PACIFIC JOURNAL OF MATHEMATICS
Vol. 178, No. 1, 1997

NON-SPLITTING INCLUSIONS OF FACTORS OF TYPE III_0

Hideki Kosaki and Takashi Sano

By generalizing constructions in Kosaki (1994) and Kosaki and Longo, we will construct an AFD type III_0 factor with uncountably many non-conjugate subfactors such that (i) each subfactor has the same flow of weights as the ambient factor, and (ii) the principal and the dual principal graphs are of a specific form. We will deal with two cases: (a) the graphs are described by the Dynkin diagram A_{4m-3}, and (b) the graphs are the ones given by a pair of a group and its subgroup (see Kosaki and Yamagami) which are simultaneous semi-direct products. Subfactors are distinguished by looking at the dual action on the type II graphs. It is also possible to distinguish subfactors by investigating automorphisms appearing in the irreducible decomposition of the relevant sector (or bimodule).

1. Introduction.

Classification of subfactors in the Powers factor of type III_λ $(0 < \lambda < 1)$ with small indices is known to be closely related to that for the AFD II_1-factor R_0 and analysis on (trace-scaling) automorphisms for related inclusions of II_∞-factors (see [**32**, **46**] for the classification when Index < 4). For the latter, the Loi invariant ([**32**, **33**]) plays an important role (see [**46**], and also [**3**, **21**, **23**, **29**, **57**] for related results). Since subfactors in R_0 are quite rigid objects and there is only small amount of freedom left for the Loi invariant, the Powers factor does not generally admit so many subfactors (with small indices).

On the other hand, an (AFD) type III_0 factor admits many subfactors (with the same flow of weights ([**4**]) as the ambient factor) due to the fact that the flow space is huge. In fact, in [**28**], the existence of an AFD type III_0 factor with uncountably many non-conjugate subfactors (with the same flow of weights) with the principal graph A_5 $(\cong \mathfrak{S}_3/\mathfrak{S}_2)$ was shown. The purpose of the present article is to generalize this result into two directions as was mentioned in the abstract.

In §3, by generalizing constructions in [**28**, **30**], we will construct an AFD type III_0 factor with the same properties, but with the principal graph

A_{4m-3} ($m = 2, 3, \cdots$) instead. A (unique) pair of AFD II_1-factors with the principal graph D_{2m} and the symmetry switching the last two vertices in the graph ([**22**]) will play important roles. Our construction also uses a two-to-one ergodic extension $(\tilde{T}, \tilde{X})$ of (T, X). Our inclusion $M \supseteq N$ (of AFD type III_0 factors with the graph A_{4m-3}) has the same flow of weights, and it is given by (T, X) (together with a ceiling functon). As in [**28**] the extention $(\tilde{T}, \tilde{X})$ used during our construction can be recovered from the inclusion data of $M \supseteq N$ (as a part of the type II principal graph together with the dual action). Therefore, by starting from (T, X) with uncountably many non-conjugate two-to-one ergodic extensions (see [**47**, p. 262], for example), we will obtain an AFD type III_0 factor with the required property.

In §4, we will show that the subfactors in §3 can be also distinguished based on the sector technique (for example [**14**, **34**, **36**]). A unique (non-trivial) automorphism among descendent sectors ([**14**]) in question is shown to be a period 2 extended modular automorphism ([**4**]). The corresponding (± 1-valued) cocycle contains information on the extention $(\tilde{T}, \tilde{X})$. This result requires the characterization of non-strongly outer automorphisms in [**3**, **29**] and a certain duality between the (Connes-Takesaki) module ([**4**]) and the modular invariant in the sense of [**24**, **55**]. The latter duality result will be proved in Appendix A.

In §5, $A_5 (\cong \mathfrak{S}_3/\mathfrak{S}_2)$ will be generalized to a general group-subgroup pair $G \supseteq H$. We need an action (for an inclusion of II_1-factors) with non-trivial Loi invariant so that it is natural to start from a pair of simultaneous semi-direct products $G = G_0 \rtimes_\mu K \supseteq H = H_0 \rtimes_\mu K$. The pair of AFD II_1-factors arising from $G_0 \supseteq H_0$ admits the obvious K-action. Not only the graphs of the inclusion (see [**1**, **31**]) but also the Loi invariant of this K-action can be described in terms of various irreducible representations. The description of the Loi invariant will be given in Appendix B. Based on this K-action and a $\#K$-to-one ergodic extension (whose cocycle takes values in K), we will construct a pair $M \supseteq N$ of AFD type III_0 factors whose type II towers are described by $G_0 \supseteq H_0$. The dual action on these towers are determined by the K-valued cocycle and the Loi invariant of the K-action. Therefore, by making use of basic properties in [**30**] of the minimal conditional expectation ([**12**, **13**, **35**, **37**]), one can compute the (type III) towers of $M \supseteq N$ in principle.

However, in §6, we will directly show that $M \supseteq N$ in §5 is of the form $P \rtimes_\gamma G \supseteq P \rtimes_\gamma H$ by making use of certain coactions and their crossed products (see [**38**] for example).

In §7, we will deal with the special case $\mathfrak{S}_n = \mathfrak{A}_n \rtimes_\mu \mathbf{Z}_2 \supseteq \mathfrak{S}_{n-1} = \mathfrak{A}_{n-1} \rtimes_\mu \mathbf{Z}_2$ (the symmetric and alternating groups). Here, everything is very explicitly calculated from the branching rule for Young diagrams. Consequently we will once again obtain an AFD type III_0 factor with uncountably many non-conjugate subfactors with the graphs determined by $\mathfrak{S}_n \supseteq \mathfrak{S}_{n-1}$.

Our basic reference for the modular theory is [52] while standard facts on the index theory can be found in the original article [19] and [7, 26, 35, 41], [42]. Some facts necessary here from recent articles are summarized in §2 for the convenience of the reader and partly to fix our notations.

The authors are grateful to the referee for many useful comments which have made the article more readable.

2. Preliminaries.

Let $M \supseteq N$ be a factor-subfactor pair (of type III in most cases) with a conditional expectation E. Throughout the article we assume that $\mathrm{Ind}\, E < \infty$ ([26, 35]) and E is a minimal conditional expectation ([12, 13, 35]). From the Jones tower $N \subseteq M \subseteq M_1 \subseteq M_2 \subseteq \cdots$ we get the increasing sequences $\{M_k \cap N'\}_k$, $\{M_k \cap M'\}_k$ of finite dimensional algebras and hence two graphs as in the II_1 case ([7, 19]). We will call them the (type III) principal and dual principal graphs.

2.1. Let θ ($\in \mathrm{Aut}(M, N)$) be an automorphism leaving N globally invariant. By the uniqueness of a minimal expectation we have $E \circ \theta = \theta \circ E$, and θ is canonically extended to an automorphism of the basic extension M_k (in such a way that the relevant Jones projections are fixed). The effect of the (extended) θ on the towers $\{M_k \cap N'\}_k, \{M_k \cap M'\}_k$ is called the Loi invariant ([32, 33]) and plays an important role in study on automorphisms for $M \supseteq N$ ([3, 21, 23, 29, 46, 57]).

2.2. Let ψ be a faithful state in N_*^+, and we set

$$\tilde{M} = M \rtimes_{\sigma^{\psi \circ E}} \mathbf{R} \supseteq \tilde{N} = N \rtimes_{\sigma^{\psi}} \mathbf{R}.$$

They are von Neumann algebras of type II_∞, and the above construction does not depend upon the choice of a state (thanks to Connes' Radon-Nikodym theorem). The crossed product of M_1 (the basic extension) relative to the modular automorphism group attached to $\psi \circ E \circ E_1$ (where $E_1 \colon M_1 \longrightarrow M$ is the dual expectation, [26]) can be identified with the basic extension $\tilde{M}_1$ of $\tilde{M} \supseteq \tilde{N}$ by the characterization of the basic extension (Proposition 1.2, [42], or see §2 of [30] for details). Furthermore, the Jones projection for $M \supseteq N$ and that for $\tilde{M} \supseteq \tilde{N}$ are the same. Iterating this procedure, we know that the Jones tower $\tilde{N} \subseteq \tilde{M} \subseteq \tilde{M}_1 \subseteq \tilde{M}_2 \subseteq \cdots$ (called the type II tower) can be obtained as the crossed product of the tower $N \subseteq M \subseteq M_1 \subseteq M_2 \subseteq \cdots$ by the relevant modular automorphism group. The dual action $\{\theta_t\}_{t \in \mathbf{R}}$ acts on the type II tower, and as was shown in Corollary 6, [30] we have

$$M_k \cap N' = \{\tilde{M}_k \cap \tilde{N}'\}_\theta. \tag{1}$$

2.3. Let us further assume that the von Neumann algebras $\tilde{M} \supseteq \tilde{N}$ of type II_∞ have the identical center ($\cong L^\infty(\Omega, d\nu)$). This means that $M \supseteq N$ is of the form $M = \mathfrak{A} \supseteq \mathfrak{B} = N$ in the sense of [**27**]. Let

$$\int_\Omega^\oplus \tilde{N}_\omega d\nu(\omega), \quad \int_\Omega^\oplus \tilde{M}_\omega d\nu(\omega), \quad \int_\Omega^\oplus (\tilde{M}_k)_\omega d\nu(\omega)$$

be the central decomposition. It is straight-forward to see that the k-th basic extension of $\tilde{M}_\omega \supseteq \tilde{N}_\omega$ is $(\tilde{M}_k)_\omega$ (Once again [**42**] can be used.) From $\{(\tilde{M}_k)_\omega \cap \tilde{N}'_\omega\}_k, \{(\tilde{M}_k)_\omega \cap \tilde{M}'_\omega\}_k$ we also get graphs (which do not depend upon ω since the centrally ergodic dual action is around). These graphs are referred to as the type II principal and type II dual principal praphs (of $M \supseteq N$) respectively.

2.4. Let α be an action of a (discrete for example) group G for $M \supseteq N$ and assume that it is canonically extended to an action of the basic extension M_k as in 2.1. The action α is called strongly outer ([**3**, **29**], or properly outer [**46**]) when, for each $g \neq e$ and j, we have: If $x \in M_j$ satisfies $yx = x\alpha_g(y)$ for $y \in N$, then $x = 0$.

(i) Such an action is completely classified by its Loi invariant ([**46**], when $M \supseteq N$ is strongly amenable and G is amenable) in the type II_1 case.

Let $\mathrm{End}(M)$ be the unital normal *-endomorphisms of M, and we set $\mathrm{Sect}(M) = \mathrm{End}(M)/\mathrm{Int}(M)$, the sectors, as in [**36**] (and the properly infiniteness of M is assumed). For $\sigma \in \mathrm{End}(M)$, the conjugate sector is defined by $\overline{[\sigma]} = [\bar{\sigma}]$ ($\bar{\sigma} = \sigma^{-1} \circ \gamma$, where γ is the canonical endomorphism attached to $M \supseteq \sigma(M)$, [**34**]), and this notion is essential in the sector theory ([**14**, **15**, **16**, **36**, **37**]). (For simplicity the class $[\sigma]$ will be denoted by σ in what follows.) The statistical dimension $d\sigma$ means the square root of the minimal index of $M \supseteq \sigma(M)$. When $M \cap \sigma(M)' = \mathbf{C}1$, σ is called irreducible. Otherwise, (but $d\sigma < \infty$) the irreducible decomposition can be performed (see [**36**]).

(ii) When $N = \rho(M)$ ($\rho \in \mathrm{End}(M)$), the strong outerness is characterized as follows ([**3**, **29**]): α is strongly outer if and only if none of $\alpha_g (g \neq e)$ appears in the sectors $(\rho\bar{\rho})^n$ ($n = 0, 1, \cdots$) as an irreducible component.

