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ESSENTIAL TANGLE DECOMPOSITION FROM THIN POSITION OF A LINK

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In this paper, we develop the idea of Thompson which treats the relationship between bridge position, incompressible meridianal planar surfaces, and thin position. We show that for a link in thin position there exits a canonical depth 1 nested tangle decomposition with incompressible 2-spheres arising from the thin position (Proposition 3.7), and we show that there is a maximal essential tangle decomposition of the link that is closely related to the thin position (Theorem 4.3).

1. Introduction.

The bridge number for a knot or link, introduced by Schubert [**11**], is a classical and well-understood link invariant. The concept of *thin position* of a knot or link was introduced by David Gabai (see [**4**]) in 1987, and has since been playing an important role in 3-dimensional topology (see for example [**5**], [**12**]). The relationship between the two was first explored by Thompson in [**15**], in which it was shown that either a knot in thin position is also in bridge position, or the knot has an incompressible meridianal planar surface properly imbedded in its complement.

In this paper, we further develop the idea of Thompson which treats the relationship between bridge position, incompressible meridianal planar surfaces, and thin position. We show that for a link in thin position there exits a canonical depth 1 nested tangle decomposition with incompressible 2-spheres arising from the thin position (Proposition 3.7), and we show that there is a maximal essential tangle decomposition of the link that is closely related to the thin position (Theorem 4.3).

2. Preliminaries.

We begin with some definitions.

For a submanifold H of a manifold K, $N(H, K)$ denotes a regular neighborhood of H in K.

Let L be an irreducible link in S^3. A *meridianal* planar surface in the complement of L is a planar surface properly imbedded in the link complement with boundary components consisting only of meridians of L.

Note that $S^3\backslash\{\text{two points}\} = S^2 \times \mathbf{R}$. We define $p : S^2 \times \mathbf{R} \to S^2$ to be the projection onto the first factor, and h to be projection onto the second factor. Assume that $h\mid_L$ is a Morse function. A meridianal 2-sphere S is said to be *bowl like* if $S = F_1 \cup F_2$ such that $F_1 \cap F_2 = \partial F_1 = \partial F_2$, F_1 is a round 2-disc contained in a level plane, $h\mid_{\circ F_2}$ is a Morse function with exactly one maximum or minimum, $p(F_1) = p(F_2)$, and $p\mid_{F_2}: F_2 \to p(F_2)$ is a homeomorphism. Further, we shall require that all punctures lie in F_1. A bowl like 2-sphere is *flat face up* (*flat face down*) if F_1 is above (below) F_2.

We recall the definition of *thin position*: Let $f_s, 0 \leq s \leq 1$ be an ambient isotopy of S^3 such that $h\mid_{f_1(L)}$ is a Morse function. Choose a regular value t_i between each pair of adjacent critical values of $h\mid_{f_1(L)}$. Define the *width of* L *with respect to* f to be the sum over i of the [number of intersections of $f_1(L)$ with $h^{-1}(t_i)$], and denote it by $w_f(L)$. Define the *width of* L, $w(L)$, to be the minimum width of L with respect to f over all f; L is in *thin position* if it is in a position which realizes its width.

We say that S is a *thin* 2-*sphere* (*thick* 2-*sphere*) for L with respect to h if $S = h^{-1}(t)$ for some t which lies between adjacent critical values x and y of h, where x is a minimum (maximum) of L lying above t and y is a maximum (minimum) of L lying below t, see Figure 2.1. We define *thin discs* (*thick discs*) contained inside a bowl like 2-sphere analogously.

We say that L is in *bridge position* if there exists some thick 2-sphere for L such that all maxima (minima) of L are above (below) the thick 2-sphere. Let L' be the part of L lying inside the bowl like 2-sphere S. We say that L' is in *bridge position* if there exists some thick 2-disc D for L' such that all maxima (minima) of L' are above (below) D.

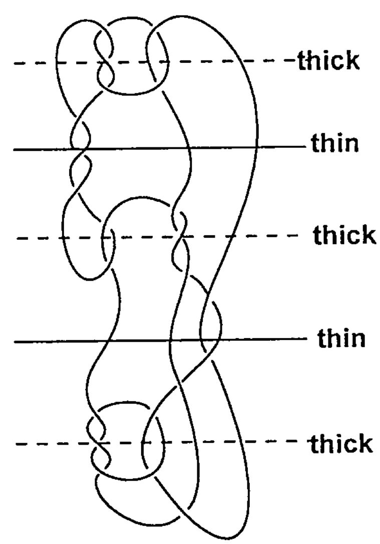

Figure 2.1.

Let L, L' be two links in S^3. We say that L is *h-equivalent* to L' if there exists an ambient isotopy f_s $(0 \le s \le 1)$ of S^3 such that $f_1(L) = L'$ and such that for every $x \in L$ we have $h(f_1(x)) = h(x)$. We remark that if L is h-equivalent to L', then L and L' are clearly the same link. Note that if L is h-equivalent to L', this implies that h is a Morse function on L' and that $w_{\text{id}}(L) = w_f(L) = w_{\text{id}}(L')$. Thus if L is in thin position, then L' is also.

