PACIFIC JOURNAL OF MATHEMATICS
Vol. 179, No. 1, 1997

UNIQUE CONTINUATION FOR A CLASS OF HIGHER ORDER ELLIPTIC OPERATORS

Laura De Carli

In this paper we prove a unique continuation theorem for a class of elliptic operators of order $m \geq 2$ with highly singular potentials using the method of Carleman estimates.

Introduction.

In this paper we establish a unique continuation theorem for solutions of a class of differential inequalities of the form:

$$|\, P(D)U(x)\,| \leq |\, V(x)U(x)\,|\,, \tag{0.1}$$

where P is an elliptic operator with constant coefficients and V is a singular potential.

We will prove that if $n > m$, P is an elliptic operator of order $m \geq 2$ whose principal part satisfies conditions that will be specified later, $V \in L^{\frac{n}{m}}(\mathbb{R}^n)$, and if $U \in H^{m,p}(\mathbb{R}^n)$, $p = \frac{2n}{m+n}$, satisfies (0.1), then U is identically zero if its support is contained in a half space whose normal direction satisfies a hypothesis involving the symbol of P. By $H^{m,p}(\mathbb{R}^n)$ we mean the space of functions with m derivatives in $L^p(\mathbb{R}^n)$.

It is well known that the above unique continuation property, (u.c.p. henceforth in this paper), for the solutions of the differential Inequality (0.1), follows from the proof of a weighted inequality of the form

$$\left\| e^{\tau\phi(x)} u \right\|_{p'} \leq C \left\| e^{\tau\phi(x)} P(D) u \right\|_{p}, \tag{0.2}$$

valid for all $u \in H^{m,p}(\mathbb{R}^n)$, a suitable weight ϕ depending on the half space, and values of the real parameter τ which are allowed to tend to $+\infty$. An estimate of the form of (0.2) is called a *Carleman-type inequality*.

There is a lot of literature concerning Carleman-type inequalities and unique continuation properties for solutions of partial differential equations. See [**H1**], [**J**], [**S1**], just to cite a few. When $m = 2$, the u.c.p. for the solutions of (0.1) has been proved in [**KRS**]. Moreover, the assumption that u vanishes on a open set of $\mathbb{R}^n$ can be replaced by the much weaker assumption that u vanishes of infinite order at some point in $\mathbb{R}^n$. See [**JK**].

In their paper the authors proved also that the hypothesis $V \in L^{\frac{n}{2}}_{\mathrm{loc}}(\mathbb{R}^n)$ is optimal, in the sense that $\frac{n}{2}$ cannot be replaced by any smaller exponent.

Let $u \in H^{m,p}(\mathbb{R}^n)$ be a solution of (0.1). Suppose that u is supported in the half space $\{x : \langle x\,\nu\rangle \geq 0\}$, ν being a unit vector of $\mathbb{R}^n$. Our paper will be devoted to the proof of (0.2), with $\phi(x) = -\langle x\,\nu\rangle$. A standard argument, that will be recalled in the next section, will prove that the u.c.p. holds in $H^{m,p}(\mathbb{R}^n)$ for the solutions of (0.1).

To prove (0.2), and to explain the assumptions that the principal part of P must satisfy, we shall make some standard reductions. For simplicity we assume that P is homogeneous, since the other cases follow from easy adaptions of this argument. If we define the *conjugate operator* of P as

$$P_\tau(D)u = e^{\tau\langle x\,\nu\rangle}P(D)e^{-\tau\langle x\,\nu\rangle} = P\,(\,D + i\tau\nu\,)\,,$$

then (0.2) will be a consequence of the following uniform inequality

$$\|\,u\,\|_{p'} \leq C\,\|\,P\,(\,D + i\tau\nu\,)\,u\,\|_{p}\,. \tag{0.3}$$

Since the inverse Fourier transform of $(P(\zeta + i\tau\nu))^{-1}$ is a fundamental solution for the operator $P(D + i\tau\nu)$, the inequality (0.3) will be a consequence of

$$\left\| \int_{\mathbb{R}^n} \frac{e^{i\langle\zeta,\,x\rangle}}{P(\zeta + i\tau\nu)}\hat{f}(\zeta)d\zeta \right\|_{p'} \leq C||f||_p, \qquad f \in \mathcal{S}(\mathbb{R}^n). \tag{0.4}$$

We now make the following important assumption:

(1) $P(D)$ has *simple complex characteristics* in the direction ν, in the sense that the polynomial $\tau \to P(\zeta + i\tau\nu)$, $\tau \in \mathbb{C}$, has only simple zeroes for each fixed $\zeta \in \mathbb{R}^n/\{0\}$.

