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A POLYHEDRAL TRANSVERSALITY THEOREM FOR ONE-PARAMETER FIXED POINT THEORY

Thomas Plavchak

The fixed point set of a piecewise linear (PL) map $h : P \times I \to P$ is the set of points where h coincides with the projection $\pi : P \times I \to P$; it is denoted by Fix(h) and is a subpolyhedron of $P \times I$. When P is a compact polyhedron, we show how to deform h (with appropriate control) to a new PL map h' so that Fix(h') is as nice as possible. Indeed it is not hard to arrange that Fix(h') have dimension ≤ 1 (Theorem A), but one would wish for a map h' such that Fix(h') is a manifold of dimension ≤ 1. This is achieved in Theorem B. If P is a PL manifold, Theorem B reduces to a standard PL transversality theorem (Theorem C).

1. Introduction.

In recent years there has been considerable interest in one-parameter fixed point theory. See, for example, [**D**$_1$], [**D**$_2$], [**DG**], [**GN**$_{1-5}$], [**GNO**], [**J**]. A basic requirement in using that theory is a "preparation theorem" of the kind described in the abstract. Indeed, several of the papers mentioned cite the present paper.

To state our theorems precisely we must set up our simplicial notation. By a *simplex* of a finite simplicial complex K in $\mathbb{R}^n$ we mean an "open simplex"; thus a k-simplex t is an open subset of a k-dimensional affine subspace of $\mathbb{R}^n$. If t is a simplex, $\mathrm{cl}(t)$ is its closure in $\mathbb{R}^n$ and $\mathrm{bd}(t)$ is its boundary. The subspace of $\mathbb{R}^n$ spanned by t is denoted by $\mathrm{span}(t)$. If the simplex s is a face of a simplex t we write $s < t$; in particular $t < t$. We note that since our simplexes are open, if $s < t$ and $s \neq t$ then $s \cap t = \emptyset$. The simplicial complexes $\{s | s < t\}$ and $\{s | s < t \text{ and } s \neq t\}$ are denoted by $\bar{t}$ and ∂t respectively. The barycenter of s is denoted by $b(s)$. When the vertices $v_0, \ldots, v_n$ of K span a simplex of K we sometimes denote that simplex by $\langle v_0, \cdots, v_n \rangle$ (except that we sometimes write v rather than $\langle v \rangle$ for a 0-simplex); when s and t are faces of $\langle v_0, \ldots, v_n \rangle$, the face spanned by the vertices of s and of t is the *join* of s and t, denoted $s * t$. If $s \in K$, the *star* of s in K is $st(s, K) = \{t \in K | s < t\}$ and the *link* of s in K is $lk(s, K) = \{t \in K | s \text{ and } t \text{ have no vertex in common and } s * t \in K\}$. If

$M \subset K$, $|M| = \cup\{s|s \in M\}$. If $f : K \to L$ is a simplicial map (between finite simplicial complexes in Euclidean spaces), $|f| : |K| \to |L|$ denotes the corresponding "simplicial map" of spaces.

Throughout the paper we make some **Standing Assumptions**: K is a finite simplicial complex with $|K| \subset \mathbb{R}^n$, K_0 is a simplicial subdivision of $K \times I$ formed without adding any new vertices, K_0' is a subdivison of a barycentric subdivision of K_0, and $f : K_0' \to K$ is a simplicial map. By simplicial approximation, any map $|K| \times I \to |K|$ is homotopic to a map with these properties. For the construction of K_0 see [**RS**, page 16]. $|\pi| : |K_0'| \to |K|$ denotes the projection $(x,t) \mapsto x$. We note that $\pi : K_0 \to K$ is simplicial and that $\mathrm{Fix}(|f|) = \{x \in |K_0| \,|\, |f|(x) = |\pi|(x)\}$.

Theorem A. $\dim(\mathrm{Fix}(|f|)) \leq 1$. *More precisely: Let t be a p-simplex of K_0' containing a fixed point of $|f|$ and let $s = f(t)$; then $\dim(s) = p$ or $p-1$.*

(i) *If $\dim(s) = p$, t contains exactly one fixed point.*

(ii) *If $\dim(s) = p-1$, then $\mathrm{Fix}(|f|) \cap t$ is an open line segment in t the boundary of which is fixed and contained in $\mathrm{bd}(t)$.*

(iii) *If $t \subset |K| \times \{0,1\}$, then t contains exactly one fixed point.*

To state our next theorem we need some notation.

If $s \in K$, we write $M_s = |\pi|^{-1}(\mathrm{cl}(s))$. This is a closed cell of the product cell complex $K \times I$ and hence it is subdivided as a subcomplex of any simplicial subdivision of $K \times I$. We write M_s' for the subcomplex of K_0' such that $M_s = |M_s'|$. Similarly (below) M_s'' corresponds to K_0'' etc. The space M_s is a closed ball of dimension $\dim(s)+1$, so M_s', M_s'' etc. are pseudomanifolds of that dimension, and ∂M_s is triangulated by $\partial M_s'$, $\partial M_s''$ etc. Note that if $t \in M_s'$ and $\dim(t) = \dim(s)$ then $lk(t, M_s')$ consist of at most two vertices. For $i = 0$ or 1, $K_{0,i}'$ denotes the subcomplex of K_0' triangulating $|K| \times \{i\}$.

Let $N \subset |K_0'|$ be the union of the stars of all (open) simplexes which meet $\mathrm{Fix}(|f|)$. Let $\mathcal{U}$ be the cover of $|K|$ by the closures of the stars of vertices of K. Now we can state our main theorem: qualitatively, it says that the fixed point set can be made as nice as could be expected; see Figure 1.

Theorem B. *There is a subdivision K_0'' of K_0' and a simplicial map $g : K_0'' \to K$, with $|g|$ $\mathcal{U}$-homotopic to $|f|$ by a homotopy rel $\mathrm{cl}(|K_0'| - N)$, such that*

(i) $\mathrm{Fix}(|g|) \subset S_0 \cup S_1$ *where* $S_1 = |\pi|^{-1}(\cup\{s \in K|lk(s,K) = \emptyset\})$, $S_0 = |\pi|^{-1}(\cup\{s \in K|lk(s,K) \neq \emptyset$ *but not connected*$\})$, *and* $\dim(S_i \cap \mathrm{Fix}(|g|)) = i$ *for* $i = 0$ *and* 1,

(ii) $\mathrm{Fix}(|g|)$ *is a* 1*-manifold placed in the following way with respect to the simplexes of K_0'': For $t \in K_0''$ and $s \in K$ such that $|\pi|(t) \subset s$ and* $t \cap \mathrm{Fix}(|g|) \neq \emptyset$,

(a) *if* $lk(s,K)=\emptyset$, $\dim(t)=\dim(s)$ *and* $t\in\partial M''_s$, *then* $|st(t,K''_0)|\cap \mathrm{Fix}(|g|)$ *is a half-open line segment with an endpoint in* t*;*

(b) *if* $lk(s,K)=\emptyset$, *and* $\dim(t)=\dim(s)$ *or* $\dim(s)+1$ *and* $t\notin \partial M''_s$, *then* $|st(t,K''_0)|\cap \mathrm{Fix}(|g|)$ *is the union of two half-open line segments;*

(c) *if* $lk(s,K)\neq\emptyset$, *then* $lk(s,K)$ *is not connected,* t *contains just one fixed point of* $|g|$, *and* $|st(t,K''_0)|\cap \mathrm{Fix}(|g|)$ *is the union of two half-open line segments.*

Moreover, if for $i=0$ *and* 1 *the points of* $(\mathrm{Fix}\,|f|)\cap|K'_{0,i}|$ *all lie in principal simplexes of* $K'_{0,i}$ *then the homotopy from* $|f|$ *to* $|g|$ *can be chosen to be rel* $|K|\times\{0,1\}$ *and* K''_0 *can be chosen to have* $K'_{0,0}$ *and* $K'_{0,1}$ *as subcomplexes.*

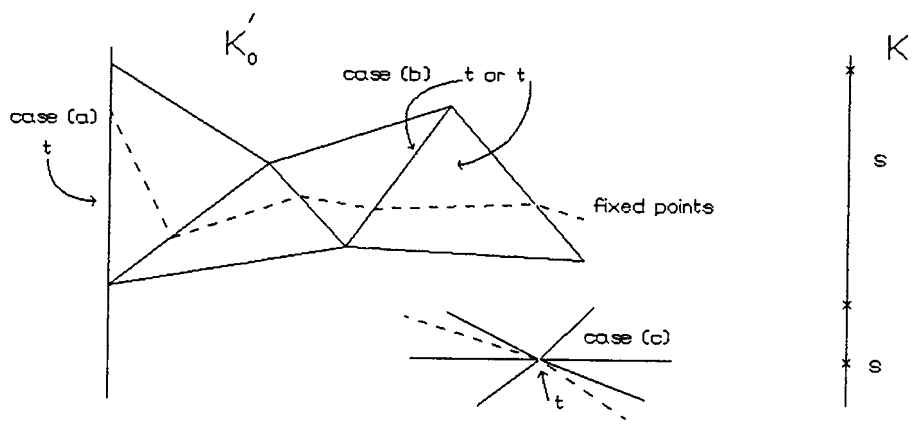

Figure 1.

