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DISTORTION THEOREMS FOR BLOCH FUNCTIONS

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A function f holomorphic on the unit disk $\mathbb{D}$ is called a Bloch function if

$$\|f\|_B = \sup\{(1-|z|^2)|f'(z)| : z \in \mathbb{D}\} < \infty.$$

For $\alpha \in [0,1]$ let $B_1(\alpha)$ denote the class of Bloch functions which have the normalization $\|f\|_B \le 1$, $f(0) = 0$ and $f'(0) = \alpha$. A type of subordination theorem is established for $B_1(\alpha)$. This theorem yields numerous sharp growth, distortion, curvature and covering theorems for $B_1(\alpha)$.

1. Introduction.

In an earlier paper [**BMY**] the authors used a type of subordination theorem to systematically derive a number of known and new results for normalized locally univalent Bloch functions. This paper has a similar theme. We show that there is an analogous subordination theorem (Theorem 1) for normalized (not necessarily locally univalent) Bloch functions. This subordination result enables us to obtain some known results from a unified perspective and also leads to new results.

Let us introduce some notation and terminology. The unit disk in the complex plane is denoted by $\mathbb{D}$. For a function f holomorphic on $\mathbb{D}$ the Bloch seminorm is given by

$$\|f\|_B = \sup\{(1-|z|^2)|f'(z)| : z \in \mathbb{D}\},$$

and f is called a Bloch function when $\|f\|_B < \infty$. Normalized classes of Bloch functions are

$$B_1 = \{f \text{ holomorphic on } \mathbb{D} : f(0) = 0, f'(0) > 0 \text{ and } \|f\|_B \le 1\}$$

and

$$B_1(\alpha) = \{f \in B_1 : f'(0) = \alpha\}.$$

The normalization $\|f\|_B \le 1$ requires that $\alpha \in [0,1]$. Also, B_1 is the disjoint union of the classes $B_1(\alpha)$ as α ranges over $[0,1]$.

There are a number of parallels between our former paper [**BMY**] and this one. But it is interesting that a complete parallelism does not hold. For instance, here we obtain some results (such as the radius of starlikeness) which do not have analogs in [**BMY**]. Sometimes analogous results hold but the proofs are different. For example, determining the variability region for $\log f'(z)$ in the class $B_\infty(\alpha) = \{f \in B_1(\alpha) : f$ is locally univalent$\}$ was fairly direct in [**BMY**]. The analogous problem of finding the variability region for $f'(z)$ in the class $B_1(\alpha)$ involves much more computation. Here we only prove a partial result (Theorem 2) which is sufficient for the subsequent applications. Another difference is illustrated by the problem of minimizing $|f(z)|$ over $B_1(\alpha)$ or $B_\infty(\alpha)$. We completely solve this for $B_1(\alpha)$ while the problem for $B_\infty(\alpha)$ was only partially resolved in [**BMY**].

Several previous papers dealt with sharp growth, distortion, curvature and covering theorems for Bloch functions. A number of results of this type were obtained by Bonk ([$\mathbf{B}_1$], [$\mathbf{B}_2$]). In particular, he found the sharp lower bound on $\operatorname{Re} f'(z)$ for $f \in B_1(1)$. A geometric method for obtaining this result was given by Minda [**M**]. Liu and Minda [**LM**] extended this geometric approach to locally univalent Bloch functions and to other classes of Bloch functions between $B_1(\alpha)$ and $B_\infty(\alpha)$. The subordination method of this paper unifies these results and gives new ones.

2. Preliminaries.

We recall some basic facts together with terminology and notation that will be needed.

We begin by introducing two invariant differential operators. Suppose f is holomorphic on $\mathbb{D}$. Then $D_j f (j = 1, 2)$ is defined by

$$D_1 f(z) = (1 - |z|^2) f'(z),$$
$$D_2 f(z) = (1 - |z|^2)^2 f''(z) - 2\bar{z}(1 - |z|^2) f'(z).$$

If $a \in \mathbb{D}$ and $T(z) = (z + a)/(1 + \bar{a}z)$, then T is a conformal automorphism of $\mathbb{D}$ sending 0 to a and $D_j f(a) = (f \circ T)^{(j)}(0)(j = 1, 2)$. In particular, $D_j f(0)$ is the ordinary jth derivative of f at the origin. These differential operators have the invariance property

$$|D_j(S \circ f \circ T)| = |D_j f| \circ T \quad (j = 1, 2),$$

where S is any euclidean motion of $\mathbb{C}$ and T is any conformal automorphism of $\mathbb{D}$.

The Bloch seminorm is given by $\|f\|_B = \sup\{|D_1 f(z)| : z \in \mathbb{D}\}$. The invariance property implies that $\|f \circ T\|_B = \|f\|_B$ for any conformal automorphism T of $\mathbb{D}$. For $z \in \mathbb{D}$ let $r(z, f)$ denote the radius of the largest schlicht disk centered at $f(z)$ in the Riemann image surface of f viewed as spread over the complex plane. The function f is Bloch if and only if $\sup\{r(z,f) : z \in \mathbb{D}\} < \infty$.

It is convenient to let $D_e(a,r) = \{z : |z-a| < r\}$ denote the euclidean disk with a center a and radius r.

Next, we recall basic facts about hyperbolic geometry on the unit disk. The hyperbolic metric on $\mathbb{D}$ is $\lambda_{\mathbb{D}}(z)|dz| = |dz|/(1-|z|^2)$. It is invariant under conformal automorphisms of $\mathbb{D}$; that is,

$$\lambda_{\mathbb{D}}(T(z))|T'(z)| = \lambda_{\mathbb{D}}(z),$$

or

$$\frac{|T'(z)|}{1-|T(z)|^2} = \frac{1}{1-|z|^2},$$

for any conformal automorphism T of $\mathbb{D}$. The associated hyperbolic distance function on $\mathbb{D}$ is

$$d_{\mathbb{D}}(a,b) = \frac{1}{2}\log\left\{\frac{1+\left|\frac{a-b}{1-\overline{a}b}\right|}{1-\left|\frac{a-b}{1-\overline{a}b}\right|}\right\} = \operatorname{artanh}\left|\frac{a-b}{1-\overline{a}b}\right|.$$

The hyperbolic disk (circle) with center a and radius $r > 0$ is $D_{\mathbb{D}}(a,r) = \{z : d_{\mathbb{D}}(a,z) < r\}(C_{\mathbb{D}}(a,r) = \{z : d_{\mathbb{D}}(a,z) = r\})$. Hyperbolic disks and circles in $\mathbb{D}$ are actually euclidean disks and circles in $\mathbb{D}$ with possibly different center and radius. A horocycle Γ in $\mathbb{D}$ based at $\lambda \in \partial\mathbb{D}$ is a euclidean circle in $\overline{\mathbb{D}}$ which is tangent to the unit circle at λ. The interior of a horocycle is called a horodisk. For $a \in \mathbb{D}$ the following are readily seen to be equivalent:

$$\begin{aligned} d_{\mathbb{D}}(a,r) &= r, \\ \left|\frac{z-a}{1-\overline{a}z}\right| &= \tanh(r), \\ \frac{|1-\overline{a}z|^2}{1-|z|^2} &= \frac{1-|a|^2}{1-\tanh^2(r)}. \end{aligned}$$

Hyperbolic geometry is transformed to any simply connected region $\Omega \neq \mathbb{C}$ as follows. The hyperbolic metric $\lambda_\Omega(w)|dw|$ on Ω is determined from

$$\lambda_\Omega(f(z))|f'(z)| = \lambda_{\mathbb{D}}(z),$$

where $f : \mathbb{D} \to \Omega$ is a conformal mapping. It is independent of the choice of the conformal mapping of $\mathbb{D}$ onto Ω. For $A, B \in \Omega$ the hyperbolic distance between them is

$$d_\Omega(A,B) = \inf \int_\gamma \lambda_\Omega(w)|dw|,$$

where the infimum is taken over all paths γ in Ω joining A and B. A conformal mapping $f : \mathbb{D} \to \Omega$ is an isometry relative to hyperbolic distance: $d_\Omega(f(a), f(b)) = d_{\mathbb{D}}(a, b)$. Hyperbolic disks (circles) in Ω are defined by $D_\Omega(a, r) = \{z : d_\Omega(a, z) < r\} (C_\Omega(a, r) = \{z : d_\Omega(a, z) = r\})$. Typically these are not euclidean disks (circles).