Without loss of generality $\nu = (1, 0, \dots, 0)$. After rescaling, we can assume $\tau = 1$. Let $P(\zeta)$ be the symbol of $P(D)$. After perhaps a change of coordinates,

$$P(\zeta) = \zeta_1^m + \sum_{j=0}^{m-1} \zeta_1^j Q_j(\zeta'),$$

where the Q_j's are homogeneous polynomials of degree $m - j$, and where we have set $\zeta = (\zeta_1, \zeta')$. The roots of $\zeta_1 \to P(\zeta)$, $\lambda_1(\zeta') + i\mu_1(\zeta'), \dots, \lambda_m(\zeta') + i\mu_m(\zeta')$, are smooth and homogeneous of degree 1 in $\mathbb{R}^{n-1}/\{0\}$. Since $P(D)$ is elliptic with real coefficients, the μ_j's vanish only at the origin, and $\lambda_j(\zeta') + i\mu_j(\zeta')$ and $\lambda_j(\zeta') - i\mu_j(\zeta')$ are both roots of $\zeta_n \to P(\zeta', \zeta_n)$. Then,

$$P(\zeta) = \Pi_{j=1}^{\frac{m}{2}}(\zeta_1 - \lambda_j(\zeta') \pm i\mu_j(\zeta')),$$

where $\mu_j(\zeta') > 0$, $\zeta' \in \mathbb{R}^{n-1}/\{0\}$, and hence

$$P(\zeta + i\nu) = \Pi_{j=1}^{\frac{m}{2}}(\zeta_1 - \lambda_j(\zeta')) + i(1 \pm i\mu_j(\zeta')).$$

The factors of the form $\zeta_1 - \lambda_j(\zeta') + i(1 + \mu_j(\zeta'))$ are the "good" factors which never vanish, but the "bad" factors $\zeta_1 - \lambda_j(\zeta') + i(1 - \mu_j(\zeta'))$ vanish on the (compact) manifolds

$$S_j = \{\zeta : \zeta_1 = \lambda_j(\zeta'),\ 1 = \mu_j(\zeta')\}, \qquad j = 1, \dots, \frac{m}{2}.$$

By (1), $S_i \cap S_j = \emptyset$ when $i \neq j$. Let $\chi_j(\zeta) \in C_0^\infty(\mathbb{R}^n)$ be a cutoff function which is $\equiv 1$ in a neighborhood of S_j and is $\equiv 0$ on a neighborhood of S_k, $k \neq j$. Let $\chi_0(\zeta) = 1 - (\chi_1(\zeta) + \cdots + \chi_{\frac{m}{2}}(\zeta))$. Since $P(\zeta + i\nu)$ does not vanish on the support of χ_0, $\dfrac{\chi_0(\zeta)}{P(\zeta + i\nu)}$ is the symbol of a pseudifferential operator of order $-m$, which can thus be extended to a bounded operator from L^p to $H^{m,p}(\mathbb{R}^n)$ (see e.g. [**H**]). By the Sobolev embedding theorem,

$$\left\| \int_{\mathbb{R}^n} \frac{e^{i\langle x,\,\zeta\rangle}\hat{f}(\zeta)}{P(\zeta + i\nu)}\chi_0(\zeta)d\zeta \right\|_{p'} \leq C||f||_p.$$

We also observe that $\zeta \to \dfrac{\zeta_1 - \lambda_j(\zeta') + i(1 - \mu_j(\zeta'))}{P(\zeta + i\nu)}\chi_j(\zeta)$, $j = 1, \dots, \frac{m}{2}$, is a L^p Fourier multiplier for every $p > 1$ (see e.g. [**H**]). We have then reduced matters to proving the following inequalities

(0.5)
$$\left\| \int_{\mathbb{R}^n} \frac{\hat{f}(\zeta)e^{i\langle x,\,\zeta\rangle}\,\chi_j(\zeta)d\zeta}{\zeta_1 - \lambda_j(\zeta) + i(1 - \mu_j(\zeta'))} \right\|_{p'} \leq C\,\|\,f\,\|_p\,, \quad j = 1, \cdots \frac{m}{2}, \quad f \in \mathcal{S}(\mathbb{R}^n).$$

Our main results can now be stated as follows.

