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PRIMITIVE LATTICE POINTS IN STARLIKE PLANAR SETS

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This article is concerned with the number $B_{\mathcal{D}}(x)$ of integer points with relative prime coordinates in $\sqrt{x}\,\mathcal{D}$, where x is a large real variable and $\mathcal{D}$ is a starlike set in the Euclidean plane. Assuming the truth of the Riemann Hypothesis, we establish an asymptotic formula for $B_{\mathcal{D}}(x)$. Applications to certain special geometric and arithmetic problems are discussed.

1. Introduction.

Let $\mathcal{D}$ denote a subset of $\mathbb{R}^2$ which is *starlike* with respect to the origin, i.e., if $\mathbf{u} \in \mathbb{R}^2$ belongs to $\mathcal{D}$, automatically $\lambda\mathbf{u} \in \mathcal{D}$ for $0 < \lambda < 1$. The distance function F of $\mathcal{D}$ is defined by

$$F(\mathbf{u}) = \inf\left\{\tau > 0 : \ \frac{\mathbf{u}}{\tau} \in \mathcal{D}\right\},$$

with the usual understanding that $\inf\emptyset = \infty$. Let us put $Q = F^2$, then Q is homogeneous of degree 2. For a large real variable x, we define $A_{\mathcal{D}}(x)$ as the number of lattice points of $\mathbb{Z}_*^2 := \mathbb{Z}^2 \setminus \{(0,0)\}$ in the "blown up" domain $\sqrt{x}\,\mathcal{D}$, i.e.,

$$A_{\mathcal{D}}(x) = \#\left(\sqrt{x}\mathcal{D} \cap \mathbb{Z}_*^2\right).$$

We make the supposition that $A_{\mathcal{D}}(x)$ satisfies an asymptotic formula

$$A_{\mathcal{D}}(x) = \sum_{r=0}^{R} c_r x^{\alpha_r} + O\left(x^{\alpha}\right), \tag{1.1}$$

with

$$\alpha_0 = 1 > \alpha_1 > \cdots > \alpha_R > \alpha\,, \quad \alpha < \tfrac{1}{2}. \tag{1.2}$$

(For a wealth of results of the form (1.1) on specific planar lattice point problems the reader may consult the monograph of Krätzel [11].)

The objective of the present paper is to study the number $B_{\mathcal{D}}(x)$ of *primitive* lattice points in $\sqrt{x}\mathcal{D}$, i.e.,

$$B_{\mathcal{D}}(x) = \#\left\{\mathbf{m} = (m_1, m_2) \in \mathbb{Z}_*^2 :\ \mathbf{m} \in \sqrt{x}\mathcal{D}\,,\ \gcd(m_1, m_2) = 1\right\}.$$

By Möbius inversion,

$$B_{\mathcal{D}}(x) = \sum_{m\in\mathbb{N}} \mu(m) A_{\mathcal{D}}\left(\frac{x}{m^2}\right), \tag{1.3}$$

where $\mu(m)$ denotes the Möbius function. By an elementary convolution argument, one can derive from the bound

$$\sum_{m\le Y} \mu(m) \ll Y\omega(Y) \tag{1.4}$$

(see Ivić [8, p. 309]), combined with (1.1) and (1.3),

$$B_{\mathcal{D}}(x) = \sum_{r:\ \alpha_r \ge \frac{1}{2}} \frac{c_r}{\zeta(2\alpha_r)} x^{\alpha_r} + O\left(x^{1/2}\omega(x)\right), \tag{1.5}$$

where

$$\omega(x) = \exp(-c(\log x)^{3/5}(\log\log x)^{-1/5})$$

with $c > 0$, is a factor familiar from the Prime Number Theorem. (1.4) and (1.5) contain the strongest information available to date concerning zero-free regions of the Riemann zeta-function. At the present state of art, it is not possible to reduce the exponent $\frac{1}{2}$ of x in the O-term of (1.5). This will be evident from Lemma 1 below (with $y = 1$), in view of the fact that the Riemann zeta-function could have zeros with real part arbitrarily close to 1.

It is therefore natural to search for stronger estimates assuming the truth of the *Riemann Hypothesis* (henceforth quoted as RH).

This problem has been attacked by Moroz [15], for the slightly simplified case that $R = 0$. He obtained the result that

$$B_{\mathcal{D}}(x) = c_0 \frac{6}{\pi^2} x + O\left(x^{\frac{2-\alpha}{5-4\alpha}+\varepsilon}\right) \qquad (\varepsilon > 0),$$

conditionally under RH.

We remark that recently Hensley [5] has recently written a paper on the subject, too, apparently unaware of Moroz's work. He used a methodically original approach but failed to sharpen the estimate.

In this paper, our ultimate goal will be to prove the following.

Theorem. *Suppose that $\mathcal{D} \subset \mathbb{R}^2$ is starlike with respect to the origin and that $A_{\mathcal{D}}(x)$ satisfies the asymptotic formula (1.1). If* RH *is true,*

$$B_{\mathcal{D}}(x) = \sum_{r:\ \alpha_r > \theta} \frac{c_r}{\zeta(2\alpha_r)} x^{\alpha_r} + O\left(x^{\theta+\varepsilon}\right),$$

for a large real variable x, arbitrary fixed $\varepsilon > 0$, and

$$\theta := \frac{4-\alpha}{11-8\alpha}.$$

Before going into technical details (which we postpone to Sections 2 and 3), we outline the essential ideas of the proof.

