)} \varphi |\nabla u_{\lambda_k,a_k} - \nabla u_{\lambda_j,a_j}|^2 \, dx \\
&\leq c \int_{B_r(0)} |\varphi| \left(|\nabla u_{\lambda_k,a_k}|^2 + |\nabla u_{\lambda_j,a_j}|^2 \right) |u_{\lambda_k,a_k} - u_{\lambda_j,a_j}| \, dx \\
&\quad + c \int_{B_r(0)} |\varphi| \left(|\nabla t(u)_{\lambda_k,a_k}|^2 + |\nabla t(u)_{\lambda_j,a_j}|^2 \right) |u_{\lambda_k,a_k} - u_{\lambda_j,a_j}| \, dx \\
&\quad + c \int_{B_r(0)} |\nabla \varphi| |\nabla u_{\lambda_k,a_k} - \nabla u_{\lambda_j,a_j}| |u_{\lambda_k,a_k} - u_{\lambda_j,a_j}| \, dx.
\end{aligned} \tag{3.15}$$

Here, by Proposition 2.1 and De Giorgi-Nash theorem, there exists $c > 0$ independent of k such that

$$\int_{B_r(0)} |\nabla u_{\lambda_k,a_k}|^2 \, dx = \lambda_k^{2-m} \int_{B_{r\lambda_k}(a_k)} |\nabla u|^2 \, dx \leq c \tag{3.16}$$

$$\int_{B_r(0)} |\nabla t(u)_{\lambda_k,a_k}|^2 \, dx = \lambda_k^{2-m} \int_{B_{r\lambda_k}(a_k)} |\nabla t(u)|^2 \, dx \leq c. \tag{3.17}$$

Combining (3.13), (3.15), (3.16) and (3.17), and since $\varphi \in C_0^\infty(B_r(0) \setminus \bigcup_i B_{r_i}(x_i))$ is arbitrary, we obtain, for some subsequence of $\{u_{\lambda_i,a_i}\}$ (we also denote it by $\{u_{\lambda_i,a_i}\}$)

$$u_{\lambda_i,a_i} \to u_\infty \quad \text{strongly in } H^1(B_r(0)).$$

Since $r > 0$ is arbitrary, by diagonal sequence argument, we obtain the result.

(c) By Proposition 2.1, we have, for $j \leq i$

$$\begin{aligned}
&\lambda_i^{2-m} \int_{B_{\lambda_i}(a)} |\nabla u|^2 \, dx + 2 \int_{\lambda_i \leq |x-a| \leq \lambda_j} |x-a|^{2-m} \left| \frac{\partial u}{\partial r} \right|^2 \, dx \\
&\leq \lambda_j^{2-m} \int_{B_{\lambda_j}(a)} |\nabla u|^2 \, dx + c(\lambda_i^\alpha + \lambda_j^\alpha).
\end{aligned} \tag{3.18}$$

By Proposition 2.1, there exists limit $\lim_{r \downarrow 0} r^{2-m} \int_{B_r(a)} |\nabla u|^2 \, dx = L$. Letting $i \to \infty$ in (3.18), we obtain

$$L + 2 \int_{|x-a| \leq \lambda_j} |x-a|^{2-m} \left| \frac{\partial u}{\partial r} \right|^2 \, dx \leq \lambda_j^{2-m} \int_{B_{\lambda_j}(a)} |\nabla u|^2 \, dx + c\lambda_j^\alpha. \tag{3.19}$$

By rescaling, we obtain

(3.20)
$$L + 2\int_{|x|\le 1} |x|^{2-m} \left|\frac{\partial u_{\lambda_j,a}}{\partial r}\right|^2 \, dx \le \lambda_j^{2-m} \int_{B_{\lambda_j}(a)} |\nabla u|^2 \, dx + c\lambda_j^{\alpha}.$$

Since $u_{\lambda_j,a} \to u_\infty$ strongly in $H^1_{\rm loc}(\mathbf{R}^m; N_0)$, taking a limit $j \to \infty$ in (3.20), we obtain

$$\int_{|x|\le 1} |x|^{2-m} \left|\frac{\partial u_\infty}{\partial r}\right|^2 \, dx = 0.$$

Therefore $\frac{\partial u_\infty}{\partial r} = 0$. □

Corollary 3.6.

