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COVERING THEOREMS FOR OPEN CONTINUOUS MAPPINGS HAVING TWO VALENCES BETWEEN ORIENTABLE SURFACES

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Let $\mathcal{X}$ be an open orientable surface with finite genus and finite number of boundary components, and let $\mathcal{Y}$ be a closed orientable surface. An open continuous function from $\mathcal{X}$ to $\mathcal{Y}$ is termed a (p, q)-map, $0 < q < p$, if it has a finite number of branch points and assumes every point in $\mathcal{Y}$ either p or q times, counting multiplicity, with possibly a finite number of exceptions. These comprise the most general class of all nontrivial functions having two valences between $\mathcal{X}$ and $\mathcal{Y}$.

In this paper we generalize and prove in a unified manner almost all the earlier covering and existence results involving (p, q)-maps between orientable surfaces. Our main tools are (i) a generalized embedding theorem (see Lyzzaik, 1995) which asserts that image surfaces of (p, q)-maps embed in p-fold closed coverings possibly having branch points off the image surfaces, and (ii) results (see Lyzzaik and Stephenson, and Lyzzaik, 1996) "modifying" general (p, q)-maps to "simplicial" ones having the same covering structures. This leads us to combinatorial results of (p, q)-maps relating their branch orders and exceptional sets of points to the valences p, q and the topological invariants of $\mathcal{X}$ and $\mathcal{Y}$. The paper ends with open questions.

0. Introduction and statements of results.

The object of this paper is to prove covering and existence theorems involving the most general class of open continuous mappings having two valences between orientable surfaces. This concludes in a unified manner most of the earlier results on special classes of these functions (see [**2**, **3**, **4**, **7**, **8**, **10**, **11**, **12**], [**13**, **14**]).

Let us first formulate the precise definition of (p, q)-maps.

Definition 1. Let p and q be integers satisfying $0 < q < p$, $\mathcal{X}$ an open orientable surface of finite genus and finite number of boundary components, and $\mathcal{Y}$ a closed orientable surface. A function $f : \mathcal{X} \to \mathcal{Y}$ is termed a (p, q)-map if f satisfies the following conditions:

(a) f is open and continuous.

(b) f admits every point of $\mathcal{Y}$, with at most finitely many exceptions, either p or q times, counting multiplicity, with at least one point admitted exactly p times.

(c) f admits only a finite number of branch points.

Further, f is termed a BQ-map if f has no exceptional points.

There will be no loss of generality in assuming throughout the paper that (p,q)-maps are sense preserving.

Because a (p,q)-map $f:\mathcal{X}\to\mathcal{Y}$ is open and continuous, a *branch point* and the *order of a branch point* will mean the same for f as for an analytic function (see [**15**]). The *total branch order* of f will be denoted by β_f. A companion of f is the integer-valued function $\nu_f(y)$ defined as the number of times, counting multiplicity, that f assumes a value y in $\mathcal{Y}$. Let $\mathcal{P}=\{y\in\mathcal{Y}:\nu_f(y)=p\}$, $\mathcal{Q}=\{y\in\mathcal{Y}:\nu_f(y)=q\}$ and $\mathcal{E}=\{y\in\mathcal{Y}:\nu_f(y)\neq p,q\}$. We call $\mathcal{P}$ the p-set, $\mathcal{Q}$ the q-set, and $\mathcal{E}$ the *exceptional set*, of f. Evidently, these sets are mutually disjoint and their union is $\mathcal{Y}$. The p-set happens to be open and to have finitely many connected components each with finitely many boundary components. Every point $e\in\mathcal{E}$, called *exceptional point*, will be assigned the value

$$\delta_f(e)=q-\nu_f(e)$$

which we call the *deficiency* of f at e.

Let $\mathcal{G}$ be a boundary component of $\mathcal{X}$. We use the concept of a boundary component of an open orientable surface as defined in [**1**], and the concept of the impression of a boundary component as defined in [**5**]. We denote by $C(f;\mathcal{G})$ the *cluster set of f at $\mathcal{G}$*, and by $C(f)$ the set-union of the cluster sets of f over all the boundary components of $\mathcal{X}$ [**5**, **6**].

Generally speaking, $\delta_f(e)$, may, unlike in [**7**], be negative. This fortunately can be avoided without losing generality by imposing a weak condition on f as in [**5**, **6**]. To illustrate this, If $\mathcal{G}$ is a boundary component of $\mathcal{X}$, then either $C(f;\mathcal{G})$ is a point or a non-degenerate continuum. View $\mathcal{X}$ as a closed orientable surface, $\widehat{\mathcal{X}}$, with a number $\eta(X)$ of pairwise disjoint points or closed topological discs, ω, removed. Observe that each point or disc ω contains the impression of one and only one boundary component $\mathcal{G}$ of $\mathcal{X}$. Choose ω a point if the cluster set of f at the corresponding boundary component $\mathcal{G}$ is a point, and a disc otherwise. It follows that f extends to a continuous map, g, of the orientable surface, $\mathcal{Z}$, obtained by taking the union of $\mathcal{X}$ and all the points ω. Since these points are isolated and $(\mathcal{X},f)$ is a covering, $g:\mathcal{Z}\to\mathcal{Y}$ is an open and continuous map [**1**, pp. 39-41]. If $C(f;\mathcal{G})$ is a point for every boundary component $\mathcal{G}$, then either $(\mathcal{Z},g)$ is a complete covering of $\mathcal{Y}$ or $g:\mathcal{Z}\to\mathcal{Y}$ is a (p,q)-map. In the first case, f is fully described as the restriction of the projection map of a p-fold covering of $\mathcal{Y}$ to all but finitely many points of the covering. In the second case, every boundary component of $\mathcal{Z}$ is a boundary component $\mathcal{G}$ of $\mathcal{X}$ and $C(g;\mathcal{G})$ $(= C(f;\mathcal{G}))$ is a non-degenerate continuum. It turns out (see [**5**, **6**]) that the latter property of g is needed to ensure that g takes a valency less than q at each exceptional point [**5**, **6**].

Definition 2. A (p, q)-map $f : \mathcal{X} \to \mathcal{Y}$ is termed *normal* if $C(f;\mathcal{G})$ is a nondegenerate continuum for every boundary component $\mathcal{G}$ of $\mathcal{X}$. Further, if f is a BQ-map then f is termed a *normal* BQ-map.

For these maps the *total deficiency* of f is defined as the sum of all the deficiences $\delta_f(e)$, $e \in \mathcal{E}$.

We will utilize some basic concepts and results from surface topology. For instance, the *genus*, *number of boundary components* and *Euler's characteristic* of a surface, denoted by $g(.)$, $\eta(.)$ and $\chi(.)$, respectively, will be used in connection with the Riemann-Hurwitz relation to establish the following theorem:

Theorem 1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a normal (p, q)-map. Then*

$$\beta_f + \delta_f = 2[(g(\mathcal{X}) - 1) - q(g(\mathcal{Y}) - 1)] + \eta(\mathcal{X}) + (p - q)\chi(\mathcal{P}).$$

This is the paper's central result. Our second result states as follows:

Theorem 2. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a normal (p, q)-map. Then*

$$\beta_f + \delta_f \geq 2[(g(\mathcal{X}) - 1) - p(g(\mathcal{Y}) - 1)] - (p - q - 1)\eta(\mathcal{X}).$$

Further, the inequality is sharp.

If $\mathcal{P}$ is simply-connected but not necessarily connected, then we have:

Theorem 3. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a normal (p, q)-map with a simply-connected p-set. Then*

$$\beta_f + \delta_f \geq 2[(g(\mathcal{X}) - 1) - q(g(\mathcal{Y}) - 1)] + p - q + \eta(\mathcal{X}).$$

Further, the inequality is strict if and only if $\mathcal{P}$ has more than one connected component.

If $\mathbf{D}$ is the unit disc, then an application of this theorem yields [**7**, Theorem 5]:

Theorem 4. *Let $f : \mathbf{D} \to \mathcal{Y}$ be a normal (p, q)-map. Then*

$$\beta_f + \delta_f \geq p + q - 1 - 2qg(\mathcal{Y}).$$

Further, the inequality is sharp for $g(\mathcal{Y}) = 0$ *or* 1.