Theorem 1. *Let $P(D)$ be an elliptic operator of order $m < n$ with constant coefficients. Let λ_j and μ_j be defined as above. Suppose that the assumption (1) is satisfied. Suppose also that*

(2) *the cospheres $\{\zeta' : \mu_j(\zeta') = 1\}$ have everywhere nonvanishing Gaussian curvature.*

Then, for every $f \in C_0^\infty(\mathbb{R}^n)$ and for $p \leq \frac{2n}{n+2}$, (0.5) holds.

Theorem 2. *Let $P(D)$ be as in Theorem 1, and let $U \in H^{m,p}(\mathbb{R}^n)$, $p = \frac{2n}{n+m}$, be a solution of the differential inequality (0.1), with $V \in L^{\frac{n}{m}}(\mathbb{R}^n)$. Suppose that the support of U is contained on one side of a hyperplane,* *and that $P(D)$ satisfies (1) in the direction ν of the exterior normal to the hyperplane containing the support of U. Suppose also that (2) holds. Then $U \equiv 0$.*

The Assumption (1) is crucial. The counterexamples of Plis [**P**] show in fact that the u.c.p. across the hyperplane $\{x : \langle x\, \nu\rangle = 0\}$ can fail for solutions of elliptic operators that do not have simple complex characteristics in the direction ν.

The Assumption (2) might be not necessary. An early result due to Hörmander [**H2**] shows in fact that the u.c.p. holds in $H^m(\mathbb{R}^n)$ for the solutions of (0.1) when $V \in L^\infty(\mathbb{R}^n)$ and (1) is satisfied. The same is probably also true in the L^p setting. However, our theorem represents a natural generalization of the case $m = 2$. When $P(D)$ is the Laplacean, we can see that the "bad factor" of the symbol of the conjugate operator $P(D+i(1,\, 0,\, \dots 0))$ is $\zeta_1 + i(1 - |\,\zeta'\,|)$, and the Condition (2) is then satisfied.

In what follows we shall use the convention that χ denotes a smooth cutoff function which is not necessarily the same at each occurrence. Also, we will denote by C a constant which may change from line to line.

This paper is a part of my doctoral thesis at UCLA. I would like to thank my teacher and adviser, Prof. C.D. Sogge for his invaluable counsel, and the referee for his constructive comments.

Section 1.

In this section we will prove our main results. First of all, we proceed as in [**KRS**] to show that Theorem 1 implies Theorem 2. Let $U \in H^{m,p}(\mathbb{R}^n)$ be a solution of (0.1). For simplicity of notation, we shall assume that U is supported in the half space $\{x = (x_1,\, x') : x_1 > 0\}$, since the argument for the other cases is similar. We have shown in the Introduction that Theorem 1 implies the following special case of (0.2):

$$||e^{-\tau x_1} u||_{L^{p'}(\mathbb{R}^n)} \leq C||e^{-\tau x_1} P(D)u||_{L^p(\mathbb{R}^n)}, \qquad u \in C_0^\infty\,(\,\mathbb{R}^n\,). \tag{1.1}$$

Since $\tau x_1 \geq 0$ on the support of U, and since $C_0^\infty(\mathbb{R}^n)$ is dense in $H^{m,p}(\mathbb{R}^n)$, it is easy to see that U satisfies (1.1).

To prove that $U \equiv 0$ it is sufficient to prove that there is a $\rho > 0$ so that $U \equiv 0$ in the strip $S_\rho = \{x \in \mathbb{R}^n : x_1 \leq \rho\}$. Take $\rho > 0$ so small that, if V is as above and C is as in (0.2),

$$C||V||_{L^{\frac{n}{m}}(S_\rho)} \leq \frac{1}{2}. \tag{1.2}$$

If one uses Hölder's inequality, along with (1.1) and (1.2), and the fact that $|P(D)U| \leq |VU|$, then one has the following string of inequalities for every

$\tau > 0$.

$$\begin{aligned}||e^{-\tau x_1}U||_{L^{p'}(S_\rho)} &\leq C||e^{-\tau x_1}P(D)U||_{L^p(\mathbb{R}^n)}\\ &\leq C||e^{-\tau x_1}VU||_{L^p(S_\rho)} + C||e^{-\tau x_1}P(D)U||_{L^p(\mathbb{R}^n/S_\rho)}\\ &\leq \frac{1}{2}||e^{-\tau x_1}U||_{L^{p'}(S_\rho)} + C||e^{-\tau x_1}P(D)U||_{L^p(\mathbb{R}^n/S_\rho)}.\end{aligned}$$