(i) *The conclusions of Lemma 3.5 hold for maps in*

$$\mathcal{TM} = \{v : \text{there exist } \lambda_i \downarrow 0,\ a_i \to a \text{ such that } u_{\lambda_i,a_i} \to v \text{ in } H^1_{\rm loc}(\mathbf{R}^m)\}.$$

Here $a \in \Omega$ and u is a $\mathcal{F}$-minimizing map in $H^1_\varphi(\Omega; N_0)$.

(ii) *Define $\mathcal{TM}^l$ $(l \ge 0)$ inductively as follows: $\mathcal{TM}^0 = \mathcal{TM}$, $\mathcal{TM}^l = \{v :$ there exist $u \in \mathcal{TM}^{l-1}$, $\lambda_i \downarrow 0$ and $a_i \to a \in \mathbf{R}^m$ such that $u_{\lambda_i,a_i} \to v$ in $H^1_{\rm loc}(\mathbf{R}^m)\}$ for $l \ge 1$. Then the conclusions of Lemma 3.5 hold for maps in $\mathcal{TM}^l$ for $l \ge 0$.*

Proof. (i) First we show the monotonicity property for the maps in $\mathcal{TM}$. Let $v \in \mathcal{TM}$, $b \in \mathbf{R}^m$ and $0 < R_1 < R_2$. Then there exist sequences $\{\lambda_i\}, \lambda_i \downarrow 0$ and $\{a_i\}, a_i \to a$ such that $u_{\lambda_i,a_i} \to v$ in $H^1_{\rm loc}(\mathbf{R}^m)$. By Proposition 2.1, we have

$$\begin{aligned}&(\lambda_i R_1)^{2-m} \int_{B_{\lambda_i R_1}(a_i+\lambda_i b)} |\nabla u|^2 \, dx \\ &\quad + \int_{\lambda_i R_1 \le |x-a_i-\lambda_i b| \le \lambda_i R_2} |x - a_i - \lambda_i b|^{2-m} \left|\frac{\partial u}{\partial r}\right|^2 \, dx \\ &\le (\lambda_i R_2)^{2-m} \int_{B_{\lambda_i R_2}(a_i+\lambda_i b)} |\nabla u|^2 \, dx + c((\lambda_i R_1)^\alpha + (\lambda_i R_2)^\alpha).\end{aligned}$$

From this, we obtain:

$$\begin{aligned}&R_1^{2-m} \int_{B_{R_1}(b)} |\nabla u_{\lambda_i,a_i}|^2 \, dx + 2\int_{R_1 \le |x-b| \le R_2} |x-b|^{2-m} \left|\frac{\partial u_{\lambda_i,a_i}}{\partial r}\right|^2 \, dx \\ &\le R_2^{2-m} \int_{B_{R_2}(b)} |\nabla u_{\lambda_i,a_i}|^2 \, dx + c((\lambda_i R_1)^\alpha + (\lambda_i R_2)^\alpha).\end{aligned}$$

Letting $i \to \infty$, we obtain the following monotonicity inequality:

$$R_1^{2-m} \int_{B_{R_1}(b)} |\nabla v|^2 \, dx + 2 \int_{R_1 \le |x-b| \le R_2} |x-b|^{2-m} \left| \frac{\partial v}{\partial r} \right|^2 \, dx \le R_2^{2-m} \int_{B_{R_2}(b)} |\nabla v|^2 \, dx.$$

Next observe that for $v \in \mathcal{TM}$, $\lambda > 0$ and $b \in B_1(0)$ we have $v_{\lambda,b} \in \mathcal{TM}$ since $(u_{\lambda_i,a_i})_{\lambda,b} = u_{\lambda_i\lambda,\lambda_i b+a_i}$. (Note that even in the case $a_i \equiv a$, $\lambda_i b + a_i = \lambda_i b + a \ne a$. For this reason, we have considered the case $a_i \to a$ and not the case $a_i \equiv a$.)