We will pay special attention to the following subclass of (p, q)-maps:

Definition 3. A normal (p, q)-map $f : \mathcal{X} \to \mathcal{Y}$ is termed *BQ-map* if its exceptional set $\mathcal{E}$ is empty.

If $\eta(\mathcal{X}) = 1$, then a geometric characterization of normal BQ-maps [6] yields:

Theorem 5. *Let* $f : \mathcal{X} \to \mathcal{Y}$ *be a normal BQ-map with* $\eta(\mathcal{X}) = 1$. *Then* $g(\mathcal{X}) \geq g(\mathcal{Y})$ *and*

$$\beta_f = 2[(g(\mathcal{X}) - 1) - p(g(\mathcal{Y}) - 1)] - p + q + 1 + 2n(p - q),$$

where n *is an integer satisfying* $0 \leq n \leq g(\mathcal{Y})$.

The following is somehow a converse of the above result:

Theorem 6. *Let* α, β, n, *and* b *be non-negative integers satisfying* $n \leq \beta \leq \alpha$, $b \geq p - q - 1$ *if* $n < \beta$ *and* $b \geq p - q + 1$ *otherwise, and*

$$b = 2[(\alpha - 1) - p(\beta - 1)] - p + q + 1 + 2n(p - q).$$

Then there exist an open orientable surface $\mathcal{X}$ *with* $g(\mathcal{X}) = \alpha$ *and* $\eta(\mathcal{X}) = 1$, *a closed orientable surface with* $g(\mathcal{Y}) = \beta$, *and a normal BQ-map* $f : \mathcal{X} \to \mathcal{Y}$ *so that* $\beta_f = b$.

Further use of the geometry of normal BQ-maps [6] gives:

Theorem 7. *Let* $f : \mathcal{X} \to \mathcal{Y}$ *be a normal BQ-map. Then*

$$a \leq \beta_f \leq b$$

where

$$a = 2[(g(\mathcal{X}) - 1) - p(g(\mathcal{Y}) - 1)] - (p - q - 1)\eta(\mathcal{X})$$

and

$$\begin{aligned} b = \min\{&2[(g(\mathcal{X}) - 1) - q(g(\mathcal{Y}) - 1)] + (p - q + 1)\eta(\mathcal{X}), \\ &(p - q + 1)(2[(g(\mathcal{X}) - 1) - q(g(\mathcal{Y}) - 1)] + (p - q)\eta(\mathcal{X}))\}. \end{aligned}$$

The inequalities are best possible if $q = p - 1$ *or*

$$b = 2[(g(\mathcal{X}) - 1) - q(g(\mathcal{Y}) - 1)] + (p - q + 1)\eta(\mathcal{X}).$$

An immediate consequence of this extends a result of Srebro and Wajnryb [14, Mapping Theorem]:

Theorem 8. *A necessary condition for a normal BQ-map $f : \mathcal{X} \to \mathcal{Y}$ to exist is that*

$$g(\mathcal{X}) \geq q(g(\mathcal{Y}) - 1) + 1 - (p - q)\eta(\mathcal{X})/2.$$

The paper is organized as follows. §1 is devoted to some geometric examples of (p, q)-maps which will help us to illustrate and assert the sharpness of some results. In §2, we establish the notation and introduce the preliminary results that will be deployed in the paper. The proofs of Theorems 1, 2, 3 and 4 will be presented in §3, Theorems 5 and 6 in §4, and Theorems 7 and 8 in §5. The paper concludes with open questions in §6.

1. Examples.

This section makes available a variety of examples that will motivate the results of the paper and assert their sharpness.

Example 1. Let $(\widetilde{\mathcal{Y}}, \pi)$ be a p-fold covering of a closed orientable surface $\mathcal{Y}$. We modify this covering as follows. Denote by $\widetilde{\Delta}_1$, $\widetilde{\Delta}_2$, $\dots$, $\widetilde{\Delta}_{p-q}$, pairwise disjoint subregions of $\widetilde{\mathcal{Y}}$ each of which is homeomorphic under π to a closed Jordan subregion Δ of $\mathcal{Y}$. Let L be a simple arc joining two points a and b in the interior of Δ, and let $\widetilde{L}_1$, $\widetilde{L}_2, \dots , \widetilde{L}_{p-q}$, be the lifts of L in the respective regions $\widetilde{\Delta}_1$, $\widetilde{\Delta}_2, \dots , \widetilde{\Delta}_{p-q}$. Cut $\widetilde{\mathcal{Y}}$ along each arc $\widetilde{L}_j$, and identify the edges of these cuts crosswise in the usual manner leading to two branch points, $\tilde{a}$ and $\tilde{b}$, over a and b, respectively, each of order $p - q - 1$. Denote by $\widehat{\mathcal{Y}}$ the resulting surface, and by $\hat{\pi}$ the continuous map coinciding with π on $\widetilde{\mathcal{Y}} \setminus (\bigcup_{j=1}^{p-q} \widetilde{L}_j)$. Then $(\widehat{\mathcal{Y}}, \hat{\pi})$ is also a p-fold covering of $\mathcal{Y}$ with $\hat{\pi}$ having a total order given by the Riemann-Hurwitz relation as

$$\beta_{\hat{\pi}} = 2\left[\left(g\left(\widehat{\mathcal{Y}}\right) - 1\right) - p(g(\mathcal{Y}) - 1)\right].$$

Now, let D be a closed Jordan subregion of $\mathcal{Y}$ containing a in its interior and b in its exterior. By the above construction, D has a unique lift, $\widehat{D}$, under $\hat{\pi}$ that contains $\tilde{a}$ in its interior and $\tilde{b}$ in its exterior, and is of multiplicity $p - q$. Denote by $\mathcal{X}$ the subsurface of $\widehat{\mathcal{Y}}$ obtained by removing $\widehat{D}$, and by f the restriction of $\hat{\pi}$ to $\mathcal{X}$. Then $\mathcal{X}$ is an open surface satisfying $g(\mathcal{X}) = g(\widehat{\mathcal{Y}})$ and $\eta(\mathcal{X}) = 1$, and f is a (p, q)-map. Obviously, f assumes every value of D q times and $\mathcal{Y} \setminus D$ p times, counting multiplicity. Further, f has zero deficiency δ_f and total branch order $\beta_f = \beta_{\hat{\pi}} - p + q - 1$. It follows that

$$\beta_f = 2[(g(\mathcal{X}) - 1) - p(g(\mathcal{Y}) - 1)] - p + q + 1.$$

Example 2. A repeated use of the above procedure in Example 1 yields (i) a p-fold covering $(\widehat{\mathcal{Y}}, \hat{\pi})$ of $\mathcal{Y}$, and (ii) a covering subsurface $(\mathcal{X}, f)$ of $(\widehat{\mathcal{Y}}, \hat{\pi})$ satisfying the following properties:

(a) $f : \mathcal{X} \to \mathcal{Y}$ is a normal (p, q)-map.

(b) f assumes every value in a pairwise disjoint union of closed Jordan regions $D_1, D_2, \dots, D_n$, exactly q times, and p times otherwise.

(c) $g(\mathcal{X}) = g(\widehat{\mathcal{Y}})$, $\eta(\mathcal{X}) = n$, and $\beta_f = \beta_{\hat{\pi}} - (p-q-1)n$.