Let F be a compact surface. A *compression body* W is a 3-manifold obtained from $F \times [0,1]$ by attaching 2-handles along mutually disjoint simple closed curves in $F \times \{1\}$ and attaching some 3-handles so that $\partial_- W = \partial W - (\partial_+ W \cup \partial F \times [0,1])$ has no 2-sphere components, where $\partial_+ W$ is a subsurface of ∂W which corresponds to $F \times \{0\}$. It is known that W is irreducible (Lemma 2.3 of [**2**]).

3. Canonical Depth 1 Nested 2-spheres.

In general, for a link L in S^3, the *exterior* of L, denoted by $E(L)$, is the closure of $S^3 - N(L, S^3)$. Let L be a link in thin position. Assume further that L is not in bridge position. In this section, we show that there is a canonical decompostion of the link by a system of incompressible 2-spheres which is associated with the thin position (Proposition 3.7). Note that this sharpens the result of Thompson [**15**].

Since L is not in bridge position, there is at least one thin level 2-sphere. Let S be a member of the isotopy class of the highest such thin level 2-spheres, and $P =$ the corresponding planar surface in $E(L)$. The closure of the part of $E(L)$ above (below) P we denote by M_0 (M_1).

Claim 3.1. P is incompressible above.

Proof. Since S is the highest thin 2-sphere, the critical points above S are a number of minima followed by a number of maxima. Between the highest such minima and the lowest such maxima we take a thick 2-sphere F. Above F, L is just a bunch of arcs, between F and S is the same, with some vertical segments as well. The closure of the part of M_0 above (below) F we denote by $M_{0,1}$ $(M_{0,2})$. We again use F to denote the surface $F \cap M_0$, slightly abusing notation. Denote by B_1 the union of closures of $\partial M_0 \backslash N(\partial F; \partial M_0)$ contained in $\partial M_{0,1}$, and by B_2 the closure of the component of $\partial M_0 \backslash N(\partial F; \partial M_0)$ contained in $\partial M_{0,2}$. Note that $M_{0,i}$ is a compression body with $\partial_- M_{0,i} = B_i$. Thus F is a Heegaard surface for $(M_0;\, B_1,\, B_2)$, see [**3**].

Assume that there exists a compressing disc, D, for P above. By [**3**] we may assume that D has been isotoped to intersect F in one circle. We may assume that D contains components of L inside and outside, since otherwise D is parallel to a piece of P. Since D intersects F in a circle, note that there are maxima of L contained on the inside and on the outside of D. Also, there is at least one minima of L above P by definition of thin 2-sphere. Hence, there is a pair of maxima and minima separated by D. We push the maxima to lie slightly below F, and at the same time push the minima above F, see Figure 3.1. This lowers the width of the link by 4, contradicting thin position. This contradiction proves the claim. □

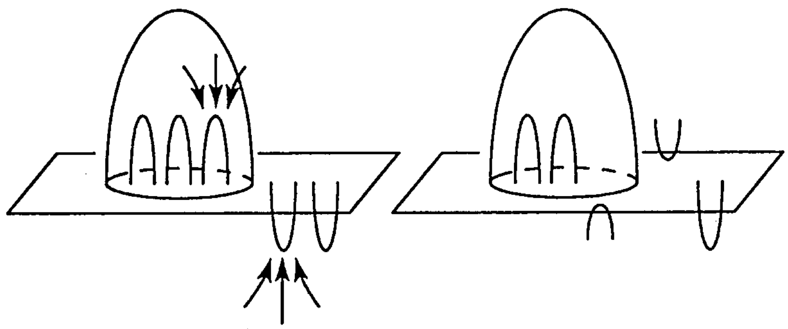

Figure 3.1.

Note that P may be compressible below. Let C be the maximal compression body (modulo isotopy) for $(M_1,\ P)$, c.f. [**2**]. Then C is obtained from $N(P; M_1) = P \times [0,1]$ by adding 2-handles constructed from compressing discs for $P \times \{0\}$, say $D_{1,1}, \ldots, D_{1,m_1}$, and capping off the 2-sphere boundary component.

Claim 3.2. We may assume that the discs $D_{1,1}, \ldots, D_{1,m_1}$ are non-nested.

Proof. The following argument takes place entirely within M_1. Assume that $D_{1,2}$ lies on the inside of $D_{1,1}$. Now $\partial D_{1,1} \cup \partial D_{1,2}$ bound a (punctured) annulus $A \subset P \times \{0\}$; we choose some arc α lying in A joining $\partial D_{1,1}$ to $\partial D_{1,2}$.

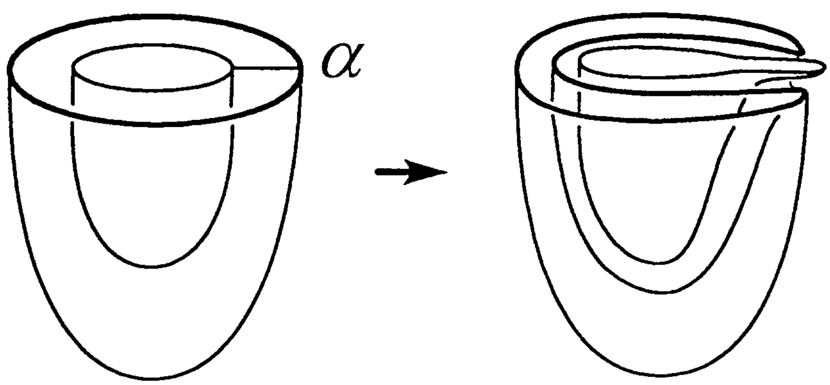

Figure 3.2.