We first prove the assertion (a) for the map $v \in \mathcal{TM}$. Fix $r > 0$. Let $v \in \mathcal{TM}$ as above, $\{\lambda_i'\}, \lambda_i' \downarrow 0$ and $a_i' \to a'$. Consider the rescaled maps $v_{\lambda_i',a_i'}$.

By the monotonicity inequality for the maps in $\mathcal{TM}$ proved above, there exists subsequence of $\{v_{\lambda_i',a_i'}\}$ (we also denote it by $\{v_{\lambda_i',a_i'}\}$) such that $v_{\lambda_i',a_i'} \rightharpoonup v_\infty$ weakly in $H^1(B_r(0))$ and $v_{\lambda_i',a_i'} \to v_\infty$ strongly in $L^2(B_r(0))$.

Let $\zeta \in B_r(0)$ and $\rho < \operatorname{dist}(\zeta, \partial B_r(0))$; small be such that $\rho^{-m} \int_{B_\rho(\zeta)} |v_\infty - (v_\infty)_{\rho,\zeta}|^2 \, dx < \epsilon_0$. Then, as in the proof of Lemma 3.5 (a), there exists i_0 such that for $i \ge i_0$, we have

$$\rho^{-m} \int_{B_\rho(\zeta)} |v_{\lambda_i',a_i'} - (v_\infty)_{\rho,\zeta}|^2 < \epsilon_0.$$

Since $u_{\lambda_k,a_k} \to v$ in $H^1_{\text{loc}}(\mathbf{R}^m)$, there exists $k(i)$ such that for $k \ge k(i)$

$$\rho^{-m} \int_{B_\rho(\zeta)} |(u_{\lambda_k,a_k})_{\lambda_i',a_i'} - (v_\infty)_{\rho,\zeta}|^2 \, dx < \epsilon_0.$$

Since $(u_{\lambda_k,a_k})_{\lambda_i',a_i'} = u_{\lambda_k\lambda_i',\lambda_k a_i'+a_k}$, we get

$$(\lambda_k\lambda_i'\rho)^{-m} \int_{B_{\lambda_k\lambda_i'\rho}(\lambda_k a_i'+a_k+\lambda_k\lambda_i'\zeta)} |u - (v_\infty)_{\rho,\zeta}|^2 \, dx < \epsilon_0$$

for such i and k.

Then by the same argument in the proof of Lemma 3.5 (a), u is Hölder continuous in $B_{\lambda_k\lambda_i'\rho/2}(\lambda_k a_i' + a_k + \lambda_k\lambda_i'\zeta)$, and there exists constant $c > 0$ such that $|(u_{\lambda_k,a_k})_{\lambda_i',a_i'}(x) - (u_{\lambda_k,a_k})_{\lambda_i',a_i'}(y)| \le c|x-y|^\delta$ for $x, y \in B_{\rho/2}(\zeta)$. From this, $\{(u_{\lambda_k,a_k})_{\lambda_i',a_i'}\}_{k \ge k(i)}$ is equi-continuous and, by Arzela-Ascoli, there exists a subsequence (we also denote it by the same sequence) such that $(u_{\lambda_k,a_k})_{\lambda_i',a_i'} \to v_i$ uniformly in $B_{\rho/2}(\zeta)$ for some v_i as $k \to \infty$. But $u_{\lambda_k,a_k} \to v$ in $H^1_{\text{loc}}(\mathbf{R}^m)$, so we have $v_i \equiv v_{\lambda_i',a_i'}$. By the uniform convergence, $v_{\lambda_i',a_i'}$ also satisfies $|v_{\lambda_i',a_i'}(x) - v_{\lambda_i',a_i'}(y)| \le c|x-y|^\delta$ for $x, y \in B_{\rho/2}(\zeta)$ and $i \ge i_0$. Again by Arzela-Ascoli, we have $v_{\lambda'_i,a'_i} \to v_\infty$ uniformly in $B_{\rho/2}(\zeta)$. From this and the same argument in the proof of Lemma 3.5 (a) shows that the conclusion of Lemma 3.5 (a) also holds for maps in $\mathcal{TM}$.