Henceforth, we generally use d_h to denote the hyperbolic distance and $D_h(a, r)$ the hyperbolic disk with center a and radius r when reference to the specific region Ω is clear. In fact, we usually use this notation when $\Omega = \mathbb{D}$.

When $D = D_e(a, r)$ is a euclidean disk, then hyperbolic geometry on D is simple to understand. The function $f(z) = a + rz$ is a conformal map of $\mathbb{D}$ onto D; this is a stretching followed by a translation. The hyperbolic metric on D is

$$\lambda_D(w)|dw| = \frac{r|dw|}{r^2 - |w - a|^2}.$$

Hyperbolic circles and horocycles in D are euclidean circles.

The final topic is curvature. The euclidean curvature of a path $\Gamma : w = w(t)$ in $\mathbb{C}$ at $w = w(t)$ is

$$\kappa_e(w, \Gamma) = \frac{1}{|w'(t)|} \ \mathrm{Im} \left\{ \frac{w''(t)}{w'(t)} \right\}.$$

If f is holomorphic in $\mathbb{D}$ and $f'(z) \neq 0$, then the euclidean curvature of the image path $f \circ \gamma$ at $f(z)$, where $\gamma : |z| = r$, is

$$\kappa_e(f(z), f \circ \gamma) = \frac{1 + \mathrm{Re} \left\{ \frac{zf''(z)}{f'(z)} \right\}}{|zf'(z)|}.$$

3. Extremal functions.

We present basic facts about certain two-sheeted branched coverings of $\mathbb{D}$ onto other disks. These functions are extremal for all of the results in this paper.

The function

$$F(z) = -\frac{3\sqrt{3}}{4} z^2$$

is a two-sheeted branched covering of $\mathbb{D}$ onto $D_e(0, \frac{3\sqrt{3}}{4})$ with $F(0) = 0$, $F'(0) = 0$ and $\|F\|_B = 1$. The latter holds since

$$(1 - |z|^2)|F'(z)| = \frac{3\sqrt{3}}{2} |z|(1 - |z|^2) = M(|z|),$$

where

$$M(t) = \frac{3\sqrt{3}}{2} t(1-t^2)$$

is increasing on $[0, 1/\sqrt{3}]$, decreasing on $[1/\sqrt{3}, 1]$ and $M(1/\sqrt{3}) = 1$. Thus for $z \in \mathbb{D}$, $(1-|z|^2)|F'(z)| \leq 1$ with equality if and only if $|z| = 1/\sqrt{3}$. Note that $(1-|z|^2)|F'(z)|$ is constant on circles centered at the origin and F maps $[0, 1)$ decreasingly onto $(-3\sqrt{3}/4, 0]$ and $(-1, 0]$ increasingly onto $(-3\sqrt{3}/4, 0]$.

Henceforth, we let $m : [0, 1] \to [0, 1/\sqrt{3}]$ be the inverse function for the restriction of M to the interval $[0, 1/\sqrt{3}]$. The function m is increasing with $m(0) = 0$, $m(1) = 1/\sqrt{3}$ and

$$\frac{3\sqrt{3}}{2} m(\alpha)(1 - m^2(\alpha)) = \alpha$$

for $\alpha \in [0, 1]$.

A whole class of extremal functions is obtained by precomposing F with certain conformal automorphisms of $\mathbb{D}$ and then normalizing the function at the origin. For $a \in (-1, 1)$ the function

$$T_a(z) = \frac{z-a}{1-az}$$

is a conformal automorphism of $\mathbb{D}$ so

$$F \circ T_a(z) = -\frac{3\sqrt{3}}{4} \left(\frac{z-a}{1-az} \right)^2$$

has Bloch seminorm 1. From

$$(F \circ T_a)'(z) = \frac{3\sqrt{3}}{2} \frac{(1-a^2)(a-z)}{(1-az)^3}$$

we obtain

$$(F \circ T_a)'(0) = \frac{3\sqrt{3}}{2} a(1-a^2) = M(a).$$

For each $\alpha \in [0, 1]$ there is a unique $a \in [0, 1/\sqrt{3}]$ with $M(a) = \alpha$; in fact, $a = m(\alpha)$. We define

$$F_\alpha(z) = F \circ T_{m(\alpha)}(z) - F \circ T_{m(\alpha)}(0)$$

$$= \frac{3\sqrt{3}}{4}\left[m^2(\alpha) - \left(\frac{z - m(\alpha)}{1 - m(\alpha)z}\right)^2\right].$$

The function F_α belongs to $B_1(\alpha)$ and is a two-sheeted branched covering of $\mathbb{D}$ onto $D_e\left(\frac{3\sqrt{3}}{4}m^2(\alpha), \frac{3\sqrt{3}}{4}\right)$ with

$$F'_\alpha(z) = \frac{3\sqrt{3}}{2}\frac{(1 - m^2(\alpha))(m(\alpha) - z)}{(1 - m(\alpha)z)^3}.$$

Note that $F'_\alpha(m(\alpha)) = 0, F_\alpha$ is increasing on $[0, m(\alpha)]$, maps this interval onto $[0, \frac{3\sqrt{3}}{4}m^2(\alpha)]$, decreasing on $\left[m(\alpha), \frac{2m(\alpha)}{1+m^2(\alpha)}\right]$ and maps this interval onto $\left[0, \frac{3\sqrt{3}}{4}m^2(\alpha)\right]$. Next,

$$F'_\alpha(z) = F'(T_{m(\alpha)}(z))T'_{m(\alpha)}(z),$$

and since

$$|T'_{m(\alpha)}(z)| = \frac{1 - |T_{m(\alpha)}(z)|^2}{1 - |z|^2},$$

we get

$$(1 - |z|^2)|F'_\alpha(z)| = (1 - |T_{m(\alpha)}(z)|^2)|F'(T_{m(\alpha)}(z))|.$$

Consequently, $(1 - |z|^2)|F'_\alpha(z)| \le 1$ with equality if and only if $|T_{m(\alpha)}(z)| = 1/\sqrt{3}$; that is, $d_h(m(\alpha), z) = \operatorname{artanh}(1/\sqrt{3})$. Also, $(1-|z|^2)|F'_\alpha(z)|$ is constant on hyperbolic circles centered at $m(\alpha)$. We note that $F_0 = F$.