A small regular neighborhood of α in M_1 is a ball β. Using β, isotope $D_{1,2}$ until it intersects $D_{1,1}$ in an arc, and such that the piece inside $D_{1,2}$ and outside of $D_{1,1}$ is a tiny ball lying completely inside of M_1. This isotoped $D_{1,2}$ we call $D'_{1,2}$. The piece of $D'_{1,2}$ lying interior to $D_{1,1}$ we call D. The piece of $D_{1,1}$ lying interior to $D'_{1,2}$ we call E. Change $D_{1,1}$ into $D'_{1,1}$ by an isotopy over $D'_{1,2}$ in such a way that $D'_{1,1}$ is just $D_{1,1} \backslash E$ union a push off of D, see Figure 3.2. This completes the proof of the claim. □

Note that the $D_{1,i}$ form, together with punctured subdiscs of $P \times \{0\}$, punctured 2-spheres $S_{1,1}, \ldots, S_{1,m_1}$. Then we remark that $C = M_1 \setminus$ the interior of the $S_{1,i}$'s.

Claim 3.3. Maintaining the condition that the $D_{1,i}$ are non-nested, there is an ambient isotopy f_s such that the link $L' = f_1(L)$ is h-equivalent to L, $f_s\,|_{M_0} = id_{M_0}, \forall s$ and the $S_{1,i}$ can be chosen to be flat face up bowl like 2-spheres.

Proof. Again, the following argument takes place entirely within M_1. Assume the interior of $D_{1,1}$ is non-convex. Take the 2-sphere $S_{1,1}$ and shrink it by ambient isotopy f_s to an extremely tiny 2-sphere $S'_{1,1}$, with shrinking of the piece of L inside of it similarly. It may be necessary to "untwist" $S_{1,1}$ in order to do this, but we can "retwist" it when it is very small. Note that this may bring the link out of thin position. Choose a small circle c lying in $P \times \{0\}$ such that the projection of the part of L lying inside of $S_{1,1}$ lies completely inside of c. We now construct the vertical cylinder $C_{1,1} = c \times \mathbf{R}$ in the $S^2 \times \mathbf{R}$ structure (recall the Morse function inducing thin position was really a height function on $S^3 \backslash \{\text{two points}\}$) of M_1. By a strictly horizontal ambient isotopy, we may assume that the part of L lying outside of $S_{1,1}$ doesn't intersect $C_{1,1}$. Now, isotope the link $f_1(L)$ so that each point lies at the same height as the original link L, and so that it does not intersect the cylinder $C_{1,1}$. But then we can restretch $S'_{1,1}$, forming $S''_{1,1}$, so that its critical points lie on the same level as $S_{1,1}$, but $S''_{1,1}$ is contained in $C_{1,1}$. Since the same critical points lie on the same level as before, thin position has not been altered. Furthermore, since everything was done by ambient isotopy the link remains unaltered as well; it is clear that the isotopies can be chosen so as to form an h-equivalence. The resultant link we call L'. Now choose in place of $D_{1,1}$ the disc $D'_{1,1}$ formed by taking some area of the sides of $C_{1,1}$ together with a horizontal disc lying below the lowest minimum of L' inside $C_{1,1}$. If we alter this disc by a tiny isotopy in order to bulge the bottom downward and slightly slope the sides, we obtain the desired disc. Successive repetition of this procedure on each of the $D_{1,i}$'s completes the proof of the claim. □

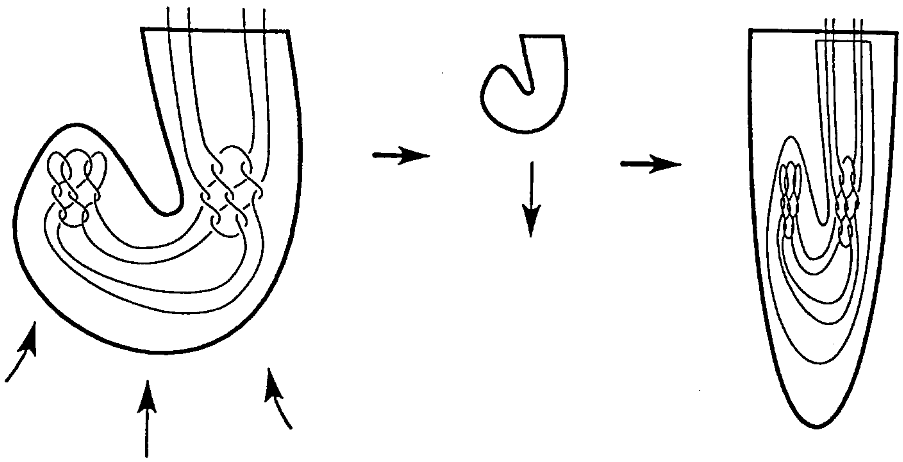

Figure 3.3.

Claim 3.4. The $S_{1,i}$ are unique up to an isotopy in the link exterior.

Proof. This is a consequence of the uniqueness of the maximal compression body C, see [**2**]. □

Claim 3.5. $\cup\, S_{1,i}$ is incompressible in E(L).