Next, we recall the definition of PL manifold transversality. Let X be a PL manifold and let $h: X\times I\to X$ be a PL map. We denote the graph of h by Γ_h. We say h is *transverse* to $|\pi|: X\times I\to X$ if h has no fixed points on $(\partial X)\times I$ and each point of $\Gamma_{|\pi|}\cap\Gamma_h$ has a regular neighborhood in $(X\times I\times X, \Gamma_{|\pi|}, \Gamma_h, \Gamma_{|\pi|}\cap\Gamma_h)$ which is PL homeomorphic to a regular neighborhood of 0 in either $(\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n; \mathbb{R}^n\times\mathbb{R}\times 0, 0\times\mathbb{R}\times\mathbb{R}^n, 0\times\mathbb{R}\times 0)$ or $(\mathbb{R}^n\times\mathbb{R}_+\times\mathbb{R}^n; \mathbb{R}^n\times\mathbb{R}_+\times 0, 0\times\mathbb{R}_+\times\mathbb{R}^n, 0\times\mathbb{R}_+\times 0)$.

Theorem C. *If* $|K|$ *is a PL manifold then the proof of Theorem B yields a map* $|g|$ *which is transverse to* $|\pi|$.

2. Proof of Theorem A.

Lemma 2.1 is used to locate fixed points, only in Lemma 2.1 are simplexes closed.

Lemma 2.1. *Let A and B be closed simplexes with* $\dim(A) \geq \dim(B)$ *and let $f : A \to B$ be a linear extension of a map of the vertices of A onto the vertices of B. If $g : A \to B$ is a continuous function, then there is a point $x \in A$ such that $f(x) = g(x)$.*

Proof. First, assume $\dim(A) = \dim(B)$. Then f is a homeomorphism and so $g \circ f^{-1} : B \to B$ is defined and continuous. By the Brouwer Fixed Point Theorem there is a point $b \in B$ such that $g \circ f^{-1}(b) = b$; let $x = f^{-1}(b)$. Then $f(x) = b = g \circ f^{-1}(b) = g(x)$.

If $\dim(A) > \dim(B)$, then there is a face C of A such that $f(C) = B$ and $\dim(C) = \dim(B)$. Thus there is a point x in C such that $f(x) = g(x)$. $\square$

Lemma 2.2. *Let t be a p-simplex in $\mathbb{R}^n$, $p \geq 1$, v a vertex of t, and l a line in $\mathbb{R}^n$. If $l \cap t$ contains at least two points, then there is a proper face t' of t having v as a vertex and such that $t' \cap l \neq \emptyset$.*

Proof. Because l meets t in at least two points, $l \subset \mathrm{span}(t)$; therefore $l \cap \mathrm{bd}(t)$ contains at least two points. Let s be the face of t opposite v. If no proper face of t having v as a vertex meets l, then every point in $l \cap \mathrm{bd}(t)$ is either in s or a face of s. So $l \subset \mathrm{span}(s)$, but $l \subset \mathrm{span}(s)$ contradicts $\mathrm{span}(s) \cap t = \emptyset$. $\square$

Lemma 2.3. *Let t be a p-simplex in $\mathbb{R}^n$, $p \geq 2$, v a vertex of t, and P a plane in $\mathbb{R}^n$. If $P \cap t$ contains at least three non-collinear points, then there is a face t' of t such that v is a vertex of t',* $\dim(t') \leq p - 2$, *and $t' \cap P \neq \emptyset$.*

Proof. Let p_0, p_1, and p_2 in $P \cap t$ be three non-collinear points. The line $l_{0,1}$ spanned by p_0 and p_1 meets $\mathrm{bd}(t)$ at two points x_0 and x_1, and the line $l_{0,2}$ meets $\mathrm{bd}(t)$ at some point x_3. The point x_3 is not on the line spanned by x_0 and x_1 because $p_2 \notin l_{0,1}$. Thus there are three non-collinear points of P in $\mathrm{bd}(t)$. $\square$

Let s be the face of t opposite v. If no proper face of t having v as a vertex meets P, then every point in $P \cap \mathrm{bd}(t)$ is either in s or a face of s. So $P \subset \mathrm{span}(s)$, contradicting $\mathrm{span}(s) \cap t = \emptyset$. Thus P meets a proper face t_1 of t having v as a vertex.

Assume $\dim(t_1) = p-1$. Now $\dim(P \cup \mathrm{span}(t_1)) = \dim(P) + \dim(\mathrm{span}(t_1)) - \dim(P \cap \mathrm{span}(t_1))$. Since $\dim(P \cup \mathrm{span}(t_1)) \leq p$ and $\dim(P) = 2$, $\dim(P \cap$

$\mathrm{span}(t_1)) \geq 1$. Thus there is a line l lying in P and meeting t_1 in at least two points. By Lemma 2.2 there is a proper face t' of t_1 having v as a vertex and such that $l \cap t' \neq \emptyset$. Thus there is a face t' of t having v as a vertex such that $\dim(t') \leq p-2$ and $t' \cap P \neq \emptyset$.

Lemma 2.4. *If t is a simplex of K_0' and $|\pi|(t)$ is contained in the simplex s of K, then there is a vertex v of t such that $|\pi|(v) \in s$.*

Proof. Let K_0^1 be the first barycentric subdivision of K_0 and let $t'' \in K_0^1$ and $t' \in K_0$ be such that $t \subset t'' \subset t'$. Since s and $\pi(t')$ are simplexes of K and both contain $|\pi|(t)$, $s = \pi(t')$. Since $\mathrm{bd}(t) \subset \mathrm{cl}(t'')$, either t'', and so t', contains a vertex of t or each vertex of t is in $\mathrm{bd}(t'')$. If t' contains a vertex v of t, then $|\pi|(v) \in s$.

Suppose each vertex of t is in $\mathrm{bd}(t'')$. Since $t'' \in K_0^1$, $t'' = \langle b(s_0), \dots, b(s_p)\rangle$ where $s_0, \dots, s_p \in K_0$ and $s_0 < \cdots < s_p$. Now $t'' \subset s_p$ so $t \subset s_p$, thus $t' = s_p$. Let $v' = b(t')$. Let τ be the face of t'' opposite v'. Now if each vertex of t is contained in τ or a face of τ, then $t \subset \mathrm{cl}(\tau)$, but $\mathrm{cl}(\tau) \cap t'' = \emptyset$ contradicting $t \subset t''$. Thus there is a proper face t_0 of t'' having v' as a vertex and containing a vertex v of t. Because $v' = b(t')$, $t_0 \subset t'$. Now $v \in t_0 \subset t'$ implies $|\pi|(v) \in |\pi|(t')$ and $|\pi|(t')$ is s. □

Lemma 2.5. *Let t be a simplex of K_0', $s \in K$, $|\pi|(t) \subset s$, and let v be a vertex of t such that $|\pi|(v) \in s$. If t' is a face of t and v is a vertex of t', then $|\pi|(t') \subset s$.*

Proof. (By induction on $\dim(t')$.) Let $t' = \langle v, v_1, \dots, v_n\rangle$ and let $p \in t'$. Now $p = av_n + bx$ where x is a point in $\langle v, v_1, \dots, v_{n-1}\rangle$. By the inductive hypothesis $|\pi|(\langle v, v_1, \dots, v_{n-1}\rangle) \subset s$. Since $|\pi|(t) \subset s$, $|\pi|(v_n) \in \mathrm{cl}(s)$. Thus $|\pi|(p) \in s$. □

Proof of Theorem A. Since t contains a fixed point, $|\pi|(t) \subset s$. Since f is simplicial, $\dim(f(t)) \leq \dim(t)$. So $\dim(s) \leq \dim(t)$. Because $|\pi|(t) \subset s$, $t \in M_s'$. So $\dim(t) \leq \dim(s) + 1$.

First, assume $\dim(s) = p$ and x_0, x_1 are fixed points of $|f|$ in t. Let v be a vertex of t such that $|\pi|(v) \in s$ (see Lemma 2.4). Now if t' is a proper face of t having v as a vertex, then t' cannot contain a fixed point of $|f|$ because if v is a vertex of t' then $|\pi|(t') \subset s$, and if $\dim(t') < \dim(t)$, then $f(t') \neq s$. But by Lemma 2.2 the line l determined by x_0 and x_1 intersects a proper face of t having v as a vertex and since $l \cap \mathrm{cl}(t)$ is fixed by $|f|$ this face contains a fixed point of $|f|$. This contradicts the fact that no proper face of t having v as a vertex can contain a fixed point.

Next, assume $\dim(s) = p-1$. Then there is a proper face t_0 of t such that $f(t_0) = s$. Since $|\pi|(\mathrm{cl}(t_0)) \subset \mathrm{cl}(s)$, a proper face of t contains a fixed point.

So t contains a line segment l of fixed points the boundary of which is fixed and contained in $\mathrm{bd}(t)$. Suppose x is a fixed point of $|f|$ and x is not on l. Let v be a vertex of t such that $|\pi|(v) \in s$. Now if t_0 is a $(p-2)$-face of t having v as a vertex, then t_0 cannot contain a fixed point of $|f|$, because if v a vertex of t_0 then $|\pi|(t_0) \subset s$, and if $\dim(t_0) = p-2$ then $f(t_0) \neq s$. But by Lemma 2.3 the plane P determined by l and x meets a $(p-2)$-face of t having v as a vertex, and since $P \cap \mathrm{cl}(t)$ is fixed by $|f|$ this face contains a fixed point of $|f|$. This contradicts the fact that no $(p-2)$-face of t having v as a vertex can contain a fixed point.