For future reference we record several observations. Set $\Delta(\alpha) = D_h(m(\alpha), \operatorname{artanh}(1/\sqrt{3}))$. In euclidean terms $\Delta(\alpha) = D_e\left(\frac{2m(\alpha)}{3-m^2(\alpha)}, \frac{\sqrt{3}(1-m^2(\alpha))}{3-m^2(\alpha)}\right)$. The boundary of $\Delta(\alpha)$ meets the real axis in the points $-\frac{1-\sqrt{3}m(\alpha)}{\sqrt{3}-m(\alpha)} \in [-1/\sqrt{3}, 0]$ and $\frac{1+\sqrt{3}m(\alpha)}{\sqrt{3}+m(\alpha)} \in [1/\sqrt{3}, \sqrt{3}/2]$. The function $T_{m(\alpha)}$ maps $\Delta(\alpha)$ conformally onto $D_e(0, 1/\sqrt{3})$. For any unimodular constant λ

$$\Delta_\lambda(\alpha) = D_h(\lambda m(\alpha), \operatorname{artanh}(1/\sqrt{3})) = \{\lambda z : z \in \Delta(\alpha)\}$$

is the rotation of $\Delta(\alpha)$ through the angle $\arg \lambda$.

The functions F_α satisfy a differential identity.

Lemma 1. *For $\alpha \in [0, 1]$ and $z \in \overline{\Delta(\alpha)}$*

$$|D_2F_\alpha(z)| = \frac{3\sqrt{3}}{2}[1 - m^2(|D_1F_\alpha(z)|)][1 - 3m^2(|D_1F_\alpha(z)|)].$$

Proof. Since $F_\alpha = S \circ F \circ T_{m(\alpha)}$, where S is a euclidean motion, $|D_j F_\alpha(z)| = |D_j F|(T_{m(\alpha)}(z))$ $(j = 1, 2)$. Because $T_{m(\alpha)}$ maps $\overline{\Delta(\alpha)}$ onto $\overline{\Delta(0)}$, it suffices to establish the result when $\alpha = 0$. Now,

$$D_1 F_0(z) = -\frac{3\sqrt{3}}{2} z(1 - |z|^2),$$

so $|D_1 F_0(z)| = M(|z|)$. This gives $m(|D_1 F_0(z)|) = |z|$ when $|z| < 1/\sqrt{3}$. Since

$$D_2 F_0(z) = \frac{3\sqrt{3}}{2}(1 - |z|^2)(3|z|^2 - 1),$$

we have

$$\begin{aligned} |D_2 F_0(z)| &= \frac{3\sqrt{3}}{2}(1 - |z|^2)(1 - 3|z|^2) \\ &= \frac{3\sqrt{3}}{2}[1 - m^2(|D_1 F_0(z)|)][1 - 3m^2(|D_1 F_0(z)|)] \end{aligned}$$

for $|z| < 1/\sqrt{3}$. □

There is an important auxiliary function associated with each F_α. For $\alpha \in [0, 1]$ set

$$\begin{aligned} G_\alpha(z) &= (1 - m(\alpha)z)^2 F'_\alpha(z) \\ &= \frac{3\sqrt{3}(1 - m^2(\alpha))}{2} \frac{m(\alpha) - z}{1 - m(\alpha)z} \\ &= -\frac{3\sqrt{3}(1 - m^2(\alpha))}{2} T_{m(\alpha)}(z). \end{aligned}$$

The function G_α is a Möbius transformation. For $d_h(m(\alpha), z) = r, |G_\alpha(z)| = \frac{3\sqrt{3}}{2}(1 - m^2(\alpha)) \tanh(r)$. Therefore, G_α is a conformal mapping of $D_h(m(\alpha), r)$ onto $D_e(0, \frac{3\sqrt{3}}{2}(1 - m^2(\alpha)) \tanh(r))$. In particular, G_α maps $\Delta(\alpha)$ conformally onto $D_e(0, \frac{3}{2}(1 - m^2(\alpha)))$. Observe that G_α is decreasing on $\Delta(\alpha) \cap \mathbb{R}$ with

$$\begin{aligned} G_\alpha\left(-\frac{1 - \sqrt{3}m(\alpha)}{\sqrt{3} - m(\alpha)}\right) &= \frac{3}{2}(1 - m^2(\alpha)), \\ G_\alpha\left(\frac{1 + \sqrt{3}m(\alpha)}{\sqrt{3} + m(\alpha)}\right) &= -\frac{3}{2}(1 - m^2(\alpha)). \end{aligned}$$

4. Subordination theorem.

We make use of a slight variant of the customary notion of subordination. Suppose both k and K are holomorphic on $\mathbb{D}$ with $k(0) = K(0)$ and Δ is an open disk in $\mathbb{D}$ with $0 \in \Delta$. We say k is subordinate to K on Δ relative to the origin, written $k \prec_0 K$, if there is a holomorphic function φ defined on Δ with $\varphi(\Delta) \subset \Delta$, $\varphi(0) = 0$ and $K \circ \varphi(z) = k(z)$ for $z \in \Delta$. If D is any hyperbolic disk (relative to hyperbolic geometry on Δ) with center 0, then $k \prec_0 K$ implies $k(D) \subset K(D)$ since the function φ must map D into itself. If K is univalent on Δ and a point of ∂D is sent by k to a point of $K(\partial D)$, then φ must be a conformal automorphism of D which fixes 0. If the function φ fixes any point of Δ distinct from the origin, then φ is the identity function. A variant of Schwarz' Lemma implies that $|\varphi'(0)| \leq 1$ with equality if and only if φ is a conformal automorphism of Δ fixing the origin. In particular, $\varphi'(0) = 1$ if and only if φ is the identity function.

We also need a second variation of subordination. In this situation Δ is an open disk in $\mathbb{D}$ with $0 \in \partial\Delta$. In this context we write $k \prec_0 K$ on Δ if there is a holomorphic function φ defined on $\Delta \cup \{0\}$ with $\varphi(\Delta) \subset \Delta$, $\varphi(0) = 0$ and $K \circ \varphi(z) = k(z)$ for $z \in \Delta \cup \{0\}$. Julia's Lemma applied to the disk Δ and boundary point 0 yields $\varphi'(0) > 0$. In this situation we cannot conclude $|\varphi'(0)| \leq 1$ as when 0 is an interior point of Δ. But whenever we employ this type of subordination we will always have the additional hypothesis that $k'(0) = K'(0) \neq 0$, so $\varphi'(0) = 1$ is always valid by the chain rule. But now $\varphi'(0) = 1$ does not necessarily imply that φ is the identity function. For $\varphi'(0) = 1$ Julia's Lemma implies that if D is any horodisk (relative to hyperbolic geometry on Δ) based at the origin (in other words, D is an open euclidean disk which is internally tangent to $\partial\Delta$ at the origin), then $\varphi(D) \subset D$ and if φ maps one boundary point of D (other than the origin) to another boundary point of D, then φ is a conformal automorphism of Δ fixing the boundary point 0. If φ fixes an interior point of Δ, then φ is the identity function.

Theorem 1. *Let $f \in B_1(\alpha)$. Then for any unimodular constant λ*

$$\begin{aligned}(1 - m(\alpha)\overline{\lambda}z)^2 f'(z) &\prec_0 (1 - m(\alpha)\overline{\lambda}z)^2 F_\alpha'(\overline{\lambda}z) \\ &= \frac{3\sqrt{3}(1 - m^2(\alpha))}{2} \cdot \frac{m(\alpha) - \overline{\lambda}z}{1 - m(\alpha)\overline{\lambda}z}\end{aligned}$$

on $\Delta_\lambda(\alpha)$.

Proof. We need only establish this result in the case $\lambda = 1$ since the general case follows from applying this case to the function $\overline{\lambda}f(\lambda z)$ which also belongs to $B_1(\alpha)$.