Proof. Using the notation of Claim 3.1, F is a Heegaard surface for $E(L)\setminus$ the interior of the union of the $S_{1,i}$. Now the proof is analogous to that of Claim 1. □

Remark 3.6. By the theorem in the appendix, each of the $S_{1,i}$ is incompressible in E(L).

We summarize the results of Claims 3.1-3.4 and Remark 3.6 in the statement:

Proposition 3.7. *If a link L has thin position differing from bridge position, then there exists an ambient isotopy f_s, such that $L' = f_1(L)$ is h-equivalent to L, and L' has a tangle decomposition by a finite number of non-trivial non-nested flat face up bowl like 2-spheres each of which is incompressible in the link complement. In this decomposition we have a tangle "on top," (above P) with all of the incompressible 2-spheres below it connected by vertical strands, as per Figure 3.4.*

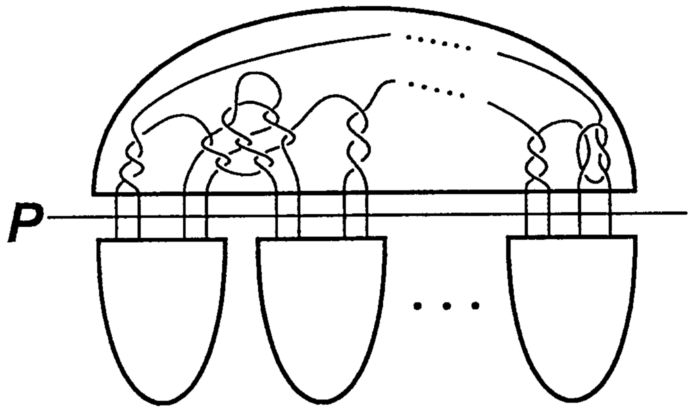

Figure 3.4.

4. Essential Tangle Decomposition.

In this section, we further develop the consideration in Section 3 for the inside of the $S_{1,i}$'s. We note that the main idea of this argument, in which ST-complexity is used, is borrowed from [**13**].

From this point on, we make the assumption that the link L has already been altered to satisfy the conclusion of Proposition 3.7. We now move to consideration of the piece $L_{1,i}$ of the link L lying inside of $S_{1,i}$.

It may be the case that $L_{1,1}$ is in bridge position. If so, we move to the consideration of $S_{1,2}$. Thus we assume that $L_{1,1}$ is not in bridge position. Let $P_{1,1}$ be the highest thin disc for $L_{1,1}$, and denote by $M_{1,1}$ the exterior of $L_{1,1}$ lying inside of $S_{1,1}$ and below $P_{1,1}$.

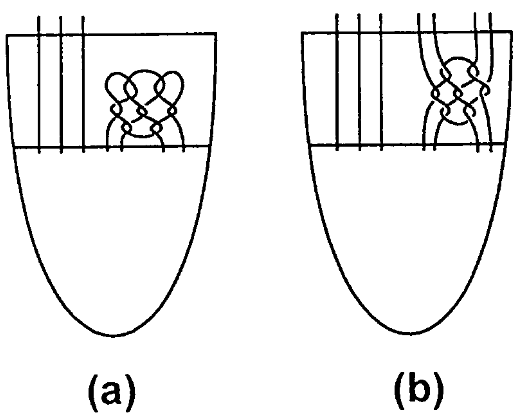

Figure 4.1.

We now have two possibilities:
1) $P_{1,1}$ is compressible above (Figure 4.1(a)), and
2) $P_{1,1}$ is incompressible above (Figure 4.1(b)).

In case (1), the argument of Claim 3.1 demonstrates that all maxima and minima between P and $P_{1,1}$ are contained on the inside of the compressing disc D. In addition, by following the above argument, we see that we may choose D in such a way as to form, together with a component of $P_{1,1}\backslash\partial D$, a flat face down bowl like 2-sphere, call it $S_{2,0}$ (see Figure 4.2(a)). This 2-sphere may be compressible in $M_{1,1}$.

In case (2), an argument analogous to that of Claim 3.1 shows that there is no vertical annulus separating two critical points of L between P and $P_{1,1}$, so this piece of $L_{1,1}$ looks just like Figure 4.2(b).

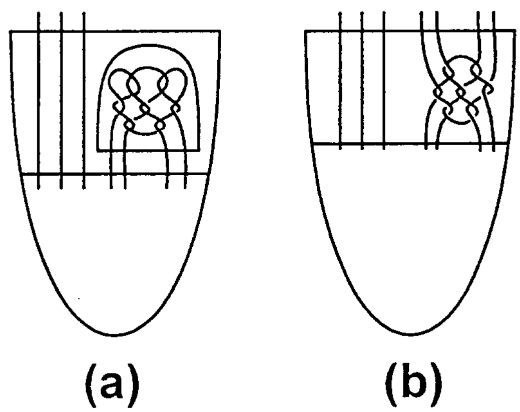

Figure 4.2.

In either case, $P_{1,1}$ is compressible below, at least by a disc attached to the boundary of $P_{1,1}$ and encasing the piece of $L_{1,1}$ below $P_{1,1}$. Using the above method, we find the maximal compression body $C_{1,1}$ for $M_{1,1}$. By applying the argument of Claim 3.3, we may suppose that $\partial_- C_{1,1}$ consists of a disc and a system of flat face up bowl like 2-spheres, say $S_{2,1}, \dots, S_{2,n_2}$. Then we move to the consideration of $S_{2,1}$ and so on.