Hence,

$$||e^{-\tau x_1}U||_{L^{p'}(S_\rho)} \leq 2C||e^{-\tau x_1}P(D)U||_{L^p(\mathbb{R}^n/S_\rho)},$$

and consequently,

$$||e^{\tau(\rho - x_1)}U||_{L^{p'}(S_\rho)} \leq 2C||P(D)U||_{L^p(\mathbb{R}^n)}.$$

Since the above inequality holds for every $\tau > 0$, this forces $U \equiv 0$ in S_ρ.

Proof of Theorem 1. Consider the operators

$$T_j f(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f(y)e^{i\langle x-y,\,\zeta\rangle}}{(\zeta_1 - \lambda_j(\zeta')) + i(1-\mu_j(\zeta'))}\chi(\zeta)\,d\zeta\,dy,$$
$$f \in \mathcal{S}(\mathbb{R}^n), \quad j = 1, \dots, \frac{m}{2}.$$

(0.5) can thus be rewritten as

$$\| T_j f \|_{p'} \leq C \| f \|_p, \qquad f \in \mathcal{S}(\mathbb{R}^n), \quad j = 1, \dots, \frac{m}{2}.$$

For simplicity, we will omit the subscript j from now on. The change of variables $\zeta_1 \to \zeta_1 + \lambda(\zeta')$ allows to write

$$Tf(x) = \int_{\mathbb{R}^n} f(y)\left(\int_{\mathbb{R}^n} \frac{e^{i(\langle x-y,\,\zeta\rangle + \lambda(\zeta')(x_1-y_1))}}{\zeta_1 + i(1-\mu(\zeta'))}\chi(\zeta_1,\,\zeta')\,d\zeta\right)dy\ ,$$

and if we set $\zeta_1\,(1-\mu(\zeta'))^{-1} = t$, and we assume that $\chi(t(1-\mu(\zeta')),\,\zeta') = \chi(t)\chi(\zeta')$,

$$Tf(x) = \int_{\mathbb{R}^n} f(y)\int_{\mathbb{R}^{n-1}} e^{i(\langle x'-y',\,\zeta'\rangle + \lambda(\zeta')(x_1-y_1))}h((x_1-y_1)(1-\mu(\zeta'))\chi(\zeta'))\,d\zeta',$$

where

$$h(s) = \int_{\mathbb{R}} \frac{e^{its}}{t+i}\chi(t)dt. \tag{1.3}$$

Define the "frozen operator"

$$T_{x_1-y_1}g(x') = \int_{\mathbb{R}^{n-1}} \hat{g}(\zeta')h((x_1-y_1)(1-\mu(\zeta')))\, e^{i(\langle x',\,\zeta'\rangle+\lambda(\zeta')(x_1-y_1))}\chi(\zeta')d\zeta'.$$

If we prove that $T_{x_1-y_1}$ extends to a bounded operator from $L^p(\mathbb{R}^{n-1})$ to $L^{p'}(\mathbb{R}^{n-1})$ with norm $O(|\,x_1-y_1\,|^{-\beta})$,

$$\beta \geq 1 + \frac{1}{p'} - \frac{1}{p}, \tag{1.4}$$

then we are done. In fact, $Tf(x) = \int_{\mathbb{R}} T_{x_1-y_1}f(y_1,\,x')dy_1$, and hence

$$\begin{aligned} \|\, Tf(x_1,\,.)\,\|_{L^{p'}(\mathbb{R}^{n-1})} &\leq \int_{\mathbb{R}} \|\, T_{x_1-y_1}f(y_1,\,.)\,\|_{L^{p'}(\mathbb{R}^{n-1})}\, dy_1 \\ &\leq C\int_{\mathbb{R}} \|\, f(y_1,\,.)\,\|_{L^p(\mathbb{R}^{n-1})}\, |\,x_1-y_1\,|^{-\beta}\, dy_1. \end{aligned} \tag{1.5}$$

