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ON CLASSIFICATION OF HEEGAARD SPLITTINGS AND TRIANGULATIONS

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In this paper we consider Heegaard splittings of 3-manifolds. By using Gabai's concept of thin position on the 1-skeleton of some polyhedral decomposition, together with Casson-Gordon's concept of strong irreducibility, we prove the Main Theorem (4.0). This theorem will allow us to classify the Heegaard splittings of manifolds whose polyhedral decompositions are particularily nice, which we demonstrate via examples. Specifically, we use it to classify Heegaard splittings of several hyperbolic spaces, including the figure-8 knot complement (Example 6.4) and the genus 2 case of the 5_2-knot complement (Example 6.7).

1. Introduction.

In this paper, we continue the line of reasoning of [**6**], resulting in a theorem which we use to classify Heegaard splittings of several hyperbolic spaces. We note that in the spaces mentioned above, the Heegaard splittings coincide with the well–known unknotting tunnels, lending further creedence to the conjecture of [**8**].

In particular, the Main Theorem essentially says that all Heegaard splittings satisfying a certain technical property for a manifold with a "good" polyhedral decomposition can either be classified or the problem can be simplified in one of two ways. The manifolds arising from the simplification procedure may not be "good," so that a complete classification may not be possible. However, if M is a good manifold with a single boundary component of genus n, we can classify its Heegaard splittings of genus $n+1$ using this theorem.

In Section 2 we give the required definitions. We prove some technical lemmas in Section 3, the most important of which (3.4) follows the argument of ([**9**, 3.1]). In Section 4 we prove the Main Theorem. We give an independent result in Section 5; demonstrating that the manifolds having the necessary technical properties form an infinite class.

We give several examples of the use of the Main Theorem in Section 6; we classify Heegaard splittings of several hyperbolic manifolds obtained from the cusped census of [11], and discuss the "bad" case. We note that 22 of the manifolds in this census having four or less ideal tetrahedra have easy to find "good" IPD's, and thus that, theoretically, we can apply this theorem to each of them. As the number of exceptional cases is quite large for those with 3 or more edges, we will be content with classifying genus 2 Heegaard splittings in such examples.

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2. Definitions.

Throughout this paper we use the notation $N(*)$ to refer to a regular neighborhood of *, $^\circ *$ to refer to the interior of *, and the notation $\sharp(*)$ to refer to the number of components of *.

For a closed surface F, we refer to the number g such that $\chi(F) = 2 - 2g$ as the *genus of* F. Note that a nonorientable surface may have fractional genus, for example, genus$(\mathbf{R}P^2) = \frac{1}{2}$.

A *compression body* H is constructed by adding 2-handles to a (*closed connected surface*) $\times I$ along a collection of disjoint simple closed 2-sided curves on (surface) $\times\{0\}$, and capping off any resulting 2-sphere boundary components with 3-balls. The component (surface) $\times\{1\}$ of ∂H is denoted $\partial_+ H$ and the surface $\partial H \setminus \partial_+ H$, which may or may not be connected, is denoted $\partial_- H$. If $\partial_- H = \emptyset$, then H is a *handlebody*.

For a compact manifold M, a 3-tuple of manifolds $(H_0, H_1; S)$ is called a *Heegaard splitting* of M if H_0, H_1 is a pair of compression bodies with the property that $M = H_0 \cup H_1$ and $H_0 \cap H_1 = \partial_+ H_0 = \partial_+ H_1 = S$, for some closed connected surface S embedded in M. The surface S is called the *splitting surface* of the Heegaard splitting $(H_0, H_1; S)$, but we shall sometimes refer to either S or (H_0, H_1) as the Heegaard splitting. Two Heegaard splittings of M are considered *equivalent* if their splitting surfaces are isotopic. We note that $(H_0, H_1; S)$ is equivalent to $(H_1, H_0; S)$.

A *spine*, X, of a compression body H is a properly embedded 1–complex such that $H = N(\partial_- H \cup X)$. Let H be a compression body. Once and for all pick a point $p \in {}^\circ H$ and a point p_k in each component of $\partial_- H$. Let X

be a spine for H chosen so that:
(0) $X \cap \partial_- H = \cup\{p_k\}$,
(1) $X \cap p = \emptyset$ if H is not a handlebody, and
(2) $X \backslash (p \cup (\cup\{p_k\}))$ is a collection of open arcs.
Then we call the number $\sharp(X)$ the *complexity* of X, denoted $c(X)$. It is elementary to check that:

$$c(X) = \text{genus}(\partial_+ H) - \sum \text{genus}(\partial_- H) + \sharp(\partial_- H) - 1.$$

Let (H_0, H_1) be a Heegaard splitting for a manifold M. The *spinal complexity* of (H_0, H_1) is the pair $\{c(X_0), c(X_1)\}$ arranged in lexicographical order. The pair $\{a, b\}$ (with $a \le b$) is said to be *less than* the pair $\{c, d\}$ (with $c \le d$) if both $a \le c$ and $b \le d$, and at least one of the inequalities is strict. A Heegaard splitting (H_0, H_1) of M is said to have *lower spinal complexity* than $(\tilde{H}_0, \tilde{H}_1)$ of $\tilde{M}$ if $\{c(X_0), c(X_1)\} < \{c(\tilde{X}_0), c(\tilde{X}_1)\}$, see Figure 0.

An *elementary stabilization* $E(H_0,\, H_1)$ of S is the splitting surface obtained by taking the connected sum of pairs $(M, S)\ \sharp\ (S^3, T^2)$, for T^2 the standard unknotted torus in S^3. A *stabilization* of $(H_0,\, H_1)$ is a Heegaard splitting $E^k(H_0,\, H_1)$, such that $E^i(H_0,\, H_1)$ is an elementary stabilization of $E^{i-1}(H_0,\, H_1)$. A Heegaard splitting is *stabilized* if it is an elementary stabilization of another splitting. We note that this is equivalent to the existence of proper discs $D_i \subset H_i$ such that $\partial D_0 \cap \partial D_1 = \{\text{one point}\}$.

Following [**9**], we will say that a splitting surface is *reducible* if there exists an essential simple closed (two-sided) curve $c \subset S$ which bounds imbedded discs in both H_0 and H_1. A splitting surface is *weakly reducible* if there exist essential discs $D_0 \subset H_0$ and $D_1 \subset H_1$ with $\partial D_0 \cap \partial D_1 = \emptyset$. If S is reducible then it is clearly weakly reducible. If S is not weakly reducible, then S is said to be *strongly irreducible*. Again, if S is strongly irreducible, S is clearly irreducible.

Remark 2.0. If S is a splitting surface for an irreducible manifold M with genus$(S) > 1$, then it is well known that "S is reducible," and "S is stabilized" are equivalent statements. A proof of this can be found in [**9**].

Since all Heegaard splittings of manifolds of genus 0, $\frac{1}{2}$, and 1 are classified by [**10**], [**5**], and [**1**], we shall from this point assume that any manifold in question is of genus at least $\frac{3}{2}$, and thus that "reducible" and "stabilized" are equivalent.