Now suppose $t \subset |K| \times \{0,1\}$. Since $t \subset s \times \{0,1\}$, t is a face of a $(p+1)$-simplex contained in $s \times I$. So $p+1 \leq \dim(s)+1$. Because $f(t) = s$, $p \geq \dim(s)$. Thus $\dim(s) = p$ and t contains just one fixed point. □

3. Proof of Theorem B.

Lemma 3.1. *Let $t,\ t' \in K_0'$, $s,\ s' \in K$, $|\pi|(t) \subset s$, and $|\pi|(t') \subset s'$.*

(a) *If $t < t'$, then $s < s'$.*

(b) *If $s < s'$ and $t * t' \in K_0'$, then $|\pi|(t * t') \subset s'$.*

(c) *If $t * t' \in K_0'$, then $s < s'$ or $s' < s$.*

Proof of (a). Let $A, B \in K_0$ be such that $t \subset A$ and $t' \subset B$. Then $\pi(A) = s$ and $\pi(B) = s'$. Now $|st(A, K_0)|$ is an open subset of $|K_0|$, $|st(A, K_0)|$ contains t, and each point of t is a limit point of t', so $B \in st(A, K_0)$. Thus $A < B$ and so $|\pi|(A) < |\pi|(B)$, i.e., $s < s'$.

Proof of (b). Let $A \in K_0$ be such that $t * t' \subset A$. Since $t < t * t'$, $|\pi|(t) \subset s$, and $|\pi|(t * t') \subset |\pi|(A)$, by Part (a), $s < \pi(A)$. Similarly $s' < \pi(A)$. By Lemma 2.4 there is a vertex v of $t * t'$ such that $|\pi|(v) \in \pi(A)$. But if v is a vertex of t, then $|\pi|(v) \in \mathrm{cl}(s)$ and if v is a vertex of t', then $|\pi|(v) \in \mathrm{cl}(s')$. Since $s < s'$, in either case $|\pi|(v) \in \mathrm{cl}(s')$. Thus $|\pi|(v)$ is in the simplex $|\pi|(A)$ and in the closure of the simplex s'. This means $\pi(A) < s'$. From above, $s' < \pi(A)$. So $\pi(A) = s'$. Hence $|\pi|(t * t') \subset s'$.

Proof of (c). Let A, B, and C be simplexes of K_0 such that $t \subset A$, $t' \subset B$, and $t * t' \subset C$. Since $t * t' \subset \mathrm{cl}(C)$, $t \subset \mathrm{cl}(C)$. So $A < C$. Similarly $B < C$. So $A * B \in K_0$. Since $t * t' \subset A * B$, $A * B = C$.

Let X, Y, and Z be simplexes of K_0^1 be such that $t \subset X$, $t' \subset Y$, and $t * t' \subset Z$. From the last paragraph $X * Y \in K_0^1$ and $X * Y = Z$. Let $X = \langle b(T_0), b(T_1), \ldots, b(T_{p-1}), b(A)\rangle$ and let $Y = \langle b(S_0), b(S_1), \ldots, b(S_{q-1}), b(B)\rangle$ where $T_0 < T_1 < \cdots < T_{p-1} < A$ and $S_0 < S_1 < \cdots < S_{q-1} < B$. Now $X * Y \in K_0^1$ implies $A < B$ or $B < A$. So $\pi(A) < \pi(B)$ or $\pi(B) < \pi(A)$, that is, $s < s'$ or $s' < s$. □

Proposition 3.2. *Let the simplex t_0 in K_0' contain a fixed point of $|f|$. Then each component of $(\mathrm{Fix}(|f|) \cap |st(t_0, K_0')|) - t_0$ is contained in a simplex of $st(t_0, K_0')$.*

Proof. Suppose $t \in st(t_0, K_0')$, $t \neq t_0$, and t contains a fixed point of $|f|$. Suppose that t' is a proper face of t, t_0 is a proper face of t', and t' contains a fixed point of $|f|$. Then $\mathrm{cl}(t)$ contains three non-collinear fixed points of $|f|$: a, b, and c. Thus the fixed point set of $|f|$ contains a 2-dimensional subset: $\mathrm{cl}(t) \cap \mathrm{span}(\{a, b, c\})$. But this contradicts Theorem A. Thus t' contains no fixed points of $|f|$. □

Lemma 3.3. *Let M and N be complexes and let L be a subcomplex of M. Suppose $f : M \to N$ and $g : M \to N$ are simplicial maps that agree on L. Furthermore suppose for each $x \in |M| - |L|$ there are simplexes s, t, and s_0 in N such that $s_0 < s$, $s_0 < t$, $|f|(x) \in t$, and $|g|(x) \in \mathrm{cl}(s)$. Then $|f|$ is $\mathcal{U}$-homotopic to $|g|$ rel $|L|$.*

Proof. Use Lemma 1, page 124 in [**Br**]. □

Let M be a simplicial complex and L a subcomplex of M. Define the *simplicial subdivision of M modulo the subcomplex L*, denoted by M_L', as follows: The vertices of M_L' are the vertices of M along with the vertices $b(t)$ where t is a simplex in $M - L$. For $p \geq 1$, the p-simplexes of M_L' are of the form $\langle v_0, \ldots, v_q, b(t_{q+1}), \ldots, b(t_p)\rangle$ where $\langle v_0, \ldots, v_q\rangle$ is a q-simplex in L, $t_{q+1}, \ldots, t_p$ are distinct simplexes in $M - L$, and $\langle v_0, \ldots, v_q\rangle \lneqq t_{q+1} < \cdots < t_p$. Here q can equal 0 and p can equal q.

Proposition 3.4. *Assume t_0 is a p-simplex of K_0', t_0 contains a fixed point of $|f|$, $f(t_0) = s_0$, and $\dim(s_0) = p$. Assume $lk(t_0, M_{s_0}')$ consists of two vertices v_1, v_2 and $f(v_1) \neq f(v_2)$ are not vertices of s_0. Let L be the subcomplex of K_0' consisting of t_0 and all simplexes that do not have t_0 as a face. Then there is a simplicial map $|g| : |(K_0')_L'| \to |K|$ homotopic to $|f|$ rel $|L|$ such that $\mathrm{Fix}(|g|) \cap |st(t_0, (K_0')_L')|$ is a 1-manifold.*

Proof. Define g on the vertices of $(K_0')_L'$ as follows: If a is a vertex of L, $g(a) = f(a)$; if t_0 is a proper face of t and $t \neq v_i * t_0$ for $i = 1$ or 2, then let $g(b(t))$ be any vertex of s_0; for $i = 1$ and 2 let $g(b(v_i * t_0)) = f(v_i)$.

Let $\sigma = \langle a_0, a_1, \ldots, a_l, b(t_{l+1}), \ldots, b(t_m)\rangle$ be a simplex of $(K_0')_L'$. By definition, $g(a_j)$ is a vertex of $f(t_{l+1})$ for $0 \leq j \leq l$, and, for $l + 1 \leq j \leq m$, $g(b(t_j))$ is $f(v_1)$ or $f(v_2)$ or a vertex of s_0. If for each j, $g(b(t_j)) < s_0$, then, since $s_0 < f(t_{l+1})$, $g(\sigma)$ is a face of $f(t_{l+1})$. If for some j_0, $g(b(t_{j_0})) = f(v_1)$ or $f(v_2)$, then $j_0 = l + 1$ and $t_{j_0} = v_1 * t_0$ or $t_{j_0} = v_2 * t_0$. So $g(\sigma) < f(v_1) * s_0$ or $g(\sigma) < f(v_2) * s_0$. For $i = 1$ and 2, $f(v_i) * s_0$ is a simplex of K because $f(v_i) \in lk(s_0, K)$. Thus $g(\sigma)$ is a simplex of K and so $g : (K_0')_L' \to K$ is simplicial. Furthermore, since $\sigma \subset t_m$, $|f|(\sigma) \subset f(t_m)$; because $t_0 < t_m$, $s_0 < f(t_m)$; and $g(\sigma)$ is a subset of either $\mathrm{cl}(s_0)$ or $\mathrm{cl}(f(v_1)*s_0)$ or $\mathrm{cl}(f(v_2) * s_0)$. By Lemma 3.3, $|g| : |(K_0')_L'| \to |K|$ is $\mathcal{U}$-homotopic to $|f|$ rel $|L|$.

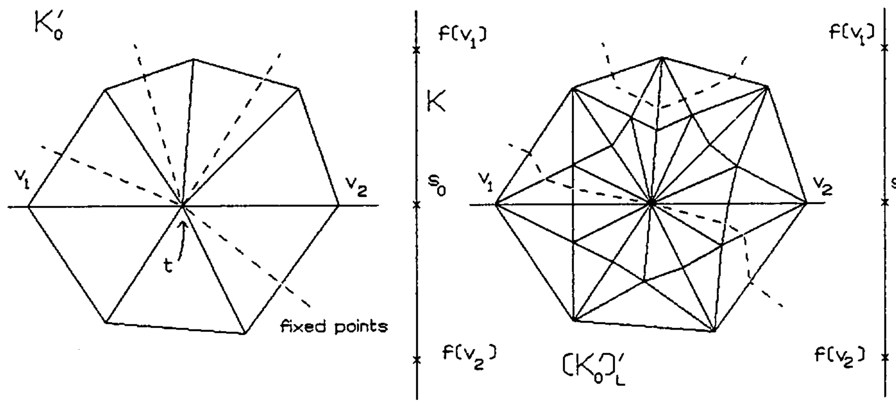

Figure 2.