By the Hardy-Littlewood-Sobolev inequality,

$$x_1 \to \int_{\mathbb{R}} \|\, f(y_1,\,.)\,\|_{L^p(\mathbb{R}^{n-1})}\, |\,x_1-y_1\,|^{-\beta}\, dy_1 \in L^{p'}(\mathbb{R})$$

when β is as in (1.4), and

$$\begin{aligned} &\left\| \int_{\mathbb{R}} \|\, f(y_1,\,.)\,\|_{L^p(\mathbb{R}^{n-1})}\, |\,x_1-y_1\,|^{-\beta}\, dy_1 \right\|_{L^{p'}(\mathbb{R},\,dx_1)} \\ &\leq C\left(\int_{\mathbb{R}} \|\, f(y_1,\,.)\,\|^p_{L^p(\mathbb{R}^{n-1})}\, dy_1\right)^{\frac{1}{p}} = C\,\|\, f\,\|_p\,. \end{aligned}$$

By (1.5) and the above, $\|\, Tf\,\|_{p'} \leq C\,\|\, f\,\|_p$.

We are thus left to estimate $\|\, T_{x_1-y_1}g\,\|_{L^{p'}(\mathbb{R}^{n-1})}$. If we let $\Sigma = \{\zeta' \in \mathbb{R}^{n-1} \,:\, \mu(\zeta') = 1\}$, and use polar coordinates associated to Σ,

$$\begin{aligned} &T_{x_1-y_1}g(x') \\ &= \int_0^{+\infty} r^{n-2}\int_{\Sigma} g(r\omega)h((x_1-y_1)(1-r))\chi(r\omega)\, e^{ir(\langle x',\,\omega\rangle+(x_1-y_1)\lambda(\omega))}d\omega dr. \end{aligned}$$

Without loss of generality, $r^{n-2}\chi(r\omega) = \chi(r)\chi(\omega)$. Then,

$$T_{x_1-y_1}g(x') = \int_0^{+\infty} \chi(r)h((x_1-y_1)(1-r))I_{x_1-y_1,\,r}g(x')\, dr, \tag{1.6}$$

where we have set

$$I_{x_1-y_1,\,r}\, g(x') = \int_{\Sigma} \hat{g}(r\omega)e^{ir(\langle x',\,\omega\rangle+(x_1-y_1)\lambda(\omega))}\chi(\omega)d\sigma(\omega). \tag{1.7}$$

By Minkowsky's inequality,

$$\| T_{x_1-y_1}\, g \|_{L^{p'}(\mathbb{R}^{n-1})} \le \int_0^{+\infty} \chi(r)\,|\, h((x_1-y_1)(1-r))\,|\, \| I_{x_1-y_1,\,r} g \|_{L^{p'}(\mathbb{R}^{n-1})}\, dr.$$

If $I_{x_1-y_1,\,r}$ maps $L^p(\mathbb{R}^{n-1})$ boundedly into $L^{p'}(\mathbb{R}^{n-1})$, then

$$\| T_{x_1-y_1} g \|_{L^{p'}(\mathbb{R}^{n-1})} \le C\, \| g \|_{L^p(\mathbb{R}^{n-1})} \int_0^{+\infty} \chi(r)\,|\, h((x_1-y_1)(1-r))\,|\, dr,$$

and, by the definition of h, $\| T_{x_1-y_1} g \|_{L^{p'}(\mathbb{R}^{n-1})} \le C\,|\, x_1-y_1\,|^{-1}\, \| g \|_{L^p(\mathbb{R}^{n-1})}$, which is even better than what we need.

Let $\beta \in C_0^\infty(\mathbb{R})$ be an even and nonnegative function whose support is concentrated in a neighborhood of the origin and such that $\int_{\mathbb{R}} \beta(t)dt = 1$. Let

$$I^\epsilon_{x_1-y_1,\,r}\, g(x') = \frac{1}{\epsilon} \int_{\mathbb{R}^{n-1}} \hat{g}(\eta) \beta\left(\frac{|\, r-\mu(\eta)\,|}{r\epsilon}\right) e^{i(\langle x',\,\eta\rangle + (x_1-y_1)\lambda(\eta))} \chi(\eta) d\eta.$$

If we prove that

(i) $\left\| I^\epsilon_{x_1-y_1,\,r}\, g \right\|_{L^{p'}(\mathbb{R}^{n-1})} \le C\, \| g \|_{L^p(\mathbb{R}^{n-1})}$, $g \in \mathcal{S}(\mathbb{R}^{n-1})$, with C independent of ϵ,