Let R be a closed surface contained in the boundary of a 3-manifold M. Let U_0, U_1 be a pair of compression bodies defining a Heegaard splitting of M, and assume that $R \subset \partial U_0$. Note that there is some $R' \subset \partial U_0$ (R' can be empty, or have multiple components) so that $U_0 = N(R \cup R') \cup$ *1-handles*.

Let f be a homeomorphism $N(R) \to R \times I$ and $p : R \times I \to R$ the projection onto the first factor.

Let M_1, M_2 be two manifolds each with non-empty boundary and with Heegaard splittings (U_0, U_1), (V_0, V_1) respectively. Let R_1, R_2 be two homeomorphic surfaces such that $R_1 \subset \partial U_0 \subset \partial M_1$ and $R_2 \subset \partial V_0 \subset \partial M_2$, and let $f_i, p_i, i = 1, 2$, be the corresponding functions respectively.

Define an equivalence relation $\sim$ on $M_1 \cup M_2$ as follows:

1) If x_i, y_i are points such that $x_i, y_i \in N(R_i)$ and $p_i f_i(x_i) = p_i f_i(y_i)$ then $x_i \sim y_i,\ i = 1, 2$.

2) If $x \in R_1, y \in R_2$ and $g(x) = y$, where $g : R_1 \to R_2$ is the homeomorphism between the surfaces, then $x \sim y$.

Furthermore we can arrange that the attaching discs on $R_1 \times I$ $(R_2 \times I)$ for the one handles in U_0 (V_0) respectively, have disjoint images in R_1 (R_2) and hence do not get identified to each other. Now set:

$$M = (M_1 \cup M_2)/\sim, \quad H_0 = (U_0 \cup V_1)/\sim, \quad H_1 = (U_1 \cup V_0)/\sim .$$

Note that $H_0 = V_1 \cup N(R_1') \cup (1 - \mathit{handles})$ and $H_1 = U_1 \cup N(R_2') \cup (1 - \mathit{handles})$ (the 1-handles connect $\partial_+ V_1$ to $\partial N(R_1')$ ($\partial_+ U_1$ to $\partial N(R_2')$ respectively)) so that H_0, H_1 are compression bodies defining a Heegaard splitting for M. This Heegaard splitting is called the *amalgamation* of the Heegaard splittings (U_0, U_1) of M_1 and (V_0, V_1) of M_2 along R_1, R_2. We note that Figure 1 shows the amalgamation process.

Let M be a 3-manifold. Pick a closed subset $\partial_0 M$ of ∂M, and set $\partial_1 M = \partial M \backslash \partial_0 M$. Let $(H_0,\ H_1;\ S)$ be a genus g Heegaard splitting of M such that $\partial_- H_i = \partial_i M$. Choose a minimal set of defining discs for the compression bodies, so we need at exactly one 3-handle for a handlebody, and none for a compression body which isn't a handlebody.

The handle description defines a Morse function $h : M \to [0,\ 1]$. The splitting surface will occur as the inverse image of a regular value of h. We arrange the singular values of h, i.e.

$$0 < a_1 < \ldots < a_k < b_1 < \ldots < b_l < 1,$$

so that passing through a critical point labelled with an a_i corresponds to adding a 1–handle, passing a b_i corresponds to adding a 2–handle, and $h^{-1}(c)$ is isotopic to S for $a_k < c < b_1$. Thus we have the following:

The leaf of the foliation corresponding to $h^{-1}(0)$ is just $\partial_- H_0 = \partial_0 M$, if $\partial_0 M \neq \emptyset$, or the minimal point if H_0 is a handlebody. All leaves $h^{-1}(r)$ are isotopic for $0 < r < a_1$, but $h^{-1}(a_1)$ is a singular surface in which two points have been pinched together, forming a 1-handle, see Figure 2. Similarly, each of the $h^{-1}(a_i)$ corresponds to a singular surface in which two points on the previous leaves have been pinched together, increasing the genus of the leaves in the foliation by one, or lowering the number of components of $h^{-1}(r)$, so that for $a_k < r < b_1$, $h^{-1}(r)$ is a genus g surface. On the other hand, $h^{-1}(b_1)$ is a singular level in which a circle has been pinched into a point, forming a 2-handle, and the picture is the same as in Figure 2, except turned upside down. Then for $b_1 < r < b_2$, $h^{-1}(r)$ is a genus $g-1$ surface. Note that $h^{-1}(1)$ is the maximal point in H_1, if H_1 is a handlebody, or $h^{-1}(1) = \partial_1 M$ if $\partial_1 M \neq \emptyset$.

Let T be a polyhedral decomposition for M such that:

(0) $M\setminus$ (2-*skeleton of* T) is a union of balls, $T_3^0, \dots T_3^{n_3}$,

(1) ∂M appears as a union of (punctured) discs in ∂T_3^k,

(2) (1-*skeleton of* T)$\setminus \partial M$ is a union of open arcs, $T_1^0, \dots T_1^{n_1}$,

(3) (2-*skeleton of* T)$\setminus \partial M$ is a union of discs, $T_2^0, \dots T_2^{n_2}$, such that each T_2^k is a zerogon (i.e., its boundary lies in ∂M), a monogon (i.e., its boundary consists of some T_1^i union a part of ∂M), a bigon, or a triangle.

Then T is said to be an *idealized polyhedral decomposition*, or *IPD*, for M.

Denote by $T_j = \cup_{k=1}^{n_j} T_j^k$ the j–skeleton of T, that is, each T_3^k is a polyhedron, T_2^k is a zerogon, monogon, bigon, or triangle, and T_1^k is an arc.

We note that all Heegaard splittings are classified for manifolds M with T having $\sharp(T_1) \leq 1$ (see [**6**]), so we assume that $\sharp(T_1) > 1$.

Let $I_1, \dots, I_n$ be the critical values of T_1 (with respect to the Morse function h induced by the Heegaard splitting S), where $0 < I_1 < \dots < I_n < 1$. Let x_i be regular values of $h\mid_M, h\mid_{T_2}$ such that $0 < x_0 < I_1,\ I_n < x_n < 1$, and $I_i < x_i < I_{i+1}$ for $0 < i < n$. Then each $h^{-1}(x_i)$ is a level surface S_i. Define the width $w(T)$ of T to be the number of intersections of $\cup S_i$ with T_1, that is, $w(T) = \sum_i \sharp(S_i \cap T_1)$. If T has been isotoped to have minimal width, we say that T is in *thin position.* Henceforth we assume that T is in thin position.

If each edge T_1^k in the 1-skeleton of T has either no critical points, or if all critical points are maxima with respect to the Morse foliation induced by S, (or all minima), then S is said to be *weakly rigid.* If there exists some i so that T_1^i has more than 1 critical point, and no T_1^j has exactly one critical point which is a minimum, then S is called a *Gabaic* Heegaard surface.

Let M be a 3-manifold and T an IDP for M. We denote an arc embedded in a face T_2^k as *normal* if its endpoints lie on distinct edges of T_2^k. An arc imbedded in a side is *abnormal* if both its endpoints lie on the same edge. (Arcs with an endpoint on ∂M do not arise in the following arguments.) A curve lying in T_2 is called *simple* if it does not intersect T_1.