Continuing in this fashion, we decompose the interior of each of the $S_{i,j}$ with bowl like 2-spheres, the interior of each of which contains at most one thick disc up to isotopy, and perhaps a collection of bowl like 2-spheres, see Figure 4.3. The above $S_{i,j}$'s we rename $S_1, \dots, S_n$.

Let $S = S_1 \cup \dots \cup S_n$ be the union of bowl like 2-spheres obtained as above. Let $C_0, \dots, C_n$ be the closure of the components of $S^3\backslash S$, such that C_0 lies exterior to all of the S_j, and C_i is the component lying directly inside of S_i. Then, by construction, S satisfies the following.

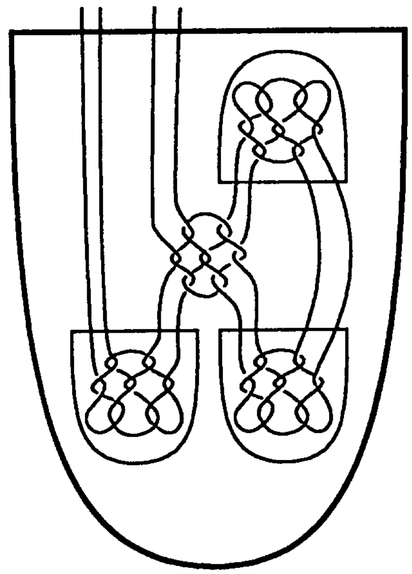

Figure 4.3.

Property 1. For each C_i, $i > 0$, either 1) there is both a maximum and minimum of L in C_j, or 2) there is nor a maximum and minimum of L in C_j, and there exists a level disc F_i such that every flat face down (up) bowl like 2-sphere directly inside S_i lies above (below) F_i, and every maximum (minimum) of L in C_i lies above (below) F_i.

We note that, if one exists, F_i is a thick 2-disc for C_i. We also note that F_i is a Heegaard splitting surface for $C_i \cap E(L)$. Denote by the *complexity of* S_i, $c(S_i)$, the number of intersections of F_i and L. The (Scharlemann-Thompson) ST-complexity of S, denoted by $c(S)$, we define to be the list of integers $\{c(S_i)\}_{i=1}^{n}$, together with the following order.

We arrange the integers in this set in monotonically non-increasing order, and compare the ordered multi-sets lexicographically. For example, the multi-set $\{3,3,5,3,2,1\}$ is less than the multi-set $\{2,2,5,3,4\}$ since $\{5,3,3,3,2,1\}$ precedes $\{5,4,3,2,2\}$ in lexicographic order.

Let D_u be a disc in C_i such that $\partial D_u = \alpha_u \cup \beta_u$ with α_u a sub-arc of L containing a single critical point which is a maximum, and $\beta_u = D_u \cap F_i$. Similarly, let D_b be a disc in C_i such that $\partial D_b = \alpha_b \cup \beta_b$ with α_b a sub-arc of L containing a single critical point which is a minimum, and $\beta_b = D_b \cap F_i$. Assume that ${}^\circ\beta_u \cap^\circ \beta_b = \emptyset$. Then the maximum-minimum arc pair (α_u, α_b) is called a *bad pair of arcs* for S_i.

Similarly, let α_u, α_b be a pair of arcs in C_i each containing a single critical point which is a maximum (minimum) respectively, and $\beta_u, \beta_b \subset F_i$ be two arcs whose interiors don't intersect, such that $\partial\alpha_u = \partial\beta_u$, $\partial\alpha_b = \partial\beta_b$. We say that α_u, α_b is a *weakly bad pair of arcs* if there exists discs D_u, D_b such that ${}^\circ D_u \cap {}^\circ D_b = \emptyset$, $\partial D_u = \alpha_u \cup \beta_u$, $\partial D_b = \alpha_b \cup \beta_b$, $N(\beta_u, D_u)$ is above F_i, and $N(\beta_b, D_b)$ is below F_i. Note that a bad pair of arcs is obviously weakly bad.

Since L is in thin position, we have:

Property 2. There does not exist a weakly bad pair of arcs for S_i.

Recall that we have decomposed L into tangles with the collection $S = S_1 \cup \ldots \cup S_n$ of bowl like 2-spheres.

Proposition 4.1. *In general, let $S = S_1 \cup \ldots \cup S_n$ be a collection of bowl like* 2*-spheres satisfying Properties* 1 *and* 2*. Suppose that $S_i \in S$ is compressible. Then after an ambient isotopy of L, there exists another collection S' of bowl like* 2*-spheres satisfying Property 1 and having a lower ST-complexity.*

Proof. Let $C_1, \ldots, C_n$ be as above. By an innermost disc argument, and changing suffix if necessary, we may assume that D is a compressing disc for S_1 and that $D \cap S = \partial D$. Since the argument is symmetric, we assume that S_1 is flat face up. By changing suffix if necessary, we may suppose that $S_2, \ldots, S_m$ lie directly inside of S_1. Let C_j be the component which contains D. Since F_j is a Heegaard surface for $C_j \cap E(L)$, we may assume, by the argument of Claim 3.1, that $D \cap F_j$ is a circle. Again by the argument of Claim 3.1 and Property 2, we have:

(*) All maxima and minima inside of C_j are contained on either the inside or the outside of D.