2.5. An N-M bimodule (or correspondence) $\mathcal{Y} = {}_N\mathcal{Y}_M$ means a Hilbert space equipped with commuting normal representations of N and the opposite algebra of M ([**43**, **51**]). Sectors are closely related to bimodules ([**36**]), and in fact one has to deal with bimodules in the II_1 case. Here, contragredient bimodules ($\bar{\mathcal{Y}}$ ($= {}_M(\bar{\mathcal{Y}})_N$) is the conjugate Hilbert space of $\mathcal{Y}$; $m \cdot \bar{\xi} \cdot n = \overline{n^* \cdot \xi \cdot m^*}$) should be considered instead of conjugate sectors, and the ordinary composition of sectors (as endomorphisms) is replaced by the notion of the relative tensor product ([**51**]). Let us briefly recall Ocneanu's description on graphs ([**39**, **40**], see also [**58**]). Let $\mathcal{X}$ be the (basic)

N-M bimodule ${}_NL^2(M)_M$, and consider the following sequences (that are obtained by the induction-restriction procedure (see [39, 40])):

$${}_NL^2(N)_N, \mathcal{X}, \mathcal{X} \otimes_M \bar{\mathcal{X}}, \mathcal{X} \otimes_M \bar{\mathcal{X}} \otimes_N \mathcal{X}, \cdots$$

$${}_ML^2(M)_M, \bar{\mathcal{X}}, \bar{\mathcal{X}} \otimes_N \mathcal{X}, \bar{\mathcal{X}} \otimes_N \mathcal{X} \otimes_M \bar{\mathcal{X}}, \cdots$$

The principal and dual principal graphs describe the irreducible decomposition of the above sequences. The sectors $(\rho\bar{\rho})^k$ $(k = 0, 1, \cdots)$ actually correspond to the M-M bimodules appearing (alternatively) in the second sequence, and hence the irreducible sectors in $\bigsqcup_k (\rho\bar{\rho})^k$ correspond to the irreducible M-M bimodules (or the "even vertices" in the dual principal graph) in [39, 40]. Let $\mathcal{Y}$ be an N-M bimodule. Since the (left) N-action of $\mathcal{Y}$ is sitting in the commutant of the right M-action, $\mathcal{Y}$ gives us an inclusion of factors whose minimal index is called the dimension of $\mathcal{Y}$. Finally we point out that an automorphism gives rise to an M-M bimodule with dimension one, or equivalently, a sector with statistical dimension one (and vice versa). Namely, $\alpha \in \mathrm{Aut}(M)$ defines the M-M bimodule $\mathcal{H}_\alpha$ by $\mathcal{H}_\alpha = L^2(M);\ m_1 \cdot \xi \cdot m_2 = \alpha(m_1) J_M m_2^* J_M \xi$. Notice that an inner perturbation of α does not change the unitary equivalence class of $\mathcal{H}_\alpha$.

3. Inclusions with the graph A_{4m-3}.

In this section, by generalizing the methods in [28, 30], we will construct an AFD type III_0 factor with many subfactors with the principal graph A_{4m-3} $(m = 2, 3, \cdots)$.

Let $A \supseteq B$ be a (unique) inclusion of AFD type II_1 factors with the principal graph D_{2m} (see [39, 40, 44, 45]). Let π be a period 2 automorphism in $\mathrm{Aut}(A, B)$ with non-trivial Loi invariant (see 2.1), that is, (the extended) π switches the last two vertices of the graph D_{2m} ([22]). (See Figure 1 below.)

Let $M_0 \rtimes_{\theta_0} \mathbf{Z}$ be a discrete decomposition of an AFD type III_0 factor. Hence, M_0 is an AFD II_∞ von Neumann algebra with a trace tr_{M_0} and $tr_{M_0} \circ \theta_0 = tr_{M_0}(e^{-f}\cdot)$ with a positive element f in the center $\mathcal{Z}(M_0)$. Let $\mathcal{Z}(M_0) = L^\infty(X, \mu)$ and we assume that $\theta_0|_{\mathcal{Z}(M_0)}$ is induced by a (non-singular ergodic) transformation T on X. (Therefore, f is a measurable function on (X, μ) and the flow of weights of $M_0 \rtimes_{\theta_0} \mathbf{Z}$ is obtained from the ceiling function f together with the base transformation (T, X).)

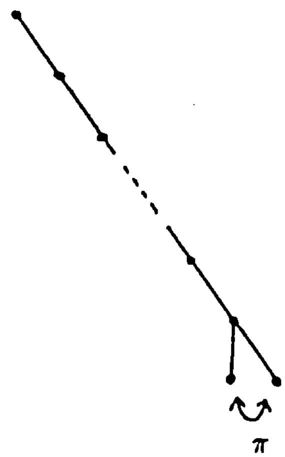

Figure 1.

Let

$$M_0 = \int_X^{\oplus} M_0(\omega) d\mu(\omega), \quad tr_{M_0} = \int_X^{\oplus} tr_\omega d\mu(\omega)$$

be the central decomposition together with the disintegration of the trace tr_{M_0}. Assume that the automorphism θ_0 corresponds to a field $\{\theta_0(\omega)\}_{\omega \in X}$ of isomorphisms $(\theta_0(\omega) : M_0(\omega) \to M_0(T\omega))$. The assumption $tr_{M_0} \circ \theta_0 = tr_{M_0}(e^{-f} \cdot)$ means

$$tr_{T\omega}(\theta_0(\omega)\cdot) \frac{d\mu(T\omega)}{d\mu(\omega)} = e^{-f(\omega)} tr_\omega(\cdot) \tag{2}$$

by the standard argument.

Let us choose and fix a two-to-one ergodic extension $(\tilde{T}, \tilde{X})$ of (T, X):

$$\tilde{X} = X \times \mathbf{Z}_2 \ \text{ (with the obvious product measure)},$$
$$\tilde{T}^n(\omega, i) = (T^n\omega, i + \varphi_{\omega,n}),$$

where $\varphi : (\omega, n) \in X \times \mathbf{Z} \to \varphi_{\omega,n} \in \mathbf{Z}_2$ is a cocycle (see [59], especially Corollary 3.8).

By using the above cocycle φ together with $(A \supseteq B, \pi)$, we now construct an automorphism on

$$A \otimes M_0 = \int_X^{\oplus} A \otimes M_0(\omega) d\mu(\omega).$$

Define the automorphism θ by

$$\theta(x)(\omega) = (\pi^{\varphi_{T^{-1}\omega,1}} \otimes \theta_0(T^{-1}\omega))(x(T^{-1}\omega))$$

for $x \in A \otimes M_0$. We obviously have $\theta \in \mathrm{Aut}(A \otimes M_0, B \otimes M_0)$. Let tr_A be the unique II_1 trace on A. Because of $\mathrm{tr}_A \circ \pi = \mathrm{tr}_A$, one easily gets

$$(\mathrm{tr}_A \otimes \mathrm{tr}) \circ \theta = (\mathrm{tr}_A \otimes \mathrm{tr})(e^{-f} \cdot)$$

by using (2). We now set

$$M = (A \otimes M_0) \rtimes_\theta \mathbf{Z} \supseteq N = (B \otimes M_0) \rtimes_\theta \mathbf{Z}.$$

Let E_B be the unique normal conditional expectation from A onto B. The tensor product $E_B \otimes \mathrm{Id}_{M_0} \colon A \otimes M_0 \to B \otimes M_0$ commutes with θ because of $\pi \circ E_B = E_B \circ \pi$. Thus, $E_B \otimes \mathrm{Id}_{M_0}$ lifts to a normal conditional expectation $E \colon M \to N$ and

$$\begin{aligned} \mathrm{Ind}\ E &= \mathrm{Ind}\ (E_B \otimes \mathrm{Id}_{M_0}) \quad \text{(see 2.1 of [30])} \\ &= [A : B] = 4\cos^2(\pi/(4m-2)). \end{aligned}$$

Notice that

$$\begin{aligned} \mathcal{Z}(A \otimes M_0) = \mathcal{Z}(M_0) \ &= \ L^\infty(X), \\ \theta|_{\mathcal{Z}(A \otimes M_0)} &= \theta_0|_{\mathcal{Z}(M_0)}. \end{aligned}$$

Therefore, the flow of weights of M is the same as that of $M_0 \rtimes_{\theta_0} \mathbf{Z}$. The same is obviously true for N.

In what follows we will assume that $f = \alpha_0 1, \alpha_0 > 0$, that is, the flow of weights of $M_0 \rtimes_{\theta_0} \mathbf{Z}$ has the constant ceiling function α_0. As in 2.2, we set

$$\tilde{M} = M \rtimes_{\sigma^{\psi \circ E}} \mathbf{R} \supseteq \tilde{N} = N \rtimes_{\sigma^\psi} \mathbf{R}.$$

It is well-known ([48], see also the proof of Lemma 5) that

$$\begin{aligned} \mathcal{Z}(\tilde{M}) &= L^\infty(X \times [0, \alpha_0)), \\ \tilde{M} &= \int_{X \times [0,\alpha_0)}^{\oplus} \tilde{M}_{\omega,t} d\mu(\omega) dt, \\ \tilde{M}_{\omega,t} &= (A \otimes M_0)(\omega) = A \otimes M_0(\omega). \end{aligned}$$

Of course the similar properties are valid for N, and in particular $\tilde{M} \supseteq \tilde{N}$ have the identical center $(= L^\infty(X \times [0, \alpha_0)))$. The dual action $\{\theta_t\}_{t \in \mathbf{R}}$ for $\tilde{M}$ (and also for $\tilde{N}$) is described by θ in the well-known fashion. (In the "vertical direction" θ_t looks like a translation, and the "base automorphism" is θ.)

Let

$$B \subseteq A \subseteq A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$$

be the Jones tower with the Jones projections $e_0 = e_B \in A_1, e_i \in A_{i+1}$. The automorphism π is uniquely extended to that of A_i (still denoted by

π) subject to the condition $\pi(e_j) = e_j, j \geq 0$. Thanks to the compatibility between the basic construction and taking a crossed product (by a **Z**-action) as in 2.2, we know:

(i) The k-th extension of $M \supseteq N$ is

$$M_k = (A_k \otimes M_0) \rtimes_\theta \mathbf{Z}.$$

where θ is defined as before (by using the extended π).

(ii) The k-th extension $\tilde{M}_k$ of $\tilde{M} \supseteq \tilde{N}$ is the crossed product of M_k relative to the modular automorphism group (the dual action is described as above) and:

$$\mathcal{Z}(\tilde{M}_k) = L^\infty(X \times [0, \alpha_0)) \quad \text{with} \quad (\tilde{M}_k)_{\omega,t} = A_k \otimes M_0(\omega).$$

Thus, in our case the type II tower $\tilde{N} \subseteq \tilde{M} \subseteq \tilde{M}_1 \subseteq \cdots$ gives rise to the following field (over $X \times [0, \alpha_0)$):

$$(\tilde{M}_k)_{\omega,t} \cap (\tilde{N})'_{\omega,t} = (A_k \otimes M_0(\omega)) \cap (B \otimes M_0(\omega))' \;=\; A_k \cap B'$$
$$\tilde{M}_k \cap \tilde{N}' = (A_k \cap B') \otimes L^\infty(X \times [0, \alpha_0)).$$

In particular, the type II principal graph is D_{2m}.

By using the Jones projections $e_j (= \pi_\theta(e_j \otimes id_{M_0})$, where π_θ denotes the standard imbedding of $A_{j+1} \otimes M_0$ into the crossed product $\cdots$ recall (i)) $\in M_{j+1}$, we have

$$(1 - e_0 \vee e_1 \vee \cdots \vee e_{2m-4})((A_{2m-3} \otimes M_0) \cap (B \otimes M_0)') = L^\infty(X \times \{0, 1\}),$$

$$(1 - e_0 \vee e_1 \vee \cdots \vee e_{2m-4})(\tilde{M}_{2m-3} \cap \tilde{N}') = L^\infty(X \times [0, \alpha_0) \times \{0, 1\}).$$

In fact, $A_{2m-3} \cap B'$ is the direct sum of several matrix algebras and two copies of $\mathbf{C}$ (corresponding to the last two vertices of D_{2m}), and the projection $1 - e_0 \vee e_1 \vee \cdots \vee e_{2m-4}$ kills all the matrix algebras. Recall the description of θ_t in terms of the "base automorphism" θ (containing $\varphi_{T^{-1}\omega,1}$) before. Whenever the extension $\tilde{T}$ switches the two sheets (i.e., $\varphi_{T^{-1}\omega,1} = 1 \neq 0$), $\pi^{\varphi_{T^{-1}\omega,1}}$ switches the last two vertices of D_{2m} and hence θ (considered on $(1 - e_0 \vee e_1 \vee \cdots \vee e_{2m-4})((A_{2m-3} \otimes M_0) \cap (B \otimes M_0)') = L^\infty(X \times \{0, 1\})$) switches the two sheets. This means that, on the above abelian algebra $L^\infty(X \times [0, \alpha_0) \times \{0, 1\})$, the dual action $\{\theta_t\}$ looks like:

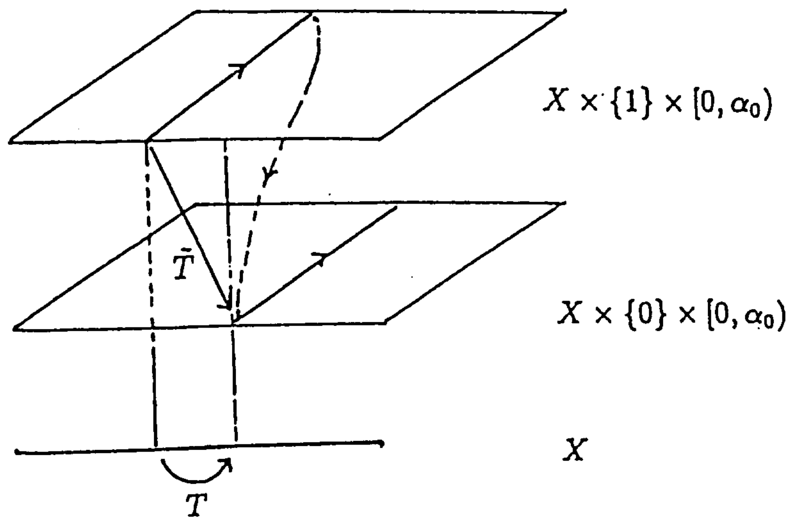

Figure 2.