Let r be a regular value of h on both T_1 and M. Suppose there is an abnormal arc α of $h^{-1}(r) \cap T_2^k$. Then α together with a piece γ of T_1 bounds a disc D in T_2^k. We say that D is *bad* if $^\circ D \cap h^{-1}(r)$ is empty or consists of simple closed curves. If D is bad, γ is *above* $h^{-1}(r)$ if it lies on the side of $h^{-1}(r)$ containing $h^{-1}(1)$; otherwise it is *below*. A bad disc lies above or below $h^{-1}(r)$ according to whether γ lies above or below, see Figure 3.

Let F be a closed 2-sided surface embedded in M. Assume that F intersects T_2 in normal arcs and simple curves, and that for at least one i, $F \cap T_1^i > 2$. Then F is called a *Gabai surface*.

If F is a 2-sided surface for which $F \cap T_2$ contains only normal arcs and $F \cap T_3^k$ is a collection of discs for each k, then F is called a *normalized surface*.

We say that a set $S_1, \dots, S_n$ of connected, normalized surfaces is a *base* for (M, T) if any connected normalized surface in (M, T) is represented (up to isotopy preserving number of intersections with each T_1^i) by exactly one of the S_i. If M contains a base with respect to T, then T is called a *good* IPD, otherwise it is *bad*. Similarly, a manifold possessing a good IPD is called *good*, otherwise it is *bad*.

If F is a 2-sided, closed, incompressible surface in M, then F is said to be a *Haken surface* in M. Let M be a manifold, and F a collection of disjoint non-boundary parallel Haken surfaces for M. Then the submanifolds $cl(M \backslash N(F))$ are called *Haken submanifolds* of M.

3. Preliminary Results.

Theorem 3.0. *Let S be a Heegaard splitting surface for the manifold M which is weakly rigid with respect to the idealized polyhedral decomposition T. Then S is either a regular neighborhood of a subset of T_1, together with a neighborhood of some components of ∂M, or S is stabilized.*

Proof. [**6**, 2.3]. □

Lemma 3.1 (**Thompson**). *Let S be a Heegaard splitting surface for a 3-manifold M. Let Δ be a collection of compressing discs in H_0 and set $M' = H_1 \cup N(\Delta)$. If there exists an essential disc $(D, \partial D)$ in $(M', \partial M' \backslash \partial M)$, then S is weakly reducible.*

Proof. Note that M' has a natural Heegaard splitting along S, since to one side we have H_1, to the other $S \times I$ union some 2-handles. By [**2**, 1.1] we know that if a manifold has a compressible boundary component, then the Heegaard splitting is weakly reducible. Apply this to $\partial M'$. Since $\partial M'$ is compressible, the Heegaard splitting of M' is weakly reducible, and thus S is weakly reducible when considered at the splitting surface for M. □

Lemma 3.2 (**Scharlemann-Thompson**). *Suppose that S gives a Heegaard splitting of a 3-manifold M into compression bodies H_0 and H_1, and $F \subset S$ is a compact subsurface so that every component of ∂F bounds a disc in M disjoint from $^\circ F$. Then either ∂F bounds a collection of discs in a single compression body H_i, or S is weakly reducible.*

Proof. [**9**, 2.6]. □

Lemma 3.3. *Let F be a Haken surface for M. Then F can be isotoped so as to be a normalized surface.*

Proof. This follows from the argument of [**6**, 1.1]. □

Lemma 3.4. *Let S be a Gabaic Heegaard splitting surface for M. Then S is equivalent via isotopy to a Gabai surface.*

We will use the following subclaim, which uses the thin position assumption on T.

Subclaim 3.5 (**Gabai**). *Let r be a regular value of h on both T_1 and M. Then $h^{-1}(r) \cap T_2$ cannot contain abnormal arcs α_0, α_1 cutting off bad discs D_0 and D_1 such that D_0 is above $h^{-1}(r)$ and D_1 is below.*

Proof. [**3**, §4]. □

Proof of Lemma 3.4. There is at least one minimum on T_1. Consider the highest such; without loss of generality we may assume that this occurs on T_1^0 at critical level I_p. T_1^0 has at least one maximum above I_p. In addition, all arcs of T_1 above I_p contain either a single maximum or no critical points. Note that exchanging the heights of the maxima of two arcs lying above I_p does not alter the width of T_1, see Figure 4. Thus we may assume that critical level I_{p+1} lying immediately above I_p is a maxima for T_1^0.

For every r such that $I_p < r < I_{p+1}$, T_1^0 intersects $h^{-1}(r)$ in at least 3 points. For r very close to I_p, $h^{-1}(r) \cap T_2$ will contain some bad discs below $h^{-1}(r)$. For r very close to I_{p+1}, $h^{-1}(r) \cap T_2$ will contain some bad discs above $h^{-1}(r)$. Again as in [**3**, §4], we can conclude that either for some regular value r of h (on T_1, T_2, and M), $I_p < r < I_{p+1}$ there are disjoint bad discs both above and below $h^{-1}(r)$, or that there exists a regular value r, $I_p < r < I_{p+1}$ such that $h^{-1}(r) \cap T_2$ contains no bad discs on either side. By Subclaim 3.5, the first case cannot occur, hence the second case must hold.

Case 1. $a_k < r < b_1$.

Then $h^{-1}(r)$ is isotopic to S, and we have the desired result.

Case 2. $a_{i-1} < r < a_i$ for some i, or $b_i < r < b_{i+1}$.

The cases are symmetric, so we take $a_{i-1} < r < a_i$. Note that S is constructed from $h^{-1}(r)$ by 1-surgery along arcs lying in $h^{-1}((r,1])$. We can assume that the ends of these arcs are disjoint from $N(h^{-1}(r) \cap T_2)$, so that $h^{-1}(r) \cap T_2$ persists into S. Also we may assume that these arcs intersect T_2 transversely, so that no abnormal arcs are added in the process of the 1–surgery. This is the desired result. □

4. Main Result.

Main Theorem 4.0. *Let T be a good IPD for the manifold M. Then we can list a finite collection C of (isotopy classes of) surfaces in M together with a finite collection N of 3-submanifolds of M such that if S is a Gabaic Heegaard surface, then one of the following is true:*

(a) $S \in C$,

(b) *S is induced by a Heegaard splitting S' of $M' \in N$, and S' has lower spinal complexity than S,*

(c) *S is weakly reducible, and thus is obtained from amalgamation of Heegaard splittings of a (some) Haken submanifold(s) of M.*

Assume that M isn't a compression body, since all Heegaard splittings of compression bodies are classified by [**6**].

Without loss of generality, we assume that the surface S satisfies the conclusion of Lemma 3.4.

Let μ be the collection of all normal arcs, and let $U \subset h^{-1}(r)$ be a regular neighborhood of μ in $h^{-1}(r)$, see Figure 5. Then each component of ∂U is compressible in $M \backslash U$. We call the surface obtained from these compressions $\bar{S}$, and the compressing discs Δ. Note that $\bar{S}$ is a normalized surface, see Figure 6.

By 3.2 we may assume that either all discs in Δ lie inside of H_i for $i = 0$ or $i = 1$, or that S is weakly reducible.

If S is weakly reducible, then either S is reducible, or by the argument of [**4**, 1.1], S is an amalgamation of Heegaard splittings of Haken submanifolds along a finite collection of Haken surfaces in M. We thus assume that S is not weakly reducible.