Then we have the following cases:

Case 1. The disc D compresses S_1 outside.

In this case, we note that C_j lies outside of S_1, and C_1 lies inside.

Case 1.1. No maxima or minima in C_j are contained on the inside of D.

Note that this contains the case when there are no maxima and minima of L in C_j.

Case 1.1A. No flat face up bowl like 2-spheres are contained in the interior of D.

Then D contains only a collection of flat face down 2-spheres connected to S_1 by vertical strands. In this case, we push these interior bowl like 2–spheres into the interior of C_1. This does not alter the complexity $c(S_1)$, but does remove intersections between F_j and L, lowering the complexity $c(S_j)$ as per Figure 4.4. Thus the complexity of S is lowered.

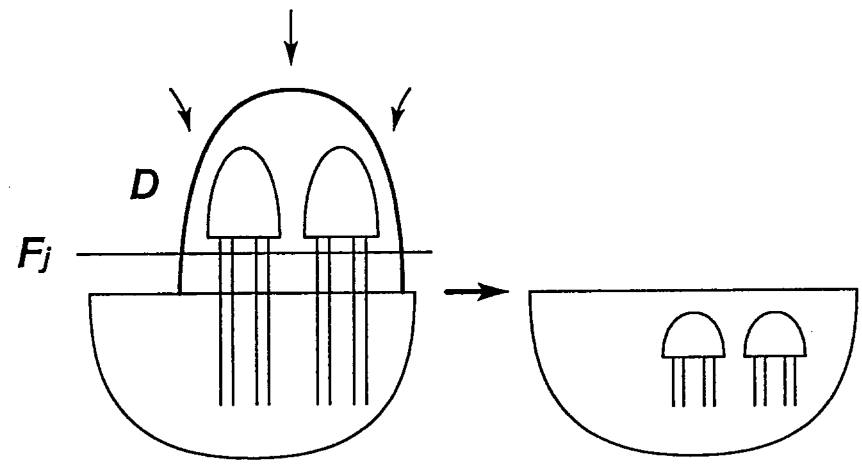

Figure 4.4.

Case 1.1B. Some flat face up bowl like 2-sphere(s) are contained in the interior of D.

Then we first use an isotopy to pull these above the flat face of S_1, see Figure 4.5(a) & (b). We can use the argument of Claim 3.3 to assure that the interior of D fits into a vertical cylinder. Now use D, together with a subdisc of the flat face of S_1, to form a new flat face down bowl like 2-sphere S_{n+1}. Note that adding S_{n+1} to our collection of 2-spheres lowers the complexity, as per Figure 4.5(b) & (c), though it leaves S_1 compressible. Then we may use the argument of 1.1A to further reduce the complexity.

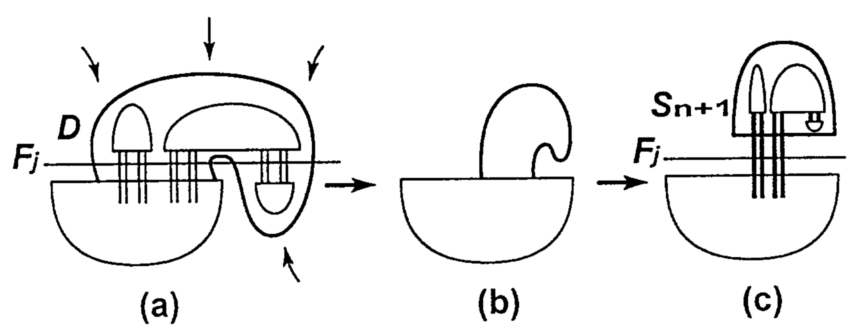

Figure 4.5.

Case 1.2. All maxima and minima fall on the inside of D.

Then the argument of either 1.1A or 1.1B suffices to lower the complexity.

Case 2. The disc D compresses S_1 inside.

Then we note that $C_j = C_1$. We denote the component of $S^3 \backslash S$ lying directly outside of S_1 as C_k.

Case 2.1. No maxima or minima in C_1 are contained on the inside of D.

Note that this contains the case when there are no maxima and minima of L in C_1.

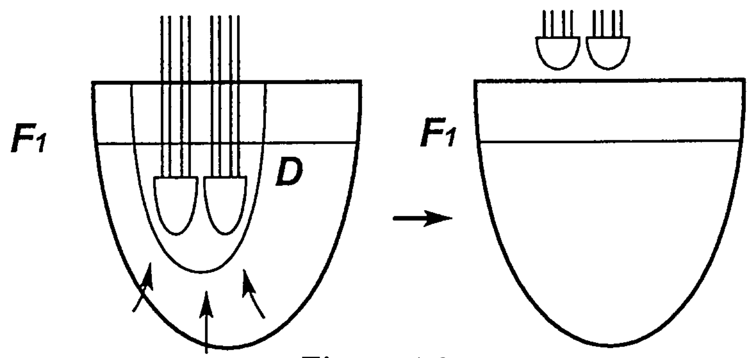

Figure 4.6.

Case 2.1A. No flat face down bowl like 2-spheres are contained in the interior of D.