Let $t_0 = \langle a_0, \dots, a_p\rangle$ and for $i = 1$ and 2 let $s_i = f(v_i) * s_0$, let γ_i be the unique $(p+2)$-simplex of M_{s_i}' having v_i*t_0 as a face, let $g(b(\gamma_i)) = f(a_{j_i})$, and let $\sigma_i = \langle a_0, \dots, \hat{a}_{j_i}, \dots, a_p, b(v_i*t_0), b(\gamma_i)\rangle$. Then $g(\sigma_i) = s_i$ and $|\pi|(\sigma_i) \subset s_i$ for $i = 1$ and 2. So $\mathrm{cl}(\sigma_i)$ contains a fixed point of $|g|$. No proper face of σ_i contains a fixed point of $|g|$. [Proof: Using Theorem 1 no face of $\langle a_0, \dots, \hat{a}_{j_i}, \dots, a_p\rangle$ contains a fixed point of $|g|$. Any face of σ_i having $b(v_i*t_0)$ as a vertex but not $b(\gamma_i)$ is projected into s_0, but the image under $|g|$ of any such face is not s_0. Any face having $b(\gamma_i)$ as a vertex but not $b(v_i*t_0)$ is projected into s_i, but the image under $|g|$ of such a face is a face of s_0. If $t = \langle a_0, \dots, \hat{a}_l, \dots, \hat{a}_{j_i}, \dots, a_p, b(v_i * t_0), b(\gamma_i)\rangle$ where $0 \le l \le p$, $l \ne j_i$, then $|\pi|(t) \subset s_i$ but $g(t) \ne s_i$.] Thus for $i = 1$ and 2, $\sigma_i * t_0$ is a simplex of $(K_0')_L'$ and $\sigma_i * t_0$ contains a line segment of fixed points of $|g|$ with endpoints in σ_i and t_0.

Let $\mu = \langle a_0, \dots, a_p, b(t_{p+1}), \dots, b(t_q)\rangle$ be a simplex in $st(t_0, (K_0')_L')$ and suppose μ contains a fixed point of $|g|$. Let $g(\mu) = s'$. Because $\mu \subset t_q$ and $|\pi|(\mu)$ is a subset of s', $|\pi|(t_q) \subset s'$, that is, $t_q \in M_{s'}'$. If for each j, $p+1 \le j \le q$, t_j is not the simplex $v_1 * t_0$ or the simplex $v_2 * t_0$, then $g(\mu) = s_0$. So $s' = s_0$ and $t_q \in M_{s_0}'$. But $t_0 < t_q$, so $q = p$ and $\mu = t_0$. If for some j_0, $t_{j_0} = v_i * t_0$ for $i = 1$ or 2, then $g(\mu) = s_i$, so $s_i = s'$ and

$t_q \in M'_{s_i}$. But γ_i is the unique simplex in M'_{s_i} having $v_i * t_0$ as a proper face, thus $\mu = \sigma_i * t_0$ for $i = 1$ or 2 [note that if $\mu = \langle a_0, \dots, a_p, b(v_i * t_0)\rangle$, then μ has no fixed points because $|\pi|(\mu) \subset s_0$ but $g(\mu) = s_i$]. Thus the only simplexes in $st(t_0, (K'_0)'_L)$ containing fixed points of $|g|$ are $\sigma_1 * t_0$, $\sigma_2 * t_0$, and t_0. Since $f(v_1) \neq f(v_2)$, $\sigma_1 \neq \sigma_2$ and so $\mathrm{Fix}(|g|) \cap |st(t_0, (K'_0)'_L)|$ is a 1-manifold. □

Condition 1. *A subdivision K''_0 of K'_0, a simplicial map $g : K''_0 \to K$, and a simplex $t_0 \in K''_0$ containing a fixed point of $|g|$ satisfy Condition* 1 *if for each component A of $(\mathrm{Fix}(|g|) \cap |st(t_0, K''_0)|) - t_0$, if $t_A \in K''_0$ is the simplex of $st(t_0, K''_0)$ that contains A (see Proposition 3.2) and v is a vertex in $lk(t_A, K''_0)$, then $g(v)$ is a vertex of $g(t_A)$.*

Addendum 3.5. *The simplex t_0, the subdivision $(K'_0)'_L$, and the map g satisfy Condition* 1.

Proof. Let A be a component of $(|\,\mathrm{Fix}(|g|)| \cap |st(t_0, (K'_0)'_L)|) - t_0$ and let the simplex t_A in $st(t_0, (K'_0)'_L)$ be the carrier of A. By construction $t_A = \sigma_1 * t_0$ or $t_A = \sigma_2 * t_0$. Suppose $t_A = \sigma_1 * t_0$ and v is a vertex in $lk(t_A, (K'_0)'_L)$. Because $b(v_1 * t_0)$ is a vertex of $\sigma_1 * t_0$, $b(v_1 * t_0)$ is not a vertex in $lk(t_A, (K'_0)'_L)$. If $b(v_2 * t_0) \in lk(t_A, (K'_0)'_L)$, then $b(v_2 * t_0) * t_A$ is a simplex of $(K'_0)'_L$. Since $b(v_2 * t_0)$ is not a vertex of t_A, $b(v_2 * t_0) * t_A$ is a $(p+3)$-simplex in $(M'_{s_1})'_L$ but this contradicts $\dim((M'_{s_1})'_L) = p + 2$. Thus $v \neq b(v_i * t_0)$ for $i = 1$ and 2. Since $t_0 < \sigma_1$, $v \in lk(t_0, (K'_0)'_L)$. By definition of g, $g(v)$ is a vertex of s_0. Since $s_0 < s_1$ and $g(t_A) = s_1$, $g(v)$ is a vertex of $g(t_A)$. □

Lemma 3.6. *Assume t_0 is a p-simplex of K'_0, t_0 contains a fixed point of $|f|$, $f(t_0) = s_0$, and $\dim(s_0) = p$. Assume $lk(t_0, M_{s_0})$ consists of two vertices v_1 and v_2 and $f(v_1) \neq f(v_2)$. Let L be the subcomplex of K'_0 consisting of t_0 and all simplexes that do not have t_0 as a face. If there is an arc c_n in $lk(s_0, K)$ with endpoints $f(v_1)$ and $f(v_2)$ consisting of n 1-simplexes, then there is a simplicial map $|g| : |(K'_0)'_L| \to |K|$ $\mathcal{U}$-homotopic to $|f|$ rel $|L|$ such that $lk(t_0, (M'_{s_0})'_L)$ consists of two vertices v'_1 and v'_2, and there is an arc c_{n-1} in $lk(s_0, K)$ with endpoints $g(v'_1)$ and $g(v'_2)$ consisting of $n - 1$ 1-simplexes.*

Proof. Let c_n be the arc $\langle f(v_1)\rangle \cup \langle f(v_1), b_1\rangle \cup \langle b_1\rangle \cup \ldots \cup \langle b_{n-1}, f(v_2)\rangle \cup \langle f(v_2)\rangle$. Define g on the vertices of $(K'_0)'_L$ as follows: if a is a vertex of L, $g(a) = f(a)$; if t is a simplex of K'_0 and t_0 is a proper face of t and $t \neq v_i * t_0$ for $i = 1$ or 2, then let $g(b(t))$ be any vertex of s_0; let $g(b(v_1 * t_0)) = b_1$ and $g(b(v_2 * t_0)) = f(v_2)$.

Let $\sigma = \langle a_0, \dots a_l, b(t_{l+1}), \dots, b(t_m)\rangle$ be a simplex of $(K'_0)'_L$. By definition, $g(a_j) < f(t_{l+1})$ for $0 \le j \le l$ and, for $l + 1 \le j \le m$, $g(b(t_j)) = b_1$ or $f(v_2)$ or is a vertex of s_0. If for each j, $g(b(t_j))$ is a vertex of s_0, then, since

$s_0 < f(t_{l+1})$, $g(\sigma)$ is a face of $f(t_{l+1})$. If for some j_0, $g(b(t_{j_0})) = b_1$, then $j_0 = l+1$ and $t_{j_0} = v_1 * t_0$. So $g(\sigma) < \langle f(v_1), b_1 \rangle * s_0$ because for $j > l + 1$ $g(b(t_j)) < s_0$, for $j \leq l$ $g(a_j) < f(v_1) * s_0$, and $g(b(v_1 * t_0)) = b_1$. If for some j_0, $g(b(t_{j_0})) = f(v_2)$, then $j_0 = l+1$ and $t_{j_0} = v_2 * t_0$. So $g(\sigma) < f(v_2) * s_0$, because for $j > l+1$ $g(b(t_j)) < s_0$, while for $j \leq l$ $g(a_j) < f(v_2) * s_0$ and $g(b(v_2 * t_0)) = f(v_2)$. Since $f(v_2) \in lk(s_0, K)$ and $\langle f(v_1), b_1 \rangle \in lk(s_0, K)$ in all cases $g(\sigma)$ is a simplex of K. So $g : (K_0')_L' \to K$ is simplicial. Furthermore, since $\sigma \subset t_m$, $|f|(\sigma) \subset f(t_m)$; because $t_0 < t_m$, $s_0 < f(t_m)$; and $g(\sigma)$ is a subset of either $\mathrm{cl}(f(t_m))$ or $\mathrm{cl}(\langle f(v_1), b_1 \rangle * s_0)$ or $\mathrm{cl}(f(v_2) * s_0)$. So by Lemma 3.3 $|g|$ is $\mathcal{U}$-homotopic to $|f|$ rel $|L|$.

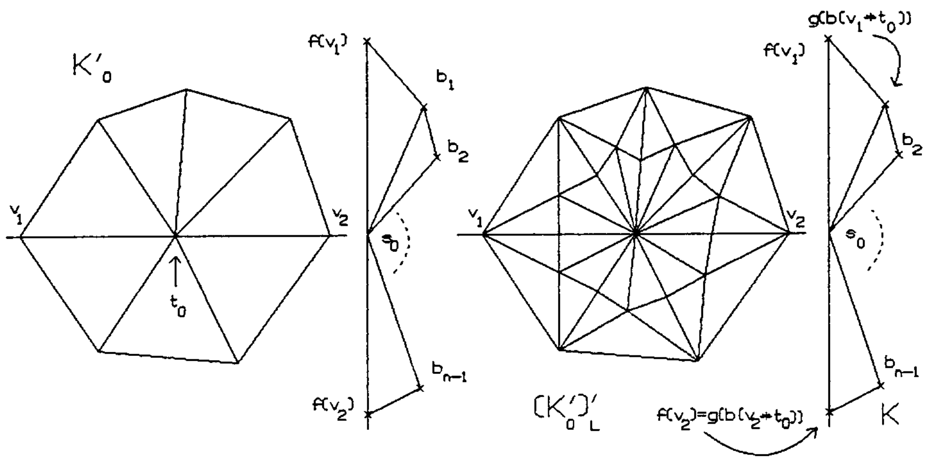

Figure 3.