Let $\bar{H}_0$, $\bar{H}_1$ be the manifolds obtained from H_0 and H_1 respectively by compression along Δ. Without loss of generality we assume that $\Delta \subset H_0$, so that $\bar{H}_0 = H_0 \backslash N(\Delta)$, $\bar{H}_1 = H_1 \cup N(\Delta)$. Note that $\bar{H}_0$ is a union of compression bodies with $\partial_- \bar{H}_0 = \partial_- H_0$.

We thus consider "all" possible normalized surfaces $\bar{S}$ in M having at least three intersections with some T_1^i. Assume that each such $\bar{S}$ is obtained from compression of a Heegaard surface to one side as above. Color the components of $M\backslash\bar{S}$ black and white alternately. Without loss of generality assume that $\bar{H}_0$ is black and $\bar{H}_1$ is white.

In the following argument, Case n assumes that the hypothesis of Case i is false for $i < n$.

Case 0. $M\backslash\bar{S}$ contains more than one white component.

Then $\bar{H}_1$ contains at least two components. But $\bar{H}_1 = H_1 \cup N(\Delta)$, implying that H_1 contained multiple components, a contradiction.

Case 1. $M\backslash\bar{S}$ contains a black component which is not a compression body.

This implies that $\bar{H}_0$ is not a union of compression bodies, a contradiction.

Case 2. The white component of $M\backslash\bar{S}$ is not a compression body, and the black component(s) are spineless, i.e., a (perhaps disconnected) surface $\times I$.

Reconstructing H_0, we must add 1-handles along the cores of the discs Δ. If $\bar{H}_0$ contains an $F \times I$ component for which F isn't boundary parallel, then, since S was connected, we can arrange via edge slides (see [4]) that at least one 1-handle runs from $F \times \{0\}$ to $F \times \{1\}$. But $F \times I$ union a 1-handle running between both boundary surfaces isn't a compression body. This in turn implies that H_0 isn't a compression body, a clear contradiction.

Thus $\bar{S}$ is a collection of boundary parallel surfaces. But $\sharp(\bar{S} \cap T_1^i) > 2$ for some i, so that there are at least two parallel surfaces in $\bar{S}$. Choose one such pair, and call the one closer to the boundary $\bar{S}'$. Then to both sides of $\bar{S}'$, there is a component of $M\backslash\bar{S}$ homeomorphic to $\bar{S}' \times I$. But one of these two must be colored white, contradicting the fact that the white component is not a compression body.

Case 3. The white component of $M\backslash\bar{S}$ is not a compression body, and the black component(s) have non-trivial spine X_0.

Then we consider the manifold $M' = M\backslash N(X_0)$. Note that any Heegaard splitting of M having spine X_0 induces a Heegaard splitting of M' , and vice versa. In addition, such an induced Heegaard splitting of M' has lower spinal complexity.

Case 4. $M\backslash\bar{S}$ contains more than one black component.

Then Δ is non-trivial since H_0 is connected. If $\partial_+\bar{H}_1$ is compressible, then S was weakly reducible by 3.1, a contradiction. Thus $\bar{H}_1 =$ (surface) $\times\, I$, and the original Heegaard splitting can be reconstructed by adding a single 1-handle to $\bar{H}_0$ along an arc corresponding to a vertical arc in the I-bundle structure of $\bar{H}_1$ by [4, 2.4].

Case 5. $\bar{S}$ divides M into two compression bodies.

If $\bar{S}$ is weakly reducible, let $D_i \subset \bar{H}_i$ be weakly compressing discs for $\bar{S}$. If Δ was non-trivial, D_1 is an essential disc fulfilling the hypothesis of 3.1, so that S was weakly reducible, a contradiction. Thus we assume that $\Delta = \emptyset$, and thus that $\bar{H}_i = H_i$. But in this case $\bar{S} = S$, so that S is also weakly reducible, again a contradiction.

Hence $\bar{S}$ isn't weakly reducible. Since M isn't a compression body, there is a non-trivial spine X_0 for H_0. As per Case 3, a Heegaard splitting of M having spine X_0 is induced naturally by a Heegaard splitting of $M' = M \backslash N(X_0)$. But M' is a compression body, so its Heegaard splittings have already been classified by [**4**, 2.5].

Note that if $\bar{S}$ contains three parallel surfaces, we may apply Case 0. If there exists a finite base for (M, T), there are only finitely many normalized surfaces not containing a parallel trio. Thus, in applying the above argument, we only need to check finitely many surfaces. For each such surface, there are two possible colorings, and thus finitely many possibile outcomes.

This is the desired result. □

5. Good Idealized Polyhedral Decompositions.

Let F be a surface intersecting T_2 in normal arcs and T_3 in discs. We shall denote F by the n-tuple $(\sharp(F \cap T_1^0), \dots, \sharp(F \cap T_1^{n-1}))$. Note that any normalized surface has a unique notation (modulo isotopy preserving number of intersections with each T_1^i) as such an n-tuple. Conversely, any such n–tuple either represents a normalized surface, a 1-sided surface, or else no surface F with the above properties exists, see Figure 7.

Theorem 5.0. *Let M be a 3-manifold, and assume that M has an IPD with only one edge T_1^0. Then (M, T) is good.*

Proof. This follows from the argument of [**6**, 1.1]. □

Theorem 5.1. *Let M be a 3-manifold, and assume that M has an IPD having two edges T_1^0, T_1^1. Then (M, T) is good.*

Proof. We denote each surface F intersecting T_2 in normal arcs and T_3 in discs with a pair of non-negative integers (a, b) as above.

Such surfaces F cannot intersect zerogonal faces, so we may assume that no such faces exist.

If a monogonal face exists, say with edge T_1^1, then $\sharp(F \cap T_1^1) = 0$, so we may for the purposes of the argument remove this edge from the manifold, and consider the manifold $M' = M \backslash N(T_1^1)$, which has an IPD with only one edge. Then we apply Theorem 5.0 to (M', T'), and pull the argument back into M in the natural way. We may thus assume that there are no monogonal faces in T.

If T contains a bigonal face with both T_1^0 and T_1^1 edges, then $\sharp(F \cap T_1^0) = \sharp(F \cap T_1^1)$. We may thus for the purposes of this argument assume that T has only one edge, and apply 5.0. Thus we assume that T contains no bigonal face with different edges.

Any other bigonal faces give no information concerning normalized surfaces, so we may ignore them for the purposes of this argument. We then assume that all faces of T are triangular. A face with (i)–T_1^0 edges and (j)–T_1^1 edges we denote [i,j]. Note that $j = 3 - i$, so there are four types of faces: [0,3], [1,2], [2,1], and [3,0], see Figure 8.

The case when all faces are of type [0,3] or [3,0] is contained in 5.0. We thus assume we have faces of type:

Case 0. [1,2] only.

Then (0,1) and (2,1) are either normalized surfaces or 1-sided. (A simple linear algebra argument shows that such F exist.) Assume the former. Any normalized surface can be denoted $(2n, m)$, where $m \geq n$, since each face in T_2 looks like Figure 9. A normalized surface with n parallel copies of (2,1) and $m - n$ parallel copies of (0,1) is denoted $(2n, m)$, demonstrating that (0,1) and (2,1) is a base.