Set $g(z) = (1 - m(\alpha)z)^2 f'(z)$. Then $g(0) = \alpha$. For $d_h(m(\alpha), z) = r$,

$$\begin{aligned} |g(z)| &= \frac{|1 - m(\alpha)z|^2}{1 - |z|^2}(1 - |z|^2)|f'(z)| \\ &\leq \frac{|1 - m(\alpha)z|^2}{1 - |z|^2} = \frac{1 - m^2(\alpha)}{1 - \tanh^2(r)}. \end{aligned}$$

This shows that g maps $D_h(m(\alpha), R)$ into $D_e\left(0, \frac{1-m^2(\alpha)}{1-\tanh^2(R)}\right)$ for $0 < R < \infty$. We know that G_α is a conformal map of $D_h(m(\alpha), R)$ onto the disk $D_e\left(0, \frac{3\sqrt{3}}{2}(1 - m^2(\alpha))\tanh(R)\right)$. Since $g(0) = G_\alpha(0) = \alpha$, the relation $g \prec_0 G_\alpha$ will hold on $D_h(m(\alpha), R)$ when

$$\frac{3\sqrt{3}}{2}\tanh(R) = \frac{1}{1 - \tanh^2(R)};$$

that is, when $M(\tanh(R)) = 1$. This holds precisely when $\tanh(R) = 1/\sqrt{3}$, or

$$R = \operatorname{artanh}(1/\sqrt{3}) = \log\left(\frac{1 + \sqrt{3}}{2}\right) = 0.658478\ldots.$$

Note that $f \in B_1(1)$ implies $f''(0) = 0$. It follows that in case $\alpha = 1$ we have $g'(0) = G_1'(0) = -2/\sqrt{3} \neq 0$ in addition to $g \prec_0 G_1$. □

Remark. The relationship $g \prec_0 G_\alpha$ fails to hold on any larger hyperbolic disk centered at $m(\alpha)$. This can be seen as follows. Recall that $\frac{1+\sqrt{3}m(\alpha)}{\sqrt{3}+m(\alpha)}$ is a boundary point of $\Delta(\alpha)$. For $\frac{1+\sqrt{3}m(\alpha)}{\sqrt{3}+m(\alpha)} \leq r < 1$ the boundary of the variability region $\{f'(r) : f \in B_1(\alpha)\}$ is the circle $\{w : |w| = \frac{1}{1-r^2}\}$ [**B**1, Satz 2.2.1]. Thus, for each r with $\frac{1+\sqrt{3}m(\alpha)}{\sqrt{3}+m(\alpha)} \leq r < 1$ there is a function $f \in B_1(\alpha)$ with $(1 - r^2)|f'(r)| = 1$. For such r and an associated function $f \in B_1(\alpha)$,

$$\begin{aligned} |g(r)| &= (1 - m(\alpha)r)^2|f'(r)| \\ &= \frac{(1 - m(\alpha)r)^2}{1 - r^2} = \frac{1 - m^2(\alpha)}{1 - \tanh^2(r)}. \end{aligned}$$

Since G_α maps $D_h(m(\alpha), r)$ conformally onto $D_e(0, \frac{3\sqrt{3}}{2}(1 - m^2(\alpha))\tanh(r))$ and

$$\frac{1 - m^2(\alpha)}{1 - \tanh^2(r)} > \frac{3\sqrt{3}}{2}(1 - m^2(\alpha))\tanh(r)$$

for $1/\sqrt{3} < \tanh(r)$ (because $M(t) < 1$ for $1/\sqrt{3} < t < 1$), it follows that $g \prec_0 G_\alpha$ cannot hold on $D_h(m(\alpha), r)$ for any $r > \operatorname{artanh}(1/\sqrt{3})$.

5. Applications of subordination.

We begin by using direct consequences of subordination to establish sharp growth, distortion and covering theorems for the classes $B_1(\alpha), \alpha \in [0,1]$.

Theorem 2. *Suppose* $f_1 \in B_1(\alpha), \alpha \in [0,1]$.

(i) *For* $|z| \leq \frac{1+\sqrt{3}m(\alpha)}{\sqrt{3}+m(\alpha)}$

$$\operatorname{Re} f'(z) \geq F'_\alpha(|z|) = \frac{3\sqrt{3}}{2}\frac{(1-m^2(\alpha))(m(\alpha)-|z|)}{(1-m(\alpha)|z|)^3}$$

with equality at $z = \mathrm{re}^{i\theta}$, $r \in \left(0, \frac{1+\sqrt{3}m(\alpha)}{\sqrt{3}+m(\alpha)}\right)$ *if and only if* $f(z) = e^{i\theta}F_\alpha(e^{-i\theta}z)$. *In particular, for* $|z| \leq m(\alpha)$

$$|f'(z)| \geq F'_\alpha(|z|) \geq 0$$

with equality at $z = \mathrm{re}^{i\theta}$ *as above.*

(ii) *For* $z \in \mathbb{D}$

$$|f'(z)| \leq \begin{cases} F'_\alpha(-|z|) = \frac{3\sqrt{3}}{2}\frac{(1-m^2(\alpha))(m(\alpha)+|z|)}{(1+m(\alpha)|z|)^3} & \text{if } |z| < \frac{1-\sqrt{3}m(\alpha)}{\sqrt{3}-m(\alpha)}, \\ \frac{1}{1-|z|^2} & \text{if } \frac{1-\sqrt{3}m(\alpha)}{\sqrt{3}-m(\alpha)} \leq r < 1. \end{cases}$$

Equality holds at $z = -\mathrm{re}^{i\theta}, r \in \left(0, \frac{1-\sqrt{3}m(\alpha)}{\sqrt{3}-m(\alpha)}\right)$, *if and only if* $f(z) = e^{i\theta}F_\alpha(e^{-i\theta}z)$.

Proof. Set $g(z) = (1-m(\alpha)z)^2 f'(z)$. The proof below is valid when $\alpha \in [0,1)$; the simple modification required when $\alpha = 1$ in case (i) is indicated.

(i) By making use of the rotational invariance of the class $B_1(\alpha)$, it suffices to establish (i) when $z = x \in \left(0, \frac{1+\sqrt{3}m(\alpha)}{\sqrt{3}+m(\alpha)}\right)$ and show that equality holds if and only if $f = F_\alpha$. For x so restricted, inequality (i) will follow from

$$\operatorname{Re}\ g(x) \geq G_\alpha(x)$$

with equality if and only if $g = G_\alpha$.

We now establish this result for g. Let δ_x be the hyperbolic circle (relative to hyperbolic geometry on $\Delta(\alpha)$) with center 0 which passes through x. Since $g \prec_0 G_\alpha$ on $\Delta(\alpha)$, g maps the circle δ_x into the closed disk bounded by the circle $G_\alpha(\delta_x)$. Note that $G_\alpha(\delta_x)$ is a hyperbolic circle (relative to hyperbolic geometry on $D_e\left(0, \frac{3}{2}(1-m^2(\alpha))\right)$) with hyperbolic center $G_\alpha(0) = \alpha$ and is symmetric about $\mathbb{R}$. Since G_α is decreasing on $\Delta(\alpha) \cap \mathbb{R}$, the point of $G_\alpha(\delta_x)$ with the smallest real part is $G_\alpha(x)$. Consequently, Re $g(x) \geq G_\alpha(x)$ and if equality holds we must actually have $g(x) = G_\alpha(x)$. But then $g = G_\alpha \circ \varphi$

implies $\varphi(x) = x$ and so φ is the identity function since it fixes both 0 and x. Thus, equality implies $g = G_\alpha$.

A simple modification must be made in the case $\alpha = 1$. The hyperbolic circle δ_x must be replaced by the horocycle (relative to hyperbolic geometry on $\Delta(\alpha)$) based at the origin that passes through x. In other words, when $\alpha = 1$ we take δ_x to be the circle through 0 and x which is symmetric about the real axis.