Theorem 1. *The inclusion* $M \supseteq N$ *of AFD type* III_0 *factors constructed in this section satisfies:* (i) *M and N have the same flow of weights* (*the flow built under the constant ceiling function* α_0 *together with the base transformation* (X, T)), (ii) *the principal graph of* $M \supseteq N$ *is* A_{4m-3}, (iii) *the type* II *principal graph of* $M \supseteq N$ *is* D_{2m}, *and* (iv) *a given two-to-one ergodic extension* $(\tilde{X}, \tilde{T})$ *can be recovered from* $((1 - e_0 \vee e_1 \vee \cdots \vee e_{2m-4})(\tilde{M}_{2m-3} \cap \tilde{N}'), \{\theta_t\}_{t \in \mathbf{R}})$.

Proof. Figure 2 represents the flow built under the constant ceiling function α_0 with the base transformation $(\tilde{T}, \tilde{X})$ so we have (iv). It remains to show (ii). To this end, it suffices to show that $(1-e_0 \vee e_1 \cdots \vee e_{2m-4})(M_{2m-3} \cap N')$ is one dimensional. However, it is included in $\{(1-e_0 \vee e_1 \cdots \vee e_{2m-4})(\tilde{M}_{2m-3} \cap \tilde{N}')\}_\theta$ because of $\theta_t(e_j) = e_j$ (see the proof of Corollary 7, [**30**]) and (1). This space of fixed points is one dimensional because of the ergodicity of $(\tilde{T}, \tilde{X})$ (recall (iv), i.e., Figure 2). □

The last statement (iv) means that the given two-to-one ergodic extension can be captured by inclusion data of $M \supseteq N$. As in [**28**], by using an ergodic transformation with uncountably many two-to-one ergodic extensions, we conclude:

Corollary 2. *There exists an AFD type* III_0 *factor M with uncountably many non-conjugate subfactors N such that* (i) *M and N have the same flow of weights, and* (ii) *the principal graph of* $M \supseteq N$ *is* A_{4m-3}.

4. Extended modular automorphisms appearing in sectors.

In this section we will show that the two-to-one ergodic extension (that appeared in the type II tower) in §3 can be also captured by the sector tecnique (2.4 and 2.5).

We begin by expressing the inclusion $M \supseteq N$ (in §3) in a slightly different way. The original construction used the triple $(A \supseteq B, \pi)$. One can replace this by

$$A \otimes M_2(\mathbf{C}) \supseteq B \otimes M_2(\mathbf{C}), \pi \otimes \mathrm{Ad}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In fact, the action is strongly outer (seen easily from 2.4.(ii), actually Corollary 8, [**29**]) and the tensoring does not change the Loi invariant (2.4.(i)). Therefore,

$$M = (A \otimes M_2(\mathbf{C}) \otimes M_0) \rtimes_\theta \mathbf{Z} \ \supseteq N = (B \otimes M_2(\mathbf{C}) \otimes M_0) \rtimes_\theta \mathbf{Z},$$

where M_0 is as in §3 and θ is defined as before by using

$$\left(\pi \otimes \mathrm{Ad}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)^{\varphi_{T^{-1}\omega,1}} \in \mathrm{Aut}(A \otimes M_2(\mathbf{C})).$$

Let $D(\subseteq M_2(\mathbf{C}))$ be the abelian algebra of diagonal matrices. Since θ leaves $A \otimes D \otimes M_0$ (and $B \otimes D \otimes M_0$) invariant, we can set

$$P = (A \otimes D \otimes M_0) \rtimes_\theta \mathbf{Z} \supseteq Q = (B \otimes D \otimes M_0) \rtimes_\theta \mathbf{Z}.$$

Notice that

$$\begin{aligned} \mathcal{Z}(A \otimes D \otimes M_0) &= \mathcal{Z}(B \otimes D \otimes M_0) \\ &= \mathcal{Z}(D) \otimes \mathcal{Z}(M_0) \\ &= L^\infty(X \times \mathbf{Z}_2). \end{aligned}$$

Since Ad $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ exchanges the two sheets, P and Q have the same flow of weights, and this is given by the constant ceiling function α_0 together with the base transformation $(\tilde{X}, \tilde{T})$. The inclusion $\tilde{P} \supseteq \tilde{Q}$ (defined as before) of II_∞ von Neumann algebras gives rise to the constant field (D_{2m}) over $\tilde{X} \times [0, \alpha_0)$ of principal graphs, and the dual action $\{\theta_t\}$ is described by the "base automorphism" θ. Therefore, we easily conclude:

Lemma 3. *The principal graph of $P \supseteq Q$ is D_{2m}.*

Notice that we have:

$$\begin{array}{ccc} M & \supseteq & N \\ \cup & & \cup \\ P & \supseteq & Q \end{array}.$$

Set

$$U = \mathrm{Id}_A \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \mathrm{Id}_{M_0}.$$

This is a self-adjoint unitary in $B \otimes M_2(\mathbf{C}) \otimes M_0$ ($\subseteq A \otimes M_2(\mathbf{C}) \otimes M_0$) and normalizes $A \otimes M_2(\mathbf{C}) \otimes M_0$ and $B \otimes M_2(\mathbf{C}) \otimes M_0$. We also observe $\theta(U) = U$. Therefore, $\pi_\theta(U) \in N$ ($\subseteq M$) normalizes P and Q. Hence, the automorphism

$$\alpha = \mathrm{Ad}\ \pi_\theta(U)$$

satisfies $\alpha \in \mathrm{Aut}(P, Q)$ and $\alpha^2 = 1$. Since Ad U acts non-trivially on $\mathcal{Z}(A \otimes D \otimes M_0) = \mathcal{Z}(B \otimes D \otimes M_0)$, $\alpha \in \mathrm{Aut}(P)$ and $\alpha|_Q \in \mathrm{Aut}(Q)$ are outer. The indices of $M \supseteq P$ and $N \supseteq Q$ being obviously 2, we conclude that

$$M = P \rtimes_\alpha \mathbf{Z}_2 \supseteq N = Q \rtimes_\alpha \mathbf{Z}_2.$$

Since $M \supseteq N$ are AFD type III_0 factors with the same flow of weights, they are isomorphic by the Krieger theorem. Hence, there exists a (unital normal *-) endomorphism ρ of M satisfying $N = \rho(M)$.

Lemma 4. *The dual automorphism $\hat{\alpha}$ (of period 2) $\in \mathrm{Aut}(M, N)$ appears in $\bigsqcup_k (\rho\bar{\rho})^k$.*

Proof. We assume that $\hat{\alpha}$ does not appear, i.e., $\hat{\alpha}$ is strongly outer for the inclusion $M \supseteq N$ (2.4.(ii)). We look at the tower of $M \rtimes_{\hat{\alpha}} \mathbf{Z}_2 \supseteq N \rtimes_{\hat{\alpha}} \mathbf{Z}_2$. Let $N \subseteq M \subseteq M_1 \subseteq M_2 \cdots$ be the Jones tower as usual. Then the k-th extension of $M \rtimes_{\hat{\alpha}} \mathbf{Z}_2 \supseteq N \rtimes_{\hat{\alpha}} \mathbf{Z}_2$ is $M_k \rtimes_{\hat{\alpha}} \mathbf{Z}_2$, and it is easy to see

$$(M_k \rtimes_{\hat{\alpha}} \mathbf{Z}_2) \cap (N \rtimes_{\hat{\alpha}} \mathbf{Z}_2)' = (M_k \cap N')_{\hat{\alpha}}$$

by the strong outerness (see [3]). The principal graph of $M \supseteq N$ is A_{4m-3} (Theorem 1, (ii)), and $\hat{\alpha}$ acts trivially on the tower $\{M_k \cap N'\}_k$ (which is generated by the Jones projections). Hence

$$(M_k \rtimes_{\hat{\alpha}} \mathbf{Z}_2) \cap (N \rtimes_{\hat{\alpha}} \mathbf{Z}_2)' = M_k \cap N',$$

and we conclude that the principal graph of $M \rtimes_{\hat{\alpha}} \mathbf{Z}_2 \supseteq N \rtimes_{\hat{\alpha}} \mathbf{Z}_2$ is A_{4m-3}. On the other hand, the Takesaki duality implies

$$M \rtimes_{\hat{\alpha}} \mathbf{Z}_2 \supseteq N \rtimes_{\hat{\alpha}} \mathbf{Z}_2 \cong P \supseteq Q.$$

Hence, Lemma 3 shows that the principal graph of $M \rtimes_{\hat{\alpha}} \mathbf{Z}_2 \supseteq N \rtimes_{\hat{\alpha}} \mathbf{Z}_2$ is D_{2m}, a contradiction. □

Since $M \supseteq \rho(M) = N$ has the principal graph A_{4m-3}, a unique (non-trivial) automorphism (i.e., a sector with statistical dimension 1) appears in the irreducible decomposition of $(\rho\bar{\rho})^{2m-2}$. (See 3.2 of [14] for details.) Since $\hat{\alpha}$ is an outer automorphism (i.e., $\hat{\alpha} \neq id$ as a sector), thanks to Lemma 4 this must be the above unique automorphism appearing in $(\rho\bar{\rho})^{2m-2}$.

We have already seen that the flow of weights is given by the constant ceiling function α_0 together with the base transformation $(\tilde{X}, \tilde{T})$.

Lemma 5. *The period 2 automorphism α has a non-trivial module, and*

$$(\mathrm{mod}\ \alpha)(\omega, i, t) = (\omega, i+1, t).$$

Proof. Let ψ be the dual weight of $\mathrm{tr}_A \otimes \mathrm{tr}_{M_2(\mathbf{C})} \otimes \mathrm{tr}_{M_0}$ on M. Then $\psi' = \psi|_P$ is the dual weight of $\mathrm{tr}_B \otimes \mathrm{tr}_{M_2(\mathbf{C})} \otimes \mathrm{tr}_{M_0}$, and we set $\tilde{P} = P \rtimes_{\sigma^{\psi'}} \mathbf{R}$ $(\subseteq \tilde{M} = M \rtimes_{\sigma^{\psi}} \mathbf{R})$. Since $\psi' \circ \alpha = \psi'$, the canonical extension $\tilde{\alpha} \in \mathrm{Aut}(\tilde{P})$ ([8, 9]) is charaterized by $\tilde{\alpha}(\pi_{\sigma=\sigma^{\psi}}(p)) = \pi_\sigma(\alpha(p)),\ p \in P$, and $\tilde{\alpha}(\lambda(t)) = \lambda(t)$. Notice that $\sigma_t^{\psi}(\pi_\theta(U)) = \pi_\theta(U)$. Thus, $\pi_\sigma(\pi_\theta(U))$ satisfies

$$\begin{aligned} \lambda(t)\pi_\sigma(\pi_\theta(U))\lambda(t)^* &= \pi_\sigma(\pi_\theta(U)), \\ \pi_\sigma(\pi_\theta(U)) &= \pi_\theta(U) \otimes Id_{L^2(\mathbf{R})}. \end{aligned}$$

The first equality guarantees that $\tilde{\alpha} = \mathrm{Ad}\ \pi_\sigma(\pi_\theta(U))$. The algebra $\tilde{M}$ is isomorphic to

$$\begin{aligned} &\{(A \otimes M_2(\mathbf{C}) \otimes M_0) \otimes \lambda(\mathbf{R})''\} \rtimes_{\tilde{\theta}} \mathbf{Z} \\ &\cong \{A \otimes M_2(\mathbf{C}) \otimes M_0 \otimes L^\infty(\mathbf{R})\} \rtimes_{\tilde{\tilde{\theta}}} \mathbf{Z} \end{aligned}$$

(via the Fourier transform), where $\tilde{\tilde{\theta}}$ is defined by

$$\left(\tilde{\tilde{\theta}}(x)\right)(\omega, t) = \left(\left(\pi \otimes \mathrm{Ad}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)^{\varphi_{T^{-1}\omega,1}} \otimes \theta_0(T^{-1}\omega)\right)(x(T^{-1}\omega, t - \alpha_0)),$$

$x(\omega, t) \in A \otimes M_2(\mathbf{C}) \otimes M_0(\omega)$ (see [48] for details). Under these isomorphisms, $\pi_\sigma(\pi_\theta(U))$ is mapped to

$$\pi_{\tilde{\tilde{\theta}}}\left(\mathrm{Id}_A \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \mathrm{Id}_{M_0} \otimes \mathrm{Id}_{L^2(\mathbf{R})}\right).$$

Since the automorphism induced by this unitary exchanges the two sheets of $\mathcal{Z}(A \otimes D \otimes M_0 \otimes L^\infty(\mathbf{R})) = L^\infty(X \times \mathbf{Z}_2 \times \mathbf{R})$, the result is now obvious. □

Recall that the flow of weights of P is a two-to-one extension of that of M, and hence it is given by a cocycle $(X \times [0, \alpha_0) \times \mathbf{R} \to \mathfrak{S}_2 \cong \mathbf{Z}_2)$. Lemma 5 and Lemma A.2 (also the paragraph after the lemma) in Appendix A imply:

Theorem 6. *A unique (non-trivial) automorphism appearing in the irreducible decomposition of $(\rho\bar{\rho})^{2m-2}$ is a period 2 extended modular automorphism of M. Futhermore, the cocycle $(\in \boldsymbol{Z}^1_\theta(\mathbf{R}, \mathcal{U}(\mathcal{Z}(\tilde{M}))))$ defining this automorphism is exactly the one arising from the two-to-one extention structure of the flow of weights of P.*

It is actually possible to prove this result directly from Theorem 1. However, our approach here seems to reveal more structure of the inclusion $M \supseteq N$ constructed in §3. When M_0 is a factor, our construction gives rise to a unique non-splitting inclusion of AFD type III_λ factors with $0 < \lambda = e^{-\alpha_0} < 1$ ([**32**, **46**]). Also the automorphism in the above theorem is a period 2 modular automorphism (see [**16**], especially Remark 3.8).