We thus assume that (0,1) and (2,1) are 1-sided. Then any normalized surface can be denoted $(2n, 2m)$, where $n \leq 2m$. For $m \geq n$, we have that n parallel copies of (2,2) union $m - n$ parallel copies of (0,2) corresponds to $(2n, 2m)$. Hence assume that $m < n$. Then $2m - n$ copies of (2,2) union $n - m$ copies of (4,2) corresponds to the surface $(2n, 2m)$. Thus (0,2), (2,2), and (4,2) constitute a base.

Case 1. [0,3] and [1,2].

Then any normalized surface can be denoted $(2n, 2m)$, where $n \leq 2m$, so that (0,2), (2,2), and (4,2) constitute a base as per Case 0.

Case 2. [0,3] and [2,1].

The argument is identical to Case 0.

Case 3. [1,2] and [2,1].

Then both $\sharp(F \cap T_1^0)$ and $\sharp(F \cap T_1^1)$ must be even and non-zero, so that any normalized surface can be denoted $(2n, 2m)$, $n, m > 0$, with $n \leq 2m$ and $m \leq 2n$. For $m \geq n$, $2n - m$ copies of (2,2) and $m - n$ copies of (2,4) corresponds to $(2n, 2m)$. For $m < n$, $2m - n$ copies of (2,2) and $n - m$ copies of (4,2) corresponds to $(2n, 2m)$. Thus (2,2), (2,4), and (4,2) constitute a base.

Case 4. [0,3], [1,2] and [2,1].

The argument is identical to Case 3.

Case 5. [0,3], [1,2] and [3,0].

The argument is identical to Case 1.

Case 6. [0,3], [1,2], [2,1] and [3,0].

The argument is identical to Case 3.

All other cases can be obtained from switching the roles of T_1^0 and T_1^1. This completes the proof. □

Theorem 5.2. *Let M be a 3-manifold, and assume that M has an IPD with $\sharp(T_1) = 3$ such that any triangular face T_2^k has at most two distinct edges. Then (M, T) is good.*

Proof. The proof, being analogous to 5.1 but having many more cases, is left to the reader. We note that two such manifolds are given in Examples 6.6 and 6.7. □

Conjecture 5.3. *Let M be a 3-manifold, and assume that M has an IPD such that any triangular face T_2^k has at most two distinct edges. Then (M, T) is good.*

We note, for example, that Manifold V_{3383}, obtained from the cusped census of [**11**], and having a 7 tetrahedron decomposition, is an example of a manifold with a good IPD having $\sharp(T_1) = 6$, see Figure 10. (It also has a good IPD with $\sharp(T_1) = 7$.)

6. Examples.

We consider the cusped hyperbolic manifolds of small volume obtained from the cusped census of [**11**]. We give in Figure 11 a table of the triangulations and good IPD's we found for manifolds 000–006. Note that manifold 005 may have a good IPD. If so, it cannot be obtained from the canonical triangulation by gluing "bad" faces together.

If a Heegaard splitting of a manifold with a given IPD is just a regular neighborhood of (some boundary components union some components of T_1), then we say that the splitting is *trivial* (with respect to the IPD). In particular, when we examine some of the manifolds in the census of [**11**], we will often refer to the splittings as "trivial", the IPD being tacit.

Note 6.0. In most cases, it is easy to check that Case 0 of 4.0 can be applied to any normalized surface with $\sharp(\bar{S}) > 2$. We shall tacitly eliminate all such cases. Further, we note that although we can color the pieces (as per 4.0) in two ways, we shall tacitly assume that we have chosen a coloring which eliminates application of Case 0.

Example 6.1. Manifold 000, M_{000}, the *Gieseking Manifold*, has only one irreducible (non-orientable) Heegaard splitting.

This is just Example 6.0 of [**6**].

We note that the Gieseking Manifold is a nonorientable manifold which is double covered by the figure–8 knot complement. The above result then implies that any Heegaard splitting surface for the figure–8 knot complement which can be isotoped so as to be symmetric with respect to the double cover must be induced by the pull-back of the one splitting for M_{000}. It is elementary to check that this pulled-back surface is reducible.

Example 6.2. An irreducible Heegaard splitting S of M_{001} is trivial.

If S is weakly rigid, then S is trivial by 3.0. Otherwise, S is Gabaic. The triangulation given in [**11**] has two edges, so it is good, having base $\{(2,0),(2,2)\}$, using the notation of 5.1. Then to apply 4.0 we need only to check the normalized Gabai surfaces:

(a) (4,0),
(b) (4,2), and
(c) (4,4).

For case (a), $\bar{H}_0$ is a non-trivial I-bundle over a surface of genus $\frac{3}{2}$ union a compression body, so Case 1 applies. Similarly for (b). For (c), $\bar{H}_0$ is again two pieces, one of which is homeomorphic to the original manifold. Thus Case 1 can again be applied.

Hence we must assume that S is weakly reducible. All possible candidates for Haken surfaces are (2,0) and (2,2). It is easy to check that (2,0) has an essential compressing disc. The surface (2,2) is boundary parallel. Thus any non-trivial S is almost trivial. But then it is easy to check (as per 6.1), that almost trivial S are reducible, and hence stabilized.

Example 6.3. An irreducible Heegaard splitting S of M_{002} is trivial.

By 3.0 we assume that S is not weakly rigid. Note that M_{002} has two boundary components, $\partial_0 M_{002}$, and $\partial_1 M_{002}$. If $(\partial_0 M_{002} \cup \partial_1 M_{002}) \subset H_1$, then it is possible for S to be non-Gabaic. But by reversing the flow of the Heegaard foliation (considering $1-h$ instead of h), we may assume that a non-weakly rigid Heegaard splitting of M_{002} is Gabaic.

Again the triangulation is good, having base $\{(1,0),(1,2)\}$. We only need to check the Gabai surface (2,4), since all other cases allow application of

Case 0. But it is easy to check that Case 1 can be applied to this surface. Then, as per M_{001}, S is trivial.

We note that M_{002} is a nonorientable manifold which is double covered by the 6^2_2-link complement. The above result implies that any Heegaard splitting surface for the link complement which can be isotoped so as to be symmetric with respect to the double cover must be induced by the pull-back of one of the two splittings. It is elementary to check that both such splittings are reducible.

The 6^2_2-link has 4 known Heegaard splittings of genus 2: Those given by the upper and lower tunnels, and those given by the two σ–tunnels (see [7]). It is interesting to note that the two (reducible) Heegaard splittings given by the above argument are: (1) that induced by the upper and lower tunnels (together), and (2) that induced by both σ-tunnels (together).

Example 6.4. An irreducible Heegaard splitting S of either M_{003} or M_{004} is trivial.

We note that M_{004} is the Figure–8 Knot Complement, and M_{003} is its so-called "Sister Manifold".