By the Riemann-Hurwitz relation we get:

$$\beta_{\hat{\pi}} = 2\left[\left(g\left(\widehat{\mathcal{Y}}\right) - 1\right) - p(g(\mathcal{Y}) - 1)\right],$$

which gives

$$\beta_f = 2[(g(\mathcal{X}) - 1) - p(g(\mathcal{Y}) - 1)] - (p-q-1)\eta(\mathcal{X}).$$

Example 3. Suppose $\mathcal{Y}$ is a closed orientable surface with positive genus, g. It is well known that there exists a system of loops a_1, b_1, a_2, $b_2, \dots, a_g$, b_g, in $\mathcal{Y}$ starting and ending at the same point, O, and are otherwise pairwise disjoint, whose removal reduces $\mathcal{Y}$ into a simply-connected domain (see [9, pp. 37-42]). Let $(\widehat{\mathcal{Y}}, \pi_1)$ be a q-fold covering of $\mathcal{Y}$. The branch order of π_1 is given by

$$\beta_{\pi_1} = 2\left[\left(g\left(\widehat{\mathcal{Y}}\right) - 1\right) - q(g(\mathcal{Y}) - 1)\right].$$

Choose an integer $n, 1 \le n \le g$. Let Z be the surface obtained by removing from $\mathcal{Y}$ the loops a_1, b_1, a_2, b_2, $\dots$, a_n, b_n. It is easy to see that Z is an open surface of genus, $g-n$, and with one boundary component. Denote by Z_1, Z_2, $\dots$, Z_{p-q} copies of Z. We attach these surfaces as follows. Let L be a Jordan arc ending at O and lying otherwise in Z, and let O' be the other endpoint. Cut each surface Z_j, $1 \le j \le p-q$, along the associated Jordan arc over L, and attach the resulting surfaces by identifying the edges of the cuts crosswise in the usual manner leading to a $(p-q)$-fold covering $(\widehat{Z}, \pi_2)$ of Z having a branch point of order $p-q-1$ over O'. Evidently, the surface $\widehat{Z}$ has genus $(p-q)(g-n)$ and one boundary component. We attach the surfaces $\widehat{\mathcal{Y}}$ and $\widehat{Z}$, first by making a cut in each over a Jordan arc in Z that avoids all the branch values of π_1 and π_2, and second by identifying the edges of the cuts crosswise leading to a covering surface $(\mathcal{X}, f)$ of $\mathcal{Y}$ satisfying:

(a) $f : \mathcal{X} \to \mathcal{Y}$ is a normal BQ-map.

(b) $\beta_f = p - q + 1 + \beta_{\pi_1}$.

(c) $g(\mathcal{X}) = (p-q)(g-n) + g(\widehat{\mathcal{Y}})$ and $\eta(\mathcal{X}) = 1$.

We conclude:

$$\beta_f = 2[(g(\mathcal{X}) - 1) - p(g(\mathcal{Y}) - 1)] - p + q + 1 + 2n(p-q).$$

Example 4. Let $(\widehat{\mathcal{Y}}, \pi_1)$ be the q-fold covering surface of $\mathcal{Y}$ of the previous example. Denote by D_1, D_2, $\dots$, D_n, pairwise disjoint Jordan domains in

$\mathcal{Y}$. For each $j, 1 \leq j \leq n$, let (Z_j, h_j) be the simply-connected $(p-q)$-fold covering of D_j having only one branch point, of order $p-q-1$, and one boundary component, and let L_j be a Jordan arc in D_j not passing through any of the branch values of π_1 or h_j. Make a single cut in $\widehat{\mathcal{Y}}$ over each L_j, and a cut in each Z_j over L_j, $1 \leq j \leq n$, then attach the resulting surfaces, conveniently denoted by $\widehat{\mathcal{Y}}$ and Z_j, $1 \leq j \leq n$, by identifying crosswise the edges of their cuts in the usual manner. This yields a covering surface $(\mathcal{X}, f)$ of $\mathcal{Y}$ having the following properties:

(a) $f : \mathcal{X} \to \mathcal{Y}$ is a normal BQ-map.

(b) $\beta_f = n(p-q+1) + \beta_{\pi_1}$.

(c) $g(\mathcal{X}) = g(\widehat{\mathcal{Y}})$ and $\eta(\mathcal{X}) = n$.

With β_{π_1} given as in the previous example, we conclude:

$$\beta_f = 2[(g(\mathcal{X})-1) - q(g(\mathcal{Y})-1)] + (p-q+1)\eta(\mathcal{X}).$$

Example 5. Let $\mathcal{Y}$ be the torus, π a universal covering map of $\mathcal{Y}$ having the square $\{z : 0 \leq \Re z < 1, 0 \leq \Im z < 1\}$ as a fundamental domain, $\{n_k\}_{k=1}^n$ a finite sequence of positive integers, and X_k the square $\{z : 0 < \Re z < n_k, 0 < \Im z < n_k\}$. Denote by γ_1 and γ_2 the simple closed curves

$$\pi(\{x : 0 \leq x \leq 1\}) \quad \text{and} \quad \pi(\{iy : 0 \leq y \leq 1\}),$$

respectively, and $e = \gamma_1 \cap \gamma_2$. Let $\pi_k = \pi_{|X_k}$. It is easy to see that $\pi_k : X_k \to \mathcal{Y}$ is a (p,q)-map with $p = p_k = n_k^2$, $q = q_k = n_k^2 - n_k$, p-set $\mathcal{Y} \setminus (\gamma_1 \cup \gamma_2)$, q-set $(\gamma_1 \cup \gamma_2) \setminus \{e\}$, and exceptional set $\{e\}$. Observe that $\delta_{\pi_k} = n_k - 1$, $\beta_{\pi_k} = 0$ and

$$\beta_{\pi_k} + \delta_{\pi_k} = n_k - 1 = p_k + q_k - 1 - 2q_k g(\mathcal{Y}).$$

Let L be a Jordan arc in $\mathcal{Y}$ not meeting $\gamma_1 \cup \gamma_2$. With a slight abuse of notation, we assume the coverings (X_k, π_k) pairwise disjoint. Make a single cut over L in each of these coverings, and attach the resulting coverings by identifying the edges of the cuts crosswise in the usual manner which leads to a (connected) covering surface $(\mathcal{X}, f)$ having exactly two branch points each located over one endpoint of L and has order $n-1$. It easily follows that

(a) $f : \mathcal{X} \to \mathcal{Y}$ is a normal (p,q)-map with $p = \Sigma_{k=1}^n n_k^2$ and $q = \Sigma_{k=1}^n (n_k^2 - n_k)$.

(b) $\beta_f = 2(n-1)$ and $\delta_f = \Sigma_{k=1}^n \delta_{\pi_k} = \Sigma_{k=1}^n n_k - n$.

(c) $g(\mathcal{X}) = 0$ and $\eta(\mathcal{X}) = n$.

We conclude:

$$\beta_f + \delta_f = 2[(g(\mathcal{X})-1) - q(g(\mathcal{Y})-1)] + p - q + \eta(\mathcal{X}).$$

2. Notation and preliminaries.

The object of this section is to introduce the notation, concepts and preliminary results that will be used throughout the paper. Our two main sources will be Ahlfors and Sario [**1**], Lyzzaik and Stephenson [**7**] and Lyzzaik [**5**].

Let F be an orientable surface [**1**]. If F is compact and without boundary, then F is termed *closed*. The *canonical* (*normal*) *form* of a closed orientable surface is simply a surface with finitely many attached handles. The number of handles is called the *genus* of F and is denoted by $g(F)$.

An open orientable surface, E, of finite genus, $g(E)$, and finite number of boundary components, $\eta(E)$, is obtained by removing $\eta(E)$ pairwise disjoint points or closed discs from a closed orientable surface F having the same genus. If only closed discs are removed, then the closure, $\overline{E}$, of E in F is a *bordered orientable surface* of genus $g(E)$ and number of components $\eta(E)$. Here, the surface E is called the *interior* of the bordered orientable surface $\overline{E}$.

The concept of a *boundary component* of an open orientable surface will be used in its formal sense as in [**1**, pp. 81-90]. The *impression* of a boundary component will be used as defined in [**5**]. If F and $\widetilde{F}$ are orientable surfaces and $f : \widetilde{F} \to F$ is an open continuous function, then the pair $(\widetilde{F}, f)$ is termed a *ramified covering*, or simply a *covering*, of F. It is known that f has the same local properties as the mappings $z \to z^m$, where m is a positive integer. If $m \geq 2$ at some point $x \in \widetilde{F}$, then x is termed a *branch point* of f of *order* $m-1$ and the point $y = f(x)$ a *branch value* of f.