Next we verify Lemma 3.5 (b) for maps in $\mathcal{TM}$. For this, first observe that the map $v \in \mathcal{TM}$ is a harmonic map into N_0, that is, it satisfies the equation $\Delta v + A(v)(\nabla v, \nabla v) = 0$. To see this, we only note that the left hand side of (3.17) goes to zero as $k \to \infty$ by De Giorgi-Nash theorem. Then, by the result (a), the same argument in the proof of Lemma 3.5 (b) shows that the conclusion (b) of Lemma 3.5 also holds for maps in $\mathcal{TM}$.

The proof of Lemma 3.5 (c) for maps in $\mathcal{TM}$ follows the same argument in the proof of Lemma 3.5 (c), since we already have monotonicity inequality, (a) and (b).

(ii) The proof of assertion (ii) follows from the same argument in (i) using the induction on l. □

Remark 3.7. (a) As was stated in the introduction, $\mathcal{F}$-minimizing map in $H^1_\varphi(M; N_0)$ does not have local minimizing property. So there is no sense to consider local minimizing maps to study regularity properties of $\mathcal{F}$-minimizing maps and, of course, scaled maps $u_{\lambda,a}$ for $\mathcal{F}$-minimizing map u does not have any local minimizing property. Therefore in the above proposition and corollary, we only considered the compactness properties of scaled maps (not local minimizing maps as in many other problems). This is the troublesome point in our problem.

(b) The limiting map $v \in \mathcal{TM}$, which we call *tangent map* at a in the case $a_i \equiv a$, is indeed a harmonic map into N_0. This follows from the same reason as in Corollary 3.6 (i), proof of part (b).

We are now ready to prove our main theorem Theorem A. As in many other regularity problems, we prove it by Federer's dimension reduction method [**3**], [**12**], [**13**].

Completion of the proof of Theorem A. We define the measure φ^s for $s \in \mathbf{R}$ as follows:

$$\varphi^s(E) = \inf \left\{ \sum_i r_i^s : E \subset \bigcup_{i=1}^{\infty} B_{r_i}(x_i) \right\}.$$

We prove the following

Lemma 3.8. *Let u be $\mathcal{F}$-minimizing map or $u \in \mathcal{TM}^l$, $l \geq 0$. Let $\lambda_i \downarrow 0$ and $a \in B_1(0)$. Assume $u_{\lambda_i,a} \rightharpoonup u_\infty$ weakly in $H^1(B_1; N_0)$. Let Σ_i and Σ_∞ be singular sets of $u_{\lambda_i,a}$ and u_∞, respectively. Then we have*

$$\varphi^s(\Sigma_\infty \cap B_{1/2}(0)) \geq \limsup_{i\to\infty} \varphi^s(\Sigma_i \cap B_{1/2}(0)).$$

Proof. Let $\epsilon > 0$ be given. Let $\{B_{r_i}(x_i)\}$ be a sequence of balls such that $\Sigma_\infty \cap B_{1/2}(0) \subset \bigcup_{i=1}^\infty B_{r_i}(x_i)$ and

$$\sum_{i=1}^\infty r_i^s \le \varphi^s(\Sigma_\infty \cap B_{1/2}(0)) + \epsilon.$$

By Lemma 3.5 and Corollary 3.6, for large i, $u_{\lambda_i,a}$ is continuous on $B_{1/2}(0) \setminus \bigcup_{i=1}^\infty B_{r_i}(x_i)$. Therefore we have

$$\Sigma_i \cap B_{1/2}(0) \subset \bigcup_{i=1}^\infty B_{r_i}(x_i).$$

So

$$\varphi^s(\Sigma_i \cap B_{1/2}(0)) \le \sum_{i=1}^\infty r_i^s \le \varphi^s(\Sigma_\infty \cap B_{1/2}(0)) + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we obtain the result. □

We continue the proof of Theorem A. Let Σ be a singular set of u. By Theorem 1.1, we know $\mathcal{H}^{m-2}(\Sigma) = 0$.