As per M_{001}, any such S is either weakly rigid or Gabaic, the triangulations are good, and a base in either case is the set $\{(2,2),(2,4),(4,2)\}$. The surfaces we need to check are then:

(a) (2,4): Case 5, weakly reducible,
(b) (4,2): Case 5, weakly reducible,
(c) (4,4): Case 1,
(d) (4,6): Case 4; $\bar{H}_1$ compressible $\Rightarrow$ weakly reducible,
(e) (4,8): Case 4; construct S, easy to check weakly reducible,
(f) (6,4): as per (d),
(g) (8,4): as per (e).

Again as per the argument of M_{001}, weakly reducible implies reducible, so that S must be trivial.

Conjecture 6.5. *An irreducible Heegaard splitting S of M_{005} is trivial.*

The author was unable to find a good IPD for this manifold, so this question remains open. However, even if a good IPD cannot be located, it may still be possible to prove that all Heegaard splittings of bounded genus are trivial. We note that in all cases we have checked, normalized Gabai surfaces have genus four or greater. If this can be proven to be always true, then since these surfaces have been obtained from compression of a Heegaard surface, any Heegaard splitting of genus 2 or 3 must be trivial or almost trivial.

Example 6.6. An irreducible genus 2 Heegaard splitting S of M_{006} is trivial.

First note that S must be weakly rigid or Gabaic. Also note that a good polyhedral decomposition for M exists by identifying faces having 3 distinct edges, as per Figure 11.

We now use Lemma 3.4 on the good IPD, and apply the argument of 4.0 to the surfaces of genus less than 3.

It is elementary to check that the set of normalized surfaces {(2,2,0), (2,2,2), (2,2,4), (2,4,0), (2,4,2), (2,4,4), (4,2,0), (4,2,2), (4,2,4), (4,2,6), (4,2,8), (8,4,2)} forms a base for T_{006}.[1]

We thus need to apply the argument of 1.0 to each of: (2,2,4), (2,4,0), (2,4,2), (2,4,4), (4,2,0), (4,2,2), (4,2,4), etc., that is, each of the Gabai surfaces not having three parallel copies of any base surface. Any of these of genus greater than 2 can be eliminated, since we are only interested in classifying the genus 2 case. This eliminates all of the cases except for (2,4,2) and (4,4,4).

(2,4,2): This is a surface of genus 2, and divides M into two pieces. One of the two pieces has first homology group $Z^2 \oplus Z_5$, and thus cannot be a handlebody. Thus we can apply Case 3 of 4.0. It is a simple matter to continue the arguments to demonstrate that any such Heegaard splitting must in fact be induced by two edges of the 1-skeleton. At any rate, any such Heegaard surface is of genus greater than 2.

(4,4,4): The center black piece is homeomorphic to M_{006}, so that we may apply Case 1.

Note then that a non-boundary parallel Haken surface F must be of genus 2 or greater, and that a Heegaard splitting obtained from amalgamations of Haken submanifolds must thus be of genus at least 4.

Thus any genus 2 Heegaard splitting is trivial or almost trivial. It is an easy matter to check that almost trivial Heegaard splittings are reducible (and of genus 3).

We also note that we restricted ourselves to genus 2 only for convenience. It seems that, given a little patience and a lot of time, we could check each of the exceptional surfaces and classify all Heegaard splittings.

Example 6.7. Any unknotting tunnel for the 5_2 knot is isotopic to one of the edges in the canonical triangulation of its complement.

Proof. It is elementary to check that M_{015} is the 5_2 knot complement, and

[1] We also note that we can apply Lemma 3.4 to the triangulation and glue the "bad" faces together afterwards. Applying 4.0 to the good IPD at this point allows us to eliminate several surfaces.

that it has two different good IPDs which are equivalent by a homeomorpism rotating the knot 180 degrees. Any unknotting tunnel for the knot is induced by a genus 2 Heegaard splitting. We thus classify all genus 2 Heegaard splittings, as per Example 6.6. There is only one Gabaic surface which is genus 2; it is shown after an isotopy in Figure 12. Although this is not induced by an edge in the IPD shown, it is easy to check that its spine is induced by an edge in the canonical triangulation. It in fact is a Heegaard surface for M_{015}. All other genus 2 Heegaard splittings must be rigid, hence induced by the 1-skeleton of the IPD, which coincides with the the other three edges of the canonical triangulation.

This is the desired result. □

Example 6.8. The canonical triangulation of the Whitehead link complement, W.

The Whitehead link complement has canonical triangulation as shown in Figure 13. We call the boundary components $\partial_0 W$ and $\partial_1 W$. Let S be a Heegaard splitting surface such that $\partial_i W = \partial_- H_i$. Then T_1^0 has an odd number of maxima, while T_1^1 has an odd number of minima. Assume that both have exactly one critical point. Then the Heegaard splitting is neither weakly rigid nor Gabaic. In addition, reversing the flow of the Heegaard foliation (switching H_0 and H_1) doesn't make the splitting weakly rigid or Gabaic, as per Example 6.3. Thus we cannot apply the results of this paper to this polyhedral decomposition of W. (We should also note that the canonical triangulation of W is not "good".)

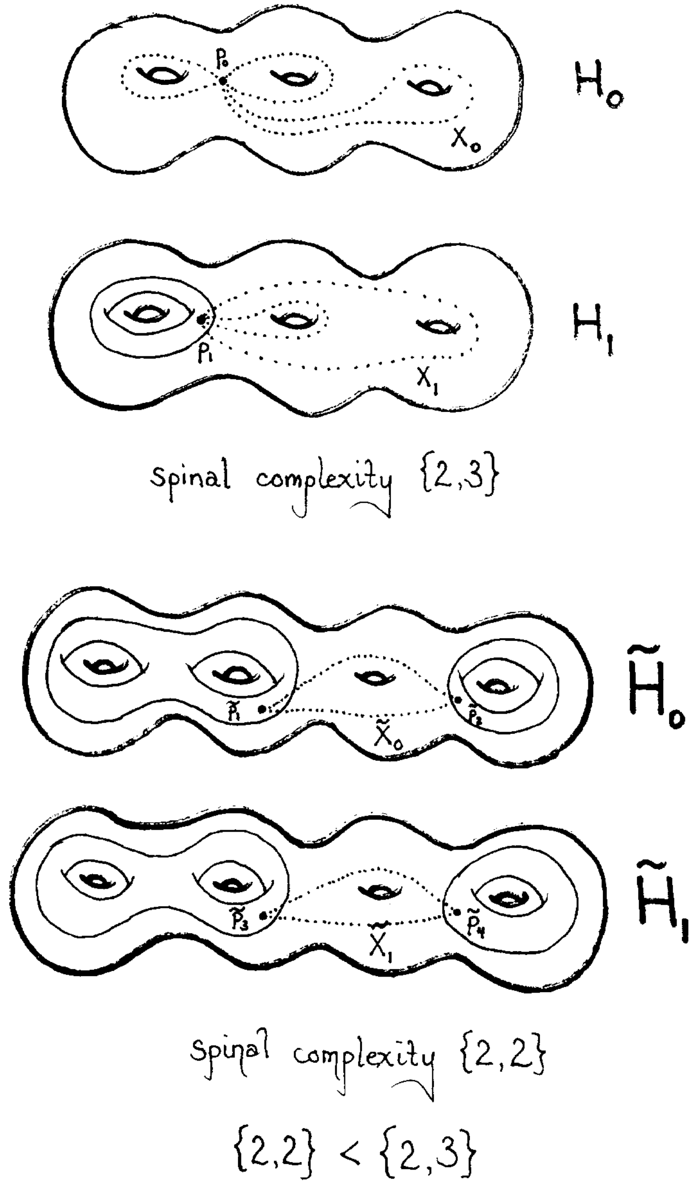
H₀
P₀
X₀
H₁
P₁
X₁
spinal complexity {2,3}
H̃₀
P̃₁
X̃₀
P̃₂
H̃₁
P̃₃
X̃₁
P̃₄
spinal complexity {2,2}
{2,2} < {2,3}

Figure 0.