Let $v_1' = b(v_1 * t_0)$ and $v_2' = b(v_2 * t_0)$. Then $lk(t_0, (M_{s_0})_L')$ consists of v_1' and v_2' and $c_{n-1} \equiv \langle g(v_1') \rangle \cup \langle g(v_1'), b_2 \rangle \cup \ldots \cup \langle b_{n-1}, g(v_2') \rangle \cup \langle g(v_2') \rangle$ is an arc in $lk(s_0, K)$ with endpoints $g(v_1)$ and $g(v_2')$ consisting of $n-1$ 1-simplexes. $\square$

Proposition 3.7. *Assume t_0 is a p-simplex of K_0', t_0 contains a fixed point of $|f|$, $f(t_0) = s_0$, and $\dim(s_0) = p$. Assume $lk(t_0, M_{s_0}')$ consists of two vertices v_1 and v_2 and that $f(v_1)$ and $f(v_2)$ are in the same component of $|lk(s_0, K)|$. Let L be the subcomplex of K_0' consisting of all simplexes that do not have t_0 as a face. Then there is a simplicial map $|g| : |(K_0')_L'| \to |K|$*

$\mathcal{U}$-homotopic to $|f|$ rel $|L|$ such that $|g|$ has no fixed points in t_0.

Proof. Since $f(v_1)$ and $f(v_2)$ are in the same component of $|lk(s_0, K)|$, by using Lemma 3.6 repeatedly if necessary, assume $f(v_1) = f(v_2)$. Define g on the vertices of $(K'_0)'_L$ as follows: If a is a vertex of L, $g(a) = f(a)$; if t_0 is a proper face of t and $t \neq v_i * t_0$ for $i = 1$ or 2 then let $g(b(t))$ be any vertex of s_0; for $i = 1$ and 2 let $g(b(v_i * t_0)) = f(v_1)$; let $g(b(t_0)) = f(v_1)$.

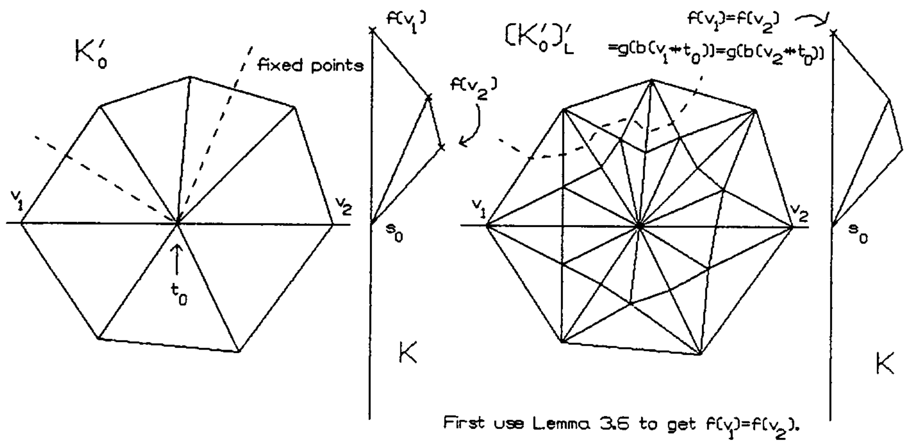

Figure 4.

Let $\sigma = \langle a_0, \ldots, a_l, b(t_{l+1}), \ldots, b(t_m)\rangle$ be a simplex of $(K'_0)'_L$. By definition, $g(a_j)$ is a vertex of $f(t_{l+1})$ for $0 \leq j \leq l$, and, for $l+1 \leq j \leq m$, $g(b(t_j)) = f(v_1)$ or a vertex of s_0. If for each j, $g(b(t_j))$ is a vertex of s_0, then, since $s_0 < f(t_{l+1})$, $g(\sigma) < f(t_{l+1})$. If for some j_0, $g(b(t_{j_0})) = f(v_1)$, then $g(\sigma)$ is a face of $f(v_1) * s_0$. In either case $g(\sigma)$ is a simplex of K. So $g : (K'_0)'_L \to K$ is a simplicial map. Furthermore, $|f|(\sigma) \subset f(t_m)$, $s_0 < f(t_m)$, and $g(\sigma)$ is a subset of either $\mathrm{cl}(f(t_m))$ or $\mathrm{cl}(f(v_1) * s_0)$. So by Lemma 3.3 $|g|$ is $\mathcal{U}$-homotopic to $|f|$ rel $|L|$.

Now $t_0 = \cup \tau_i$ where each $\tau_i = \langle a_0, \ldots, a_{q_i}, b(t_0)\rangle$ is a simplex in $(K'_0)'_L$. Because $g(b(t_0))$ is not a vertex of s_0, $g(\tau_i) \neq s_0$. Since $\tau_i \subset t_0$, $|\pi|(\tau_i) \subset s_0$. So no point of t_0 can be a fixed point of $|g|$. □

Proposition 3.8. *Assume t_0 is a p-simplex of K'_0, t_0 contains a fixed point of $|f|$, $f(t_0) = s_0$, and $\dim(s_0) = p$. Assume $lk(t_0, M_{s_0})$ consists of just one vertex v and that $f(v)$ is not a vertex of s_0. Let L be the subcomplex of K'_0 consisting of all simplexes that do not have t_0 as a face. Then there is a* *simplicial map* $|g| : |(K'_0)'_L| \to |K|$ *$\mathcal{U}$-homotopic to* $|f|$ *rel* $|L|$ *such that* $|t_0|$ *contains no fixed points of* $|g|$.

Proof. Define g on the vertices of $(K'_0)'_L$ as follows: If a is a vertex of L, $g(a) = f(a)$; if t_0 is a proper face of t and $t \neq v * t_0$, then let $g(b(t))$ be any vertex of s_0; let $g(b(v * t_0)) = f(v)$ and let $g(b(t_0)) = f(v)$.

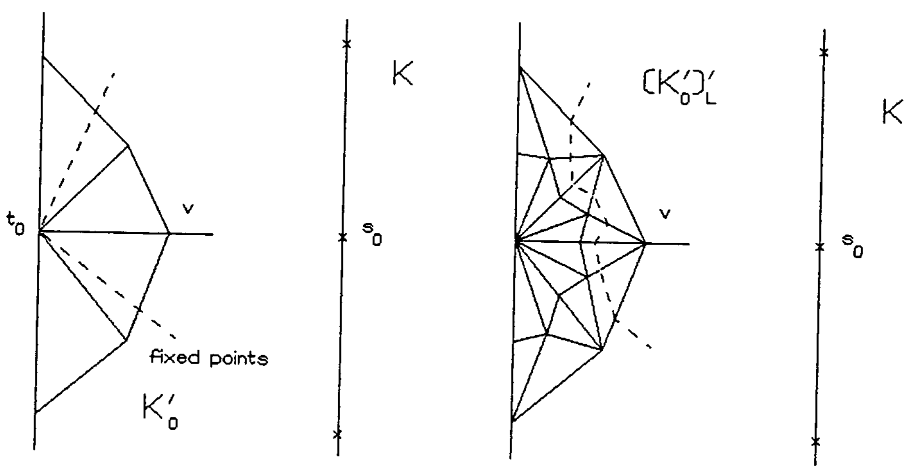

Figure 5.

Let $\sigma \in (K'_0)'_L$ be the simplex $\langle a_0, \dots, a_l, b(t_{l+1}), \dots, b(t_m)\rangle$. By definition, $g(a_j)$ is a vertex of $f(t_{l+1})$ for $0 \leq j \leq l$, and, for $l+1 \leq j \leq m$, $g(b(t_j)) = f(v)$ or is a vertex of s_0. If for each j, $g(b(t_j))$ is a vertex of s_0, then, since $s_0 < f(t_{l+1})$, $g(\sigma) < f(t_{l+1})$. If for some j_0, $g(b(t_{j_0})) = f(v)$, then $j_0 = l + 1$ and t_{j_0} is the simplex $v * t_0$ or t_0. So $g(\sigma) < f(v) * s_0$. In all cases $f(\sigma)$ is a simplex of K and so $g : (K'_0)'_L \to K$ is simplicial. Also $|f|(\sigma) \subset f(t_m)$, $s < f(t_m)$, and $g(\sigma)$ is a subset of either $\mathrm{cl}(f(t_m))$ or $\mathrm{cl}(s_0 * f(v))$, so by Lemma 3.3 $|g|$ is $\mathcal{U}$-homotopic to $|f|$ rel $|L|$.

The same proof as was used in Proposition 3.7 shows that t_0 contains no fixed points of $|g|$. □

Proposition 3.9. *Assume* t_0 *is a* $(p+1)$*-simplex of* K'_0, t_0 *contains a fixed point of* $|f|$, $f(t_0) = s_0$, $\dim(s_0) = p$, *and* $lk(s_0, K) \neq \emptyset$. *Let* L *be the subcomplex of* K'_0 *consisting of all simplexes that do not have* t_0 *as a face. Then there is a simplicial map* $|g| : |(K'_0)'_L| \to |K|$ *$\mathcal{U}$-homotopic to* $|f|$ *rel* $|L|$ *such that* t_0 *contains no fixed points of* $|g|$.