A covering $(\widetilde{F}, f)$ of an orientable surface F is termed *complete* if every point $y \in F$ has a neighbourhood N so that each connected component of $f^{-1}(N)$ is compact. It is known that the points of F will be covered by $\widetilde{F}$ equally the same number of times, counting multiplicity. If this number is n, then $(\widetilde{F}, f)$ is termed an *n-fold* (*sheeted*) *covering* of F. Another useful property of complete coverings is that they permit path lifting so that two lifts of a given path meet, if at all, only at branch points. This property yields the following result whose proof can be found in [**7**]:

Proposition 1. *Let $(\widetilde{F}, f)$ be a covering surface of an orientable surface F, and S a subset of F consisting of points that are covered equally the same number, n, of times. Then there exists an open neighborhood $\mathcal{V}$ of S so that the inverse image under f of every connected component V of $\mathcal{V}$ contains complete coverings of V having total multiplicity at least n. Moreover, if S is open, then $\mathcal{V}$ can be S itself.*

Throughout this paper and unless otherwise is specified, the function $f : \mathcal{X} \to \mathcal{Y}$ stands for a normal (p, q)-map. Recall from §1 the sets $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{E}$ and

$\mathcal{B}$ associated with the f. Some basic properties of these sets are given in [**5**, Lemma 2] as follows.

Proposition 2. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a normal (p, q)-map. Then*

(a) *$\mathcal{P}$ is an open subset whose genus is at most $min\{g(\mathcal{X}), g(\mathcal{Y})\}$.*

(b) *$\mathcal{Q} \cup \mathcal{E}$ is a union of at most $\eta(\mathcal{X})$ pairwise disjoint nondegenerate continua.*

(c) *Every connected component P of $\mathcal{P}$ has finitely many boundary components.*

(d) *$C(f) = \partial\mathcal{Q}$.*

(e) *$\nu_f(e) < q$ for every $e \in \mathcal{E}$.*

(f) *$\mathcal{E} \subset \partial\mathcal{P} \cap \partial\mathcal{Q}$.*

We will make use of special neighbourhoods of the q-set of f.

Definition 4. An open neighborhood $\mathcal{V}$ of $\mathcal{Q}$ is termed a *fattened* q-set *of* $\mathcal{Q}$ if every connected component V of $\mathcal{V}$ meets $\mathcal{Q}$ and lifts under f to a subset of $\mathcal{X}$ that contains complete coverings of V having total multiplicity exactly q.

It follows from Proposition 1 that $\mathcal{Q}$ has a fattened q-set. Further, we have:

Proposition 3. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a normal (p, q)-map. There exists a fattened q-set $\mathcal{U}$ satisfying the following properties:*

(a) *$\mathcal{U} \cap \mathcal{P} \cap \mathcal{B} = \varnothing$.*

(b) *Each connected component U of $\mathcal{U}$ has finitely many boundary components.*

(c) *Each boundary component of U has a nondegenerate impression in $\mathcal{P} \cup \mathcal{E}$.*

(d) *Each U has boundary components that are pairwise separated by $(\mathcal{Q} \cup \mathcal{E}) \cap U$.*

(e) *Each connected component O of $\mathcal{U} \cap \mathcal{P}$ is either simply-connected or a ring domain.*

(f) *If O is a ring domain, then the impressions of the boundary components of O are a common contour of U and P and a nondegenerate subcontinuum in $\mathcal{Q} \cup \mathcal{E}$.*

(g) *The number of all possible domains O is at most $\eta(\mathcal{X})$.*

This result comprises [**5**, Lemmas 5 and 6].

Definition 5. A fattened q-set satisfying the properties of Proposition 3 is termed a *refined fattened* q-set.

If f is a BQ-map (Definition 3), then refined fattened q-sets are of special nature.

Proposition 4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a BQ-map, and $\mathcal{U}$ a refined fattened q-set. Then every component U of $\mathcal{U}$ is the interior of a bordered surface so that each connected component of $U \setminus \mathcal{Q}$ is a ring domain having a common contour with U.*

We use the term *Jordan arc* for the homeomorph of a closed interval, and *Jordan curve* or *loop* for the homeomorph of the unit circle.

The following result is needed:

Proposition 5. *Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is a normal (p, q)-map. Let P be a connected component of $\mathcal{P}$, and $\gamma \subset P$ a Jordan curve dividing P into two connected components of which one is a ring domain, A. If A is a subset of a fattened q-set $\mathcal{V}$ and $A \cup \gamma$ has no branch points of f, then $f^{-1}(A)$ contains pairwise disjoint domains, $\widetilde{A}$, satisfying the following properties:*

(a) *Each $\widetilde{A}$ is a ring domain belonging to a unique boundary component of $\mathcal{X}$.*

(b) *Each $\widetilde{A}$ has a compact boundary in $\mathcal{X}$ which is a Jordan curve, $\widetilde{\gamma}$.*

(c) *The associated pairs $(\widetilde{A}, f)$ and $(\widetilde{\gamma}, f)$ are complete covering of A and γ, respectively, with equal multiplicities.*

(d) *The sum of multiplicities of the coverings $(\widetilde{A}, f)$ or $(\widetilde{\gamma}, f)$ is exactly $p - q$.*

For a proof of this result see [**5**].

A main tool for our study is the following embedding theorem whose proof can also be found in [**5**]:

Theorem A. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a normal (p, q)-map. Then there exists a p-fold covering surface $(\widetilde{\mathcal{Y}}, \pi)$ of $\mathcal{Y}$ and an embedding $\phi : \mathcal{X} \to \widetilde{\mathcal{Y}}$ so that $f \equiv \pi \circ \phi$. The projection map π inherits via ϕ the branch points of f, if exists, and possibly has some others that lie in $\widetilde{\mathcal{Y}} \setminus \phi(\mathcal{X})$ over the sets $\mathcal{E}$ and $\mathcal{Q} \setminus \mathcal{B}$. Further, the latter set of branch points (over $\mathcal{Q} \setminus \mathcal{B}$) has total order $(p - q)s - r$, where r and s are integers satisfying $0 \le s \le r \le \eta(\mathcal{X})$.*

In view of this theorem, we introduce:

Definition 6. The branch points of the projection map π lying in $\widetilde{\mathcal{Y}} \setminus \phi(\mathcal{X})$ over $\mathcal{Q}$ are termed the *auxiliary branch points* of f. Also, The branch values and total order of these points are termed the *auxiliary branch values and auxiliary branch order* of f and denoted by $\mathcal{B}'$ and β'_f, respectively.

Normal (p, q)-maps may have quite pathological q-sets. Using a special type of homotopy [**7**, **6**], termed (p, q)-*homotopy*, it was shown that every normal (p,q)-map deforms continuously to a (p,q)-map bearing the same covering properties and whose q-set is a finite set of points, Jordan arcs or loops.

Definition 7. Two (p,q)-maps f and g are said to have the same *valence structure* if there is an order preserving bijection between their branch points and a deficiency preserving bijection between their exceptional points.

Evidently, $\beta_f = \beta_g$, $\delta_f = \delta_g$ and f and g have the same valence structure.
The notion of (p,q)-homotopy is defined as follows [6, 7]:

Definition 8. A (p,q)-*homotopy* is a homotopy $\Lambda : \mathcal{X}\times[0,1] \to \mathcal{Y}$ satisfying the property that all maps $\Lambda(.,t) : \mathcal{X} \to \mathcal{Y}$, $t \in [0,1]$, are (p,q)-maps having the same valence structure.

It is immediate that (p,q)- homotopy is an equivalence relation.
We will use the following results whose proofs can be found in [6].

Theorem B. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a normal (p,q)-map. Then f is (p,q)-homotopic to a map h whose q-set $\mathcal{Q}_h$ is either empty or a disjoint union of $\mathcal{B}'$ and a finite number of open Jordan arcs or loops that do not meet $\mathcal{B}$ and start and end in $\mathcal{B}' \cup \mathcal{E}$. Further, the number of connected components of $\mathcal{Q}_h \cup \mathcal{E}$ is at most $\eta(\mathcal{X})$.*

Theorem C. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a normal BQ-map. Then f is (p,q)-homotopic to a map h whose q-set $\mathcal{Q}_h$ satisfies:*
(a) $\mathcal{Q}_h \cap \mathcal{B} = \emptyset$.
(b) *$\mathcal{Q}_h$ is a disjoint union of at most $\eta(\mathcal{X})$ continua.*
(c) *Each of the continua is either a singleton in $\mathcal{B}'$ or a union of loops starting and ending at some point of $\mathcal{B}'$ and are otherwise pairwise disjoint.*
(d) *The number of these singletons or loops in $\mathcal{Q}_h$ is at most*

$$\max\{\eta(\mathcal{X}), 2g(\mathcal{Y}) + \eta(\mathcal{X}) - 1\}.$$

.