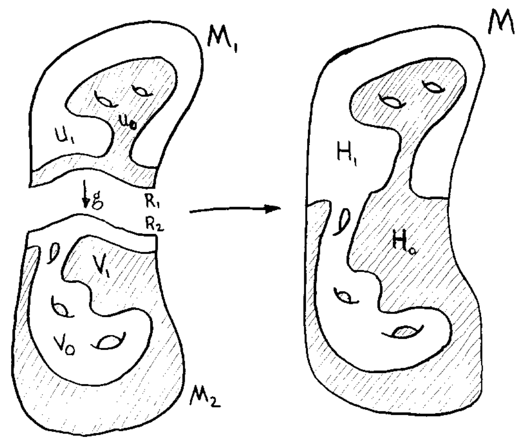

Figure 1.

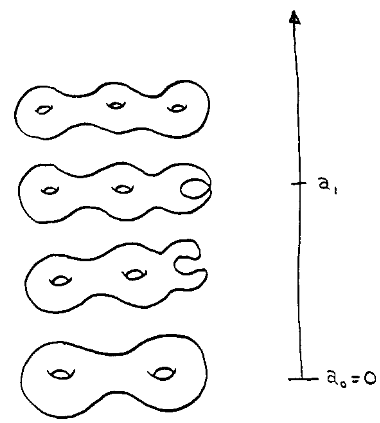

Figure 2.

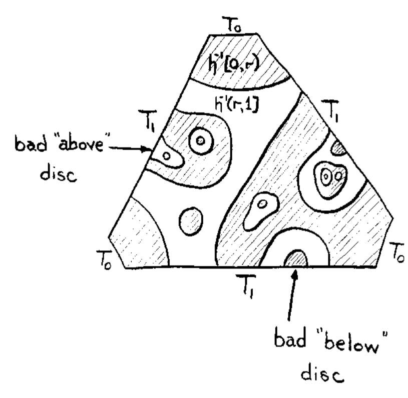

Figure 3.

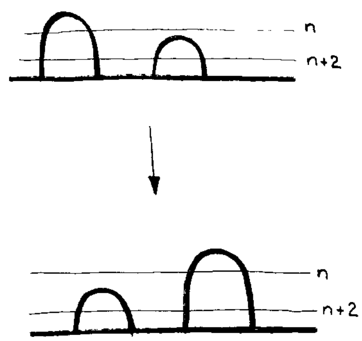

Figure 4.

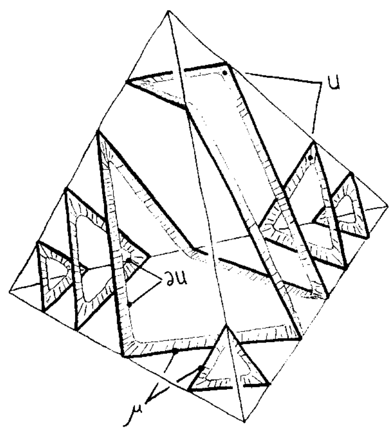

Figure 5.

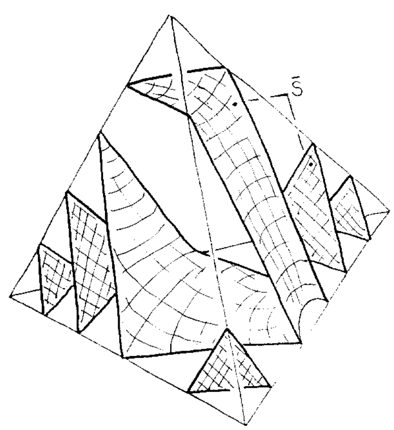

Figure 6.

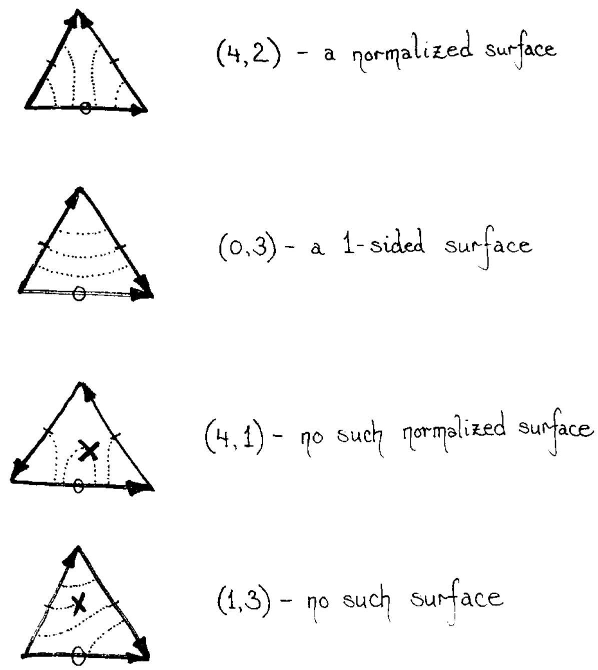

Figure 7.

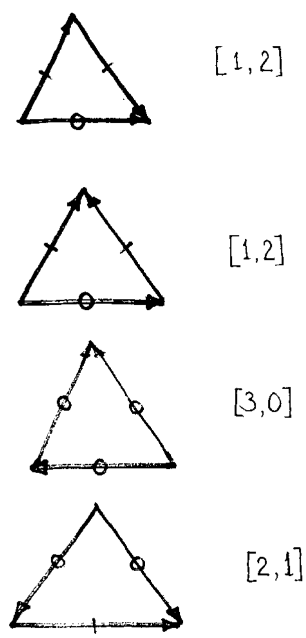

Figure 8.

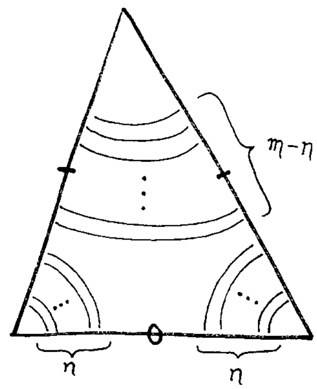

Figure 9.