5. Inclusions constructed from group-subgroup pairs.

Group-subgroup pairs give us an abundance of inclusions of AFD II_1 factors. Furthermore, it is easy to construct automorphisms for these inclusions with non-trivial Loi invariant, and one can explicitly calculate the Loi invariant by looking at irreducible representations of the finite groups in question. Based on these inclusions of II_1-factors, we will construct many non-splitting inclusions of AFD type III_0 factors.

Let G_0 be a finite group with a subgroup H_0 such that $\{h \in H_0 : ghg^{-1} \in H_0 \text{ for each } g \in G_0\} = \{e\}$ (see Proposition 3.1, [**31**]). Let μ be an action of a finite group K on G_0 leaving H_0 globally invariant. We thus have the pair of the simultaneous semi-direct products

$$G = G_0 \rtimes_\mu K \supseteq H = H_0 \rtimes_\mu K.$$

Let R_0 be the AFD II_1-factor and $\alpha : G \to \mathrm{Aut}(R_0)$ be a (unique) outer action ([**18**]). We thus get the inclusion of AFD II_1-factors

$$R_0 \rtimes_\alpha G_0 \supseteq R_0 \rtimes_\alpha H_0.$$

These factors are included in $R_0 \rtimes_\alpha G$, and $\mathrm{Ad}\ \lambda_k$ $(k \in K)$ normalizes $R_0 \rtimes_\alpha G_0$ and $R_0 \rtimes_\alpha H_0$. Hence we can set

$$\beta_k = \mathrm{Ad}\ \lambda_k|_{R_0 \rtimes_\alpha G_0} \in \mathrm{Aut}(R_0 \rtimes_\alpha G_0, R_0 \rtimes_\alpha H_0),\ k \in K.$$

The action β (of K) obviously satisfies

$$\beta_k(\pi_\alpha(x)) = \pi_\alpha(\alpha_k(x)), x \in R_0, \tag{3}$$

$$\beta_k(\lambda_{g_0}) = \lambda_{kg_0k^{-1}} = \lambda_{\mu_k(g_0)},\ g_0 \in G_0. \tag{4}$$

We can construct inclusions of AFD type III_0 factors by making use of the triple $(R_0 \rtimes_\alpha G_0, R_0 \rtimes_\alpha H_0, \beta)$ and by modifying the construction presented in §3. However, it will be more convenient (for later purposes) to replace the above triple by the following (equivalent $\cdots$ by the duality) triple:

$$\left(A = R_0 \rtimes_\alpha G_0 \rtimes_\beta K \rtimes_{\hat\beta} K,\ B = R_0 \rtimes_\alpha H_0 \rtimes_\beta K \rtimes_{\hat\beta} K,\ \pi = \hat{\hat\beta}\right).$$

Here, $\hat\beta$ is the dual coaction of β and $\hat{\hat\beta}$ is the dual action of $\hat\beta$ (see [38] for details). In this section the reader might as well regard (A, B, π) as $(R_0 \rtimes_\alpha G_0, R_0 \rtimes_\alpha H_0, \beta)$.

As in §3, let $M_0 \rtimes_{\theta_0} \mathbf{Z}$ be a discrete decomposition of an AFD type III_0 factor whose flow of weights is given by the constant ceiling function α_0 ($\mathrm{tr}_{M_0} \circ \theta_0 = e^{-\alpha_0}\,\mathrm{tr}_{M_0}$) and the base transformation (and we will keep the same notation as in §3). Let $\varphi : (\omega, n) \in X \times \mathbf{Z} \to \varphi_{\omega,n} \in K$ be a cocycle (i.e., $\varphi_{\omega,n+m} = \varphi_{T^n\omega,m}\varphi_{\omega,n}$, the right side being the product in K) such that the following extension $(\tilde T, \tilde X)$ of (T, X) is ergodic (this is a normal extension in the sense of Zimmer, [59]):

$$\tilde X = X \times K \text{ (with the obvious product measure)},$$
$$\tilde T^n(\omega, k) = (T^n\omega, k\varphi_{\omega,n}^{-1}).$$

Define the automorphism $\theta \in \mathrm{Aut}(A \otimes M_0)$ by

$$\theta^n(x)(\omega) = (\pi_{\varphi_{T^{-n}\omega,n}} \otimes \theta_0(T^{-n}\omega))(x(T^{-n}\omega))$$

(for $x(\omega) \in A \otimes M_0(\omega)$). We have $\theta \in \mathrm{Aut}(A \otimes M_0, B \otimes M_0)$, and we can set

$$M = (A \otimes M_0) \rtimes_\theta \mathbf{Z} \ \supseteq N = (B \otimes M_0) \rtimes_\theta \mathbf{Z}\ .$$

As in §3, a normal conditional expectation from M onto N is constructed, and its index is $\#G_0/\#H_0$. We can also easily see that M and N have the same flow of weights and that it is given by the constant ceiling function α_0 and the base transformation (T, X). The type II principal and dual principal graphs are seen from

$$\tilde M_k \cap \tilde N' = (A_k \cap B') \otimes L^\infty(X \times [0, \alpha_0))$$
$$\tilde M_k \cap \tilde M' = (A_k \cap A') \otimes L^\infty(X \times [0, \alpha_0)),$$

and the graphs are described based on the Mackey machine ([1, 31], see also the third paragraph in Appendix B) applied to $G_0 \supseteq H_0$. The dual action $\{\theta_t\}$ can be computed in the same way as in §3 by using the “base automorphism" θ (defined by making use of the K-valued cocycle φ and β_k $(k \in K)$). Hence, the dual action on the above towers is completely described once we know the action of β_k on $\{A_k \cap B'\}, \{A_k \cap A'\}$ (i.e., the Loi invariant). The description of the Loi invariant will be obtained in Appendix B (Lemma B.1).

6. Crossed product representation.

We would like to compute the principal and the dual principal graphs for the inclusion $M \supseteq N$ constructed in the previous section. Since $M_k \cap N', M_k \cap M'$ can be computed as the fixed point algebras under the dual action (recall (1)), it suffices to know the Loi invariant of β (as was explained at the last part of §5). However, in this section, we will directly show that $M \supseteq N$ is of the form $P \rtimes_\gamma G \supseteq P \rtimes_\gamma H$ (so that the graphs can be computed by the algorithm in [**31**] applied to $G \supseteq H$).

Let us recall that

$$A = R_0 \rtimes_\alpha G_0 \rtimes_\beta K \rtimes_{\hat{\beta}} K \supseteq B = R_0 \rtimes_\alpha H_0 \rtimes_\beta K \rtimes_{\hat{\beta}} K, \pi = \hat{\hat{\beta}}.$$

In what follows, the left regular representation of K appearing in the definition of $R_0 \rtimes_\alpha G_0 \rtimes_\beta K$ will be denoted by λ'_k. The coaction is the homomorphism

$$\hat{\beta} : \sum_{k \in K} \pi_\beta(x_k)\lambda'_k \in R_0 \rtimes_\alpha G_0 \rtimes_\beta K \longrightarrow \sum_{k \in K} \pi_\beta(x_k)\lambda'_k \otimes \lambda'_k \in (R_0 \rtimes_\alpha G_0 \rtimes_\beta K) \otimes \lambda'(K)''$$

and

$$R_0 \rtimes_\alpha G_0 \rtimes_\beta K \rtimes_{\hat{\beta}} K = \left\langle \hat{\beta}(R_0 \rtimes_\alpha G_0 \rtimes_\beta K), 1 \otimes l^\infty(K) \right\rangle'',$$

where each $f \in l^\infty(K)$ is identified with the multiplication operator m_f on $l^2(K)$ (see [**38**] for details). The bidual action $\pi = \hat{\hat{\beta}}$ is

$$\pi_k\left(\hat{\beta}(x)\right) = \hat{\beta}(x), \quad x \in R_0 \rtimes_\alpha G_0 \rtimes_\beta K,$$
$$\pi_k(1 \otimes m_f) = 1 \otimes m_{f(\cdot k)}, \quad f \in l^\infty(K).$$

We now consider the two-step inclusions

$$R_0 \rtimes_\alpha G_0 \rtimes_\beta K \rtimes_{\hat{\beta}} K \supseteq R_0 \rtimes_\alpha H_0 \rtimes_\beta K \rtimes_{\hat{\beta}} K \supseteq R_0 \rtimes_{\hat{\beta}} K.$$

Notice that the smallest algebra is actually

(5)
$$\left\langle \hat{\beta}(\pi_\beta(\pi_\alpha(R_0))) = \pi_\beta(\pi_\alpha(R_0)) \otimes 1, 1 \otimes l^\infty(K) \right\rangle'' = \pi_\beta(\pi_\alpha(R_0)) \otimes l^\infty(K).$$

By using $\lambda_g \in R_0 \rtimes_\alpha G_0$ $(g \in G_0)$ and $\lambda'_k \in R_0 \rtimes_\alpha G_0 \rtimes_\beta K$ $(k \in K)$, we set

$$\Lambda_{gk} = \hat{\beta}(\pi_\beta(\lambda_g)\lambda'_k).$$

Since $\hat{\beta}$ and π_β are homomorphisms, (4) shows that

$$gk \in G = G_0 \rtimes_\mu K \mapsto \Lambda_{gk} \in R_0 \rtimes_\alpha G_0 \rtimes_\beta K \rtimes_{\hat{\beta}} K$$

is a unitary representation (and $\Lambda_{hk} \in R_0 \rtimes_\alpha H_0 \rtimes_\beta K \rtimes_{\hat{\beta}} K$ when $g = h$ and $hk \in H = H_0 \rtimes_\mu K$). We also note:

(i)
$$\begin{aligned}\Lambda_{gk}\hat{\beta}(\pi_\beta(\pi_\alpha(x)))\Lambda^*_{gk} &= \hat{\beta}(\pi_\beta(\lambda_g)\lambda'_k\pi_\beta(\pi_\alpha(x)){\lambda'_k}^*\pi_\beta(\lambda_g)^*)\\ &= \hat{\beta}(\pi_\beta(\lambda_g)\pi_\beta(\beta_k(\pi_\alpha(x)))\pi_\beta(\lambda_g)^*)\\ &= \hat{\beta}(\pi_\beta(\lambda_g)\pi_\beta(\pi_\alpha(\alpha_k(x)))\pi_\beta(\lambda_g)^*) \text{ (by (3))}\\ &= \hat{\beta}(\pi_\beta(\alpha_g(\pi_\alpha(\alpha_k(x)))))\\ &= \hat{\beta}(\pi_\beta(\pi_\alpha(\alpha_{gk}(x)))), \quad x \in R_0,\end{aligned}$$

(ii) $\Lambda_{gk} = \pi_\beta(\lambda_g)\lambda'_k \otimes \lambda'_k$ implies $\Lambda_{gk}(1 \otimes m_f)\Lambda^*_{gk} = 1 \otimes m_{f(k^{-1}\cdot)}$, $f \in l^\infty(K)$.

