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ON THE AVERAGE GROWTH OF FOURIER COEFFICIENTS OF SIEGEL CUSP FORMS OF GENUS 2

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Let F be a Siegel cusp form of integral weight k on the Siegel modular group $Sp_2(\mathbb{Z})$ of genus 2 and denote by $a(T)$ ($T \in \mathbf{Q}^{(2,2)}$, $T > 0$ half-integral) its Fourier coefficients. It is known (see Böcherer & Raghavan, 1988 and Fomenko, 1987) that

$$(1) \qquad \sum_{\{T>0,\,\det(T)=N\}/GL_2(\mathbb{Z})} |a(T)|^2 \ll_{\varepsilon,F} N^{k-3/32+\varepsilon} \quad (\varepsilon > 0)$$

where the sum is over $GL_2(\mathbb{Z})$-classes of $T > 0$ with $\det(T) = N$.

In the present note we shall give a result on the average growth of $|a(T)|^2$, where the average is taken w.r.t. the trace.

1. Introduction.

Theorem. *One has*

$$(2) \qquad \sum_{\mathrm{tr}(T)=N} |a(T)|^2 \ll_{\varepsilon,F} N^{2k+\varepsilon} \qquad (\varepsilon > 0).$$

Estimates like (2) may certainly be viewed as curious side results, and we point out that we do not have any immediate applications. It should also be noted that by Deligne's theorem one has

$$\sum_{\mathrm{tr}(T)=N} |a(T)| \ll_{\varepsilon,F} N^{k-1/2+\varepsilon} \qquad (\varepsilon > 0);$$

in fact, the left-hand side is the N-th Fourier coefficients of $F((\begin{smallmatrix} \tau & 0 \\ 0 & \tau \end{smallmatrix}))$ ($\tau \in H$ = upper half-plane) which is a cusp form of weight $2k$ on $SL_2(\mathbb{Z})$.

The proof of (2) is easy if one exploits the same arguments as in [**3**, **4**] used there to derive the bound

$$(3) \qquad a(T) \ll_{\varepsilon,F} (\det(T))^{k/2-13/36+\varepsilon} \qquad (\varepsilon > 0)$$

for the individual Fourier coefficients of F. Note, however, that by inserting (3) into the left-hand side of (2) and using

$$\det(T) \ll (\mathrm{tr}(T))^2,\ \#\left\{T \in \mathbf{Q}^{(2,2)} | T > 0 \ \text{ half-integral}, \ \ \mathrm{tr}(T) = N\right\} \asymp N^2,$$

one only obtains

$$\sum_{\mathrm{tr}(T)=N} |a(T)|^2 \ll_{\varepsilon,F} N^{2k+23/18+\varepsilon} \qquad (\varepsilon > 0).$$

While (1) was proved using Rankin's method and Landau's theorem for the Dirichlet series

$$\sum_{\{T>0\}/GL_2(\mathbb{Z})} \frac{1}{\varepsilon(T)} |a(T)|^2/\det(T)^s$$
$$(\mathrm{Re}(s) \gg 0;\ \varepsilon(T) = \text{ number of } GL_2(\mathbb{Z})\text{-units of } T),$$

it is not clear if one could proceed in a similar way in the situation considered above. In fact, nothing seems to be known about the meromorphic continuation of the series

$$\sum_{T>0} |a(T)|^2/(\mathrm{tr}(T))^s \qquad (\mathrm{Re}(s) \gg 0),$$

and it does not seem clear what one could expect.

2. Proof.

By main result of [**3**, **4**] one has

$$a(T) \ll_{\varepsilon,F} \left(1 + \frac{|D|^{1/2+\varepsilon}}{m}\right)^{1/2} \cdot \frac{|D|^{k/2-3/4}}{m^{k/2-1}} \cdot \|\phi_m\| \quad (\varepsilon > 0) \tag{4}$$

where $D := -4\det(T)$ and ϕ_m $(m \in \mathbb{N})$ is the m-th Fourier-Jacobi coefficient of F. Therefore, writing $T = \left(\begin{smallmatrix} n & r/2 \\ r/2 & m \end{smallmatrix}\right)$ we have

$$\begin{aligned}
&\sum_{\mathrm{tr}(T)=N} |a(T)|^2 \\
&\ll_{\varepsilon,F} \sum_{\substack{m,n\in\mathbb{N},\, r\in\mathbb{Z} \\ m+n=N,\, r^2<4mn}} \left(1 + \frac{(4mn-r^2)^{1/2+\varepsilon}}{m}\right) \cdot \frac{(4mn-r^2)^{k-3/2}}{m^{k-2}} \cdot \|\phi_m\|^2 \\
&\ll \sum_{1\le m<N} \left(\sum_{r\in\mathbb{Z},\, r^2<2N^2} 1\right) \left(1 + \frac{N^{1+2\varepsilon}}{m}\right) \cdot \frac{N^{2k-3}}{m^{k-2}} \cdot \|\phi_m\|^2 \\
&\ll N^{2k-2} \sum_{1\le m<N} \|\phi\|^2/m^{k-2} + N^{2k-1+2\varepsilon} \sum_{1\le m<N} \|\phi_m\|^2/m^{k-1};
\end{aligned}$$

here in the second line we have used that $4mn - r^2 \le 4mn \le 2(m+n)^2$.

On the other hand, by the results proved in [5] the Dirichlet series

$$\sum_{m\geq 1} \|\phi_m\|^2/m^s$$

converges for $\mathrm{Re}(s) > k$, hence replacing s by $s+k-2$ (resp. $s+k-1$) we see the formula for the abcissa of convergence that

$$\sum_{1\leq m<N} \|\phi_m\|^2/m^{k-2} \ll_{\varepsilon,F} N^{2+\varepsilon}, \quad \sum_{1\leq m<N} \|\phi_m\|^2/m^{k-1} \ll_{\varepsilon,F} N^{1+\varepsilon}.$$

This proves (2).

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