Theorem D. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a normal BQ-map with $\eta(\mathcal{X}) = 1$. Then $g(\mathcal{X}) \geq g(\mathcal{Y})$ and f is (p,q)-homotopic to a map h whose q-set*
(a) *does not contain any branch value of f, and is either*
(b) *a singleton, or*
(c) *a curve given by*

$$a_1 b_1 a_1^{-1} b_1^{-1} \dots a_n b_n a_n^{-1} b_n^{-1},$$

where $1 \leq n \leq g(\mathcal{Y})$, *and*

$$a_1 b_1 a_1^{-1} b_1^{-1} \dots a_n b_n a_n^{-1} b_n^{-1} \dots a_k b_k a_k^{-1} b_k^{-1}$$

is a canonical form of $\mathcal{Y}$ *with* $a_1,\ b_1,\ \dots,\ a_k,\ b_k$ *loops starting and ending from the same point and are otherwise disjoint.*

Remark 1. It is clear from [6] that the aforementioned (p, q)-homotopies preserve the number of connected components of the p-set.

A topological invariant that we will need is the concept of Euler characteristic. Suppose W is an m-dimensional cell complex, and let $n_d(W)$ be the number of d-cells in W. The *Euler characteristic* of W is defined as

$$\chi(W) = \sum_{d=0}^{m} (-1)^d n_d(W).$$

If $m = 1$ then $\chi(W)$ is the number of 0-simplices minus the number of 1-simplices, and if $m = 2$ then $\chi(W)$ is the number of 0-simplices minus the number of 1-simplices plus the number of 2-simplices. If W is a compact orientable surface, then W affects a finite 2-dimensional cell complex, and the Euler characteristic of W is itself that of the complex. It is known that this definition is independent of the underlying complex, and that

$$\chi(W) = 2 - 2g(W) - \eta(W).$$

In particular, if W is closed then

$$\chi(W) = 2 - 2g(W).$$

Moreover, if W is an open orientable surface with finite genus and finite number of boundary components, then we can find a bordered subsurface $\overline{W_{\circ}}$ of W satisfying $g(W_{\circ}) = g(W)$ and $\eta(W_{\circ}) = \eta(W)$. This surface is topologically unique, and as such the Euler characteristic of W is defined as

$$\chi(W) = \chi\left(\overline{W_{\circ}}\right).$$

The Riemann-Hurwitz relation [9, pp. 33-34] yields:

Proposition 6. *Let* W *be an open orientable surface with finite genus, and* $\widetilde{W}$ *be a complete covering covering of* W*. Then* $g(\widetilde{W}) \geq g(W)$*.*

If W is a closed orientable surface that is divided by finitely many disjoint loops into a finite number of bordered subsurfaces $\overline{W}_1,\ \overline{W}_2,\ \dots,\ \overline{W}_s$, then

$$\chi(W) = \Sigma_{j=1}^{s} \chi(W_j).$$

Using this, we obtain:

Proposition 7. *Let W be a closed orientable surface, and let C be a finite 1-dimensional complex in W. Then*

$$\chi(W) = \chi(W \setminus C) + \chi(C).$$

Proof. It is immediate that C divides W into a finite number of surfaces W_1, $W_2, \dots, W_s$. Observe that each W_j has finitely many boundary components and contains a bordered subsurface $\overline{V}_j$ satisfying $g(V_j) = g(W_j)$ and $\eta(V_j) = \eta(W_j)$. Note that the set $W \setminus (\bigcup_{j=1}^{s} V_j)$ is then a disjoint union of finitely many bordered surfaces $U_1, U_2, \dots, U_t$. It is easy to see that each U_k contains a connected component, C_k, of C such that $U_k \setminus C_k$ is a disjoint union of finitely many ring domains each having a common contour with the some surface V_j. It is easily seen that $\chi(\overline{U}_k) = \chi(C_k)$ for each k, $1 \le k \le t$. This gives

$$\begin{aligned}\chi(W) &= \Sigma_{j=1}^{s}\chi\left(\overline{V}_j\right) + \Sigma_{k=1}^{t}\chi\left(\overline{U}_k\right)\\ &= \Sigma_{j=1}^{s}\chi(W_j) + \Sigma_{k=1}^{t}\chi(C_k)\\ &= \chi(W \setminus C) + \chi(C).\end{aligned}$$

This completes the proof. □

3. Proofs of Theorems 1, 2, 3 and 4.

Recall the sets $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{E}$ and $\mathcal{B}$ associated with f. Also recall Theorem A; namely: The covering surface $(\widetilde{\mathcal{Y}}, \pi)$ of $\mathcal{Y}$, the embedding $\phi : \mathcal{X} \to \widetilde{\mathcal{Y}}$, and the auxilliary set $\mathcal{B}'$ (Definition 5) of branch values of f.

Theorem B shows that a normal (p, q)-map f is (p, q)-homotopic to a map satisfying strict regularity conditions. Because (p, q)-homotopy preserves the valence structure, we loose no generality in assuming henceforth that f satisfies these conditions unless specified otherwise; namely that the q-set of f is a disjoint union (possibly empty) of a finite set of points $\mathcal{B}'$ and a finite number of open Jordan arcs starting and ending at points of the set $\mathcal{B}' \cup \mathcal{E}$. Further, this q-set has no branch values of f except possibly at the exceptional points.

Denote by K the 1-dimensional complex in $\mathcal{Y}$ whose 0 and 1-simplices comprise the sets $\mathcal{B}' \cup \mathcal{B} \cup \mathcal{E}$ and $\mathcal{Q} \cup \mathcal{E}$ respectively, and by $\widetilde{K}$ the 1-dimensional complex in $\widetilde{\mathcal{Y}}$ which is the lift of K under π as restricted to $\widetilde{\mathcal{Y}} \setminus \phi(\mathcal{X})$. It is immediate that (i) $\widetilde{K}$ and $\phi(\mathcal{X})$ partition $\mathcal{Y}$ and (ii) $\pi : \widetilde{K} \to K$ is a $(p - q)$-fold covering that lifts each 0-simplex $v \in K$ under π to exactly

$p-q$ 0-simplices $\tilde{\upsilon} \in \widetilde{K}$, counting multiplicity, and each 1-simplex $\sigma \in K$ to exactly $p-q$ 1-simplices $\tilde{\sigma} \in \widetilde{K}$.

Proof of Theorem 1. In addition to the above notation, we need the following:

(a) υ and $\tilde{\upsilon}$ denote the number of 0-simplices of K and $\widetilde{K}$ respectively.
(b) e and $\tilde{e}$ denote the number of 1-simplices of K and $\widetilde{K}$ respectively.
(c) β_π and $\beta_{\widetilde{K}}$ denote the total branch orders of π in $\widetilde{\mathcal{Y}}$ and $\widetilde{K}$ respectively.

Recall that each 0-simlpex $v \in K$ belongs to $\mathcal{B}' \cup \mathcal{E}$. Then v is covered by π exactly $q - \delta_f(v)$ times, counting multiplicity, in $\mathcal{X}$. Note that $\delta_f(v) = 0$ if $v \in \mathcal{B}'$. It follows that π covers v exactly $p - q + \delta_f(v)$ times, counting multiplicity, in $\widetilde{K}$. By adding over all the 0-simplices of K and using the fact that the number of 0-simplices in $\widetilde{K}$ is $\tilde{\upsilon} + \beta_{\widetilde{K}}$, we obtain:

$$\delta_f = \tilde{\upsilon} + \beta_{\widetilde{K}} - (p-q)\upsilon.$$

Since a branch point of π belongs either to $\mathcal{X}$ or $\widetilde{K}$,

$$\beta_\pi = \beta_f + \beta_{\widetilde{K}}.$$

The Riemann-Hurwitz relation gives:

$$\beta_\pi = 2[(g(\widetilde{\mathcal{Y}}) - 1) - p(g(\mathcal{Y}) - 1)].$$

Also, by Proposition 7 we have:

$$2 - 2g(\widetilde{\mathcal{Y}}) = \chi(\widetilde{\mathcal{Y}}) = \chi(\mathcal{X}) + \chi(\widetilde{K}) = 2 - 2g(\mathcal{X}) - \eta(\mathcal{X}) + \tilde{\upsilon} - \tilde{e},$$

where

$$\tilde{e} = (p-q)e$$

since each 1-simplex in K is covered by $\widetilde{K}$ under π exactly $p-q$ times.