Then D contains only a collection of flat face up 2-spheres connected to S_1 by vertical strands. In this case, we push these interior bowl like 2-spheres into the exterior of S_1. This does not alter the complexity $c(S_k)$, but does remove intersections between F_1 and L, lowering the complexity $c(S_1)$ as per Figure 4.6. Thus the complexity of S is lowered.

Case 2.1B. Some flat face down bowl like 2-sphere(s) are contained in the interior of D.

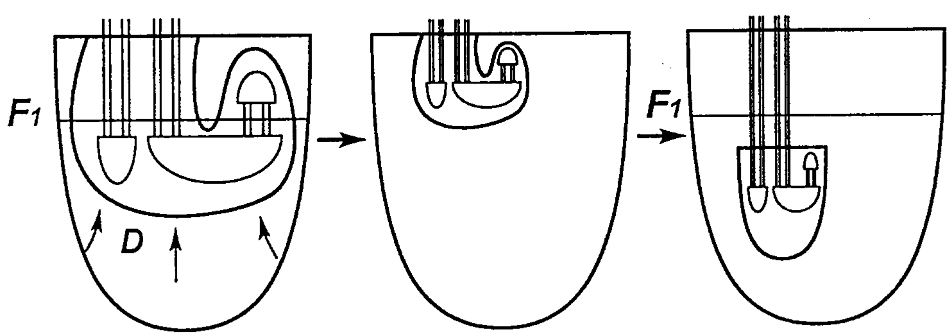

Figure 4.7.

We use the argument of Claim 3.3 to assure that the interior of D fits into a vertical cylinder. Now use D, together with a subdisc of the flat face of S_1, to form a new flat face up bowl like 2-sphere S_{n+1}. Note that adding S_{n+1} to our collection of 2-spheres lowers the complexity, as per Figure 4.7, though it leaves S_1 compressible. Then we may use the argument of 2.1A to further reduce the complexity.

Case 2.2. All maxima and minima fall on the inside of D.

Then the argument of either 2.1A or 2.1B suffices to lower the complexity.

And, finally, it is easy to see that new system of bowl like 2-spheres above satisfies Property 1, by Property (*). □

Remark. Note that possibly some components of S' in the conclusion of Proposition 4.1, say S'_1, S'_2 might be mutually parallel, that is, the region between S'_1, S'_2 contains only vertical strands.

We note that in either case, the argument may force us to pull the link out of thin position.

Proposition 4.2. *Let $L^{(n+1)}$ be the link and $S^{(n+1)}$ be the collection of bowl like* 2*-spheres obtained from n applications of Proposition 4.1. Note that here we must assume that $L^{(i)}$ satisfies Property 2 for each $i \leq n$. If $L^{(n+1)}$ does not satisfiy Property 2, then L is not in thin position.*

Proof. Let D_u, D_b be the discs corresponding to the bad pair of arcs. We reverse the operation of Proposition 4.1, pushing the discs D_u, D_b in the process. We note that we may assume the reverse process misses a tiny neighborhood of $\alpha_u, \beta_u, \alpha_b$, and β_b. Hence, this gives a weakly bad pair of discs for the original link L, contradicting thin position. □

Now we return to the link L, and the collection $S = S_1 \cup \ldots \cup S_n$ under consideration earlier.

Main Theorem 4.3. *Let L be a link in thin position, and S as above. Then there exists an ambient isotopy for L to a link L' so that there exists a collection of incompressible bowl like* 2*-spheres S' for L' such that there is a one to one correspondence between the components of $S^3 - S'$ that contain maximum* (*and minimum*) *of L' and the components of $S^3 - S$ that contain maximum* (*and minimum*) *of L.*

Proof. If S is incompressible, we are done, so suppose not. Apply Proposition 4.1 to the link L and the collection S in order to obtain the collection $S^{(1)}$ for the link $L^{(1)}$. By Proposition 4.2, we see that $(L^{(1)}, S^{(1)})$ satisfies Property 2, so that we may again apply Propositon 4.1 if necessary.

That this process ends in finitely many steps follows from the fact that the ST-complexity decreases at each application of Proposition 4.1. Let S' be the system of bowl like 2-spheres obtained in this manner. Then by Proposition 4.1 we see that S' is incompressible. And, by the proof of Proposition 4.1, we easily see that we have a one to to correspondence as in the statement of Theorem 4.3. □

5. Examples.

The first example is borrowed from Morimoto, [8].

Example 5.1. We consider the 8_{16} knot K, which is pictured in Figure 5.1(a), having bridge index 3. It is elementary to check that it has an incompressible meridianal planar surface S, as shown in 15(b). But the width of K with respect to the height function of Figure 5.1(b) is $2+4+6+4+6+4+2=28$. In the position pictured in 15(a), the width is $2+4+6+4+2=18$. The only non-trivial knots having thinner position are 2-bridge knots $(2+4+2=8)$ and 3-bridge knots which are the composition of two 2-bridge knots $(2+4+2+4+2=14)$. This knot is neither, so that 18 must be the width of K.

Note that this demonstrates that it is possible for a knot to have an incompressible, meridianal planar surface whose existence is unrelated to thin position. Thus we cannot expect to find a converse to Thompson's result (see [15]).