First of all, it will be convenient to consider the quantities

$$A^*_{\mathcal{D}}(x) := \#\{\mathbf{m} \in \mathbb{Z}^2_* :\ Q(\mathbf{m}) \le x\},$$

and

$$B^*_{\mathcal{D}}(x) := \#\{\mathbf{m} = (m_1, m_2) \in \mathbb{Z}^2_* :\ Q(\mathbf{m}) \le x\,,\ \gcd(m_1, m_2) = 1\},$$

instead of $A_{\mathcal{D}}(x),\ B_{\mathcal{D}}(x)$. Since, for every $\delta > 0$,

$$A_{\mathcal{D}}(x) \le A^*_{\mathcal{D}}(x) \le A_{\mathcal{D}}(x+\delta)\,, \qquad B^*_{\mathcal{D}}(x-\delta) \le B_{\mathcal{D}}(x) \le B^*_{\mathcal{D}}(x),$$

$A^*_{\mathcal{D}}(x)$ satisfies the asymptotic formula (1.1) as well, and the Theorem is immediate for $B_{\mathcal{D}}(x)$ if it has been proved for $B^*_{\mathcal{D}}(x)$.

Further, it is clear that

$$Q_1 := \inf_{\mathbf{u} \in \mathbb{Z}^2_*} Q(\mathbf{u}) > 0.$$

Thus we may restrict the summation in (1.3) to $1 \le m \le \sqrt{x/Q_1}$, and obtain by splitting up

$$B^*_{\mathcal{D}}(x) = \sum_{m \le y} \mu(m) A^*_{\mathcal{D}}\left(\frac{x}{m^2}\right) + \sum_{m > y} \mu(m) A^*_{\mathcal{D}}\left(\frac{x}{m^2}\right) =: S_1 + S_2, \tag{1.6}$$

where $y = y(x) < \sqrt{x/Q_1}$ is a parameter remaining at our disposition. By (1.1),

$$S_1 = \sum_{r=0}^{R} c_r x^{\alpha_r} \sum_{m \le y} \frac{\mu(m)}{m^{2\alpha_r}} + O\left(x^{\alpha} y^{1-2\alpha}\right). \tag{1.7}$$

Using the classic conditional bound (valid under RH)

$$\sum_{m\le Y}\mu(m)\ll Y^{1/2+\varepsilon'}\qquad(\varepsilon'>0), \tag{1.8}$$

summation by parts gives

$$\sum_{m\le y}\frac{\mu(m)}{m^{2\alpha_r}}=\begin{cases}\frac{1}{\zeta(2\alpha_r)}+O\left(y^{\frac12-2\alpha_r+\varepsilon'}\right) & \text{if } \alpha_r>\frac14,\\ O\left(y^{\frac12-2\alpha_r+\varepsilon'}\right) & \text{else.}\end{cases}$$

Thus (1.7) may be rewritten in the form

$$\begin{aligned}S_1=&\sum_{r:\ \alpha_r>\frac13(2+\alpha)}c_r x^{\alpha_r}\sum_{m\le y}\frac{\mu(m)}{m^{2\alpha_r}}+\sum_{r:\ \frac14<\alpha_r\le\frac13(2+\alpha)}\frac{c_r}{\zeta(2\alpha_r)}x^{\alpha_r}\\ &+\sum_{r:\ \alpha_r\le\frac13(2+\alpha)}O\left(x^{\alpha_r}y^{\frac12-2\alpha_r+\varepsilon'}\right)+O\left(x^{\alpha}y^{1-2\alpha}\right).\end{aligned}\tag{1.9}$$

To deal with S_2, an obvious possibility is to use (1.8) one more time and to apply summation by parts repeatedly. Observing that $A^*_{\mathcal{D}}(w)$ is monotone and $\ll w$, one obtains

$$S_2\ll x^{\varepsilon'}y^{1/2}\,\frac{x}{y^2}.$$

(See Moroz [**15**, formula (8)].)

The key step of the present paper is to improve this elementary estimate by a contour integration technique in the spirit of a classic paper due to Montgomery and Vaughan [**14**].

Proposition. *If the Riemann Hypothesis is true,*

$$\begin{aligned}S_2=\sum_{m>y}\mu(m)A^*_{\mathcal{D}}\left(\frac{x}{m^2}\right)=&\sum_{r:\ \alpha_r>\frac13(2+\alpha)}c_r x^{\alpha_r}\sum_{m>y}\frac{\mu(m)}{m^{2\alpha_r}}\\ &+O\left(x^{\alpha+\varepsilon'}\right)+O\left(x^{\varepsilon'}y^{1/2}\left(\frac{x}{y^2}\right)^{\frac{2+\alpha}{3}}\right)\qquad(\varepsilon'>0),\end{aligned}\tag{1.10}$$

for large real parameters x and y with $1\le y\ll x^{1/2}$.