Further, by Proposition 7 we have:

$$2 - 2g(\mathcal{Y}) = \chi(\mathcal{Y}) = \chi(\mathcal{P}) + \chi(K) = \chi(\mathcal{P}) + \upsilon - e,$$

which gives

$$\upsilon - e = 2 - 2g(\mathcal{Y}) - \chi(\mathcal{P}).$$

Using these identities successively we obtain:

$$\begin{aligned}
\beta_f + \delta_f &= \beta_\pi + \tilde{\upsilon} - (p-q)\upsilon \\
&= -(2 - 2g(\widetilde{\mathcal{Y}})) - 2p(g(\mathcal{Y}) - 1) + \tilde{\upsilon} - (p-q)\upsilon \\
&= 2(g(\mathcal{X}) - 1) - 2p(g(\mathcal{Y}) - 1) + \eta(\mathcal{X}) + \tilde{e} - (p-q)\upsilon \\
&= 2(g(\mathcal{X}) - 1) - 2p(g(\mathcal{Y}) - 1) + \eta(\mathcal{X}) - (p-q)(\upsilon - e) \\
&= 2[(g(\mathcal{X}) - 1) - q(g(\mathcal{Y}) - 1)] + \eta(\mathcal{X}) + (p-q)\chi(\mathcal{P}).
\end{aligned}$$

This ends the proof. □

One can easily verify that Theorem 1 holds in Examples 1, ... , 5 of §1.

Proof of Theorem 2. If K is the above-mentioned complex, then Proposition 7 gives

$$\chi(\mathcal{P}) = 2 - 2g(\mathcal{Y}) - \chi(K).$$

Since K decomposes into at most $\eta(\mathcal{X})$ pairwise disjoint continua, it easily follows that $\chi(K) \leq \eta(\mathcal{X})$, and

$$\chi(\mathcal{P}) \geq 2 - 2g(\mathcal{Y}) - \eta(\mathcal{X})$$

which together with the above theorem yield the desired inequality.

Examples 1 and 2 show equality, and Example 3 shows strict inequality. Therefore, the inequality is sharp and the proof is complete. □

If $\mathcal{P}$ is simply-connected, then $\chi(\mathcal{P})$ is the number of connected components of $\mathcal{P}$ and Theorem 3 follows at once from Theorem 1 and Example 5.

It is easy that the proof of Theorem 4 is a consequence of Theorem 3 and Examples 1 and 4 of [**7**]; only observe that $\mathcal{X} = \mathbf{D}$ yields $\eta(\mathcal{X}) = 1$ and $\mathcal{P}$ simply-connected [**7**, Lemma 4(a)].

4. Proofs of Theorems 5 and 6.

Proof of Theorem 5. It is immediate from Theorem D that $g(\mathcal{X}) \geq g(\mathcal{Y})$ and that $\mathcal{P}$ is a subsurface of $\mathcal{Y}$ with exactly one boundary component. By Proposition 7

$$\chi(\mathcal{P}) = 1 - 2g(\mathcal{P}) = 1 - 2(g(\mathcal{Y}) - n)$$

for some integer n, $0 \leq n \leq g(\mathcal{Y})$. Using Theorem 1, the desired inequality follows and the proof is complete. □

To illustrate the result, see Examples 1 and 3.

The proof of Theorem 6 makes use of the well known *cyclic p-covering* [**14**] which is constructed as follows. Let Z be the surface of the handle body, $B(p, \beta)$, where p and β are integers satisfying $p \geq 2$ and $\beta \geq 1$, shown in Figure 1. This body is obtained by attaching to a solid torus p congruent arms each of which is a handle body of genus $\beta - 1$. Note that Z has genus $p(\beta - 1) + 1$ and is invariant under a cyclic group, G, of order p of rotational symmetries so that the quotient space Z/G is a closed orientable surface Y of genus β. Since G acts without fixed points, the natural projection map $h : Z \to Y$ affects the desired regular p-fold regular covering (Z, h) of Y.

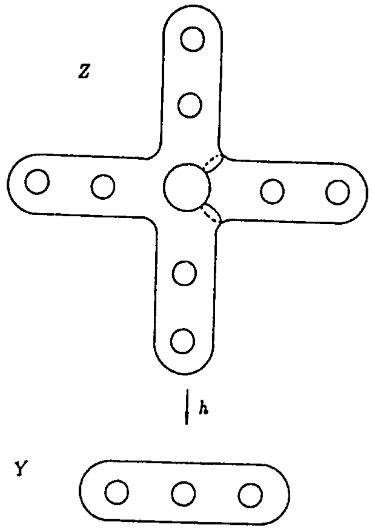

Figure 1. $Z = B(4,3)$, $Z/G = Y$.

Proof of Theorem 6. Let $\mathcal{Y}$ be a closed orientable surface of genus β, O a point in $\mathcal{Y}$, and a_j, b_j, $1 \le j \le \beta$, be loops that represent a canonical homology basis of $\mathcal{Y}$, pass through the point O and are otherwise disjoint (see Figure 2).

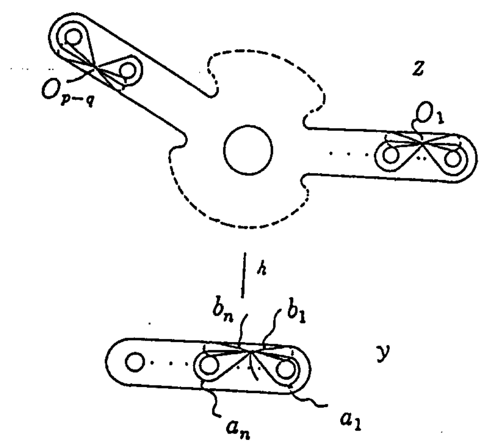

Figure 2.

We consider two cases:

(1) $n < \beta$. Let (Z, h) be the cyclic p-covering of $\mathcal{Y}$ with labelled arms H_1, H_2, $\ldots$, H_p as in Figure 2. Let $O_1, O_2, \ldots, O_{p-q}$ be the lifts of O in H_1, H_2, $\ldots$, H_{p-q} respectively. Note that the canonical basis can be labelled

so that the lifts of the loops a_j, b_j, $1 \leq j \leq n$, are as depicted in Figure 2. By Removing these loops from Z we obtain a covering surface (Z_1, h_1) of $\mathcal{Y}$ where $h_1 \equiv h_{|Z_1}$, $g(Z_1) = p(\beta - 1) + 1 - n(p-q)$, $\eta(Z_1) = p - q$ and $h_1 : Z_1 \to \mathcal{Y}$ is a normal BQ-map that assumes every point in the loops a_j, b_j, $1 \leq j \leq \beta$, q times, and p times otherwise. We modify the covering (Z_1, h_1) twice as follows.

(i) Let ℓ be a Jordan arc in $\mathcal{Y}$ that starts from O and is otherwise disjoint from the canonical basis, and let O' be the other endpoint of ℓ. Through each point O_j, $1 \leq j \leq p - q$, there exists a unique lift, $\tilde{\ell}_j$, of ℓ under h that is homeomorphic to ℓ and lies except for O_j in Z_1. Note that these lifts are pairwise disjoint. Cut Z_1 along the arcs $\tilde{\ell}_j$, and identify the edges of the cuts crosswise in the manner leading to a covering surface (Z_2, h_2) of $\mathcal{Y}$ with a branch point of order $p - q - 1$ over O'. It is easy to see that

$$\begin{aligned} g(Z_2) &= g(Z_1) + p - q - 1 \\ &= p(\beta - 1) + p - q - n(p - q) \end{aligned}$$

and $\eta(Z_2) = 1$.