Therefore, the unitary Λ_{gk} normalizes the subalgebra $R_0 \rtimes_{\hat{\beta}} K$. We now set

$$\tilde{\Lambda}_{gk} = \pi_\theta(\Lambda_{gk} \otimes 1_{M_0}) \in ((R_0 \rtimes_\alpha G_0 \rtimes_\beta K \rtimes_{\hat{\beta}} K) \otimes M_0) \rtimes_\theta \mathbf{Z} \ = M.$$

Let ℓ denote the shift in M corresponding to the generator of $\mathbf{Z}$. We then have $\ell\tilde{\Lambda}_{gk}\ell^* = \tilde{\Lambda}_{gk}$ since θ was defined by using $\pi = \hat{\hat{\beta}}$ and $\hat{\hat{\beta}}$ acts trivially to $\Lambda_{gk} = \hat{\beta}(\pi_\beta(\lambda_g)\lambda'_k)$. Hence, we have:

(iii) $$\tilde{\Lambda}_{gk}\ell\tilde{\Lambda}^*_{gk} = \ell.$$

From (i), (ii), (iii) we conclude that the unitary $\tilde{\Lambda}_{gk}$ normalizes the subalgebra

$$P = ((R_0 \rtimes_{\hat{\beta}} K) \otimes M_0) \rtimes_\theta \mathbf{Z} \ (\subseteq N \subseteq M).$$

Lemma 7. *The subalgebra P is a factor of type* III_0 *whose flow of weights is given by the constant ceiling function α_0 and the base transformation $(\tilde{T}, \tilde{X})$.*

Proof. The center of $R_0 \rtimes_{\hat{\beta}} K$ is $l^\infty(K)$ (by (5)) and hence $\mathcal{Z}((R_0 \rtimes_{\hat{\beta}} K) \otimes M_0)$ is $l^\infty(K) \otimes \mathcal{Z}(M_0) = L^\infty(\tilde{X})$. For $m_f \otimes F \in l^\infty(K) \otimes L^\infty(X)$, we compute

$$\begin{aligned}(\theta(m_f \otimes F))(\omega) &= (\pi_{\varphi_{T^{-1}\omega,1}} \otimes \theta_0(T^{-1}\omega))((m_f \otimes F)(T^{-1}\omega))\\ &= (\pi_{\varphi_{T^{-1}\omega,1}})(F(T^{-1}\omega)m_f)\\ &= F(T^{-1}\omega)m_{f(\cdot\varphi_{T^{-1}\omega,1})},\end{aligned}$$

and the base transformation is $(\tilde{T}, \tilde{X})$. □

Theorem 8. *The map* $\gamma : gk \in G = G_0 \rtimes_\mu K \mapsto \gamma_{gk} = \mathrm{Ad}\ \tilde{\Lambda}_{gk}|_P \in \mathrm{Aut}(P)$ *is an outer action, and we have*

$$M = P \rtimes_\gamma G \supseteq N = P \rtimes_\gamma H.$$

Proof. To show the outerness of $\mathrm{Ad}\ \tilde{\Lambda}_{gk}|_P$ $(gk \neq e)$ by Kallman's criterion ([20]), let us assume that $x = \sum_n x_n \ell^n \in P = ((R_0 \rtimes_{\hat{\beta}} K) \otimes M_0) \times_\theta \mathbf{Z}$ satisfies $yx = x\tilde{\Lambda}_{gk} y \tilde{\Lambda}_{gk}^*$ for each $y \in P$. When $y \in M_0$, we have $yx = xy$ from the definition of $\tilde{\Lambda}_{gk}$ and $yx_n = x_n\theta^n(y)$ for each n. Thus, $x_n = 0$ $(n \neq 0)$ by the central freeness of θ^n and $x = x_0$ belongs to $(R_0 \rtimes_{\hat{\beta}} K) \otimes \mathcal{Z}(M_0)$. Hence, x is considered as an R_0-valued function on $K \times X$ (recall (5)). When $y \in R_0$ $(\subseteq R_0 \rtimes_{\hat{\beta}} K)$, (i) implies $yx(k', \omega) = x(k', \omega)\alpha_{gk}(y)$ for each $k' \in K$ and a.e. $\omega \in X$. Since α is an outer action of G, we conclude $x(k', \omega) = 0$ (for each k' and a.e. ω) and $x = 0$ as desired.

The index between M and P is obviously $\#G_0 \times \#K = \#G$, and hence we conclude $M = P \rtimes_\gamma G$. Repeating the same argument for $H = H_0 \rtimes_\mu K$, we also get $N = P \rtimes_\gamma H$. □

We have already known how $\gamma_{gk} = \mathrm{Ad}\ \tilde{\Lambda}_{gk}|_P$ acts on $\mathcal{Z}(R_0 \rtimes_{\hat{\beta}} K) = l^\infty(K)$ ((5) and (ii) before Lemma 7). When $g \neq e$ and $k = e$, γ_g on $R_0 \rtimes_{\hat{\beta}} K \cong R_0 \otimes l^\infty(K)$ ((5)) looks like $\alpha_g \otimes \mathrm{Id}$ ((i) before Lemma 7) and α_g is of course outer. Hence, an analogous argument to the proof of Lemma 5 implies the following:

Proposition 9. *The invariants* (*in the sense of* [24, 55]) *of the* G*-action* γ *on* P *is given by* (i) $(\mathrm{mod}\ \gamma_{gk})(\omega, h, t) = (\omega, kh, t)$ *for* $(\omega, h, t) \in \tilde{X} \times [0, \alpha_0) = X \times K \times [0, \alpha_0)$ (*see Lemma 7*), (ii) $\mathrm{Ker}\ (\mathrm{mod}\ \gamma) = G_0$, *and* (iii) $N(\gamma) = 1$.

Notice that the flow (the dual action) and mod γ_{gk} commute since they correspond to the right and left multiplications in the group K.

Remark. The invariant $N(\cdot)$ being trivial, all of M, N and P admit a common Cartan subalgebra ([51]). Hence, in particular, the inclusion $M \supseteq N$ can be described by making use of an ergodic (discrete measured) relation-subrelation pair (see for example [54]). Such inclusions were studied in [6, 10, 11, 53].

7. The symmetric and alternating groups.

In this section, we will restrict ourselves to the special case ($n \geq 3$):

$$\begin{array}{ccc} G = \mathfrak{S}_n & \supset & H = \mathfrak{S}_{n-1} \\ \cup & & \cup \\ G_0 = \mathfrak{A}_n & \supset & H_0 = \mathfrak{A}_{n-1} \end{array}.$$

Here, $\mathfrak{S}_{n-1}$ (and similarly for the alternating groups) is considered as a subgroup in $\mathfrak{S}_n$ consisting of all permutations fixing n. Notice $\mathfrak{S}_n = \mathfrak{A}_n \rtimes_\mu \mathbf{Z}_2$ and $\mathfrak{S}_{n-1} = \mathfrak{A}_{n-1} \rtimes_\mu \mathbf{Z}_2$ with $\mathbf{Z}_2 = \{e, (1,2)\}$ and $\mu_{(1,2)}(g) = (1,2)g(1,2)$. Everything (in §5,6) will be explicitly calculated by using Young diagrams.

Let $M \supseteq N$ be the factor-subfactor pair constructed before. (Thanks to Theorem 8, the principal and the dual principal graphs can be computed from $\mathfrak{S}_n/\mathfrak{S}_{n-1}$, see Example 3.3, [**31**].) The type II graphs are given from $\mathfrak{A}_n/\mathfrak{A}_{n-1}$. Generally, the dual action on them can be computed from the cocycle determining the extension structure and the Loi invariant of the K-action β (see the last part of §5). In our case, since $K = \mathbf{Z}_2$ and $(\tilde{T}, \tilde{X})$ is a two-to-one ergodic extension, the description becomes particularly easy. (When $\tilde{T}$ switces the two sheets, i.e., the cocycle takes the value $1 \neq 0$, on $(A_k \cap B') \otimes L^\infty(X)$ the move determined by the Loi invariant appears.) Hence, what we have to clarify is the induction-restriction procedure for $\mathfrak{A}_n/\mathfrak{A}_{n-1}$ and the Loi invariant of the $\mathbf{Z}_2$-action on the graphs (Appendix B).

It is well-known that the irreducible representations $\hat{\mathfrak{S}}_n$ are parameterized by Young diagrams of weight n. The following facts are standard ([**17**]):
(i) The irreducible representation corresponding to a non-selfconjugate Young diagram and its conjugate provide us the same (up to unitary equivalence) irreducible representation of $\mathfrak{A}_n$ (when restricted to the subgroup $\mathfrak{A}_n$). Furthermore, for such an irreducible representation π of $\mathfrak{A}_n$, we have $\pi((1,2) \cdot (1,2)) \cong \pi$.
(ii) The representation corresponding to a selfconjugate Young diagram splits into two mutually inequivalent irreducible representations (of the same dimensions) when restricted to the subgroup $\mathfrak{A}_n$. Furthermore, $\mu_{(1,2)} =$ Ad $(1,2)$ on $\mathfrak{A}_n$ exchanges these two representations.
(iii) By looking at a half of non-selfconjugate Young diagrams and selfconjugate Young diagrams (each of them provides us two representations as in (ii)), one obtains a complete set of the irreducible representations of $\mathfrak{A}_n$. The induction-restriction procedure for $\mathfrak{A}_n/\mathfrak{A}_{n-1}$ is naturally inherited from that for $\mathfrak{S}_n/\mathfrak{S}_{n-1}$.

When $n = 3$, $\mathfrak{A}_3 = \mathbf{Z}_3$ and $\mathfrak{A}_2 = \{e\}$. Therefore, we will assume $n \geq 4$ in what follows. The homogeneous space $\mathfrak{A}_n/\mathfrak{A}_{n-1}$ is identified with

$\{1, 2, \cdots, n\}$:

$$\mathfrak{A}_n/\mathfrak{A}_{n-1} = \bigsqcup_{i=1}^{n} \{\sigma\mathfrak{A}_{n-1} : \sigma \in \mathfrak{A}_n \ \text{ with } \ \sigma(n) = i\}.$$

As the left action of $\mathfrak{A}_{n-1}$ moves the first $n-1$ points (and fixes the last point), we have the two orbits $\mathcal{O}_0, \mathcal{O}_1 : g_0 = e, g_1 = (1,2)(n-1,n) \cdots$ for the notations here and below, see Appendix B. Hence, $H_0 = \mathfrak{A}_{n-1}$ and $H_1 = \mathfrak{A}_{n-2}(= \{\sigma \in \mathfrak{A}_n : \sigma(n) = n, \ \sigma(n-1) = n-1\})$. Notice that $k = (1,2) \in K = \mathbf{Z}_2$ satisfies $kg_1k^{-1} = g_1$. Thus, $n(i,k) = i$ (i.e., k does not shuffle the orbits), and $h(i,k) = e$. Consequently, the Loi invariant of β_k for the irreducible $B - B$ bimodules $(\hat{\mathfrak{A}}_{n-1} \bigsqcup \hat{\mathfrak{A}}_{n-2})$ is just

$$\pi \mapsto \pi(k^{-1} \cdot k) \quad \left(\pi \in \hat{\mathfrak{A}}_{n-1} \bigsqcup \hat{\mathfrak{A}}_{n-2}\right).$$

When $n = 5$, by collecting a "half" of Young diagrams and splitting each self-conjugate diagram into two pieces ((ii), (iii)), we obtain the following graphs of $A = R_0 \rtimes_\alpha \mathfrak{A}_5 \supseteq B = R_0 \rtimes_\alpha \mathfrak{A}_4$ (Appendix B and p. 473, [**31**]):
The principal graph

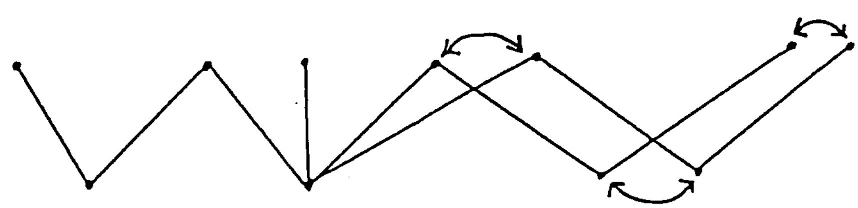

Figure 3.

The dual principal graph

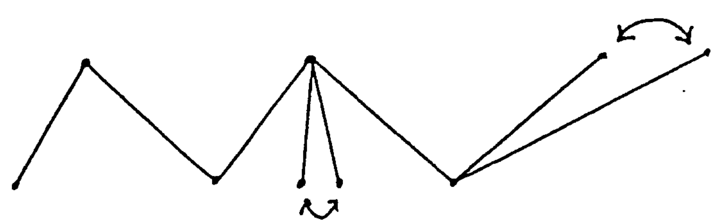

Figure 4.

Here, arrows indicate the Loi invariant, and they appear in such a way that each arrow connects the two irreducible representations described in (ii). Computations for other n's are left to the reader. When $n = 4$, the two graphs coincide and we get:

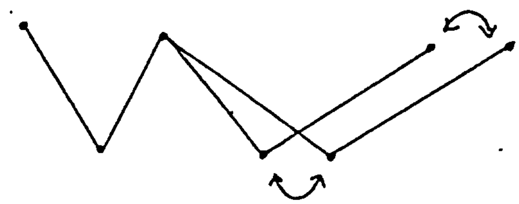

Figure 5.

Since the Loi invariant is always non-trivial (due to the presence of selfconjugate Young diagrams), the original two-to-one extension is obtained from the inclusion data of $M \supseteq N$. Hence, once again by using an ergodic transformation with uncountably many two-to-one ergodic extensions we have the following generalization of Corollary 7, [**28**]:

Theorem 10. *There exists an AFD type* III_0 *factor* M *with uncountably many non-conjugate subfactors* N *such that* (i) M *and* N *have the same flow of weights,* (ii) *the principal and the dual principal graphs for* $M \supseteq N$ *are the ones determined by* $\mathfrak{S}_n/\mathfrak{S}_{n-1}$, *and* (iii) *the type* II *principal and the dual principal graphs for* $M \supseteq N$ *are the ones determined by* $\mathfrak{A}_n/\mathfrak{A}_{n-1}$.