(ii) The second part of the inequality in (ii) is trivial since $\|f\|_B \le 1$. Actually, it is best possible as follows from the determination of the variability region for $f'(z)$ for the class $B_1(\alpha)$ [**B**$_1$, Satz 2.2.1]. Precisely, for $\frac{1-\sqrt{3}m(\alpha)}{\sqrt{3}-m(\alpha)} \le |z_0| < 1$, there exists $f \in B_1(\alpha)$ with $|f'(z_0)| = 1/(1-|z_0|^2)$.

Now, we establish the first inequality in (ii). There is nothing to prove when $\alpha = 1$, so we assume $\alpha \in [0,1)$. It suffices to prove the inequality for $z = -x, x \in \left(0, \frac{1-\sqrt{3}m(\alpha)}{\sqrt{3}-m(\alpha)}\right)$, and show that equality holds if and only if $f = F_\alpha$. This will follow from showing

$$|g(-x)| \le G_\alpha(-x)$$

for $x \in \left(0, \frac{1-\sqrt{3}m(\alpha)}{\sqrt{3}-m(\alpha)}\right)$ with equality if and only if $g = G_\alpha$.

The proof is similar to that of the inequality in part (i). Note that $-x \in \Delta(\alpha)$ and let δ_{-x} be the hyperbolic circle (relative to hyperbolic geometry on $\Delta(\alpha)$) with center 0 which passes through $-x$. As $g \prec_0 G_\alpha$ on $\Delta(\alpha)$, g maps the circle δ_{-x} into the closed disk bounded by the circle $G_\alpha(\delta_{-x})$ which is a hyperbolic circle (relative to hyperbolic geometry on $D_e\left(0, \frac{3}{2}(1-m^2(\alpha))\right)$) with hyperbolic center $G_\alpha(0) = \alpha \in [0,1)$. Since $D_e\left(0, \frac{3}{2}(1-m^2(\alpha))\right)$ is centered at the origin and the hyperbolic center of $G_\alpha(\delta_{-x})$ is nonnegative, it follows that the euclidean center of $G_\alpha(\delta_{-x})$ is also nonnegative. As G_α is decreasing on $\Delta(\alpha) \cap \mathbb{R}$ and $G_\alpha(\delta_{-x})$ is symmetric about $\mathbb{R}$, we conclude that for all w in the closed disk bounded by $G_\alpha(\delta_{-x})$, $|w| \le G_\alpha(-x)$ with equality if and only if $w = G_\alpha(-x)$. Thus, $|g(-x)| \le G_\alpha(-x)$ and equality forces $g(-x) = G_\alpha(-x)$. As in the proof of part (i), this implies $g = G_\alpha$. □

Corollary 1. *Suppose* $f \in B_1(\alpha)$.

(i) *For* $|z| \le \frac{2m(\alpha)}{1+m^2(\alpha)}$, $0 \le F_\alpha(|z|) \le |f(z)|$ *with equality at* $z = \mathrm{r}e^{i\theta}, r \in \left(0, \frac{2m(\alpha)}{1+m^2(\alpha)}\right]$, *if and only if* $f(z) = e^{i\theta}F_\alpha(e^{-i\theta}z)$. *In particular,* $f(z) \ne 0$ *for* $0 < |z| < \frac{2m(\alpha)}{1+m^2(\alpha)}$.

(ii) *For* $|z| \le \frac{1-\sqrt{3}m(\alpha)}{\sqrt{3}-m(\alpha)}$, $|f(z)| \le -F_\alpha(-|z|)$ *with equality at* $z = -\mathrm{r}e^{i\theta}, r \in \left(0, \frac{1-\sqrt{3}m(\alpha)}{\sqrt{3}-m(\alpha)}\right]$, *if and only if* $f(z) = e^{i\theta}F_\alpha(e^{-i\theta}z)$.

Proof. (i) It is sufficient to consider $z = x \in \left(0, \frac{2m(\alpha)}{1+m^2(\alpha)}\right]$ and show that equality forces $f = F_\alpha$. By using part (i) of the theorem, we obtain

$$\begin{aligned} |f(x)| \geq \text{Re } f(x) &= \int_0^x \text{Re } f'(t)dt \\ &\geq \int_0^x F'_\alpha(t)dt = F_\alpha(x). \end{aligned}$$

Equality implies Re $f'(t) = F'_\alpha(t)$ for $0 \leq t \leq x$ and so $f = F_\alpha$.
(ii) As usual, it is enough to consider $z = -x$, where $x \in \left(0, \frac{1-\sqrt{3}m(\alpha)}{\sqrt{3}-m(\alpha)}\right]$, and prove that equality implies $f = F_\alpha$. This follows by integrating the inequality in part (ii) of the theorem:

$$\begin{aligned} |f(-x)| = \left|\int_0^x f'(-t)dt\right| &\leq \int_0^x |f'(-t)|dt \\ &\leq \int_0^x F'_\alpha(-t)dt = -F_\alpha(-x). \end{aligned}$$

Equality forces $|f'(-t)| = F'_\alpha(-t)$, $0 \leq t \leq x$, and so $f = F_\alpha$. □

Corollary 2. *The radius of univalence for $B_1(\alpha)$ is $m(\alpha)$ which is also the radius of bounded turning. More precisely, if $\alpha \in (0,1]$ and $f \in B_1(\alpha)$, then f is univalent in $D_e(0,r)$ for some $r > m(\alpha)$ unless $f(z) = e^{i\theta}F_\alpha(e^{-i\theta}z)$ for some $\theta \in \mathbb{R}$.*

Proof. Recall that f is said to be of bounded turning in $D_e(0,r)$ when Re $f'(z) > 0$ in $D_e(0,r)$. The Wolff-Warschawski-Noshiro Theorem implies that a function of bounded turning is univalent. For $f \in B_1(\alpha)$ and $|z| < \frac{1+\sqrt{3}m(\alpha)}{\sqrt{3}+m(\alpha)}$,

$$\text{Re } f'(z) \geq F'_\alpha(|z|) = \frac{3\sqrt{3}(1-m^2(\alpha))(m(\alpha)-|z|)}{2(1-m(\alpha)|z|)^3}$$

with strict inequality for $z \neq 0$ unless f is a rotation of F_α. In particular, Re $f'(z) > 0$ for $|z| < m(\alpha)$, so f is univalent and of bounded turning in $D_e(0, m(\alpha))$. Since $F'_\alpha(m(\alpha)) = 0$, F_α is neither univalent nor of bounded turning in any larger disk centered at the origin. All that remains is to show that if $f(z) \neq e^{i\theta}F_\alpha(e^{-i\theta}z)$ for all $\theta \in \mathbb{R}$, then f is both univalent and of bounded turning on a strictly larger disk. If f is not a rotation of F_α, then strict inequality holds in the above inequality. In particular, for $|z| = m(\alpha)$, Re $f'(z) > 0$ which implies that there exists $r > m(\alpha)$ such that Re $f'(z) > 0$ for $z \in D_e(0,r)$. □

Corollary 3. *Suppose* $f \in B_1(\alpha)$ *and* $\alpha \in (0,1]$. *Then* $r(0,f) \geq \frac{3\sqrt{3}}{4} m^2(\alpha)$ *with equality if and only if* $f(z) = e^{i\theta} F_\alpha(e^{-i\theta} z)$ *for some* $\theta \in \mathbb{R}$.