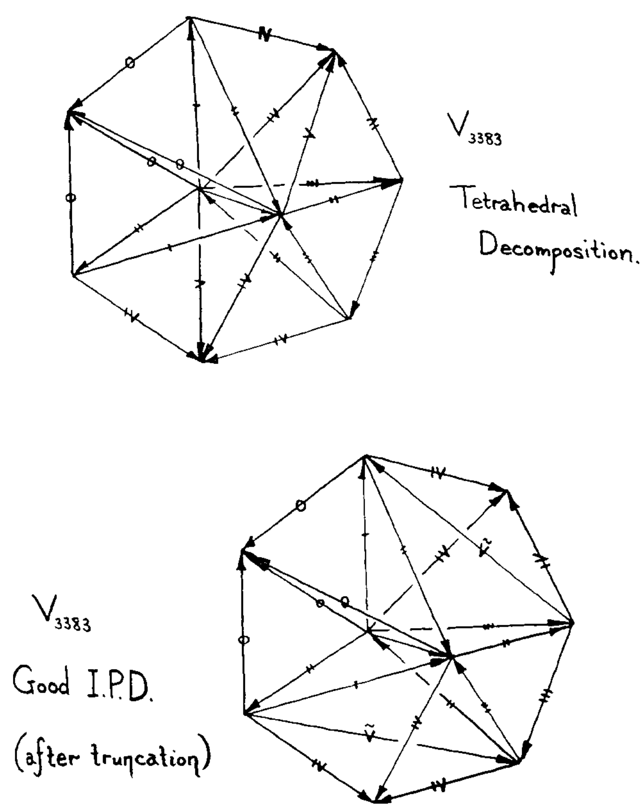

Figure 10.

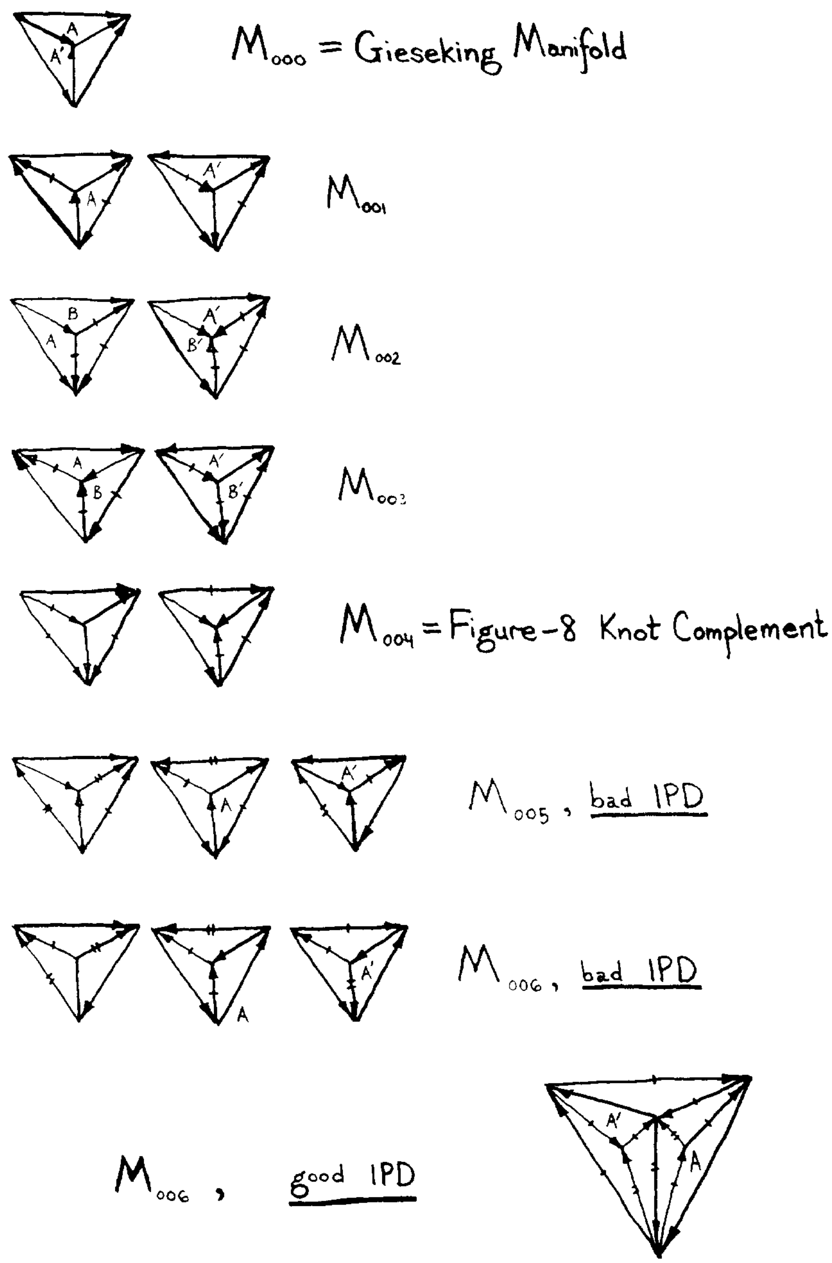
M000 = Gieseking Manifold
M001
M002
M003
M004 = Figure-8 Knot Complement
M005, bad IPD
M006, bad IPD
M006, good IPD

Figure 11.

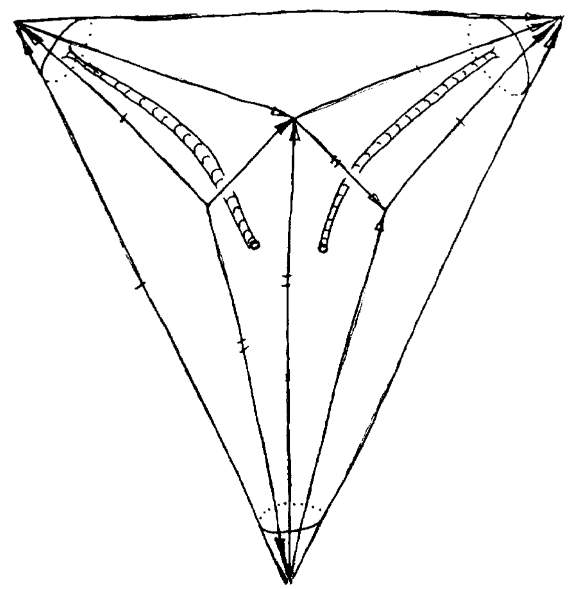

Figure 12.

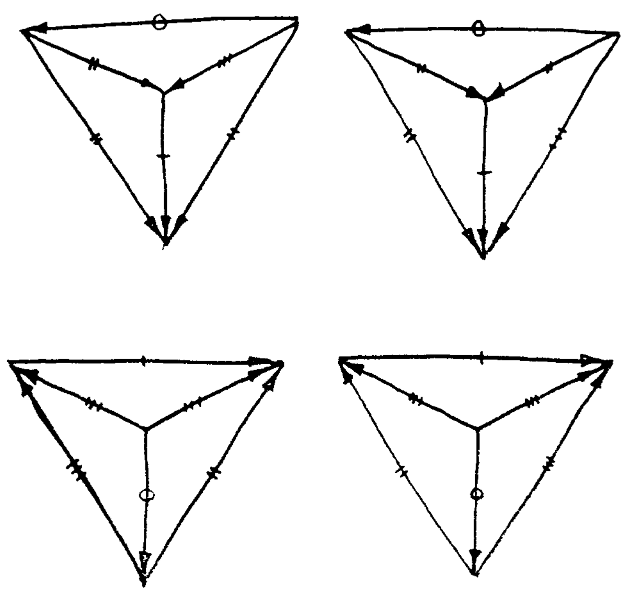

Figure 13.

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