Proof. Let w_0 be the vertex of s_0 that is the image under f of two vertices of t_0 and let b be a vertex in $lk(s_0, K)$. Define g on the vertices of $(K'_0)'_L$ as follows: If a is a vertex of L, $g(a) = f(a)$; if t_0 is a proper face of the simplex t in K_0', then let $g(b(t))$ be the vertex w_0; let $g(b(t_0))$ be the vertex b.

Let $\sigma = \langle a_0, \ldots, a_l, b(t_{l+1}), \ldots, b(t_m)\rangle$ be a simplex of $(K_0')_L'$. By definition $g(a_j)$ is a vertex of $f(t_{l+1})$ for $0 \leq j \leq l$, and, for $l+1 \leq j \leq m$, $g(b(t_j))$ is either the vertex b or the vertex w_0. If for each j, $g(b(t_j)) = w_0$, then, since $s_0 < f(t_{l+1})$, $g(\sigma) < f(t_{l+1})$. If for some j_0, $g(b(t_{j_0})) = b$, then $j_0 = l + 1$ and $t_{j_0} = t_0$. So $g(\sigma)$ is a face of $b * s_0$. In either case $g(\sigma)$ is a simplex of K. So $g : (K_0')_L' \to K$ is simplicial. Moreover, $|f|(\sigma) \subset f(t_m)$, $s_0 < f(t_m)$, and $g(\sigma)$ is a subset of either $\mathrm{cl}(f(t_m))$ or $\mathrm{cl}(s_0 * b)$, thus, by Lemma 3.3, $|g|$ is $\mathcal{U}$-homotopic to $|f|$ rel $|L|$.

Now $t_0 = \cup\tau_i$ where each $\tau_i = \langle a_0, \ldots, a_{q_i} b(t_0)\rangle$ is a simplex in $(K_0')_L'$. Because $g(b(t_0))$ is not a vertex of s_0, $g(\tau_i) \neq s_0$. Since $\tau_i \subset t_0$, $|\pi|(\tau_i) \subset s_0$. So τ_i contains no fixed points of $|g|$ and so no point of t_0 can be a fixed point of $|g|$. □

Addendum 3.10. *If $\sigma \in (K_0')_L'$ contains a fixed point of $|g|$, $|\pi|(\sigma) \subset s, \dim(s) \leq p$, and σ is not a face of t_0, then $\sigma \in L$ and $st(\sigma, K_0') \subset L$.*

Note that if t is a proper face of t_0, then, by definition of L, $t \in L$.

Proof. Let σ be the simplex $\langle a_0, \ldots, a_l, b(t_{l+1}), \ldots, b(t_q)\rangle$ and suppose $q \neq l$. Because $\sigma \subset t_q$ and $|\pi|(\sigma) \subset s$, $|\pi|(t_q) \subset s$. So $t_0 < t_q$, $|\pi|(t_q) \subset s$, and $|\pi|(t_0) \subset s_0$, by Lemma 3.1(a), $s_0 < s$. Since $\dim(s) \leq \dim(s_0)$, $s = s_0$. So $t_q \in M_{s_0}'$. But t_0 is a principal simplex of M_{s_0}' and $t_0 < t_q$, thus $t_q = t_0$ and $\sigma \subset t_0$. This contradicts the fact that t_0 contains no fixed points of $|g|$. Thus $q = l$ and $\sigma \in L$.

Suppose $\sigma * t_0 \in K_0'$. By Lemma 3.1(c), $s_0 < s$ or $s < s_0$. Since $\dim(s) \leq \dim(s_0)$, $s < s_0$. By Lemma 3.1(b), $\sigma * t_0 \in M_{s_0}'$. Because t_0 is a principal simplex of M_{s_0}' and t_0 is a face of $t_0 * \sigma$, $t_0 = \sigma * t_0$. But this contradicts the fact that σ is not a face of t_0. Thus $\sigma * t_0 \notin K_0'$. On the other hand if $t \in st(\sigma, K_0')$ and $t_0 < t$, then $t_0 * \sigma \in K_0'$. Thus $\mathrm{st}(\sigma,\ K_0') \subset L$. □

Addendum 3.11. *Assume $t \in (K_0')_L'$ contains a fixed point of $|g|$, t projects into the simplex s of K, $\dim(s) < p$, $t < t_0$ and $\mathrm{Fix}(|f|) \cap |st(t,\ K_0')|$ is a 1-manifold. If t, K_0', and $|f|$ satisfy Condition 1, then $t \in L$, $\mathrm{Fix}(|g|) \cap |st(t,\ K_0')_L'|$ is a 1-manifold and t, $(K_0')_L'$, and g satisfy Condition 1.*

Proof. The simplexes of $st(t, K_0')$ can be split into two sets $I_1 = \{t' \in K_0' | t_0$ is not a face of $t'\}$ and $I_2 = \{t' \in K_0' | t_0 < t'\}$. By assumption $\mathrm{Fix}(|f|) \cap |st(t, K_0')|$ is the union of two line segments: One of these is contained in t_0, hence the other line segment cannot meet any other simplex $t' \in I_2$ since otherwise $t' \cap \mathrm{Fix}(|f|)$ would be at least 2-dimensional. Thus the second line segment is contained in a simplex in I_1. On the other hand, $st(t, (K_0')_L') = I_1 \cup I_2'$

where $I_2' = \{\langle v_0, v_1, \dots, v_q, b(t_{q+1}), \dots, b(t_n)\rangle | t = \langle v_0, \dots, v_q\rangle\}$. Notice that $g = f$ on simplexes of I_1. It remains to find a simplex in I_2' containing a line segment in $\mathrm{Fix}(|g|)$ and to show that no other simplex in I_2' contains a fixed point.

Since $\mathrm{Fix}(|f|) \cap |st(t, K_0')|$ is a 1-manifold and t_0 contains a fixed point of $|f|$, a component, A, of $(\mathrm{Fix}(|f|) \cap |st(t, K_0')|) - t$ is contained in t_0. By Theorem A this component is a line segment with one endpoint the fixed point in t and the other endpoint a fixed point in a face t' of t_0. (Note that $t' \neq t$ else $A \subset t$.) Note that $t' * t = t_0$ because $t' * t$ is a simplex of K_0' and $A \subset t' * t$. By Lemma 2.4 there is a vertex v of t_0 such that $|\pi|(v) \in s_0$. But v is not a vertex of t, so, since $t' * t = t_0$, v must be a vertex of t'. So by Lemma 2.5 $|\pi|(t') \subset s_0$. Since t' contains a fixed point of $|f|$, $f(t') = s_0$. Thus t' is a p-face of t_0 and t' contains a fixed point of $|f|$.

Let $t = \langle a_0, \dots, a_q\rangle$ and $t_0 = \langle a_0, \dots, a_q, \dots, a_{p+1}\rangle$. Now, all but one vertex a_i of t_0 is a vertex of t'. This vertex must be a vertex of t since otherwise $t < t'$ implies $t' * t = t' \neq t_0$. So $t' = \langle a_0, \dots, \hat{a}_i, \dots, a_q, \dots, a_{p+1}\rangle$.

Let $s_1 = b * s_0$ and let γ be the unique $(p+2)$-simplex in M_{s_1}' having t_0 as a face. Since $f(t') = s_0$, $f(a_i) = w_0$ and there is a vertex a_j, $j \neq i$ of t' such that $w_0 = f(a_j)$. Let $\mu = \langle a_0, \dots, \hat{a}_i, \dots, a_q, \dots, \hat{a}_j, \dots, a_{p+1}, b(t_0), b(\gamma)\rangle$. Then μ is a $(p+1)$-simplex in $(K_0')_L'$ contained in γ, $g(\mu) = b * s_0$, $|\pi|(\mu) \subset b * s_0$ and no proper face of μ contains a fixed point of $|g|$ so μ contains a fixed point of $|g|$. Thus $t * \mu$ contains a line segment of fixed points of $|g|$ with an endpoint in t.

Suppose $\sigma \in st(t, (K_0')_L')$ contains a fixed point of $|g|$. Then $\sigma = \langle v_0, v_1, \dots, v_r, b(t_{r+1}), \dots, b(t_n)\rangle$ where $\langle v_0, v_1, \dots, v_r\rangle \in L$, $t_{r+1}, \dots, t_n \notin L$ and $\langle v_0, v_1, \dots, v_r\rangle < t_{r+1} < \cdots < t_n$. If $n = r$, then $\sigma = \langle v_0, v_1, \dots, v_n\rangle \in L$ so $\sigma = t$ or σ is a simplex of K_0' having t as a proper face but not having t_0 as a face. Now suppose $n \neq r$ and let $g(\sigma) = s'$. Because $\sigma \subset t_n$ and $|\pi|(\sigma) \subset s'$, $|\pi|(t_n) \subset s'$. So $t_n \in M_{s'}'$. Since $t_0 < t_{r+1}$ each vertex of $\langle v_0, v_1, \dots, v_r\rangle$ is a vertex of t_0 or $lk(t_0, K_0')$, so by Condition 1 $g(v_i)$ is a vertex of s_0. If $t_0 \neq t_{r+1}$, then $g(\sigma) < s_0$. So $s' < s_0$ and because s_0 is a p-simplex, t_n is at most a $(p+1)$-simplex. But this contradicts the fact that the $(p+1)$-simplex t_0 is a proper face of t_n. Thus $t_0 = t_{r+1}$ and so $g(\sigma) < b * s_0$. If $g(\sigma) \neq b * s_0$, then $\dim(t_n)$ is at most $p+1$. So $\sigma = \langle v_0, \dots, v_r, b(t_0)\rangle$. But then $|\pi|(\sigma) \subset s_0$, yet $g(\sigma)$ has b as a vertex. This contradicts the assumption that σ contains a fixed point of $|g|$. So $g(\sigma) = b * s_0$ and t_n is at most a $(p+2)$-simplex of M_{b*s_0}'. Since γ is the unique $(p+2)$-simplex in M_{b*s_0}' having t_0 as a face $t_n = \gamma$, so $\sigma = \langle v_0, v_1, \dots, v_r, b(t_0), b(\gamma)\rangle$. Since $\langle v_0, \dots, v_r\rangle < t_0$, $\sigma = \langle a_0, \dots, a_i, \dots, a_q, \dots, a_j, \dots, \hat{a}_l, \dots, a_{p+1}, b(t_0), b(\gamma)\rangle$. [Remember that $f(a_i) = f(a_j) = f(b(\gamma)) = w_0$.] Since $g(\sigma) = b * s_0$, $j = l$. So $\sigma = t * \mu$. Thus the only simplexes in $st(t, (K_0')_L')$ containing fixed points are $t * \mu,\ t$, and simplexes t_i in K_0' having t as a face but not t_0 as a face. Since $|st(t, K_0')| \cap \mathrm{Fix}(|f|)$ is a 1-manifold at most one such t_i exists. Thus $|st(t, (K_0')_L')| \cap \mathrm{Fix}(|g|)$ is a 1-manifold.