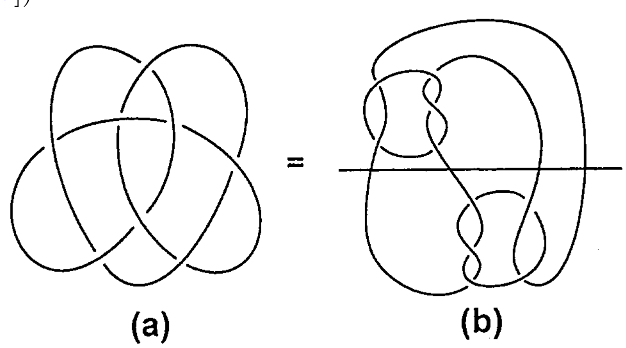

Figure 5.1.

Example 5.2. We classify thin position of prime knots up to 10 crossings and prime, irreducible links up to 9 crossings. A complete table with diagrams can be found in [10].

Note that, in the definition of thin position, $\sharp[L \cap h^{-1}(t_i)] = \sharp[L \cap h^{-1}(t_{i+1})] \pm 2$, for $i = 1, \dots, n$. The irreducibility condition implies that we cannot find an i for which $\sharp[L \cap h^{-1}(t_i)] = 0$. The prime condition implies that except for $i = 1, n$, $\sharp[L \cap h^{-1}(t_i)] \neq 2$. Thus if bridge index is 1, 2, or 3, we are already done, as the thinnest possible position is 2, $2+4+2=8$, or $2 + 4 + 6 + 4 + 2 = 18$ respectively. Any interesting example, therefore, has bridge index at least 4.

It is elementary to check that all knots and links pictured in [10] are 2- or 3-bridge with the exceptions of $9^3_{3-4}, 9^3_{15-17}, 8^4_{1-3}$, and 9^4_1. Each of these links is either 3-component for which one of the components is a 2-bridge knot, or 4 component, and thus the smallest the bridge indices can be for any of them is 4. In fact, it is easy to see from the pictures that the bridge index in each case is 4. Hence thin position has a width of either 32 (if thin position is determined by bridge index) or 28 (which requires the existence of an incompressible, meridianally planar surface in the complement) for each.

In each case, it is a rather simple matter to find a meridianal incompressible planar surface giving a thinner position than the bridge presentation. Thus width for each of these links is 28.

Example 5.3. We consider the Montesinos knot $K = M(0;\ (2, 1),\ (3, 1),\ (3, 1),\ (5, 1))$, pictured in 5.2(a) in 4-bridge presentation. Its branched double cover is the Seifert manifold S(0, 0; (2,1), (3,1), (3,1), (5,1)). That this manifold has no horizontal Heegaard splitting is easy to check using the process of [7]. Thus any Heegaard splitting is vertical, and of genus at least 3.

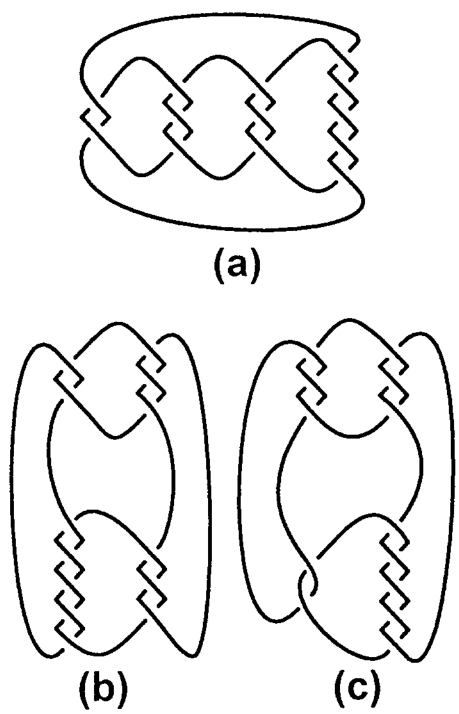

Figure 5.2.

Now we note that by [1], Lemma 3.2, K doesn't have a 3-bridge presentation, so $b(K) = 4$, and hence width is either 28 or 32. Since K is Montesinos, we can find all meridianal incompressible surfaces using [9]. It is elementary to check that two of these give essentially different tangle decompositions inducing a width of 28, demonstrating that thin position need not be unique. (See Figure 5.2(b) & (c).)

We note that we may change the constants in this example so as to produce myriads of examples.

Appendix.

Theorem . *Let $\Sigma = \bigcup_{i=1}^{n} S_i \subset M$ be a union of surfaces with $(\Sigma, \partial\Sigma) \subset (M, \partial M)$, and with Σ incompressible in $M \setminus \Sigma$. Then S_i is incompressible in M $\forall i$.*

Proof. Assume that S_1 is compressible in M, and let D be a compressing disc having the least number of intersections with Σ. Since Σ is incompressible, D must intersect some S_i, $i \neq 1$. Then $D \cap \Sigma$ is a union of simple closed curves; consider an innermost one, α. Now α bounds a disc D' in D. If D' is inessential, we can create a new disc $\tilde{D}$ by replacing D' with the sub-disc of S_i bounded by α; then $\tilde{D}$ has less intersections with Σ, contradicting choice of D. Thus D' must be essential. But then D' is a compressing disc for S_i whose interior is disjoint from Σ, that is, it is a compressing disc for Σ, a clear contradiction. □

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Note: the reference in Remark 3.6 to the theorem in the appendix appeared in the printed version as "Proposition A."