Proof. Assume there is no $\theta \in \mathbb{R}$ so that $f(z)$ is equal to $e^{i\theta} F_\alpha(e^{-i\theta} z)$. Then $|f(z)| > F_\alpha(|z|)$ for $|z| < \frac{2m(\alpha)}{1+m^2(\alpha)}$. Since $m(\alpha) < \frac{2m(\alpha)}{1+m^2(\alpha)}$, we obtain

$$\min \{|f(z)| : |z| = m(\alpha)\} > F_\alpha(m(\alpha)) = \frac{3\sqrt{3}}{4} m^2(\alpha).$$

Because f is univalent in $|z| \leq m(\alpha)$ and $f(0) = 0$, this implies that $r(0,f) > \frac{3\sqrt{3}}{4} m^2(\alpha)$. It is straightforward to check that $r(0, F_\alpha) = \frac{3\sqrt{3}}{4} m^2(\alpha)$ with the same value for all rotations of F_α. □

Subordination also yields information about the derivative. We now employ this type of information to obtain results for Bloch functions.

Theorem 3. *Suppose* $f \in B_1(\alpha)$. *Then*

$$|f''(0)| \leq -F''_\alpha(0) = \frac{3\sqrt{3}}{2} [1 - m^2(\alpha)][1 - 3m^2(\alpha)].$$

For $\alpha \in [0,1)$ *equality holds if and only if* $f(z) = \overline{\lambda} F_\alpha(\lambda z)$ *for some unimodular constant* λ.

Proof. Note that if $\alpha = 1$, then $f''(0) = 0$ and $m(1) = 1/\sqrt{3}$, so the inequality is actually a trivial identity when $\alpha = 1$. Now, we assume $\alpha \in [0,1)$ and observe there is nothing to prove when $f''(0) = 0$. Thus, we suppose $f''(0) < 0$ and prove that $-f''(0) \leq -F_\alpha(0)$ with equality if and only if $f = F_\alpha$. The general case follows by considering $\overline{\lambda} f(\lambda z)$ for an appropriate unimodular constant λ. If $g(z) = (1 - m(\alpha)z)^2 f'(z)$, then $g(0) = \alpha$ and $g'(0) = f''(0) - 2\alpha m(\alpha) < 0$. As $g \prec_0 G_\alpha$ on $\Delta(\alpha)$, there is a holomorphic self-mapping φ of $\Delta(\alpha)$ with $\varphi(0) = 0$ and $g(z) = G_\alpha \circ \varphi(z)$ for $z \in \Delta(\alpha)$. Now,

$$\begin{aligned}[f''(0) - 2\alpha m(\alpha)] &= g'(0) = G'_\alpha(0)\varphi'(0) \\ &= [F''_\alpha(0) - 2\alpha m(\alpha)]\varphi'(0).\end{aligned}$$

The two expressions in brackets are negative, so $\varphi'(0) > 0$. As $|\varphi'(0)| \leq 1$, we have $0 < \varphi'(0) \leq 1$ and so

$$2\alpha m(\alpha) - f''(0) \leq 2\alpha m(\alpha) - F''_\alpha(0),$$

or

$$-f''(0) \leq -F''_\alpha(0).$$

This establishes the inequality. If equality holds, then $\varphi'(0) = 1$ which implies that φ is the identity function and $f = F_\alpha$. $\square$

Corollary. *Suppose f is holomorphic in $\mathbb{D}$ and $\|f\|_B \le 1$. Then for $z \in \mathbb{D}$*

$$|D_2 f(z)| \le \frac{3\sqrt{3}}{2}[1 - m^2(|D_1 f(z)|)][1 - 3m^2(|D_1 f(z)|)].$$

Equality holds at point $z_0 \in \mathbb{D}$ where $D_2 f(z_0) \ne 0$ if and only if

$$f(z) = \lambda F_\alpha\left(\mu \frac{z - z_0}{z - \overline{z}_0 z}\right) + C$$

for some $\alpha \in [0, 1)$, unimodular constants λ and μ and $C \in \mathbb{C}$.

Also

$$|D_2 f(z)| \le \frac{3\sqrt{3}}{2}$$

with equality at a point $z_0 \in \mathbb{D}$ with $D_2 f(z_0) \ne 0$ if and only if

$$f(z) = \lambda F_0\left(\mu \frac{z - z_0}{1 - \overline{z}_0 z}\right) + C$$

for unimodular constants λ and μ and $C \in \mathbb{C}$.

Proof. Fix $z_0 \in \mathbb{D}$. The function $f \circ T$, where $T(z) = (z + z_0)/(1 + \overline{z}_0 z)$ satisfies $\|f \circ T\|_B = \|f\|_B \le 1$ and $|D_j(f \circ T)(0)| = |D_j f(z_0)|$ $(j = 1, 2)$. If $\alpha = |D_1 f(z_0)|$, then a rotation of $f \circ T - f(z_0)$ belongs to $B_1(\alpha)$ and the theorem applied to this function gives the desired result.

The second inequality follows immediately from the first. The function $L(s) = \frac{3\sqrt{3}}{2}[1 - m^2(s)][1 - 3m^2(s)]$ is strictly decreasing on $[0, 1]$, so $L(s) \le \frac{3\sqrt{3}}{2}$ with equality if and only if $s = 0$.

Theorem 4. *Suppose $f \in B_1(\alpha)$. Then for $|z| \le m(\alpha)$*

$$|D_2 f(z)| \le -D_2 F_\alpha(|z|)$$

or

$$|(1 - |z|^2) f''(z) - 2\overline{z} f'(z)| \le \frac{3\sqrt{3}(1 - m^2(\alpha))[1 - 3m^2(\alpha) + 4m(\alpha)|z| - (3 - m^2(\alpha))|z|^2]}{2(1 - m(\alpha)|z|)^4}.$$

Equality holds at $z = \mathrm{r}e^{i\theta}$, $r \in (0, m(\alpha)]$, if and only if $f(z) = e^{i\theta} F_\alpha(e^{-i\theta} z)$.

Proof. The corollary of Theorem 3 gives

$$|D_2 f(z)| \le L(|D_1 f(z)|).$$

The function L is strictly decreasing on $[0,1]$ and from part (i) of Theorem 2 it follows that $|z| \le m(\alpha)$ implies

$$\begin{aligned}|D_1 f(z)| &= (1-|z|^2)|f'(z)| \\ &\ge (1-|z|^2)F'_\alpha(|z|) = D_1 F_\alpha(|z|) \ge 0.\end{aligned}$$

Therefore, for $|z| \le m(\alpha)$

$$\begin{aligned}|D_2 f(z)| &\le L(D_1 F_\alpha(|z|)) \\ &= \frac{3\sqrt{3}}{2}[1-m^2(|D_1 F_\alpha(z)|)][1-3m^2(|D_1 F_\alpha(|z|)|)] \\ &= D_2 F_\alpha(|z|).\end{aligned}$$

If equality holds at $z = \mathrm{r}e^{i\theta}$, $r \in (0, m(\alpha)]$, then $|f'(z)| = F'_\alpha(|z|)$ and so $f(z) = e^{i\theta} F_\alpha(e^{-i\theta} z)$. □

Corollary 1. *Suppose $f \in B_1(\alpha)$ and $\alpha \in (0,1]$. Then for $|z| < m(\alpha)$*

$$\left|\frac{D_2 f(z)}{D_1 f(z)}\right| \le \frac{-D_2 F_\alpha(|z|)}{D_1 F_\alpha(|z|)},$$

or

$$\left|(1-|z|^2)\frac{f''(z)}{f'(z)} - 2\overline{z}\right| \le \frac{1-3m^2(\alpha)+4m(\alpha)|z|-(3-m^2(\alpha))|z|^2}{(m(\alpha)-|z|)(1-m(\alpha)|z|)}.$$

Equality holds at $z = \mathrm{r}e^{i\theta}$, $r \in (0, m(\alpha))$, if and only if $f(z) = e^{i\theta} F_\alpha(e^{-i\theta} z)$.