(ii) $I^\epsilon_{x_1-y_1,\,r}\, g \to I_{x_1-y_1,\,r}\, g$ in distribution sense as $\epsilon \to 0$,

then $\| I_{x_1-y_1,\,r}\, g \|_{L^{p'}(\mathbb{R}^{n-1})} \le C\, \| g \|_{L^p(\mathbb{R}^{n-1})}$. Recalling that the spheres of $L^{p'}(\mathbb{R}^{n-1})$ are weakly sequentially compact, from (i) follows in fact that, for every $g \in \mathcal{S}(\mathbb{R}^n)$, there exists a sequence $\{I^{\epsilon_j}_{x_1-y_1,\,r}(g)\}_{j\in\mathbb{N}}$ which converges in the weak topology of $L^{p'}(\mathbb{R}^{n-1})$. By (ii), $I^{\epsilon_j}_{x_1-y_1,\,r}\, g \to I_{x_1-y_1,\,r}\, g$, and by (i), $\| I_{x_1-y_1,\,r}\, g \|_{L^{p'}} \le C\, \| g \|_{L^p}$.

To prove (i) it is convenient to write $I^\epsilon_{x_1-y_1,\,r}$ as the composition of two operators, and study them separately.

Let

$$G^\epsilon_r g(x') = \frac{1}{\sqrt{\epsilon}} \int_{\mathbb{R}^{n-1}} \hat{g}(\eta) \beta^{\frac{1}{2}}\left(\frac{|\, r-\mu(\eta)\,|}{r\epsilon}\right) e^{i\langle x',\,\eta\rangle} \chi(\eta) d\eta,$$

$$Q^\epsilon_{x_1-y_1,\,r} g(x') = \frac{1}{\sqrt{\epsilon}} \int_{\mathbb{R}^{n-1}} \hat{g}(\eta) \beta^{\frac{1}{2}}\left(\frac{|\, r-\mu(\eta)\,|}{r\epsilon}\right) e^{i(\langle x',\,\eta\rangle + (x_1-y_1)\lambda(\eta))} \chi(\eta) d\eta.$$

Since $I^\epsilon_{x_1-y_1,\,r} = G^\epsilon_r \circ Q^\epsilon_{x_1-y_1,\,r}$, for a suitable choice of the cutoff function χ, and $(G^\epsilon_r)^* = G^\epsilon_r$, to prove that $I^\epsilon_{x_1-y_1,\,r} : L^p(\mathbb{R}^{n-1}) \to L^{p'}(\mathbb{R}^{n-1})$ is a bounded operator, it is enough to prove that G^ϵ_r and $Q^\epsilon_{x_1-y_1,\,r}$ map $L^p(\mathbb{R}^{n-1})$ boundedly into $L^2(\mathbb{R}^{n-1})$.

Consider $Q^{\epsilon}_{x_1-y_1,\,r}$ first. Since the function $\zeta' \to e^{i\lambda(\zeta')(x_1-y_1)}$ is bounded, it is a L^2 Fourier multiplier, (see e.g. [H]). Hence,

$$\left\| Q^{\epsilon}_{x_1-y_1,\,r} g \right\|_{L^2} \le \frac{1}{\sqrt{\epsilon}} \left\| \int_{\mathbb{R}^{n-1}} \hat{g}(\eta)\beta^{\frac12}\left(\frac{|\,r-\mu(\eta)\,|}{r\epsilon}\right) e^{i\langle x',\,\eta\rangle}\chi(\eta)\,d\eta \right\|_{L^2},$$

and by the Plancherel theorem,

$$\left\| Q^{\epsilon}_{x_1-y_1,\,r} g \right\|_{L^2} \le \left(\frac{1}{\epsilon}\int_{\mathbb{R}^{n-1}} |\,\hat{g}(\eta)\,|^2\,\beta\left(\frac{|\,r-\mu(\eta)\,|}{r\epsilon}\right)\chi^2(\eta)\,d\eta\right)^{\frac12}. \tag{1.8}$$

Let $Q'_{\epsilon}g$ denote the right-hand side of (1.8). In polar coordinates associated to Σ,

$$Q'_{\epsilon}g(x') = \left(\frac{1}{\epsilon}\int_0^{+\infty}\beta\left(\frac{|\,r-s\,|}{r\epsilon}\right)\int_{\Sigma}|\,\hat{g}(s\omega)\,|^2\,\chi^2(s\omega)\,d\omega ds\right)^{\frac12}.$$