As was shown in [**31**], the irreducible sectors in $\bigsqcup_k (\rho\bar{\rho})^k$ (i.e., $M-M$ bimodules) are parameterized by $\hat{G}$ when $M \supseteq N(=\rho(M))$ arises from $G \supseteq H$ (see also 2.5 and Appendix B), and the statistical dimension is actually the degree of the corresponding irreducible representation (p. 669, [**31**]).

Remark. Figure 5 (the Coxeter-Dynkin diagram $E_6^{(1)}$) appears in [**2**] (among many others). Its dual graph in the sense of [**2**] is the one for $\mathfrak{S}_4/\mathfrak{S}_3$ (i.e., the Coxeter-Dynkin diagram $E_7^{(1)}$) and possesses the obvious symmetry. In terms of bimodules (2.5) arising from the pair $C = R_0 \rtimes_\alpha \mathfrak{S}_4 \supseteq D = R_0 \rtimes_\alpha \mathfrak{S}_3$, the last vertex in $E_7^{(1)}$ comes from the signature representation $\varepsilon \in \hat{\mathfrak{S}}_4$ and represents a non-trivial one dimensional C-C bimodule $\mathcal{H} = \mathcal{H}_{\alpha_\varepsilon}$ (i.e., automorphism α_ε). The above mentioned symmetry expresses the effect of taking the relative tensor product with $\mathcal{H}$ (or equivalently, taking the composition with the automorphism α_ε in the sector picture). By Proposition 4, Corollary 6 in [**29**] and the trick in Lemma 3.3, [**15**], after an inner perturbation (see the last part of 2.5) we may and do assume $\alpha_\varepsilon \in \mathrm{Aut}(C, D)$

and $\alpha_\varepsilon^2 = Id$. The construction in p. 663, [**31**] actually shows that we can choose

$$\alpha_\varepsilon : \sum_{g\in\mathfrak{S}_4} p_g\lambda_g \in C \mapsto \sum_{g\in\mathfrak{S}_4} sgn(g)\ p_g\lambda_g \in C,$$

and the fixed point algebras C_{α_ε}, D_{α_ε} are A, B respectively. In the orbifold construction ([**5**]), a $\mathbf{Z}_3$-symmetry typically appears instead.

Finally we point out a phenomenon similar to Theorem 6 for $M \supseteq N$ in the above theorem. Let α_ε be the automorphism determined by the one dimensional M-M bimodule corresponding to the signature representation $\varepsilon(\in \hat{\mathfrak{S}}_n)$. Once again α_ε is described as above (but with the crossed product in Theorem 8) and a direct calculation shows

$$M_{\alpha_\varepsilon} = ((R_0 \rtimes_\alpha \mathfrak{A}_n \rtimes_{\hat{\beta}} \mathbf{Z}_2) \otimes M_0) \rtimes_\theta \mathbf{Z}.$$

Since $\mathcal{Z}(R_0 \rtimes_\alpha \mathfrak{A}_n \rtimes_{\hat{\beta}} \mathbf{Z}_2) \cong l^\infty(\mathbf{Z}_2)$, the flow of weights of M_{α_ε} (or equivalently, that of $M \rtimes_{\alpha_\varepsilon} \mathbf{Z}_2$) is given by the constant ceiling function α_0 and the base transformation $(\tilde{T}, \tilde{X})$ by the identical arguments as in the proof of Lemma 7. The flow of weights of $M \rtimes_{\alpha_\varepsilon} \mathbf{Z}_2$ being a two-to-one extension of that of M, α_ε must be an extended modular automorphism (see [**25**, **50**]).

Appendix A. Duality between the module and modular invariant.

The module and the modular invariant (among others) appear as invariants for the classification of actions on AFD type III factors ([**24**, **55**]). They are believed to be the "dual invariant" to the each other. In this appendix, we will show that this is indeed the case for a $\mathbf{Z}_2$-action. Computations here are implicit in [**25**, **50**], and the authors feel that duality results in more general setting deserve investigation.

Let $M = N \rtimes_\theta \mathbf{R}$ be the continuous decomposition of a type III factor M $(\tau \circ \theta_t = e^{-t}\tau,\ t \in \mathbf{R})$. For a given cocycle $\mathcal{C} = \{c_t\}_{t\in\mathbf{R}} \in \boldsymbol{Z}^1_\theta(\mathbf{R}, \mathcal{U}(\mathcal{Z}(N)))$, the extended modular automorphism $\sigma_{\mathcal{C}} = \sigma^\psi_{\mathcal{C}}$ ($\psi = \hat{\tau}$, the dual weight) $\in \mathrm{Aut}(M)$ is defined by

$$\sigma_{\mathcal{C}}(\pi_\theta(n)) = \pi_\theta(n), \quad n \in N,$$
$$\sigma_{\mathcal{C}}(\lambda(t)) = \pi_\theta(c_t)\lambda(t), \quad t \in \mathbf{R}$$

(see [**4**] for details). Let $\tilde{M} = M \rtimes_\sigma \mathbf{R}$ be the crossed product relative to the modular automorphism group $\{\sigma_s = \sigma^\psi_s\}_{s\in\mathbf{R}}$ so that $\tilde{M}$ is generated by the following three kinds of operators:

$$\pi_\sigma(\pi_\theta(n)),\ \pi_\sigma(\lambda(t)),\ \lambda'(s); \quad n \in N,\ t \in \mathbf{R},\ s \in \mathbf{R}.$$

The dual action of $\{\theta_t\}_{t\in\mathbf{R}}$ is $\{\sigma_s\}_{s\in\mathbf{R}}$ (because of $\psi = \hat{\tau}$) so that $\tilde{M}$ is isomorphic to $N \otimes B(L^2(\mathbf{R}))$ thanks to the Takesaki duality. Under this isomorphism the above three generators become

$$\begin{aligned}(\pi_\sigma(\pi_\theta(n))\xi)(r) &= \theta_{-r}(n)\xi(r),\\ (\pi_\sigma(\lambda(t))\xi)(r) &= \xi(r-t),\\ (\lambda'(s)\xi)(r) &= e^{-isr}\xi(r).\end{aligned}$$

$(\xi \in L^2(N) \otimes L^2(\mathbf{R}) \cong L^2(\mathbf{R}; L^2(N)),\ r \in \mathbf{R}.)$ We set

$$(V_{\mathcal{C}}\xi)(r) = c^*_{-r}\xi(r).$$

Obviously $V_{\mathcal{C}}$ is a unitary in $N \otimes B(L^2(\mathbf{R}))(= (N' \otimes \mathbf{C}1)')$.

Lemma A.1. *The canonical extension* $\tilde{\sigma}_{\mathrm{e}} \in \mathrm{Aut}(\tilde{M} \cong N \otimes B(L^2(\mathbf{R})))$ *in the sense of* [8, 9] *is* $\mathrm{Ad}\ V_{\mathcal{C}}$.

Proof. Because of $\psi \circ \sigma_{\mathrm{e}} = \psi$, the canonical extension $\tilde{\sigma}_{\mathrm{e}}$ is characterized by the properties $\tilde{\sigma}_{\mathrm{e}}(\pi_\sigma(\pi_\theta(n))) = \pi_\sigma(\pi_\theta(n))$, $\tilde{\sigma}_{\mathrm{e}}(\pi_\sigma(\lambda(t))) = \pi_\sigma(\pi_\theta(c_t))\pi_\sigma(\lambda(t))$, and $\tilde{\sigma}_{\mathrm{e}}(\lambda'(s)) = \lambda'(s)$. The first and third equalities follow from the obvious commutativity of $V_{\mathcal{C}}$ with $\pi_\sigma(\pi_\theta(n))$ and $\lambda'(s)$. On the other hand, since

$$\begin{aligned}(V_{\mathcal{C}}\pi_\sigma(\lambda(t))V_{\mathcal{C}}{}^*\xi)(r) &= c^*_{-r}(V^*_{\mathcal{C}}\xi)(r-t)\\ &= c^*_{-r}c_{-r+t}\xi(r-t)\\ &= \theta_{-r}(c_t)\xi(r-t)\\ &= (\pi_\sigma(\pi_\theta(c_t))\pi_\sigma(\lambda(t))\xi)(r),\end{aligned}$$

we have the second equation. □

In what follows, we will assume that σ_{e} has period 2 so that $c_t^2 = 1$ and $V = V_{\mathcal{C}}$ is a self-adjoint unitary. We have

$$\tilde{M} \rtimes_{\tilde{\sigma}_{\mathrm{e}}} \mathbf{Z}_2 = \left\{ \begin{pmatrix} X & Y \\ VYV^* & VXV^* \end{pmatrix} : X, Y \in N \otimes B(L^2(\mathbf{R})) \right\}.$$

Notice $\tilde{M} \rtimes_{\tilde{\sigma}_{\mathrm{e}}} \mathbf{Z}_2 = (M \rtimes_{\sigma_{\mathrm{e}}} \mathbf{Z}_2)\tilde{}$, the crossed product of $M \rtimes_{\sigma_{\mathrm{e}}} \mathbf{Z}_2$ relative to the modular automorphism group $\{\sigma_t^\chi\}$. Here, χ is the weight on $M \rtimes_{\sigma_{\mathrm{e}}} \mathbf{Z}_2$ naturally attached to $\psi = \hat{\tau}$ on M. The dual automorphism $\hat{\tilde{\sigma}}_{\mathrm{e}}$ (of period 2) is easily seen to be $\tilde{\hat{\sigma}}_{\mathrm{e}}$, and it is given by

$$\mathrm{Ad}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The dual action $\{\theta'_t\}$ of $\{\sigma^\chi_t\}$ is known to be $\hat{\sigma}_t \otimes \mathrm{Id}_{M_2(\mathbf{C})}$. It is straightforward to check

$$\mathcal{Z}(\tilde{M} \rtimes_{\tilde{\sigma}_e} \mathbf{Z}_2) = \left\{ \begin{pmatrix} X & VY \\ YV^* & X \end{pmatrix} : X, Y \in \mathcal{Z}(N \otimes B(L^2(\mathbf{R}))) \right\},$$

which is isomorphic to $\mathcal{Z}(N) \oplus \mathcal{Z}(N)$ via

$$\begin{pmatrix} X & VY \\ YV^* & X \end{pmatrix} \leftrightarrow 2^{-1}(X+Y) \oplus 2^{-1}(X-Y).$$

Notice

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} X & VY \\ YV^* & X \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} X & -VY \\ -YV^* & X \end{pmatrix},$$

$$\begin{aligned} (\hat{\sigma}_t \otimes \mathrm{Id}_{M_2(\mathbf{C})}) \begin{pmatrix} X & VY \\ YV^* & X \end{pmatrix} &= \begin{pmatrix} \hat{\sigma}_t(X) & \hat{\sigma}_t(VY) \\ \hat{\sigma}_t(YV^*) & \hat{\sigma}_t(X) \end{pmatrix} \\ &= \begin{pmatrix} \hat{\sigma}_t(X) & VV^*\hat{\sigma}_t(VY) \\ \hat{\sigma}_t(YV^*)VV^* & \hat{\sigma}_t(X) \end{pmatrix}. \end{aligned}$$

They correspond to the following elements (in $\mathcal{Z}(N) \oplus \mathcal{Z}(N)$) respectively:

$$2^{-1}(X-Y) \oplus 2^{-1}(X+Y),$$
$$2^{-1}(\hat{\sigma}_t(X) + V^*\hat{\sigma}_t(VY)) \oplus 2^{-1}(\hat{\sigma}_t(X) - V^*\hat{\sigma}_t(VY)).$$

Therefore, on $\mathcal{Z}(N) \oplus \mathcal{Z}(N)$ we have

$$\begin{aligned} \hat{\tilde{\sigma}}_e(X \oplus Y) &= Y \oplus X \\ \theta'_t(X \oplus Y) &= 2^{-1}\{\hat{\sigma}_t(X+Y) + V^*\hat{\sigma}_t(V)\hat{\sigma}_t(X-Y)\} \\ &\quad \oplus 2^{-1}\{\hat{\sigma}_t(X+Y) - V^*\hat{\sigma}_t(V)\hat{\sigma}_t(X-Y)\}. \end{aligned}$$

We now assume that $(\mathcal{Z}(N), \theta_t)$, the flow of weights of M, is given by an ergodic flow (X, F_t):

$$\mathcal{Z}(N) \cong L^\infty(X) \quad \text{and} \quad (\theta_t(f))(\omega) = f(F_t\omega).$$