We combine this result with (1.9) and note that the last O-terms in (1.9) and (1.10), respectively, are of the same order (apart from ε's) for

$$y=x^{\frac{4(1-\alpha)}{11-8\alpha}}.\tag{1.11}$$

This choice of y readily yields the assertion of our Theorem, since it is easily verified that, for $\alpha_r\le\frac13(2+\alpha)$,

$$x^{\alpha_r}y^{\frac12-2\alpha_r}\ll x^{\theta}.$$

2. Some Lemmas.

For $\mathrm{Re}\, s > 1$, we define the zeta-function $Z_{\mathcal{D}}(s)$ of the set $\mathcal{D}$ by the absolutely convergent Dirichlet series

$$Z_{\mathcal{D}}(s) = \sum_{\mathbf{m}\in\mathbb{Z}_*^2:\ Q(\mathbf{m})<\infty} (Q(\mathbf{m}))^{-s}.$$

We further put, for real $y \geq 1$ and a complex variable s,

$$f_y(s) = \frac{1}{\zeta(s)} - \sum_{m\leq y} \frac{\mu(m)}{m^s}. \tag{2.1}$$

This is regular in every $s \in \mathbb{C}$ which is not a zero of the Riemann zeta-function.

Lemma 1. *For a large real variable x, and any fixed $C \geq 5$,*

$$\begin{aligned} S_2 &= \sum_{m>y} \mu(m) A_{\mathcal{D}}^*\left(\frac{x}{m^2}\right) \\ &= \frac{1}{2\pi i}\int_{3-ix^C}^{3+ix^C} Z_{\mathcal{D}}(s)\, f_y(2s)\frac{x^s}{s}\,\mathrm{d}s + O\left(x^{\alpha+\varepsilon}\right) \quad (\varepsilon > 0), \end{aligned}$$

uniformly in $1 \leq y \ll \sqrt{x}$.

Proof. This clearly is a type of truncated Perron's formula. It is hard to find an explicit reference in the literature, although the argument runs on familiar lines:

Let us write the (positive and finite) values attained by $Q(\mathbf{m})$, as $\mathbf{m}$ runs through $\mathbb{Z}_*^2$, in form of a strictly increasing sequence $(\lambda_k)_{k\in\mathbb{N}}$. Put further

$$\mu_y(m) = \begin{cases} \mu(m) & \text{if } m > y, \\ 0 & \text{else,} \end{cases}$$

then it follows by the homogeneity of Q that, for $\mathrm{Re}\, s > 1$,

$$Z_{\mathcal{D}}(s)\, f_y(2s) = \sum_{\mathbf{n}\in\mathbb{Z}_*^2:\ Q(\mathbf{n})<\infty} \gamma(\mathbf{n})(Q(\mathbf{n}))^{-s} = \sum_{k=1}^{\infty} \eta_k \lambda_k^{-s}, \tag{2.2}$$

with

$$\begin{aligned} \gamma(\mathbf{n}) &:= \sum_{m|\mathbf{n}} \mu_y(m), \\ \eta_k &:= \sum_{\mathbf{n}:\ Q(\mathbf{n})=\lambda_k} \gamma(\mathbf{n}). \end{aligned}$$

Here $m|(n_1,n_2)$ means that $m|\gcd(n_1,n_2)$. For later reference, we note that, for any $\mathbf{n}=(n_1,n_2)\in\mathbb{Z}_*^2$ with $Q(\mathbf{n})<\infty$,

$$\gcd(n_1,n_2)\ll Q(\mathbf{n}). \tag{2.3}$$

To realize this, let

$$\mathbf{n}^* := \left(\frac{n_1}{\gcd(n_1,n_2)}, \frac{n_2}{\gcd(n_1,n_2)}\right),$$

then $\mathbf{n}^*$, $2\mathbf{n}^*,\ldots,\gcd(n_1,n_2)\mathbf{n}^*$ all belong to $\sqrt{2Q(\mathbf{n})}\mathcal{D}\cap\mathbb{Z}_*^2$. Therefore, by (1.1),

$$\gcd(n_1,n_2)\le A_{\mathcal{D}}(2Q(\mathbf{n}))\ll Q(\mathbf{n}).$$

Furthermore,

$$S_2=\sum_{m>y}\mu(m)\left(\sum_{Q(\mathbf{n})\le\frac{x}{m^2}}1\right)=\sum_{m,\mathbf{n}:\ Q(m\mathbf{n})\le x}\mu_y(m)=\sum_{k:\ \lambda_k\le x}\eta_k. \tag{2.4}$$

It is well-known that, for $a>0,\ a\neq 1$, and T sufficiently large,

$$\frac{1}{2\pi i}\int_{3-iT}^{3+iT}\frac{a^s}{s}\,\mathrm{d}s=\begin{cases}\chi(a)+O\left(\dfrac{a^3}{T\,|\log a|}\right), & (*)\\ O\left(a^3\right), & (**)\end{cases}$$

where χ is the characteristic function of the interval $]1,\infty[$. Of this formula, $(*)$ may be found in Apostol [**1**, p. 243], while $(**)$ is immediate by taking as a path of integration the boundary of the domain $\{s\in\mathbb{C}:\ |s|\le T,\ \mathrm{Re}\,s\le 3\}$ if $a>1$, resp., of $\{s\in\mathbb{C}:\ |s|\le T,\ \mathrm{Re}\,s\ge 3\}$ if $a<1$ (cf. Prachar [**21**, p. 379]).