Let $0 \le s < m-2$ be a real number such that $\varphi^s(\Sigma) > 0$. If there is no such s, then we complete the proof, since in this case $\Sigma = \emptyset$.

By the density theorem about the measure φ^s, see [**12**], we have $\limsup_{\lambda \downarrow 0} \lambda^{-s}\varphi^s(\Sigma \cap B_{\lambda/2}(x)) > 0$ for φ^s-a.e. $x \in \Sigma$. So there exist $a_0 \in \Sigma$ and $\lambda_i \downarrow 0$ such that

$$\lim_{i\to\infty} \lambda_i^{-s}\varphi^s(\Sigma \cap B_{\lambda_i/2}(a_0)) > 0.$$

We consider the rescaled maps u_{λ_i,a_0}.

By Lemma 3.5, there exists a subsequence (we also denote it by $\{u_{\lambda_i,a_0}\}$) such that

$$u_{\lambda_i,a_0} \to u_0 \quad \text{strongly in} \quad H^1_{\mathrm{loc}}(\mathbf{R}^m; N_0)$$

for some $u_0 \in H^1_{\mathrm{loc}}(\mathbf{R}; N_0)$ with $\frac{\partial u_0}{\partial r} = 0$.

We set $\Sigma_i^{a_0}$ = the singular set of u_{λ_i,a_0}. Then $\varphi^s(\Sigma_i^{a_0} \cap B_{1/2}(0)) = \lambda_i^{-s}\varphi^s(\Sigma \cap B_{\lambda_i/2}(a_0))$ and $\lim_{i\to\infty} \varphi^s(\Sigma_i^{a_0} \cap B_{1/2}(0)) > 0$.

By Lemma 3.8, we have $\varphi^s(\Sigma_0 \cap B_{1/2}(0)) > 0$, where Σ_0 is the singular set of u_0. Since $\frac{\partial u_0}{\partial r} = 0$, Σ_0 is a cone, i.e., $\lambda\Sigma = \Sigma$ for all $\lambda > 0$.

There are two possibilities:

(i) $s = 0$

(ii) $s > 0$.

If the case (i) occurs, we complete the proof. So we consider the case (ii). We take a new coordinate so that the radial coordinate r is x_1-direction.

There exists $a_1 \in \Sigma_0 \cap \partial B_1(0)$ such that

$$\limsup_{\lambda \downarrow 0} \lambda^{-s} \varphi^s(\Sigma_0 \cap B_\lambda(a_1)) > 0.$$

Then by the same argument as before (using Corollary 3.6), there exists a sequence $\{\lambda_i\}$, $\lambda_i \downarrow 0$ such that

$$\lim_{i\to\infty} \lambda_i^{-s} \varphi^s(\Sigma_0 \cap B_{\lambda_i}(a_1)) > 0$$

and for some subsequence of $\{u_{0\lambda_i,x_1}\}$ (we also write it as $\{u_{\lambda_i,a_1}\}$) we have $u_{0\lambda_i,a_1} \to u_1$ strongly in $H^1_{\text{loc}}(\mathbf{R}^m; N_0)$, where $u_1 \in \mathcal{TM}^1$ and $\frac{\partial u_1}{\partial r_1} = 0$. (Here r_1 is the radial with respect to the new coordinate.)

Let Σ_1 be a singular set of u_1. Then by Lemma 3.8, $\varphi^s(\Sigma_1 \cap B_1(0)) > 0$. Since $\frac{\partial u_0}{\partial x_1} = 0$, we also have $\frac{\partial u_1}{\partial x_1} = 0$.