(ii) Observe that $b - p + q + 1$ is a non-negative even integer, 2r. If $r = 0$, then we set $Z_2 \equiv \mathcal{X}$ and $h_2 \equiv f$. Otherwise, let $\ell_1, \ell_2, \dots, \ell_r$ be pairwise disjoint Jordan arcs in $\mathcal{Y}$ not meeting O' or the canonical basis. Cut Z_2 along any two lifts of each ℓ_j, $1 \leq j \leq r$, under h_2, and identify the edges of the cuts crosswise leading to $2r$ branch points each of order 1. Denote the resulting covering surface by $(\mathcal{X}, f)$. In either case, one easily sees that $f : \mathcal{X} \to \mathcal{Y}$ is a normal BQ-map, $\eta(\mathcal{X}) = 1$, $\beta_f = p - q - 1 + 2r = b$, and

$$\begin{aligned} g(\mathcal{X}) &= g(Z_2) + r \\ &= p(\beta - 1) + p - q - n(p - q) + (b - p + q + 1)/2 \\ &= \alpha. \end{aligned}$$

(2) $n = \beta$. Let (Z, h) be the cyclic q-covering of $\mathcal{Y}$. The removal of the canonical basis from $\mathcal{Y}$ leaves a simply-connected subdomain, Ω, of $\mathcal{Y}$. Let Ω_1, Ω_2, $\dots$, Ω_{p-q} be copies of Ω. The standard cutting and pasting procedure attaches these copies to each other yielding a simply-connected $(p - q)$-fold covering, $\widetilde{\Omega}$, of Ω with only a branch point of order $p - q - 1$. Attach this covering to (Z, h) by simply making a single cut in each over a Jordan arc in $\mathcal{Y}$ that avoids the branch value of h, and identify the edges of these cuts crosswise leading to a covering surface (Z_1, h_1) of $\mathcal{Y}$ so that $g(Z_1) = g(Z) = q(\beta - 1) + 1$, $\eta(Z_1) = 1$, $\beta_{h_1} = p - q + 1$ and $h_1 : Z_1 \to \mathcal{Y}$ is as above.

Observe that $b - p + q - 1$ is a non-negative even integer, 2r. Let $Z_1 \equiv \mathcal{X}$ and $h_1 \equiv f$ if $r = 0$, else we repeat the same procedure above thus resulting in a covering surface $(\mathcal{X}, f)$ similar to the above except for $\beta_f = p-q+1+2r = b$, and

$$\begin{aligned} g(\mathcal{X}) &= g(Z_1) + r \\ &= q(\beta - 1) + (b - p + q - 1)/2 \\ &= \alpha. \end{aligned}$$

This completes the proof. □

5. Proofs of Theorems 7 and 8.

The proof of Theorem 7 requires the following two lemmas:

Lemma 1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a normal BQ-map. If P is a simply-connected component of the p-set of f, then the branch points of f whose branch values lie in P have total order at least $p-q+1$. Further, this bound is best possible.*

Proof. Because P is simply-connected, P has only one boundary component, $\mathcal{G}$. Let γ be a Jordan curve dividing P into two connected components of which one is a ring domain, A, belonging to $\mathcal{G}$. Evidently, $\partial A = \gamma \cup \partial P$. Let $\mathcal{U}$ be a refined fattened q-set of f. It is easy to see that γ can be chosen so that $A \cup \gamma$ is a subset of $\mathcal{U}$. Each of P and A has complete lifts $\widetilde{P}$ and $\widetilde{A}$, respectively, under f of total multiplicity p. Obviously, each $\widetilde{A}$ is a ring subdomain of some $\widetilde{P}$. By Proposition 5, there exist lifts $\widetilde{A}$ of total multiplicity exactly $p - q$ each belonging to a boundary component of $\mathcal{X}$. Denote these by $\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_r$. Each of the remaining lifts, $\widetilde{A}_{r+1}, \widetilde{A}_{r+2}, \ldots, \widetilde{A}_s$, must be precompact in $\mathcal{X}$. For if U is the connected component of $\mathcal{U}$ containing $\overline{A}$, then by Proposition 1 U has complete lifts under f of total multiplicity q which contain the lifts $\widetilde{A}_j$, $r+1 \le j \le s$.

Suppose that $\widetilde{P}$ contains a lift $\widetilde{A}_j$, $1 \le j \le r$. Then $\widetilde{P}$ must contain some lift $\widetilde{A}_j$, $r+1 \le j \le s$ unless $\mathcal{X} = \widetilde{P}$, which contradicts the fact that f is a (p, q)-map. Denote the lifts $\widetilde{P}$ of this kind by $\widetilde{P}_1, \widetilde{P}_2, \ldots, \widetilde{P}_t$, and observe that these have total multiplicity at least $p - q + 1$.

Now a form of the Riemann-Hurwitz relation gives:

$$\chi(\widetilde{P}_j) = k_j \chi(P) - \beta_j \qquad (1 \le j \le t)$$

where k_j is the multiplicity of the covering $(\widetilde{P}_j, f)$ and β_j is the branch order of f in $\widetilde{P}_j$. But $\chi(P) = 1$ since P is connected and simply-connected, and

$$\chi(\widetilde{P}_j) = 2 - 2g(\widetilde{P}_j) - \eta(\widetilde{P}_j) \le 0$$

since $g(\widetilde{P}_j) \geq 0$ and $\eta(\widetilde{P}_j) \geq 2$. Hence $\beta_j \geq k_j (1 \leq j \leq t)$, and

$$\Sigma_{j=1}^t \beta_j \geq \Sigma_{j=1}^t k_j \geq p - q + 1.$$

This proves the desired inequality.

It remains to show that the inequality is sharp. Let $(\mathcal{X}_1, \pi_1)$ be a q-fold covering of the surface $\mathcal{Y}$, P a Jordan subdomain of $\mathcal{Y}$ that avoids all the branch values of π_1, and $(\mathcal{X}_2, \pi_2)$ a simply-connected $(p-q)$-fold covering of P. By the Riemann-Hurwitz relation it follows at once that π_2 has branch order $p-q-1$. Cut each surface $\mathcal{X}_1$ and $\mathcal{X}_2$ once over a segment in P that does meet any of the branch values of π_2, and adjoin the resulting surfaces in the usual manner by identifying them crosswise along the edges of the cuts. This yields a surface, $\mathcal{X}$, and a normal BQ-map $f : \mathcal{X} \to \mathcal{Y}$ that coincides with π_1 and π_2 in their respective subdomains in $\mathcal{X}$. Note that P forms the p-set of f, and the branch points of f with values in P have total order exactly $p-q+1$, counting multiplicity, of which $p-q-1$ account for π_1 and two for π resulting from the adjoining procedure. □

Lemma 2. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a normal BQ-map. Then*

$$\eta(\mathcal{X})/(p-q) \leq \Sigma\eta(P) \leq \eta(\mathcal{X}),$$

where the summation is taken over all the connected components P of the p-set of f. Further, the inequalities are sharp.

Proof. Fix P, and let $\mathcal{R}$ be a boundary component of P. Take γ, $\mathcal{U}$, A, and $\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_r$, exactly as in the proof of the previous lemma. The fact that P has a finite number of boundary components (Proposition 2(c)) allows us to choose the respective domains A pairwise disjoint. Let

$$R = \{\mathcal{R} : \mathcal{R} \quad \text{is a boundary component of some domain} \quad P\}$$

and

$$S = \{\mathcal{S} : \mathcal{S} \quad \text{is a boundary component of} \quad \mathcal{X}\}.$$

Suppose $A \in \mathcal{R} \in R$. Then each ring domain $\widetilde{A}_j$, $1 \leq j \leq r$, belongs to a unique boundary component $\mathcal{S}$ of $\mathcal{X}$. This establishes a relation between the sets R and S defined by $\mathcal{R} \to \mathcal{S}$. Denote by ν the inverse relation. We contend that ν is a function. This is done in the following two steps:

(i) Suppose $\nu(\mathcal{S}) = \mathcal{R}_1, \mathcal{R}_2$ and $\mathcal{R}_1 \neq \mathcal{R}_2$. Then a ring subdomain of $\mathcal{X}$ maps under f to disjoint subdomains of $\mathcal{Y}$, and we have a contradiction.