The calculations so far show $\mathcal{Z}(\tilde{M} \rtimes_{\tilde{\sigma}_e} \mathbf{Z}_2 \cong (M \rtimes_{\sigma_e} \mathbf{Z}_2)\tilde{)} \cong L^\infty(X \times \{0,1\})$ and $\tilde{\tilde{\sigma}}_e = \hat{\tilde{\sigma}}_e|_{\mathcal{Z}(\tilde{M} \rtimes_{\tilde{\sigma}_e} \mathbf{Z}_2)}$ (the module of $\hat{\sigma}_e$) is induced by the following map:

$$(x, i) \in X \times \{0,1\} \to (x, i+1)$$

(i.e., exchanging the two sheets). In the present set-up, each c_t is a function on X with the values ± 1 and satisfies the cocycle equation

$$c_{t+s}(\omega) = c_t(\omega)(\theta_t(c_s))(\omega) = c_t(\omega)c_s(F_t(\omega)).$$

We set

$$\varphi_{\omega,t} = c_t(\omega) \in \{\pm 1\}.$$

Therefore, the above equation means

$$\varphi_{\omega,t+s} = \varphi_{F_t(\omega),s}\varphi_{\omega,t},$$

that is,

$$\varphi : (\omega, t) \in X \times \mathbf{R} \to \varphi_{\omega,t} \in \{\pm 1\}$$

is a $\{\pm 1\}$-valued F_t-cocycle on X. Therefore, by identifying the multiplicative group $\{\pm 1\}$ with the symmetric group $\mathfrak{S}_2$ in the obvious way, the following $\{\tilde{F}_t\}_{t\in\mathbf{R}}$ defines a flow on $X \times \{0, 1\}$:

$$\tilde{F}_t(\omega, i) = (F_t(\omega), \varphi_{\omega,t}(i)).$$

Under the Takesaki duality, $\hat{\sigma}_t$ (the bidual of θ_t) corresponds to $\theta_t \otimes \mathrm{Ad}(\lambda(t)^*)$. In paticular, for $X \in \mathcal{Z}(N)(\subseteq N \otimes B(L^2(R)))$, we have

$$\hat{\sigma}_t(X) = \theta_t(X).$$

We also compute

$$\begin{aligned}(V^*\hat{\sigma}_t(V)\xi)(r) &= c_{-r}\theta_t(c^*_{-t-r})\xi(r)\\ &= c_t\xi(r).\end{aligned}$$

Thus, $V^*\hat{\sigma}_t(V)(\in \mathcal{Z}(N) \cong L^\infty(X))$ corresponds to the function

$$\omega \in X \to c_t(\omega) = \varphi_{\omega,t} \in \{\pm 1\}.$$

Now it is clear that the dual action θ'_t on $\mathcal{Z}(\tilde{M} \rtimes_{\tilde{\sigma_e}} \mathbf{Z}_2) \cong L^\infty(X \times \{0, 1\})$ is induced by $\{\tilde{F}_t\}_{t\in\mathbf{R}}$. Hence we have shown:

Lemma A.2. *Let (X, F_t) be a flow of weights of M. Then the flow of weights of $M \rtimes_{\sigma_e} \mathbf{Z}_2$ is a two-to-one extension of (X, F_t). The $\mathfrak{S}_2$-valued F_t-cocycle φ on X defining the extension is given by*

$$\varphi_{\omega,t} = c_t(\omega) \in \{\pm 1\} \cong \mathfrak{S}_2.$$

Furthermore, the module of the dual automorphism $\hat{\sigma}_e$ (of period 2) is

$$(\mathrm{mod}\ \hat{\sigma}_e)(\omega, i) = (\omega, i + 1).$$

Notice that the bidual $\hat{\hat{\sigma}}_e$ has the same invariants (in the sense of [**24**, **55**]) as σ_e. Therefore, when one starts from a period 2 automorphism β on M with a non-trivial module, the dual automorphism $\hat{\beta}$ turns out to an extended modular automorphism. Since mod β commutes with the flow, the assumption mod $\beta \neq 1$ means that the flow of weights of M can be expressed as a two-to-one extension and that mod β exchanges the two sheets. The cocycle defining the extended modular automorphism $\hat{\beta}$ is the one describing the two-to-one extension.

Appendix B. The Loi invariant of the K-action β.

Let $A = R_0 \rtimes_\alpha G_0 \supseteq B = R_0 \rtimes_\alpha H_0$ and $\beta : K \to Aut(A, B)$ be as in §5. In this appendix, we will compute the Loi invariant of β, that is, the effect of the (extended) automorphism $\beta_k,\ k \in K$, on the towers $\{A_j \cap B'\}_j$ and $\{A_j \cap A'\}_j$.

Recall that the Jones tower $B \subseteq A \subseteq A_1 \subseteq A_2 \subseteq \cdots$ is given by

$$\begin{aligned} A_1 &= (R_0 \otimes l^\infty) \rtimes_\alpha G_0, \\ A_2 &= (R_0 \otimes B(l^2)) \rtimes_\alpha G_0, \\ A_3 &= (R_0 \otimes B(l^2) \otimes l^\infty) \rtimes_\alpha G_0, \\ A_4 &= (R_0 \otimes B(l^2) \otimes B(l^2)) \rtimes_\alpha G_0, \\ &\cdots, \end{aligned}$$

where $l^\infty = l^\infty(G_0/H_0),\ l^2 = l^2(G_0/H_0)$, and $l^\infty \hookrightarrow B(l^2)$, are the natural imbeddings. (The characterization result in [42] can be used.) Here, the (extended) action α_g means

$$\alpha_g \otimes \mathrm{Ad}\ \rho_g \otimes \cdots \otimes \mathrm{Ad}\ \rho_g$$

with $(\rho_g \xi)(g' H_0) = \xi(g^{-1} g' H_0),\ \xi \in l^2$. Thus, the towers of the relative commutants are given by

$$\begin{aligned} A_1 \cap B' &= (l^\infty)^{H_0}, \\ A_2 \cap B' &= (B(l^2))^{H_0}, \\ A_3 \cap B' &= (B(l^2) \otimes l^\infty)^{H_0}, \\ A_4 \cap B' &= (B(l^2) \otimes B(l^2))^{H_0}, \\ &\cdots, \end{aligned}$$

and

$$\begin{aligned} A_2 \cap A' &= (B(l^2))^{G_0}, \\ A_3 \cap A' &= (B(l^2) \otimes l^\infty)^{G_0}, \\ A_4 \cap A' &= (B(l^2) \otimes B(l^2))^{G_0}, \\ A_5 \cap A' &= (B(l^2) \otimes B(l^2) \otimes l^\infty)^{G_0}, \\ &\cdots. \end{aligned}$$

Notice that four kinds of fixed point algebras have appeared. The algebras $(B(l^2) \otimes \cdots \otimes B(l^2))^{G_0}$ and $(B(l^2) \otimes \cdots \otimes B(l^2))^{H_0}$ simply represent the spaces of intertwiners of the product representation $\rho \otimes \cdots \otimes \rho$ and its restriction to

H_0 respectively. The natural action of G_0 on the homogeneous space G_0/H_0 being transitive, $(B(l^2)\otimes\cdots\otimes B(l^2)\otimes l^\infty)^{G_0}$ is identified with $(B(l^2)\otimes\cdots\otimes B(l^2))^{H_0}$ in the obvious way. To describe the remaining fixed point algebra $(B(l^2)\otimes\cdots\otimes B(l^2)\otimes l^\infty)^{H_0}$, we need to look at the double coset space $H_0\backslash G_0/H_0$. Let $\mathcal{O}_0=\{H_0\},\mathcal{O}_1,\cdots,\mathcal{O}_m$ be the orbits of the natural action of H_0 on G_0/H_0. Choose and fix $g_iH\in\mathcal{O}_i$ for each $i\in\{0,1,\cdots,m\}$ (with $g_0=e$), and set

$$H_i=g_iH_0g_i^{-1}\cap H_0,\quad i=0,1,\cdots,m.$$

It is elementary to see

$$(6)\qquad (B(l^2)\otimes\cdots\otimes B(l^2)\otimes l^\infty)^{H_0}\cong\sum_{i=0}^{m}{}^{\oplus}(B(l^2)\otimes\cdots\otimes B(l^2))^{H_i},$$

where $\sum_{gH_0}x_{gH_0}\otimes\delta_{gH_0}$ ($x_{gH_0}\in B(l^2)\otimes\cdots\otimes B(l^2)$ and $\delta_{gH_0}=e_{gH_0,gH_0}$, a diagonal rank 1 projection) is identified with $\sum_i^{\oplus}x_{g_iH_0}$. Therefore, (i) the irreducible A-A bimodules (arising from $A\supseteq B$, [**39**, **40**]) are parametrized by $\hat{G}_0$, (ii) the irreducible A-B (and B-A) bimodules are parametrized by $\hat{H}_0$, (iii) the irreducible B-B bimodules are parametrized by $\bigsqcup_i\hat{H}_i$. More careful analysis on involved identifications actually shows that the induction-restriction procedure for these bimodules is exactly the one for corresponding representations, i.e., the Mackey procedure (see [**1**, **31**] for details, and [**53**, **56**]).

To compute the Loi invariant of $\beta_k,k\in K$, we begin by determining its extension (fixing the Jones projections) to A_j. We set

$$(\rho'_k\xi)(gH_0)=\xi(k^{-1}gkH_0);\quad k\in K,\ \xi\in l^2.$$

Note that this is well-defined because of $k^{-1}H_0k=H_0$ and that

$$(7)\qquad \rho_g\circ\rho'_k=\rho'_k\circ\rho_{k^{-1}gk}\quad(k\in K,\ g\in G_0).$$

Define the extension (still denoted by β_k) by using

$$\mathrm{Ad}\ \rho'_k\otimes\cdots\otimes\mathrm{Ad}\ \rho'_k$$

on $B(l^2)\otimes\cdots\otimes B(l^2)$ or $B(l^2)\otimes\cdots\otimes B(l^2)\otimes l^\infty(\subseteq A_j)$. This is exactly what we wanted since the above product action leaves each of the following Jones projections (of $A\supseteq B$) invariant:

$$\begin{aligned}e_0&=\delta_{H_0}\ (\cong\ 1_{R_0}\otimes\delta_{H_0}),\\ e_1&=\sum_{gH_0}e_{gH_0,gH_0}\otimes e_{gH_0,gH_0},\end{aligned}$$

$$e_2 = 1 \otimes \sum_{gH_0,\ g'H_0} e_{gH_0, g'H_0},$$
$$\cdots .$$

Choose and fix a minimal projection p in $(B(l^2) \otimes \cdots \otimes B(l^2))^{G_0}$. Let $\pi_p \in \hat{G}_0$ be the corresponding irreducible representation:

$$\mathcal{H}_{\pi_p} = p(l^2 \otimes \cdots \otimes l^2), \quad \text{the representation space of } \ \pi_p,$$

$$\pi_p(g) = \rho_g \otimes \cdots \otimes \rho_g|_{\mathcal{H}_{\pi_p}}.$$

Consider $\pi_{p'}$ with $p' = \beta_k(p) = (\mathrm{Ad}\ \ \rho'_k \otimes \cdots \otimes \mathrm{Ad}\ \ \rho'_k)(p)$. Because of (7) the surjective isometry $\rho'_k \otimes \cdots \otimes \rho'_k|_{\mathcal{H}_{\pi_p}} : \mathcal{H}_{\pi_p} \to \mathcal{H}_{\pi_{p'}}$ intertwines $\pi_p(k^{-1}gk)$ and $\pi_{p'}(g)$. Thus $\pi_{p'}$ and $\pi_p(k^{-1} \cdot k)$ are unitarily equivalent, and the Loi invariant of β_k against the irreducible A-A bimodules (i.e., the "even levels" in $\{A_j \cap A'\}_j$) is described by

$$\pi \in \hat{G}_0 \mapsto \pi(k^{-1} \cdot k) \in \hat{G}_0.$$

Similarly the one for the irreducible A-B and B-A bimodules is described by

$$\pi \in \hat{H}_0 \mapsto \pi(k^{-1} \cdot k) \in \hat{H}_0.$$

The description of the Loi invariant for B-B bimodules requires more careful computations. For each $i \in \{0, 1, \cdots, m\}$ and $k \in K$, $kg_ik^{-1}H_0$ belongs to one of the orbits. We set

$$kg_ik^{-1}H_0 \in \mathcal{O}_{n(i,k)} \quad (n(i,k) \in \{0, 1, \cdots, m\}),$$
$$kg_ik^{-1} = h(i,k)g_{n(i,k)}\tilde{h}(i,k) \quad (h(i,k), \tilde{h}(i,k) \in H_0).$$

For example, $h(i,k)$ is not uniquely determined, but its ambiguity falls into $H_{n(i,k)}$.