Therefore, by (2.2) and (2.4),

$$\begin{aligned}&\frac{1}{2\pi i}\int_{3-ix^C}^{3+ix^C}Z_{\mathcal{D}}(s)\,f_y(2s)\frac{x^s}{s}\,\mathrm{d}s\\ &=S_2+\sum_{k:\ |\lambda_k-x|\ge 1}O\left(\frac{|\eta_k|}{\lambda_k^3x^2\,|\log\lambda_k-\log x|}\right)+\sum_{k:\ |\lambda_k-x|<1}O\left(|\eta_k|\right).\end{aligned} \tag{2.5}$$

By the mean-value theorem,

$$|\log\lambda_k-\log x|^{-1}\le\frac{\max(\lambda_k,x)}{|\lambda_k-x|}\ll\frac{\lambda_k x}{|\lambda_k-x|},$$

thus the first error term sum here is

$$\ll \frac{1}{x} \sum_{k=1}^{\infty} |\eta_k| \, \lambda_k^{-2} \ll 1,$$

since the series in (2.2) converges absolutely for $\operatorname{Re} s > 1$. Further,

$$|\eta_k| \le \sum_{\mathbf{n}:\ Q(\mathbf{n})=\lambda_k} |\gamma(\mathbf{n})| \ll \lambda_k^{\varepsilon'} \sum_{\mathbf{n}:\ Q(\mathbf{n})=\lambda_k} 1,$$

for any $\varepsilon' > 0$, in view of (2.3) and the definition of $\gamma(\mathbf{n})$. Thus the second error term sum in (2.5) is

$$\ll x^{\varepsilon'} \sum_{\mathbf{n}:\ |Q(\mathbf{n})-x|<1} 1 \ll x^{\varepsilon'} \left(A_{\mathcal{D}}^{*}(x+1) - A_{\mathcal{D}}^{*}(x-1)\right) \ll x^{\alpha+\varepsilon'},$$

in view of (1.1). This proves Lemma 1. □

The key point to prove the Proposition will be to have at hand the following estimates for the growth of the complex function $Z_{\mathcal{D}}(s)$ in the vertical direction.

Lemma 2.
(i) *For any $\sigma_1 > \alpha$, there exists a positive real number $\omega < 1$ such that*

$$Z_{\mathcal{D}}(\sigma + it) \ll |t|^{\omega},$$

uniformly in $\sigma \ge \sigma_1$, $|t| \ge 1$.

(ii) *For a real parameter $T \ge 4$, fixed $\varepsilon' > 0$, and any fixed β with*[1] *$\frac{2+\alpha}{3} \le \beta < 1$, it follows that*

$$\int_{T}^{2T} |Z_{\mathcal{D}}(\beta + it)| \, \mathrm{d}t \ll T^{1+\varepsilon'}.$$

Proof. Let us rewrite (1.1) in the form

$$P_{\mathcal{D}}^{*}(x) := A_{\mathcal{D}}^{*}(x) - \sum_{r=0}^{R} c_r x^{\alpha_r} \ll x^{\alpha}. \tag{2.6}$$

Let further X denote a positive real number which is not attained by $Q(\mathbf{n})$ as $\mathbf{n}$ runs through $\mathbb{Z}_{*}^{2}$. Using Stieltjes integral calculus, we conclude that,

[1] For $\beta \ge 1$, the estimate is trivial and not needed later.

for $\mathrm{Re}\, s > 1$,

$$
\begin{aligned}
Z_{\mathcal{D}}(s) - \sum_{0<Q(\mathbf{n})\le X} (Q(\mathbf{n}))^{-s} &= \int_X^\infty w^{-s}\,\mathrm{d}\left(\sum_{r=0}^R c_r w^{\alpha_r} + P^*_{\mathcal{D}}(w)\right) \\
&= \sum_{r=0}^R c_r\,\alpha_r \int_X^\infty w^{-s+\alpha_r-1}\,\mathrm{d}w + \int_X^\infty w^{-s}\,\mathrm{d}P^*_{\mathcal{D}}(w) \\
&= \sum_{r=0}^R c_r\,\frac{\alpha_r}{s-\alpha_r} X^{\alpha_r - s} - X^{-s}P^*_{\mathcal{D}}(X) + s\int_X^\infty w^{-s-1}P^*_{\mathcal{D}}(w)\,\mathrm{d}w.
\end{aligned}
\tag{2.7}
$$

In this identity we choose $0 < X < Q_1$ and let $X \to Q_1-$ to obtain

$$
Z_{\mathcal{D}}(s) = \sum_{r=0}^R c_r\,\frac{s}{s-\alpha_r} Q_1^{\alpha_r - s} + s\int_{Q_1}^\infty w^{-s-1}P^*_{\mathcal{D}}(w)\,\mathrm{d}w.
\tag{2.8}
$$