Proof. From part (i) of Theorem 2

$$|D_1 f(z)| \ge D_1 F_\alpha(|z|) \ge 0$$

for $|z| < m(\alpha)$. The theorem then gives

$$\left|\frac{D_2 f(z)}{D_1 f(z)}\right| \le \frac{D_1 F_\alpha(|z|)}{|D_1 f(z)|}\frac{(-D_2 F_\alpha(|z|))}{D_1 F_\alpha(|z|)} \le \frac{-D_2 F_\alpha(|z|)}{D_1 F_\alpha(|z|)}$$

and for $0 < |z| < m(\alpha)$ equality forces $|f'(z)| = F'_\alpha(|z|)$ and so $f(z) = e^{i\theta} F_\alpha(e^{-i\theta} z)$. □

Corollary 2. *Suppose $f \in B_1(\alpha)$ and $\alpha \in (0,1]$. Then for $|z| < m(\alpha)$*

$$\mathrm{Re}\left\{\frac{z f''(z)}{f'(z)}\right\} \ge \frac{|z| F''_\alpha(|z|)}{F'_\alpha(|z|)}$$

$$= -\frac{|z|(1 - 3m^2(\alpha) + 2m(\alpha)|z|)}{(m(\alpha) - |z|)(1 - m(\alpha)|z|)}.$$

Equality holds at $z = \mathrm{re}^{i\theta}$, $r \in (0, m(\alpha))$, *if and only if* $f(z) = e^{i\theta} F_\alpha(e^{-i\theta} z)$. *In particular, the radius of convexity for* $B_1(\alpha)$ *is*

$$R_c(\alpha) = \frac{m(\alpha)}{1 - m^2(\alpha) + \sqrt{1 - m^2(\alpha) + m^4(\alpha)}}.$$

Proof. From the preceding corollary we obtain

$$\mathrm{Re}\left\{\frac{zf''(z)}{f'(z)} - \frac{2|z|^2}{1 - |z|^2}\right\} = \mathrm{Re}\left\{\frac{zD_2 f(z)}{(1 - |z|^2)D_1 f(z)}\right\} \geq \frac{|z|F''_\alpha(|z|)}{F'_\alpha(|z|)} - \frac{2|z|^2}{1 - |z|^2}$$

which is the desired inequality. The statement concerning equality follows from Corollary 1. Now,

$$1 + \mathrm{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} \geq 1 + \frac{|z|F''_\alpha(|z|)}{F'_\alpha(|z|)}$$

and the right-hand side is positive for $|z| < R_c(\alpha)$ and vanishes for $|z| = R_c(\alpha)$. This yields the radius of convexity result. □

Remark. For $\alpha = 1$ we have $R_c(1) = \frac{\sqrt{7}-2}{\sqrt{3}} = 0.372824\ldots$ while the radius of univalence is $\frac{1}{\sqrt{3}} = 0.57735\ldots$.

Theorem 5. *Suppose* $f \in B_1(\alpha)$ *and* $\alpha \in (0, 1]$. *Then for* $|z| \leq m(\alpha)$

$$\begin{aligned} &\left|zf''(z) - \frac{2m(\alpha)|z|}{1 - m(\alpha)|z|} f'(z)\right| \\ &\leq \frac{2m(\alpha)|z|}{1 - m(\alpha)|z|} F'_\alpha(|z|) - |z|F''_\alpha(|z|) \\ &= \frac{3\sqrt{3}(1 - m^2(\alpha))^2|z|}{(1 - m(\alpha)|z|)^4}. \end{aligned}$$

Equality holds at $z = \mathrm{re}^{i\theta}$, $r \in (0, m(\alpha))$, *if and only if* $f(z) = e^{i\theta} F_\alpha(e^{-i\theta} z)$.

Proof. It is enough to consider $z = x \in (0, m(\alpha))$ and prove that

$$\left|f''(x) - \frac{2m(\alpha)}{1 - m(\alpha)x} f'(x)\right| \leq \frac{2m(\alpha)}{1 - m(\alpha)x} F'_\alpha(x) - F''_\alpha(x)$$

with equality if and only if $f = F_\alpha$. For $g(z) = (1 - m(\alpha)z)^2 f'(z)$, the preceding inequality is equivalent to

$$|g'(x)| \leq -G'_\alpha(x)$$

for $x \in (0, m(\alpha))$ with equality if and only if $g = G_\alpha$. We will establish this inequality.

We begin by establishing a stronger inequality that will be used in the proof of Theorem 6. Namely,

$$\frac{|g'(z)|}{\frac{9}{4}(1-m^2(\alpha))^2 - |g(z)|^2} \leq \frac{|G'_\alpha(z)|}{\frac{9}{4}(1-m^2(\alpha))^2 - |G_\alpha(z)|^2}$$

for $z \in \Delta(\alpha)$. The hyperbolic metric on $D(\alpha) = D_e(0, \frac{3}{2}(1-m^2(\alpha)))$ is

$$\lambda_{D(\alpha)}(w)|dw| = \frac{\frac{3}{2}(1-m^2(\alpha))|dw|}{\frac{9}{4}(1-m^2(\alpha))^2 - |w|^2}.$$

Since G_α is a conformal mapping of $\Delta(\alpha)$ onto $D(\alpha)$,

$$\lambda_{\Delta(\alpha)}(z) = \lambda_{D(\alpha)}(G_\alpha(z))|G'_\alpha(z)|,$$

where $\lambda_{\Delta(\alpha)}(z)|dz|$ is the hyperbolic metric on $\Delta(\alpha)$. Now, g maps $\Delta(\alpha)$ into $D(\alpha)$ because $g \prec_0 G_\alpha$. The Principle of Hyperbolic Metric gives

$$\lambda_{D(\alpha)}(g(z))|g'(z)| \leq \lambda_{\Delta(\alpha)}(z)$$

with equality if and only if g is a conformal mapping of $\Delta(\alpha)$ onto $D(\alpha)$. Therefore,

$$\lambda_{D(\alpha)}(g(z))|g'(z)| \leq \lambda_{\Delta(\alpha)}(G_\alpha(z))|G'_\alpha(z)|$$

for $z \in \Delta(\alpha)$ which yields the desired inequality. In particular, for $z = x \in \Delta(\alpha) \cap \mathbb{R}$ we have

$$\frac{|g'(x)|}{\frac{9}{4}(1-m^2(\alpha))^2 - |g(x)|^2} \leq \frac{-G'_\alpha(x)}{\frac{9}{4}(1-m^2(\alpha))^2 - G^2_\alpha(x)}.$$

As $0 < G_\alpha(x) \leq |g(x)|$ for $0 < x < m(\alpha)$ with equality if and only if $g = G_\alpha$ (see the proof of part (i) of Theorem 2), the preceding inequality implies $|g'(x)| \leq -G'_\alpha(x)$ for $0 < x < m(\alpha)$ with strict inequality unless $|g(x)| = G_\alpha(x)$; that is, unless $g = G_\alpha$. This completes the proof. □

Remark. The theorem gives

$$\left|\frac{zf''(z)}{f'(z)} - \frac{2m(\alpha)|z|}{1-m(\alpha)|z|}\right| \leq \frac{2m(\alpha)|z|}{1-m(\alpha)|z|} - \frac{|z|F''_\alpha(|z|)}{F'_\alpha(|z|)}$$

for $f \in B_1(\alpha)$, $\alpha \in [0,1]$ and $|z| < m(\alpha)$. But this is weaker than the analogous inequality obtained from Corollary 1 of Theorem 4.