To see that $(K_0')_L',\ g$, and t satisfy Condition 1, suppose v is a vertex in $lk(t * \mu, (K_0')_L')$. Then $v = a_j$ or $v = b(t_r)$ where $t_r \in K_0'$ and $t_0 < t_r$. By definition of g, $g(v)$ is a vertex of $s_0 * b$. □

The next lemma holds true for the construction of $(K_0')_L'$ and g as given in Propositions 3.4, 3.7 and 3.8 and Lemma 3.6. The proof is given for the construction in Proposition 3.4 but is similar, almost word for word, for the other constructions of g and $(K_0')_L'$.

Lemma 3.12. *If $\sigma \neq t_0$ is a simplex of $(K_0')_L'$ and σ contains a fixed point of $|g|$ and σ projects into a simplex s of K of dimension $\leq p$, then σ is also a simplex of K_0'; furthermore $st(\sigma, K_0')_L' \subset K_0'$.*

Proof. Let $\sigma = \langle a_0, \ldots a_l, b(t_{l+1}), \ldots, b(t_q)\rangle$ and suppose $q \neq l$. Because $\sigma \subset t_q$, $|\pi|(t_q) \subset s$. Also $t_0 < t_q$ and $|\pi|(t_0) \subset s_0$, so by Lemma 3.1(a), $s_0 < s$. Since $\dim(s) \leq \dim(s_0)$, $s = s_0$. So $t_q \in M_{s_0}'$. But t_0 is a face of exactly three simplexes of M_{s_0}', namely $t_0,\ v_1 * t_0$, and $v_2 * t_0$. So $\sigma = \langle a_0, \ldots a_l, b(v_i * t_0)\rangle$ for $i = 1$ or 2. But $|\pi|(\sigma) \subset s_0$ yet $g(\sigma) \neq s_0$ because $g(b(v_i * t_0))$ is not a vertex of s_0 for $i = 1$ or 2. This contradicts the fact that σ contains a fixed point of $|g|$. Thus $q = l$ and $\sigma \in L$.

Suppose $\sigma * t_0 \in K_0'$. By Lemma 3.1(c) $s_0 < s$ or $s < s_0$. Since $\dim(s) \leq \dim(s_0)$, $s < s_0$. So by Lemma 3.1(b) $\sigma * t_0 \in M_{s_0}'$. Since $t_0 < \sigma * t_0$, $\sigma * t_0 = v_1 * t_0$ or $v_2 * t_0$ or t_0. Since both σ and t_0 contain a fixed point of $|f|$, $\sigma * t_0$ contains a line segment of fixed points of $|f|$. By Theorem A $\sigma * t_0 \neq t_0$. Since $|\pi|(v_i * t_0) \subset s_0$ and $f(v_i)$ is not a vertex of s_0 for $i = 1$ or 2, the simplexes $v_1 * t_0$ and $v_2 * t_0$ do not contain fixed points of $|f|$, thus $\sigma * t_0 \neq v_1 * t_0$ or $v_2 * t_0$. This means $\sigma * t_0 \notin K_0'$ or $\sigma = t_0$.

Now if t is a simplex in K_0' and $\sigma \neq t_0$ and $\sigma < t$ and $t_0 < t$, then $t_0 * \sigma < t$ which contradicts the fact that $t_0 * \sigma \notin K_0'$; so if $t \in st(\sigma, K_0')$, then t_0 is not a face of t so $t \in L$. □

Proposition 3.13. *There is a subdivision K_0'' of K_0' and a simplicial map $|g| : |K_0''| \to |K|$ $\mathcal{U}$-homotopic to $|f|$ such that*

(i) *if t is a $(p+1)$-simplex of K_0'' and t projects into a p-simplex s of K and t contains a fixed point of $|g|$, then $lk(s, K) = \emptyset$,*

(ii) *if t is a p-simplex of K_0'' and t projects into a p-simplex s of K and t contains a fixed point of $|g|$ and $lk(s, K) \neq \emptyset$, then*

  (a) *$lk(s, K)$ is not connected,*

  (b) *$\mathrm{Fix}(|g|) \cap |st(t, K_0'')|$ is a 1-manifold,*

(c) *Condition* 1 *is satisfied by* t, K_0'', *and* g.

Proof. We work by induction on the skeleta of K.

Assume there is a subdivision K_0'' of K_0' and a simplicial map $|g| : |K_0''| \to |K|$ $\mathcal{U}$-homotopic to $|f|$ such that over the $(q-1)$-skeleton of K, statements (i) and (ii) hold. The induction starts trivially with the (empty) (-1)-skeleton.

If t_0 is a $(q+1)$-simplex of K_0'' and t_0 projects into the q-simplex s_0 of K and t_0 contains a fixed point of $|g|$ and $lk(s_0, K) \neq \emptyset$, then by Proposition 3.9, there is a subdivision $(K_0'')'$ of K_0'' and a simplicial map $|g'| : |(K_0'')'| \to |K|$ homotopic to $|g|$ such that t_0 contains no fixed points of $|g'|$. By Addendum 3.10 if t is a simplex of K_0'' and t contains a fixed point of $|g'|$ and t projects into the q-skeleton of K, then t is also a simplex of K_0'. By definition of $|g'|$, $|g|$ and $|g'|$ agree on t. Thus if x is a fixed point of $|g'|$ and x projects into the q-skeleton of K, then x is a fixed point of $|g|$. So no new fixed points are created over the q-skeleton of K.

Furthermore, suppose t is a p-simplex of $(K_0'')'$, $p \leq q-1$, and t contains a fixed point of $|g'|$. If t is not a face of t_0 then by Addendum 3.10 each simplex of $st(t, (K_0'')')$ is also a simplex of K_0''. So if $\mathrm{Fix}(|g|) \cap |st(t, K_0'')|$ is a 1-manifold, then $\mathrm{Fix}(|g'|) \cap |st(t, (K_0'')')|$ is a 1-manifold; if Condition 1 is satisfied by t, K_0'' and g, then Condition 1 is satisfied by t, $(K_0'')'$ and g'. And if t is a face of t_0, then, by Addendum 3.11, $\mathrm{Fix}(|g'|) \cap |st(t, (K_0'')')|$ is a 1-manifold and t, $(K_0'')'$, and g' satisfy Condition 1. Apply Proposition 3.9 to each $(q+1)$-simplex t of K_0'' such that t projects into a q-simplex s of K, t contains a fixed point of $|g|$ and $lk(s, K) \neq \emptyset$, subdividing the previously defined K_0'' each time. After a finite number of applications of Proposition 3.9, there is a subdivision K_0'' of K_0' and a simplicial map $|g| : |K_0''| \to |K|$ $\mathcal{U}$-homotopic to $|f|$ such that for $p = 0, 1, \ldots, q$ statement (i) holds and for $p = 0, 1, \ldots, q-1$ statement (ii) holds.

Now suppose t_0 is a q-simplex of K_0'' and t_0 projects into the q-simplex s_0 of K and t_0 contains a fixed point of $|g|$. If v is a vertex in $lk(t_0, M'_{s_0})$ and $g(v)$ is a vertex of s_0, then the $(q+1)$-simplex $v * t_0$ contains a line segment of fixed points of $|g|$ and so $lk(s_0, K) = \emptyset$. Thus if $lk(s_0, K) \neq \emptyset$, then $g(lk(t_0, K_0'')) \subset lk(s_0, K)$. If $g(lk(t_0, K_0''))$ is contained in a component of $|lk(s_0, K)|$ then, either by Proposition 3.7 or by Proposition 3.8, there is a subdivision $(K_0'')'$ of K_0'' and a simplicial map $|g'| : |(K_0'')'| \to |K|$ $\mathcal{U}$-homotopic to $|g|$ such that t_0 contains no fixed points of $|g'|$. By Lemma 3.12, if t is a simplex of $(K_0'')'$ and t contains a fixed point of $|g'|$ and t projects into a simplex of dimension $\leq q$, then t corresponds to a simplex of K_0'' and each simplex in $st(t, K_0'')$ corresponds to a simplex of K_0'. Since $|g|$ and $|g'|$ agree on $st(t, K_0'')$, no new fixed points are added over the q-skeleton of K and statement (ii) holds for $p = 0, 1, \ldots q-1$. Apply either Proposition 3.7

or Proposition 3.8 to each q-simplex t of K_0'' such that t projects into a q-simplex s of K, t contains a fixed point of $|g|$ and $g(lk(t, M_s'))$ is contained in a component of $|lk(s, K)|$, subdividing the previously defined $(K_0'')'$ each time. Thus there is a subdivision K_0'' of K_0' and a simplicial map $|g| : |K_0''| \to |K|$ $\mathcal{U}$-homotopic to $|f|$ such that statement (i) holds for $p = 0, 1, \dots, q$, statement (ii) holds for $p = 0, 1, \dots, q-1$, and if t is a q-simplex of K_0'' and t projects into a q-simplex s of K and t contains a fixed point of $|g|$ and $lk(s, K) \neq \emptyset$, then $lk(t, M_s')$ consists of two vertices v_1 and v_2 and $g(v_1)$ and $g(v_2)$ are in different components of $lk(s, K)$.