In view of (2.6), this provides a meromorphic continuation of $Z_{\mathcal{D}}(s)$ to the half-plane $\mathrm{Re}\, s > \alpha$, with simple poles at $s = \alpha_r,\ r = 0, \dots, R$. At the same time, (2.8) shows that

$$
Z_{\mathcal{D}}(\sigma + it) \ll |t|\,,
$$

uniformly in $\sigma \ge \sigma_0$, $|t| \ge 1$, where $\sigma_0 > \alpha$ is arbitrary but fixed. Since, by absolute convergence, $Z_{\mathcal{D}}(\sigma + it)$ is uniformly bounded in every half-plane $\sigma \ge \sigma_2 > 1$, a Phragmén-Lindelöf argument[2] establishes part (i) of Lemma 2, if we put $\sigma_0 = \frac{1}{2}(\alpha + \sigma_1)$ for arbitrary given $\sigma_1 > \alpha$.

To show (ii), we apply the identity derived in (2.7) one more time, with

$$
T^\xi \le X \le 2T^\xi\,, \quad \xi := \frac{3}{2(1-\alpha)},
\tag{2.9}
$$

$$
s = \beta + it\,, \quad T \le t \le 2T.
$$

This is clearly justified by analytic continuation. We obtain

$$
Z_{\mathcal{D}}(\beta + it) = S_X(t) + O\left(T^{-1}X^{1-\beta}\right) + O\left(TX^{-\beta+\alpha}\right),
\tag{2.10}
$$

with

$$
S_X(t) := \sum_{\mathbf{m}\in\mathbb{Z}^2_*:\ Q(\mathbf{m})\le X} (Q(\mathbf{m}))^{-\beta - it}.
$$

[2] For a classic reference, see Landau [**13**, p. 229].

Integration over $T \le t \le 2T$ gives

(2.11)
$$\int_T^{2T} |Z_{\mathcal{D}}(\beta + it)| \, \mathrm{d}t \ll \int_T^{2T} |S_X(t)| \, \mathrm{d}t + O\left(X^{1-\beta}\right) + O\left(T^2 X^{-\beta+\alpha}\right).$$

By Cauchy's inequality[3] ,

$$\begin{aligned}\left(\int_T^{2T} |S_X(t)| \, \mathrm{d}t\right)^2 &\ll T \int_T^{2T} |S_X(t)|^2 \, \mathrm{d}t \\ &\ll T \sum_{Q(\mathbf{m}) \le Q(\mathbf{n}) \le X} (Q(\mathbf{m})\, Q(\mathbf{n}))^{-\beta} \left| \int_T^{2T} \left(\frac{Q(\mathbf{n})}{Q(\mathbf{m})}\right)^{it} \mathrm{d}t \right|.\end{aligned}$$

For $Q(\mathbf{m}) < Q(\mathbf{n})$, the integrals in this sum can be estimated by

$$\begin{aligned}\left|\int_T^{2T} \exp\left(it(\log Q(\mathbf{n}) - \log Q(\mathbf{m}))\right) \mathrm{d}t\right| &\le \frac{2}{\log Q(\mathbf{n}) - \log Q(\mathbf{m})} \\ &\le \frac{2Q(\mathbf{n})}{Q(\mathbf{n}) - Q(\mathbf{m})}.\end{aligned}$$

Along with the trivial bound, this gives

(2.12)
$$\begin{aligned}\left(\int_T^{2T} |S_X(t)| \, \mathrm{d}t\right)^2 &\ll T \sum_{Q(\mathbf{n}) \le X} (Q(\mathbf{n}))^{-\beta} \\ &\times \left(\sum_{\mathbf{m}:\ Q(\mathbf{m}) \le Q(\mathbf{n})} (Q(\mathbf{m}))^{-\beta} \left(\max\left(\frac{1}{T},\ \frac{Q(\mathbf{n}) - Q(\mathbf{m})}{Q(\mathbf{n})}\right)\right)^{-1} \right).\end{aligned}$$

We now keep $\mathbf{n} \in \mathbb{Z}_*^2$ fixed for the moment and split up the inner sum over $\mathbf{m}$: For that purpose, we define a sequence $(\delta_j)_{j=0}^J$ by $\delta_j = 2^j Q(\mathbf{n})\, T^{-1}$, with J such that $\frac{1}{8} Q(\mathbf{n}) < \delta_J \le \frac{1}{4} Q(\mathbf{n})$. We distinguish three cases according to the relative size of $Q(\mathbf{n}) - Q(\mathbf{m})$.

First of all (Case 1),

$$\begin{aligned}Q(\mathbf{n}) - Q(\mathbf{m}) < \delta_0 \quad &\iff \quad \frac{1}{T} > \frac{Q(\mathbf{n}) - Q(\mathbf{m})}{Q(\mathbf{n})} \\ &\iff \quad Q(\mathbf{m}) > Q(\mathbf{n}) \left(1 - \frac{1}{T}\right),\end{aligned}$$

[3]Here and in what follows, $\mathbf{m}$ and $\mathbf{n}$ denote elements of $\mathbb{Z}_*^2$.