Without loss of generality, $\chi^2(s\omega)=\chi(\omega)\chi(s)$. Since $\hat{g}(s\omega)=s^{-n-1}\hat{g}_s(\omega)$, where $g_s(\omega)=g\left(\frac{\omega}{s}\right)$, if we let $t=\frac{r-s}{r}$,

$$Q'_{\epsilon}g(x') = \left(\frac{1}{\epsilon r^{2(n-1)}}\int_{\mathbb{R}}\beta\left(\frac{|\,t\,|}{\epsilon}\right)\frac{\chi(t)}{(1-t)^{2(n-1)}}\int_{\Sigma}|\,\hat{g}_{r(1-t)}(\omega)\,|^2\,\chi(\omega)\,d\omega dt\right)^{\frac12}.$$

By the Assumption (2), Σ has nonvanishing Gaussian curvature on the support of χ. We can apply Theorem **A** in the Appendix to obtain the following string of inequalities:

$$\begin{aligned}
&\|\,Q'_{\epsilon}g\,\|_{L^2(\mathbb{R}^{n-1})}\\
&\le C\left(\frac{1}{\epsilon r^{2(n-1)}}\int_{\mathbb{R}}\beta\left(\frac{|\,t\,|}{\epsilon}\right)\|\,g_{r(1-t)}\,\|^2_{L^p(\mathbb{R}^{n-1})}\frac{\chi(t)}{(1-t)^{2(n-1)}}dt\right)^{\frac12}\\
&= Cr^{\frac{n-1}{p'}}\,\|\,g\,\|_p\left(\frac{1}{\epsilon}\int_{\mathbb{R}}\chi(t)(1-t)^{\frac{2(n-1)}{p'}}\beta\left(\frac{|\,t\,|}{\epsilon}\right)\,dt\right)^{\frac12}.
\end{aligned}$$

The family of functions $\left\{\frac{1}{\epsilon}\beta\left(\frac{|\,t\,|}{\epsilon}\right)\right\}_{\epsilon>0}$ converges to the Dirac distribution δ_0 in distribution sense. Hence,

$$\|\,Q'_{\epsilon}g\,\|_{L^2(\mathbb{R}^{n-1})} \le Cr^{\frac{n-1}{p'}}\,\|\,g\,\|_{L^p(\mathbb{R}^{n-1})}. \tag{1.9}$$

The same technique applies to estimate $\|\,G^{\epsilon}_r g\,\|_{L^2(\mathbb{R}^{n-1})}$. (i) is thus proved.

We now prove (ii). For $\phi,\,g\in\mathcal{S}(\,\mathbb{R}^n\,)$,

$$\langle I^{\epsilon}_{x_1-y_1,\,r}g,\,\phi\rangle = \frac{1}{\epsilon}\int_{\mathbb{R}^{n-1}}\hat{g}(\eta)\hat{\phi}(\eta)\beta\left(\frac{|\,r-\mu(\eta)\,|}{r\epsilon}\right)e^{i(x_1-y_1)\lambda(\eta)}\chi(\eta)d\eta.$$

If we set $\eta = r\zeta'$,

$$\begin{aligned}&\langle I^{\epsilon}_{x_1-y_1,\,r}\,g,\,\phi\rangle\\&=\frac{r^{n-1}}{\epsilon}\int_{\mathbb{R}^{n-1}}\hat g(r\zeta')\hat\phi(r\zeta')\beta\left(\frac{|\,1-\mu(\zeta')\,|}{\epsilon}\right)e^{ir(x_1-y_1)\lambda(\zeta')}\chi(r\zeta')d\zeta',\end{aligned}$$

and we use polar coordinates associated to Σ,

$$\frac{1}{\epsilon}\int_{\mathbb{R}}\chi(t)\beta\left(\frac{|\,1-t\,|}{\epsilon}\right)\int_{\Sigma}\hat g(rt\omega)\hat\phi(rt\omega)e^{irt(x_1-y_1)\lambda(\omega)}\chi(rt\omega)\,d\omega dt.$$