If $s - 1 \leq 0$, we stop. Otherwise (this is the case $s > 1$), since $\varphi^s(\Sigma_1 \cap B_1(0) \setminus \mathbf{R} \times \{0\}) = \varphi^s(\Sigma_1 \cap B_1(0)) > 0$, there exists $a_2 \in \Sigma_2 \cap \partial B_1^{m-1}(0)$, where $B_1^{m-1}(0)$ is the unit ball in $\mathbf{R}^{m-1} = \{(0, x_2, \ldots, x_m)\}$, such that

$$\limsup_{\lambda \downarrow 0} \lambda^{-s} \varphi^s(\Sigma_1 \cap B_\lambda(a_2)) > 0$$

and there exists $u_2 \in \mathcal{TM}^2$ such that $\frac{\partial u_2}{\partial x_1} = \frac{\partial u_2}{\partial x_2} = \frac{\partial u_2}{\partial r} = 0$ (for some suitable choice of coordinate).

We continue this procedure n-times. Then we have $u_l \in H^1_{\text{loc}}(\mathbf{R}^m; N_0) \cap \mathcal{TM}^l$ $(1 \leq l \leq n)$ with $\frac{\partial u_l}{\partial x_i} = \frac{\partial u_l}{\partial r} = 0$ for $i = 1, \ldots, l$ (for suitable choice of coordinate). We can repeat the argument until we have $s - n \leq 0$.

In order to obtain constructed u_n, it is necessary $s - n + 1 > 0$. Since $s < m - 2$, we obtain $m - 1 > n$, i.e., $n \leq m - 2$.

If $n = m-2$, then $\Sigma_n =$ singular set of $u_n \supset \mathbf{R}^{m-2} = \{(x^1, \ldots, x^{m-2}, 0, 0)\}$. This is a contradiction since $\mathcal{H}^{m-2}(\Sigma_n) = 0$. Therefore we have $n \leq m - 3$ and $s \leq n \leq m - 3$. Since s is an any number satisfying $s < \dim \Sigma$, we obtain $\dim \Sigma \leq m - 3$.

Finally, we consider the case $m = 3$. We assume that there exists a limit point $x_0 \in B_r(x_0) \subset\subset \Omega$ of Σ, that is, there exist distinct points $x_i \in \Sigma \cap B_r(x_0)$ such that $x_i \to x_0$ as $i \to \infty$. Put $\lambda_i = |x_i - x_0|$ and consider the sequence $\{u_{\lambda_i,x_0}\}$. We remark that singular set of $u_{\lambda_i,x_0} \cap \partial B_1(0) \neq \emptyset$. By Lemma 3.5, we may assume $u_{\lambda_i,x_0} \to u_\infty$ strongly in $H^1_{\text{loc}}(\mathbf{R}^m; N_0)$ for some $u_\infty \in \mathcal{TM}$. We may also assume that $x_i - x_0/|x_i - x_0| \to \zeta \in \partial B_1(0)$. Since x_i is a singularity of u, by Lemma 3.2, we have

$$\frac{1}{\lambda_i r} \int_{B_{\lambda_i r}(x_i)} |\nabla u|^2 \, dx \geq \bar{\epsilon}$$

and, by rescaling

$$\frac{1}{r}\int_{B_r\left(\frac{x_i-x_0}{|x_i-x_0|}\right)} |\nabla u_{\lambda_i,x_0}|^2\ dx \geq \bar{\epsilon} > 0$$

for all small $r > 0$.

Letting $i \to \infty$, we obtain

$$\frac{1}{r}\int_{B_r(\zeta)} |\nabla u_\infty|^2\ dx \geq \bar{\epsilon}$$

for all small $r > 0$.

Therefore ζ is a singularity of u_∞. Since Σ_∞ (= singular set of u_∞) is a cone, we have $\mathcal{H}^1(\Sigma_\infty) > 0$. This is a contradiction since $\mathcal{H}^1(\Sigma_\infty) = 0$. Therefore Σ is a discrete set in the case $m = 3$. □

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