thus the contribution of these $\mathbf{m}$ to the inner sum in (2.12) is

$$\begin{aligned} &\ll (Q(\mathbf{n}))^{-\beta} T \left(A^*_{\mathcal{D}}\left(Q(\mathbf{n})\right) - A^*_{\mathcal{D}}\left(Q(\mathbf{n})\left(1-\frac{1}{T}\right)\right)\right) \\ &\ll \ (Q(\mathbf{n}))^{1-\beta} + T(Q(\mathbf{n}))^{-\beta+\alpha}. \end{aligned} \tag{2.13}$$

Further (Case 2), for $0 \le j \le J$,

$$Q(\mathbf{n}) - Q(\mathbf{m}) \in [\delta_j,\ 2\delta_j[\quad \Longleftrightarrow \quad Q(\mathbf{n}) - 2\delta_j < Q(\mathbf{m}) \le Q(\mathbf{n}) - \delta_j,$$

thus the corresponding portion of the inner sum in (2.12) is

$$\begin{aligned} &\ll \frac{(Q(\mathbf{n}))^{1-\beta}}{\delta_j}\left(A^*_{\mathcal{D}}\left(Q(\mathbf{n})-\delta_j\right) - A^*_{\mathcal{D}}\left(Q(\mathbf{n})-2\delta_j\right)\right) \\ &\ll (Q(\mathbf{n}))^{1-\beta} + \delta_j^{-1}(Q(\mathbf{n}))^{1-\beta+\alpha}. \end{aligned}$$

Summing this over $j = 0, \dots, J$ gives

$$O\left(J(Q(\mathbf{n}))^{1-\beta}\right) + O\left(\delta_0^{-1}(Q(\mathbf{n}))^{1-\beta+\alpha}\right) \ll T^{\varepsilon}(Q(\mathbf{n}))^{1-\beta} + T(Q(\mathbf{n}))^{\alpha-\beta}. \tag{2.14}$$

Finally (Case 3), the portion of the inner sum in (2.12) corresponding to the $\mathbf{m}$'s with $Q(\mathbf{n}) - Q(\mathbf{m}) \ge 2\delta_J$ is

$$\ll \sum_{\mathbf{m}:\ Q(\mathbf{m}) \le Q(\mathbf{n})} (Q(\mathbf{m}))^{-\beta} = \int_{\frac{1}{2}}^{Q(\mathbf{n})} u^{-\beta}\, \mathrm{d}A^*_{\mathcal{D}}(u) \ll (Q(\mathbf{n}))^{1-\beta}. \tag{2.15}$$

We now combine the upper bounds (2.13), (2.14), and (2.15), and use them in (2.12) to conclude that

$$\begin{aligned} \left(\int_T^{2T} |S_X(t)|\, \mathrm{d}t\right)^2 \ &\ll\ T \sum_{Q(\mathbf{n}) \le X} (Q(\mathbf{n}))^{-\beta}\left(T^{\varepsilon}(Q(\mathbf{n}))^{1-\beta} + T(Q(\mathbf{n}))^{\alpha-\beta}\right) \\ &= T^{1+\varepsilon}\int_{\frac{1}{2}}^{X} w^{1-2\beta}\, \mathrm{d}A^*_{\mathcal{D}}(w)\ +\ T^2 \ll T^{1+\varepsilon}X^{2-2\beta} + T^2. \end{aligned}$$

Combining this with (2.11), we obtain

$$\int_T^{2T} |Z_{\mathcal{D}}(\beta + it)|\, \mathrm{d}t \ll T^{1/2+\varepsilon}X^{1-\beta} + T + X^{1-\beta} + T^2X^{-\beta+\alpha}.$$

It is easy to see that the choice of X according to (2.9) is optimal, and that the bound obtained is $\ll T^{1+\varepsilon}$ for $\beta \ge \frac{2+\alpha}{3}$. Thus the proof of Lemma 2 is complete. □

Lemma 3. *If* RH *is true, the function* $f_y(s)$ *defined in* (2.1) *satisfies*

$$f_y(\sigma + it) \ll y^{\frac{1}{2}-\sigma+\varepsilon'} \left(|t|^{\varepsilon'} + 1\right) \qquad (\varepsilon' > 0 \ \text{fixed}),$$

uniformly in $\sigma_1 \le \sigma \le \sigma_2$, $y \ge 1$, *for arbitrary* $\sigma_2 > \sigma_1 > \frac{1}{2}$.

Proof. This key lemma of the Montgomery-Vaughan method is meanwhile well-known. See, e.g., Nowak and Schmeier [**20**], or Baker [**2**, Lemma 1]. □