The family of functions $\left\{\frac{1}{\epsilon}\beta\left(\frac{|\,1-t\,|}{\epsilon}\right)\right\}_{\epsilon>0}$ converges to the Dirac distribution δ_1 in distribution sense. Hence,

$$\begin{aligned}\lim_{\epsilon\to 0^+}=\langle I^{\epsilon}_{x_1-y_1,\,r}g,\,\phi\rangle &= \int_{\Sigma}\hat g(r\omega)\hat\phi(r\omega)e^{ir(x_1-y_1)\lambda(\omega)}\chi(\omega)\,d\sigma\omega\\&=\int_{\mathbb{R}^{n-1}}I_{x_1-y_1,\,r}g(x')\phi(x')\,dx'=\langle I_{x_1-y_1,\,r}g,\,\phi\rangle.\end{aligned}$$

This concludes the proof of the theorem. □

Remark. If Σ does not have everywhere nonvanishing Gaussian curvature, but has has at least k nonvanishing principal curvatures, the thesis of Theorem 1 holds for $p \le \frac{2(k+2)}{k+4}$. The proof of the above result is the same as the proof of Theorem 1, where one uses Theorem **B** in the appendix in place of Theorem **A**.

Appendix.

Let S denote a smooth hypersurface of $\mathbb{R}^n$, $n \ge 3$, equiped with a smooth compactly supported measure $d\mu$. Let $J : S \to S^N$ be the usual Gauss map taking each point on S to the outward unit normal at that point. We say that S has everywhere *nonvanishing Gaussian curvature* if the differential of the Gauss map dJ is always nonsingular. We say that S has at least k *nonvanishing principal curvatures* if the rank of dJ is always $\ge k$

Theorem A. *If S has everywhere nonvanishing Gaussian curvature, the following inequality holds for $p \le \frac{2(n+1)}{n+3}$:*

$$\left(\int_S\left|\,\hat f(\xi)\,\right|^2 d\mu(s)\right)^{\frac12}\le C\|f\|_{L^p(\mathbb{R}^n)},\qquad f\in C_0^\infty\,(\,\mathbb{R}^n\,).$$

Proof. See e.g. [S], p. 60. □

From a theorem of Littman [L], and one of Greenleaf [G], one can prove the following generalization of Theorem A:

Theorem B. *If S has at least k nonvanishing principal curvature everywhere, the following inequality holds for $p \leq \frac{2(k+2)}{k+4}$:*

$$\left(\int_S \left| \hat{f}(\xi) \right|^2 d\mu(\xi) \right)^{\frac{1}{2}} \leq C \|f\|_{L^p(\mathbb{R}^n)}, \qquad f \in C_0^\infty(\mathbb{R}^n).$$

References

[DC] L. De Carli, *Unique continuation for higher order elliptic operators*, Ph.D. thesis, UCLA, 1993.

[G] A. Greenleaf, *Principal curvatures in harmonic analysis*, Ind. Univ. Math. J., **30** (1981), 519-537.

[H] L. Hörmander, *The analysis of linear partial differential operators*, Vol. 4, Springer Verlag, 1983.

[H1] ______, *Uniqueness theorems for second order elliptic differential operators*, Comm. P.D.E., **8** (1983), 21-64.

[H2] ______, *On the uniqueness of the Cauchy problem*, Math. Scand., **6** (1958), 213-225.

[J] D. Jerison, *Carleman inequalities for the Dirac and Laplace operator and unique continuation*, Adv. Math., **63** (1986), 118-134.

[JK] D. Jerison and C. Kenig, *Unique continuation and absence of positive eigenvalues for Schrödinger operators*, Ann. of Math., **121** (1985), 463-494.

[KRS] C.E. Kenig, A. Ruiz and C.D. Sogge, *Uniform Sobolev estimates and unique continuation for second order constant coefficient differential operators*, Duke Math. J., **55**(**2**) (1987), 329-347.

[L] W. Littman, *Fourier transforms of surface-carried measures and differentiability of surface averages*, Bull. Am. Math. Soc., **69** (1963), 766-770.

[P] A. Plis, *A smooth linear elliptic differential equation without any solution in a sphere*, Comm. Pure Appl. Math., **14** (1961), 599-617.

[S] C.D.Sogge, *Fourier integrals in classical analysis*, Cambridge University Press, 1993.

[S1] ______, *Oscillatory integrals and unique continuation for second order elliptic differential equations*, J. Am. Math. Soc., **2** (1989), 491-516.

Received September 28, 1993 and revised November 20, 1995.

Universita' degli studi di Napoli
80126 Napoli, Italy
E-mail address: decarli@matna2.dma.unina.it