If t_0 is a q-simplex of K_0'' and t_0 projects into a q-simplex s_0 of K and t_0 contains a fixed point of $|g|$ and $lk(s_0, K) \neq \emptyset$ but not connected, then by Proposition 3.4 there is a subdivision $(K_0'')'$ of K_0'' and a simplicial map $|g'| : |(K_0'')'| \to |K|$ homotopic to $|g|$ such that $\mathrm{Fix}(|g'|) \cap |st(t_0, (K_0'')')|$ is a 1-manifold. By Addendum 3.5, Condition 1 is satisfied by v_0, $(K_0'')'$ and g'. By Lemma 3.12, if $t \neq t_0$ is a simplex of $(K_0'')'$ and t projects into a simplex s of K of dimension $\leq q$ and t contains a fixed point of $|g'|$, then t is also a simplex of K_0''. Since $|g'|$ and $|g|$ agree on t, no new fixed points have been added. Furthermore each simplex in $st(t, (K_0'')')$ is also a simplex in K_0''. By definition of $|g'|$, $|g'|$ and $|g|$ agree on $|st(t, (K_0'')')|$. So if $\dim(t) = \dim(s)$ and $\mathrm{Fix}(|g|) \cap |st(t, K_0'')|$ is a 1-manifold then $\mathrm{Fix}(|g'|) \cap |st(t, (K_0'')')|$ is a 1-manifold. If Condition 1 is satisfied by t, K_0'' and g, then Condition 1 is satisfied by t, $(K_0'')'$ and g'.

Apply Proposition 3.4 to each q-simplex t of K_0'' such that t projects into a q-simplex s of K, t contains a fixed point of $|g|$ and $lk(s, K) \neq \emptyset$ but not connected, subdividing the previously defined $(K_0'')'$ each time.

So there is a subdivision K_0'' of K_0' and a simplicial map $|g| : |K_0''| \to |K|$ $\mathcal{U}$-homotopic to $|f|$ such that for $p = 0, 1, \dots, q$ statements (i) and (ii) hold. Thus, by induction there is a subdivision K_0'' of K_0' and a simplicial map $|g|$ $\mathcal{U}$-homotopic to $|f|$ satisfying (i) and (ii). □

Proof of Theorem B. Let K_0'' and $g : K_0'' \to K$ be the subdivision and map given by Proposition 3.13. Let t be a simplex of K_0'', let $|\pi|(t) \subset s$ in K and suppose t contains a fixed point of $|g|$. By Theorem A either $\dim(t) = \dim(s)$ or $\dim(t) = \dim(s) + 1$. If $\dim(t) = \dim(s) + 1$, by Proposition 3.13, $lk(s, K) = \emptyset$. If $\dim(t) = \dim(s)$ but $lk(s, K) \neq \emptyset$, then, by Proposition 3.13, $lk(s, K)$ is not connected. So $\mathrm{Fix}(|g|) \subset S_0 \cup S_1$. If $\dim(t) = \dim(s) + 1$, then by Theorem A, t contains a line segment of fixed points with endpoints in $\mathrm{bd}(t)$, also $st(t, K_0'') = \{t\}$. If $\dim(t) = \dim(s)$, then, by Theorem A, t contains exactly one fixed point. Suppose $\dim(t) = \dim(s)$ and $lk(s, K) = \emptyset$. Because $\dim(t) = \dim(s)$, $lk(t, M_s'')$ consists of one or two vertices v. Because $g(v * t)$ is a simplex of K having s as a face and $lk(s, K) = \emptyset$, $g(v)$ is a vertex of s. So $v * t$ contains a line segment of fixed points of $|g|$ with one endpoint in t. So $\mathrm{Fix}(|g|) \cap |st(t, K_0'')|$ is a 1-manifold or half-open line segment with endpoint in t and $t \in \partial M_s''$. Thus $\dim(S_1 \cap \mathrm{Fix}(|g|)) = 1$. Suppose $\dim(t) = \dim(s)$ but $lk(s, K) \neq \emptyset$. Since $lk(s, K) \neq \emptyset$, by Proposition 3.13, no simplex in $st(t, M_s'')$ other than t contains a fixed point of $|g|$. So $\dim(\mathrm{Fix}(|g|) \cap S_0) = 0$ and by Proposition 3.13(ii) (b) $\mathrm{Fix}(|g|) \cap |st(t, K_0'')|$ is a 1-manifold.

The reader will observe that our method achieves the claims in the final paragraph of Theorem B. □

4. Proof of Theorem C.

Lemma 4.1. *Let U^{n+1} and V^{n+1} be linear subspaces of the real vector space W^{2n+1} and let $U \cap V$ have dimension* 1*. Then* $(W, U, V, U \cap V) \cong (\mathbb{R}^{2n+1}; \mathbb{R}^{n+1} \times 0, 0 \times \mathbb{R}^{n+1}, 0^n \times \mathbb{R} \times 0^n)$.

Proof. Pick a basis, starting in $U \cap V$, extending to U and also to V, then to W. □

Lemma 4.2. *If in addition $T^{2n} \leq W^{2n+1}$ is such that* $\dim(U \cap T) = \dim(V \cap T) = n$ *and* $\dim(U \cap V \cap T) = 0$*, then* $(W, T, U, V, U \cap V) \cong (\mathbb{R}^{2n+1}; \mathbb{R}^n \times 0 \times \mathbb{R}^n, \mathbb{R}^{n+1} \times 0, 0 \times \mathbb{R}^{n+1}, 0 \times \mathbb{R} \times 0)$.

Proof. $U \cap V \cap T = \{0\}$. Start with a basis in $U \cap V$, extend to $(U \cap V) \dot{+} (U \cap T) \dot{+} (U \cap T)$. □

Lemma 4.2 implies:

Lemma 4.3. *Let W_+ be the closed half space on one side of T. Then* $(W_+, T, U_+, V_+, (U \cap V)_+) \cong (\mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}^n;\ \mathbb{R}^n \times 0 \times \mathbb{R}^n,\ \mathbb{R}^n \times \mathbb{R}_+ \times 0,\ 0 \times \mathbb{R}_+ \times \mathbb{R}^n,\ 0 \times \mathbb{R}_+ \times 0)$.

Proof of Theorem C. Let the PL manifold $|K|$ have dimension n. It follows from Theorem B that $\mathrm{Fix}(|g|)$ consists of straight-line segments each of which connects two points in the interiors of n-dimensional faces of an $(n+1)$-simplex of K_0''; see Figure 6. Transversality over a neighborhood of a follows from Lemma 4.3, and over a neighborhood of b from Lemma 4.1. Transversality over a neighborhood of c also follows from Lemma 4.3, for if we have another $(W_-,\ T,\ \tilde{U}_-,\ \tilde{V}_-,\ (\tilde{U} \cap \tilde{V})_-) \cong (\mathbb{R}^n \times \mathbb{R}_- \times \mathbb{R}^n,\ \mathbb{R}^n \times 0 \times \mathbb{R}^n,\ \mathbb{R}^n \times \mathbb{R}_- \times 0,\ 0 \times \mathbb{R}_- \times \mathbb{R}^n, 0 \times \mathbb{R}_- \times 0)$ we can piece the two together to get a PL homeomorphism $(W, T, U_+ \cup \tilde{U}_-,\ V_+ \cup \tilde{V}_-,\ (U \cap V)_+ \cup (\tilde{U} \cap \tilde{V})_-) \cong (\mathbb{R}^{2n+1};\ \mathbb{R}^n \times 0 \times \mathbb{R}^n, \mathbb{R}^{n+1} \times 0,\ 0 \times \mathbb{R}^{n+1},\ 0 \times \mathbb{R} \times 0)$. Thus the graph of $|g|$ is transverse to the graph of $|\pi|$. □

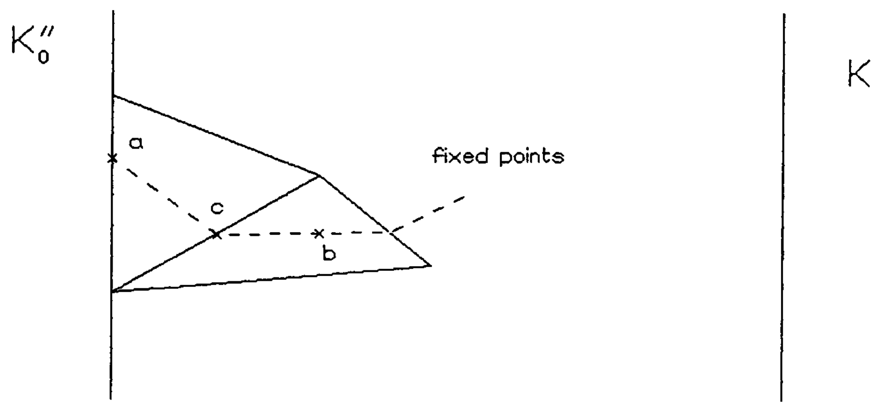

Figure 6.

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Binghamton University
Binghamton, NY 13902-6000
E-mail address: plavcha@ibm.net