(ii) We claim that for every $\mathcal{S} \in S$ there exists $\mathcal{R} \in R$ so that $\mathcal{R} = \nu(\mathcal{S})$. By Proposition 2(b,c), $\mathcal{U}$ consists of a finite number of connected components

U each containing a finite number of ring domains A. Using Propositions 1 and 5, we conclude that $f^{-1}(\mathcal{U})$ is a disjoint union of a precompact subset of $\mathcal{X}$ and the aforementioned ring domains $\widetilde{A}_1$, $\widetilde{A}_2$, ..., $\widetilde{A}_r$, for all A. It follows that if $\nu(\mathcal{S}_o)$, for some $\mathcal{S}_o \in S$, is undefined, then there exists a ring domain $\widetilde{A}_o \in \mathcal{S}_o$ so that $f^{-1}(\mathcal{U}) \bigcap \widetilde{A}_o$ is empty. Choose $y_o \in C(f; \mathcal{S}_o)$. Then $y_o \in \partial\mathcal{Q} \subset \mathcal{Q}$ (Proposition 2(d)). Evidently, there exists $x_o \in \widetilde{A}_o$ so that $f(x_o) \in \mathcal{U}$. If $f(x_o) \in \mathcal{Q}$, then $f^{-1} \circ f(x_o)$ contains exactly q points in $f^{-1}(\mathcal{U})$ in addition to x_o. If $f(x_o) \in \mathcal{P}$, then $f(x_o) \in A$, for some ring domain A, and $f^{-1} \circ f(x_o)$ contains exactly p points in $f^{-1}(\mathcal{U})$ in addition to x_o. In either case we have a contradiction.

Therefore, $\nu : S \to R$ is a function. Note that S contains at most $\eta(\mathcal{X})$ elements and R contains $\Sigma_P \eta(P)$ elements. But each $\nu^{-1}(\mathcal{R})$, $\mathcal{R} \in R$, consists of at least one and at most $p - q$ elements in S. This yields the inequlity whose sharpness is easy to conclude. □

Proof of Theorem 7. The left-hand inequality follows at once from Theorem 2. To prove the right-hand inequality, we prove first

$$\beta_f \leq 2[(g(\mathcal{X}) - 1) - q(g(\mathcal{Y}) - 1)] + (p - q + 1)\eta(\mathcal{X}).$$

Let P be a connected component of $\mathcal{P}$. Observe that $\chi(P)$ is nonpositive unless P is simply-connected in which case $\chi(P) = 1$, and $\mathcal{P}$ has at most $\eta(\mathcal{X})$ connected components (see Proposition 5). It follows that $\chi(\mathcal{P})$ is at most $\eta(\mathcal{X})$, and by Theorem 1 the inequality holds.

Now we prove:

$$\beta_f \leq (p - q + 1)(2[(g(\mathcal{X}) - 1) - q(g(\mathcal{Y}) - 1)] + (p - q)\eta(\mathcal{X})).$$

Again, since $\mathcal{P}$ has finitely many connected components

$$\chi(\mathcal{P}) = \Sigma_P \chi(P)$$

where $\chi(P) = 2 - 2g(P) - \eta(P)$. Suppose $\mathcal{P}$ has s simply-connected components and t planar but not simply-connected components. Then

$$\chi(\mathcal{P}) \leq 2(t + s) - \Sigma_P \eta(P)$$

since $\chi(P)$ is negative if $g(P)$ is positive. As an immediate consequence of Lemma 1 we have $(p - q + 1)s \leq \beta_f$, or

$$s \leq \beta_f / (p - q + 1).$$

Note that the number of boundary components of the planar domains (including the simply-connected ones) is at least $2t + s$ and at most $\Sigma_P \eta(P)$. Then by Lemma 2

$$2t + s \leq \eta(\mathcal{X}).$$

The above two inequalities give:

$$2(t+s) \leq \eta(\mathcal{X}) + \beta_f/(p-q+1),$$

and consequently

$$\chi(\mathcal{P}) \leq \eta(\mathcal{X}) + \beta_f/(p-q+1) - \Sigma_P \eta(P).$$

This gives:

$$\begin{aligned}\eta(\mathcal{X}) + (p-q)\chi(\mathcal{P}) \leq (p-q)(\eta(\mathcal{X}) + \beta_f/(p-q+1)) + \\ (\eta(\mathcal{X}) - (p-q)\Sigma_P \eta(P)).\end{aligned}$$

Using Lemma 2, again, we get:

$$\eta(\mathcal{X}) + (p-q)\chi(P) \leq (p-q)(\eta(\mathcal{X}) + \beta_f/(p-q+1)).$$

Applying this with Theorem 1 we obtain:

$$\begin{aligned}\beta_f \leq\ & 2[(g(\mathcal{X})-1) - q(g(\mathcal{Y})-1)] \\ & +(p-q)(\eta(\mathcal{X}) + \beta_f/(p-q+1)),\end{aligned}$$

which is equivalent to the desired inequality. This proves the double inequality. The sharpness results follow at once from Examples 2, 4 and [**14**, Existence Theorem 1]. This completes the proof. □

As mentioned earlier, Theorem 8 follows at once from Theorem 7. The condition in the former theorem is also sufficient if $q = p - 1$. This is the result of Srebro and Wajnryb [**14**, Mapping Theorem] who proved the above theorem only under the latter assumption. The condition however, is not sufficient in general; for instance, according to Theorem 5 there are no normal BQ-maps $f : \mathcal{X} \to \mathcal{Y}$ satisfying $g(\mathcal{X}) = 0 < 1 = g(\mathcal{Y})$, $\eta(\mathcal{X}) = 1$, $p = 4$ and $q = 2$, or, $g(\mathcal{X}) = g(\mathcal{Y})$, $\eta(\mathcal{X}) = 1$ and arbitrary values p and q, for which above the inequality holds.

6. Open questions.

This section concludes the paper with a number of open questions.

Let $f : \mathcal{X} \to \mathcal{Y}$ be a normal (p, q)-map. Recall that $\mathcal{P}$ is the p-set of f.

Question 1. For fixed surfaces $\mathcal{X}$ and $\mathcal{Y}$, what is the minimum value of the Euler characteristic of $\mathcal{P}$? This value is 1 if $\mathcal{X}$ is the unit disc and $\mathcal{Y}$ the Riemann sphere or the torus, as can be concluded from Examples 1, 2 and 4 of [**7**]. Examples 1 through 5 also provide further examples for which this value is known for specific choices of $\mathcal{X}$ and $\mathcal{Y}$.

Question 1 can also be reformulated as follows.

Question 2. For fixed surfaces $\mathcal{X}$ and $\mathcal{Y}$, what will be the sharp lower bound of $\beta_f + \delta_f$ if it is not the bound concluded in Theorem 2?

A similar question regarding Theorem 3 also arises.

Question 3. Theorem 4 (or [7, Theorem 5]) gives a sharp lower bound for $\beta_f + \delta_f$ if $\mathcal{Y}$ is the Riemann sphere or the torus. What will be the sharp lower bound if $g(\mathcal{Y}) \geq 2$? Is it $p + q - 1 + 2sg(\mathcal{X})$ with $s = k(p-q) - p$, where k is the smallest integer for which $k(p-q) \geq p$ (see [7])?

Regarding BQ-maps we have:

Question 4. Does the converse of Theorem 5 hold? If not, can one relax the hypotheses of Theorem 6?

Question 5. For fixed surfaces $\mathcal{X}$ and $\mathcal{Y}$, if the upper bound for β_f in Theorem 7 is not sharp, then what is the sharp upper bound?

Srebro and Wajnryb [14] constructed BQ-maps, $q = p - 1$, with

$$\beta_f = 2[(g(\mathcal{X}) - 1) - q(g(\mathcal{Y}) - 1)] + (p - q + 1)\eta(\mathcal{X})$$

and

$$\beta_f = (p - q + 1)(2[(g(\mathcal{X}) - 1) - q(g(\mathcal{Y}) - 1)] + (p - q)\eta(\mathcal{X}))$$

thus proving the sharpness of the upper bound of β_f in Theorem 7. For general BQ-maps however, the author failed to construct an example that achieves the latter equality in general. This leads to the following:

Question 6. Is there a normal BQ-map f satisfying

$$\beta_f = (p - q + 1)(2[(g(\mathcal{X}) - 1) - q(g(\mathcal{Y}) - 1)] + (p - q)\eta(\mathcal{X}))?$$

With the basic theory of (p, q)-maps between orientable surfaces is now established, it is appropiate to propose extending our results to (p, q)-maps into non-orientable surfaces, and between manifolds of higher dimensions. The only attempt in the direction of the latter question is Srebro's [11] which is restricted to $(p, p - 1)$-maps. Relevant to this work we ask the following [7]:

Question 7. Are there open, continuous, locally one-to-one maps from R^n to itself that have exactly two non-consecutive valences?

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