3. Proof of the Proposition.

We put

$$\beta := \frac{2+\alpha}{3} + \varepsilon'' \tag{3.1}$$

with $\varepsilon'' \ge 0$ as small as we please, such that [4]

$$\beta \notin \{\alpha_0, \dots, \alpha_R\}.$$

We start from Lemma 1 and shift the line of integration to $\mathrm{Re}\, s = \beta$, applying the residue theorem. In view of clause (i) of Lemma 2 and Lemma 3, the horizontal segments contribute

$$\ll x^{3-C} \int_\beta^3 \left| Z_{\mathcal{D}}\left(\sigma + ix^C\right) \, f_y(2\sigma + 2ix^C)\right| \, \mathrm{d}\sigma \ll x^{3-C+C(\omega+\varepsilon')} \ll \ 1$$

for C sufficiently large. Furthermore, by clause (ii) of Lemma 2 and Lemma 3,

$$\begin{aligned} &\int_{\beta-ix^C}^{\beta+ix^C} Z_{\mathcal{D}}\left(s\right) f_y(2s) x^s \frac{\mathrm{d}s}{s} \\ &\ll \ x^\beta y^{\frac{1}{2}-2\beta+\varepsilon'} \left(1 + \sum_{T=2^{-j}x^C, \ j=1,2,\dots} T^{\varepsilon'-1} \int_T^{2T} \left|Z_{\mathcal{D}}\left(\beta+it\right)\right| \, \mathrm{d}t\right) \\ &\ll \ x^{\beta+2C\varepsilon'} y^{\frac{1}{2}-2\beta+\varepsilon'}. \end{aligned}$$

Collecting results, we arrive at

$$S_2 = \sum_{r:\ \alpha_r > \beta} \mathrm{Res}_{\,s=\alpha_r} \left(Z_{\mathcal{D}}\left(s\right) f_y(2s) \frac{x^s}{s}\right) + O\left(x^{\alpha+\varepsilon'}\right) + O\left(x^\varepsilon y^{1/2} \left(\frac{x}{y^2}\right)^{\frac{2+\alpha}{3}}\right).$$

Since, by (2.8),

$$\mathrm{Res}_{\,s=\alpha_r} \left(Z_{\mathcal{D}}\left(s\right) f_y(2s) \frac{x^s}{s}\right) = c_r x^{\alpha_r} \sum_{m>y} \frac{\mu(m)}{m^{2\alpha_r}}$$

for $\alpha_r > \beta$, this completes the proof of the Proposition and thereby that of our Theorem.

[4] ε'' is only needed to deal with the case that $\frac{2+\alpha}{3}$ is equal to one of $\alpha_0, \dots, \alpha_R$.

4. Some applications to special problems.

4.1. Convex domains with nonzero curvature of the boundary. The most "generic" example is probably a convex planar domain $\mathcal{D}$ whose boundary $\partial\mathcal{D}$ is sufficiently smooth[5] and has nonzero curvature throughout. We further suppose that the origin is an inner point of $\mathcal{D}$. Under these conditions, a very deep and rather recent result of Huxley [6] says that

$$A_{\mathcal{D}}(x) = \mathrm{area}(\mathcal{D})x + O\left(x^{\frac{23}{73}} (\log x)^{\frac{315}{146}}\right) . \tag{4.1}$$

Using this with our Theorem, we obtain for the number of primitive lattice points in $\sqrt{x}\mathcal{D}$ (conditionally under RH),

$$B_{\mathcal{D}}(x) = \frac{6}{\pi^2}\mathrm{area}(\mathcal{D})x + O\left(x^{\frac{269}{619}+\varepsilon}\right).$$

However, for this special problem, Huxley and the author [7] have established the better error term $O\left(x^{\frac{5}{12}+\varepsilon}\right)$. (Numerically, $\frac{269}{619} = 0.434571\ldots$, while $\frac{5}{12} = 0.416666\ldots$.) This result does not depend on (4.1), but was derived using the mean-square bound

$$\int_0^x \left(A_{\mathcal{D}}\left(u\right) - \mathrm{area}(\mathcal{D})u\right)^2 \,\mathrm{d}u \ll x^{3/2}.$$

For the case that $\mathcal{D}$ is the unit disk (or any origin-centered rational ellipse), a recent idea of Baker [2] can be modified to prove (under RH) that

$$B_{\mathcal{D}}(x) = \frac{6}{\pi}x + O\left(x^{\frac{3}{8}+\varepsilon}\right) .$$

(See also [7] for a bit more details.)

4.2. Sums and differences of relative prime k-th powers. For a fixed natural number $k \ge 3$, we ask for the average order of the arithmetic functions $r_k^+(n)$, $r_k^-(n)$, and $\rho_k^+(n)$, $\rho_k^-(n)$, which are defined, respectively, by

$$r_k^{\pm}(n) := \{(u,v) \in \mathbb{Z}^2 :\ |u|^k \pm |v|^k = n\},$$
$$\rho_k^{\pm}(n) := \{(u,v) \in \mathbb{Z}^2 :\ |u|^k \pm |v|^k = n,\ \gcd(u,v) = 1\}.$$

From a geometric viewpoint, these functions are associated with the starlike planar domains

$$\mathcal{D}^{\pm} := \{(u,v) \in \mathbb{R}^2 :\ 0 < |u|^k \pm |v|^k \le 1\}.$$

[5]To be precise, it suffices that the curvature of $\partial\mathcal{D}$, as a function of the arclength, is twice continuously differentiable.