Let $x = \sum_{gH_0} x_{gH_0} \otimes \delta_{gH_0} \in (B(l^2) \otimes \cdots \otimes B(l^2) \otimes l^\infty)^{H_0}$. For each $h \in H_0$ we compute

$$(\mathrm{Ad}\ \ \rho_h \otimes \cdots \otimes \mathrm{Ad}\ \ \rho_h)(x) = \sum_{gH_0} (\mathrm{Ad}\ \ \rho_h \otimes \cdots \otimes \mathrm{Ad}\ \ \rho_h)(x_{gH_0}) \otimes \delta_{hgH_0}.$$

Thus, the H_0-invariance means

$$x_{hgH_0} = (\mathrm{Ad}\ \ \rho_h \otimes \cdots \otimes \mathrm{Ad}\ \ \rho_h)(x_{gH_0}). \tag{8}$$

On the other hand, because $(\mathrm{Ad}\ \ \rho'_k)(\delta_{gH_0}) = \delta_{kgk^{-1}H_0}$, we have

$$\beta_k(x) = \sum_{gH_0} (\mathrm{Ad}\ \ \rho'_k \otimes \cdots \otimes \mathrm{Ad}\ \ \rho'_k)(x_{gH_0}) \otimes \delta_{kgk^{-1}H_0}.$$

When $g = g_i$, we have $kg_ik^{-1}H_0 = h(i,k)g_{n(i,k)}\tilde{h}(i,k)H_0 = h(i,k)g_{n(i,k)}H_0$ and the "coefficient" of $\delta_{h(i,k)g_{n(i,k)}H_0}$ is

$$(\mathrm{Ad}\ \rho'_k \otimes \cdots \otimes \mathrm{Ad}\ \rho'_k)(x_{g_iH_0}).$$

Thanks to (8), the coefficient of $\delta_{g_{n(i,k)}H_0}$ is

$$(\mathrm{Ad}\ \rho_{h(i,k)^{-1}} \otimes \cdots \otimes \mathrm{Ad}\ \rho_{h(i,k)^{-1}})(\mathrm{Ad}\ \rho'_k \otimes \cdots \otimes \mathrm{Ad}\ \rho'_k)(x_{g_iH_0}).$$

Consequently, via the isomorphism (6), β_k on $\sum_i^{\oplus}(B(l^2) \otimes \cdots \otimes B(l^2))^{H_i}$ is described by the following:

(i) $\beta_k((B(l^2) \otimes \cdots \otimes B(l^2))^{H_i}) = (B(l^2) \otimes \cdots \otimes B(l^2))^{H_{n(i,k)}}$,

(ii) $\beta_k(x) = (\mathrm{Ad}\ \rho_{h(i,k)^{-1}} \otimes \cdots \otimes \mathrm{Ad}\ \rho_{h(i,k)^{-1}})(\mathrm{Ad}\ \rho'_k \otimes \cdots \otimes \mathrm{Ad}\ \rho'_k)(x)$ for $x \in (B(L^2) \otimes \cdots \otimes B(l^2))^{H_i}$.

In fact, by (7) we get

$$\rho_g(\rho_{h(i,k)^{-1}}\rho'_k) = \rho_{h(i,k)^{-1}}\rho_{h(i,k)gh(i,k)^{-1}}\rho'_k = (\rho_{h(i,k)^{-1}}\rho'_k)\rho_{k^{-1}h(i,k)^{-1}gh(i,k)^{-1}k},$$

and $k^{-1}h(i,k)H_{n(i,k)}h(i,k)^{-1}k = H_i$. For a minimal projection $p \in (B(l^2) \otimes \cdots \otimes B(l^2))^{H_i}$ $(\pi_p \in \hat{H}_i)$ and $p' = \beta_k(p)$ $(\pi_{p'} \in \hat{H}_{n(i,k)})$, $\pi_{p'}$ is unitarily equivalent to $\pi_p(k^{-1}h(i,k) \cdot h(i,k)^{-1}k) \in \hat{H}_{n(i,k)}$ as before, and the Loi invariant of β_k is given by

$$\pi \in \hat{H}_i \mapsto \pi(k^{-1}h(i,k) \cdot h(i,k)^{-1}k) \in \hat{H}_{n(i,k)}.$$

Recall that an ambiguity for choosing $h(i,k)$ came from $H_{n(i,k)}$. Therefore, the above right side uniquely determines a unitary equivalence class (in $\hat{H}_{n(i,k)}$).

Summing up the arguments so far, we have obtained the following description of the Loi invariant:

Lemma B.1. *The Loi invariant of β_k, $k \in K$, is described by* (i) $\pi \in \hat{G}_0 \mapsto \pi(k^{-1} \cdot k) \in \hat{G}_0$ *for* $\hat{G}_0$ *(the irreducible A-A bimodules),* (ii) $\pi \in \hat{H}_0 \mapsto \pi(k^{-1} \cdot k) \in \hat{H}_0$ *for* $\hat{H}_0$ *(the irreducible A-B or B-A bimodules), and* (iii) $\pi \in \hat{H}_i \mapsto \pi(k^{-1}h(i,k) \cdot h(i,k)^{-1}k) \in \hat{H}_{n(i,k)}$ *for* $\bigsqcup_i \hat{H}_i$ *(the irreducible B-B bimodules).*

References

[1] D. Bisch, *Subfactors arising from groups and their representations,* unpublished manuscript.

[2] M. Choda, *Duality for finite bipartite graphs (with applications to* II_1 *factors),* Pacific J. Math., **158** (1993), 49-65.

[3] M. Choda and H. Kosaki, *Strongly outer actions for an inclusion of factors,* J. Funct. Anal., **122** (1994), 315-322.

[4] A. Connes and M. Takesaki, *The flow of weights on factors of type* III, Tôhoku J. Math., **29** (1977), 473-575.

[5] D. E. Evans and Y. Kawahigashi, *Orbifold subfactors from Hecke algebras,* Commun. Math. Phys., **165** (1994), 445-484.

[6] J. Feldman, C. Sutherland and R. Zimmer, *Subrelations of ergodic relations,* Ergod. Th. and Dynam. Sys., **9** (1989), 239-269.

[7] F. Goodman, de la Harpe and V.F.R. Jones, *Coxeter Dynkin Diagrams and Towers of Algebras,* Springer, 1989.

[8] U. Haagerup and E. Størmer, *Equivalence of normal states on von Neumann algebras and the flow of weights,* Adv. in Math., **83** (1990), 180-262.

[9] ———, *Pointwise inner automorphisms of von Neumann algebras,* J. Funct. Anal., **92** (1990), 177-201.

[10] T. Hamachi, *Ergodic equivalence relations-subrelations,* preprint (research announcement).

[11] T. Hamachi and H. Kosaki, *Orbital factor map,* Ergod. Th. and Dynam. Sys., **13** (1993), 515-532.

[12] J.F. Havet, *Espérance conditionnelle minimale,* J. Operator Theory, **24** (1990), 33-55.

[13] F. Hiai, *Minimizing indices of conditional expectations onto a subfactor,* Publ. RIMS, Kyoto Univ., **24** (1988), 673-678.

[14] M. Izumi, *Applications of fusion rules to classification of subfactors,* Publ. RIMS, Kyoto Univ., **27** (1991), 953-994.

[15] ———, *Goldman's type theorem for index* 3*,* Publ. RIMS, Kyoto Univ., **28** (1992), 833-843.

[16] ———, *On type* II *and type* III *principal graphs for subfactors,* Math. Scand., **73** (1993), 307-319.

[17] G. James and A. Kerber, *The Representation Theory of the Symmetric Group,* Addison-Wesley Publishing Company, 1881.

[18] V.F.R. Jones, *Actions of finite groups on the hyperfinite type* II_1 *factor,* Mem. Amer. Math. Soc., **237** (1980).

[19] ———, *Index of subfactors,* Invent. Math., **66** (1983), 1-25.

[20] R. Kallmann, *A generalization of free action,* Duke Math. J., **36** (1969), 781-789.

[21] Y. Kawahigashi, *Automorphisms commuting with a conditional expectation onto a subfactor with finite index,* J. Operator Theory, **28** (1992), 127-145.

[22] ———, *On flatness of Ocneanu's connections on the Dynkin diagrams and classification of subfactors,* J. Funct. Anal., **127** (1995), 63-107.

[23] ———, *Centrally trivial automorphisms and an analogue of Connes' $\chi(M)$ for subfactors,* Duke Math. J., **71** (1993), 93-118.

[24] Y. Kawahigashi, C. Sutherland and M. Takesaki, *The structure of automorphism groups of an injective factor and the cocycle conjugacy of discrete amenable actions,* Acta Math., **169** (1992), 105-130.

[25] Y. Kawahigashi and M. Takesaki, *Compact abelian group actions on injective factors,* J. Funct. Anal., **66** (1992), 112-128.

[26] H. Kosaki, *Extension of Jones' theory on index to arbitrary factors,* J. Funct. Anal., **66** (1986), 123-140.

[27] ______, *Index theory for type* III *factors,* in "Mappings of Operator Algebras (Proc. US-Japan Seminar at Philadelphia, 1988)", Birkhäuser, (1990), 227-231.

[28] ______, *AFD factor of type* III_0 *with many isomorphic index* 3 *subfactors,* J. Operator Theory, **32** (1994), 17-29.

[29] ______, *Automorphisms in the irreducible decomposition of sectors,* in "Quantum and Non-commutative Analysis", Kluwer Academic Publishers, (1993), 305-316.

[30] H. Kosaki and R. Longo, *A remark on the minimal index of subfactors,* J. Funct. Anal., **107** (1992), 458-470.

[31] H. Kosaki and S. Yamagami, *Irreducible bimodules associated with crossed product algebras,* Internat. J. Math., **3** (1992), 661-676.

[32] P.H. Loi, *On automorphisms of subfactors,* J. Funct. Anal., **141** (1996), 275-293.

[33] ______, *On the derived tower of a certain inclusion of type* III_λ *factors of index* 4*,* Pacific J. Math., **165** (1994), 321-345.

[34] R. Longo, *Simple injective subfactors,* Adv. in Math., **63** (1987), 152-172.

[35] ______, *Index of subfactors and statistics of quantum fields* I, Commun. Math. Phys., **126** (1989), 217-247.

[36] ______, *Index of subfactors and statistics of quantum fields* II, Commun. Math. Phys., **130** (1990), 285-309.

[37] ______, *Minimal index and braided subfactors,* J. Funct. Anal., **109** (1992), 98-112.

[38] Y. Nakagami and M. Takesaki, *Duality for crossed products of von Neumann algebras,* LNM 731, Springer, 1979.

[39] A. Ocneanu, *Quantized groups, string algebras, and Galois theory,* in "Operator Algebras and Applications, Vol. II", Cambridge Univ. Press, 1988.

[40] ______, *Quantum Symmetry, Differential Geometry of Finite Graphs, and Classification of Subfactors,* Univ. of Tokyo Seminary Notes, 1991.

[41] M. Pimsner and S. Popa, *Entropy and index of subfactors,* Ann. Sci. École Norm. Sup., **19** (1986), 57-106.

[42] ______, *Iterating the basic construction,* Trans. Amer. Math. Soc., **310** (1988), 127-133.

[43] S. Popa, *Correspondences,* preprint.

[44] ______, *Classification of subfactors: Reduction to commuting squares,* Invent. Math., **101** (1990), 19-43.

[45] ______, *Classification of amenable subfactors of type* II, Acta Math., **172** (1994), 352-445.

[46] ______, *Classification of actions of discrete amenable groups on amenable subfactors of type* II*,* preprint.

[47] D. Rudolph, *Counting the relatively finite factors of a Bernoulli shift,* Israel J. Math., **30** (1978), 255-263.

[48] J.L. Sauvageot, *Sur le type du produit croisé d'une algèbre de von Neumann par un groupe localement compact,* Bull. Soc. Math. France, **105** (1977), 349-368.

[49] ______, *Sur le produit tensoriel relatif d'espaces de hilbert,* J. Operator Theory, **9** (1983), 237-252.

[50] Y. Sekine, *Flows of weights of crossed products of type* III *factors by discrete groups,* Publ. RIMS, Kyoto Univ., **26** (1990), 655-666.

[51] ______, *Cartan subalgebras in fixed point subalgebras of finite group actions,* Math. Scand., **70** (1992), 281-292.

[52] S. Strătilă, *Modular Theory in Operator Algebras,* Abacus Press, 1981.

[53] C. Sutherland, *Subfactors and ergodic theory,* in "Current Topics in Operator Algebras (Proc. Satellite Conference of ICM-90)", World Scientific, 1991.

[54] ______, *Nested pairs of equivalence relations and von Neumann algebras,* preprint.

[55] C. Sutherland and M. Takesaki, *Actions of discrete amenable groups on injective factors of type* $\mathrm{III}_\lambda, \lambda \neq 1$, Pacific J. Math., **137** (1987), 405-444.

[56] A. Wasserman, *Coactions and Yang-Baxter equations for ergodic actions and subfactors,* in "Operator Algebras and Application, Vol. II", Cambridge Univ. Press, 1988.

[57] C. Winsløw, *Approximately inner automorphisms on inclusions of type* III_λ*-factors,* Pacific J. Math., **166** (1994), 385-400.

[58] S. Yamagami, *Modular theory in bimodules,* J. Funct. Anal., **125** (1994), 327-357.

[59] R. Zimmer, *Extensions of ergodic group actions,* Ill. J. Math., **20** (1976), 373-409.

Received December 10, 1992 and revised April 12, 1994.

KYUSHU UNIVERSITY
EUKUOKA, 810, JAPAN

AND

YAMAGATA UNIVERSITY
YAMAGATA, 990, JAPAN