It is known from classic results of Krätzel [9], [10], [11] that

$$\sum_{1\le n\le T} r_k^{\pm}(n) = A_{\mathcal{D}^{\pm}}(T^{2/k}) = c_0^{\pm}(k)T^{2/k} + c_1^{\pm}(k)T^{1/(k-1)} + O\left(T^{\frac{1}{k}-\frac{1}{k^2}}\right), \tag{4.2}$$

with

$$c_0^{+}(k) = \frac{2\Gamma^2(\frac{1}{k})}{k\Gamma(\frac{2}{k})}, \qquad c_0^{-}(k) = \frac{\Gamma^2(\frac{1}{k})}{k\cos(\frac{\pi}{k})\Gamma(\frac{2}{k})},$$
$$c_1^{+}(k) = 0, \qquad c_1^{-}(k) = 4\zeta\left(\frac{1}{k-1}\right)k^{-1/(k-1)}.$$

Our Theorem readily implies (provided that RH is true)

$$\sum_{1\le n\le T} \rho_k^{\pm}(n) = B_{\mathcal{D}^{\pm}}(T^{2/k})$$
$$= \frac{6}{\pi^2}c_0^{\pm}(k)T^{2/k} + \frac{c_1^{\pm}(k)}{\zeta\left(\frac{k}{k-1}\right)}T^{1/(k-1)} + O\left(T^{\frac{7k+1}{k(7k+4)}+\varepsilon}\right).$$

Again the estimate can be improved slightly, making use of more precise representations of the error term in (4.2) (see [19]).

4.3. Primitive Pythagorean triangles. Let us define as a *primitive Pythagorean triangle* any triple of natural numbers (u, v, w) satisfying

$$u^2 + v^2 = w^2, \quad u \le v, \quad \gcd(u, v, w) = 1.$$

For a large real parameter $\mathcal{A}$, let $p(\mathcal{A})$ denote the number of primitive Pythagorean triangles with area less than $\mathcal{A}$. The problem to establish an asymptotic formula for $p(\mathcal{A})$ has been attacked by Lambek and Moser [12], Wild [22], Duttlinger and Schwarz [4], Müller, Nowak and Menzer [17], and Müller and Nowak [16]. According to Lambek and Moser [12], it is known that

$$p(\mathcal{A}) = \sum_{k=0}^{\infty}(-1)^k B_{\mathcal{D}}\left(\sqrt{\mathcal{A}}\,2^{-k}\right), \tag{4.3}$$

where

$$\mathcal{D} := \{(u, v) \in \mathbb{R}^2 : \; uv(u^2 - v^2) < 1, \; 0 < v < u\}.$$

In [16] it has been shown that

$$A_{\mathcal{D}}(x) = c_0 x + c_1 x^{2/3} + O\left(x^{7/22}(\log x)^{45/22}\right),$$

with

$$c_0 = \frac{\Gamma^2\left(\frac{1}{4}\right)}{4\sqrt{2\pi}}, \qquad c_1 = -\left|\zeta\left(\frac{1}{3}\right)\right| (1 + 2^{-1/3}).$$

Applying the most recent version of Huxley's lattice point theorems [6], it is straightforward to sharpen this error term to $O\left(x^{\frac{23}{73}} (\log x)^{\frac{315}{146}}\right)$. Using this with our Theorem and (4.3), one obtains (conditionally under RH)

$$p(\mathcal{A}) = c_0^* \mathcal{A}^{1/2} + c_1^* \mathcal{A}^{1/3} + O\left(\mathcal{A}^{\frac{269}{1238}+\varepsilon}\right)$$

with

$$c_0^* = \frac{\Gamma^2(\frac{1}{4})}{\sqrt{2\pi^5}}, \qquad c_1^* = -\frac{\left|\zeta\left(\frac{1}{3}\right)\right| \, (1 + 2^{-1/3})}{\zeta\left(\frac{4}{3}\right) \, (1 + 4^{-1/3})},$$

which improves upon all earlier results of this kind. (Numerically, $\frac{269}{1238} = 0.217285\ldots$, while the best exponent in the error term known before was $\frac{37}{164} = 0.225609\ldots$.)

4.4. Primitive lattice points in special asteroid-shaped domains. As a last somewhat "exotic" example we consider starlike sets

$$\mathcal{D}_a := \{(u, v) \in \mathbb{R}^2 : \ |u|^a + |v|^a \le 1\}$$

where a is a fixed real number with $0 < a < 1$. It was known already to van der Corput [3] that

$$A_{\mathcal{D}_a}(x) = \mathrm{area}\,(\mathcal{D}_a)\, x + \sum_{1 \le r < \frac{1}{a}(1-2\lambda)} c_r(a) x^{(1-ar)/2} \ + \ O\left(x^{\lambda}\right),$$

with

$$c_r(a) = \frac{8(-1)^r \zeta(-ar)\, \Gamma\left(1 + \frac{1}{a}\right)}{r!\, \Gamma\left(1 + \frac{1}{a} - r\right)},$$

for $\lambda = \frac{1}{3}$. (Cf. also [18] for a generalization.) Appealing again to Huxley's work [6], this can be readily established for every $\lambda > \frac{23}{73}$. Thus our Theorem implies that (if RH is true)

$$B_{\mathcal{D}_a}(x) = \frac{6}{\pi^2}\, \mathrm{area}\,(\mathcal{D}_a)\, x + \sum_{1 \le r < \frac{81}{619a}} \frac{c_r(a)}{\zeta(1-ar)} x^{(1-ar)/2} \ + \ O\left(x^{\frac{269}{619}+\varepsilon}\right).$$

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