Corollary 1. *Suppose $f \in B_1(\alpha)$ and $\alpha \in (0,1]$. Then for $|z| \le m(\alpha)$*

$$\begin{aligned}&|f(z) - (1-m(\alpha)|z|)zf'(z)| \\ &\le F_\alpha(|z|) - (1-m(\alpha)|z|)|z|F'_\alpha(|z|) \\ &= \frac{3\sqrt{3}(1-m^2(\alpha))^2|z|^2}{4(1-m(\alpha)|z|)^2}.\end{aligned}$$

Equality holds at $z = \mathrm{r}e^{i\theta}$, $r \in (0, m(\alpha)]$, if and only if $f(z) = e^{i\theta}F_\alpha(e^{-i\theta}z)$.

Proof. As usual, it is enough to establish the inequality for some $z = x \in (0, m(\alpha)]$ and demonstrate equality for $f = F_\alpha$. Note that

$$\begin{aligned}&\frac{d}{dx}[f(x) - x(1-m(\alpha)x)f'(x)] \\ &= 2m(\alpha)xf'(x) - x(1-m(\alpha)x)f''(x),\end{aligned}$$

so the theorem gives

$$\begin{aligned}&\left|\frac{d}{dx}[f(x) - x(1-m(\alpha)x)f'(x)\right| \\ &\le \frac{d}{dx}[F_\alpha(x) - x(1-m(\alpha)x)F'_\alpha(x)]\end{aligned}$$

for $0 \le x \le m(\alpha)$. Note that the right hand side of this inequality is nonnegative. By integrating this inequality over $[0,x]$ we obtain

$$|f(x) - (1-m(\alpha)x)xf'(x)| \le F_\alpha(x) - (1-m(\alpha)x)xF'_\alpha(x)$$

with equality if and only if $f = F_\alpha$. □

Corollary 2. *Suppose $f \in B_1(\alpha)$ and $\alpha \in (0,1]$. Then for $|z| \le m(\alpha)$*

$$\begin{aligned}&\left|\frac{zf'(z)}{f(z)} - \frac{1}{1-m(\alpha)|z|}\right| \le \frac{1}{1-m(\alpha)|z|} - \frac{|z|F'_\alpha(|z|)}{F_\alpha(|z|)} \\ &= \frac{(1-m^2(\alpha))|z|}{(1-m(\alpha)|z|)(2m(\alpha)-(1+m^2(\alpha))|z|)}.\end{aligned}$$

In particular, for $|z| \le m(\alpha)$

$$\begin{aligned}&\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \ge \frac{|z|F'_\alpha(|z|)}{F_\alpha(|z|)} \\ &= \frac{2(m(\alpha)-|z|)}{(1-m(\alpha)|z|)(2m(\alpha)-(1+m^2(\alpha))|z|)}.\end{aligned}$$

In both inequalities equality holds at $z = \mathrm{r}e^{i\theta}$, $r \in (0, m(\alpha))$, *if and only if* $f(z) = e^{i\theta} F_\alpha(e^{-i\theta} z)$. *The radius of starlikeness for the class* $B_1(\alpha)$ *is* $m(\alpha)$.

Proof. Recall that $f(z) \neq 0$ for $0 < |z| < m(\alpha) < \frac{2m(\alpha)}{1+m^2(\alpha)}$ (see Corollary 1(i) of Theorem 2). Therefore, the preceding corollary gives

$$\left|\frac{zf'(z)}{f(z)} - \frac{1}{1 - m(\alpha)|z|}\right| \leq \frac{F_\alpha(|z|)}{|f(z)|}\left[\frac{1}{1 - m(\alpha)|z|} - \frac{|z|F'_\alpha(|z|)}{F_\alpha(|z|)}\right]$$

for $|z| \leq m(\alpha) < \frac{2m(\alpha)}{1+m^2(\alpha)}$. Since $0 < F_\alpha(|z|) \leq |f(z)|$ for $0 < |z| < \frac{2m(\alpha)}{1+m^2(\alpha)}$, the first inequality together with the statement about equality follows from Corollary 1(i) of Theorem 2. The second inequality follows immediately from the first. This latter inequality shows that $\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0$ for $|z| < m(\alpha)$, so f is starlike on $D_e(0, m(\alpha))$. As $\frac{zF'_\alpha(z)}{F_\alpha(z)}$ vanishes for $z = m(\alpha)$, the radius of starlikeness for the class $B_1(\alpha)$ is $m(\alpha)$. □

As our final application of the subordination theorem we determine a sharp lower bound on the euclidean curvature of the image of circles centered at the origin with radius at most $1/\sqrt{3}$ for functions $B_1(1)$.

Lemma 2. *For* $0 < r < 1/\sqrt{3}$ *the function*

$$h(t) = \frac{\frac{\sqrt{3}+r}{\sqrt{3}-r} t - \frac{\sqrt{3}}{2(\sqrt{3}-2r)}}{t^2}$$

is strictly increasing on the interval $I = \left[\frac{1-\sqrt{3}r}{1-\frac{r}{\sqrt{3}}}, 1\right]$. *In particular, for* $t \in I$

$$h(t) \geq h\left(\frac{1 - \sqrt{3}r}{1 - \frac{r}{\sqrt{3}}}\right)$$

with strict inequality unless $t = \frac{1-\sqrt{3}r}{1-\frac{r}{\sqrt{3}}}$.

Proof. We show that $h'(t) > 0$ for $t \in I$. Now,

$$h'(t) = \frac{\frac{\sqrt{3}}{\sqrt{3}-2r} - \frac{\sqrt{3}+r}{\sqrt{3}-r} t}{t^3}$$

will be positive on I provided the numerator is. The numerator is a linear function with negative slope and t-intercept

$$t_0 = \frac{3 - \sqrt{3}r}{3 - \sqrt{3}r - 2r^2}.$$

Since $t_0 > 1$ when $0 < r < 1/\sqrt{3}$, we conclude that $h'(t) > 0$ for $t \in I$. □

Theorem 6. *Suppose $f \in B_1(1)$, $r \in (0, 1/\sqrt{3})$ and γ is the positively oriented circle $|z| = r$. Then for $z \in \gamma$*

$$\begin{aligned}\kappa_e(f(z), f \circ \gamma) &\geq \kappa_e(F_1(|z|), F_1 \circ \gamma) \\ &= \frac{\left(1 - \frac{|z|}{\sqrt{3}}\right)^2 \left(1 - \frac{4}{\sqrt{3}}|z| - |z|^2\right)}{|z|(1 - \sqrt{3}|z|)^2}.\end{aligned}$$

Equality holds at $z = \mathrm{r}e^{i\theta}$, $r \in (0, 1/\sqrt{3})$, if and only if $f(z) = e^{i\theta} F_1(e^{-i\theta} z)$.

Proof. The euclidean curvature of $f \circ \gamma$ at the point $f(z)$ is

$$\kappa_e(f(z), f \circ \gamma) = \frac{1 + \operatorname{Re}\left\{\frac{zf''(z)}{f'(z)}\right\}}{|zf'(z)|}.$$

It suffices to establish the inequality for $z = r \in (0, 1/\sqrt{3})$ and show that equality holds if and only if $f = F_1$.

Set $g(z) = \left(1 - \frac{z}{\sqrt{3}}\right)^2 f'(z)$. Straightforward calculation shows that

$$\kappa_e(f(r), f \circ \gamma) = \frac{\left(1 - \frac{r}{\sqrt{3}}\right)^2}{r} \frac{\operatorname{Re}\left\{\frac{\sqrt{3}+r}{\sqrt{3}-r} + \frac{rg'(r)}{g(r)}\right\}}{|g(r)|}.$$

From the proof of Theorem 5 in case $\alpha = 1$ we have

$$\begin{aligned}|g'(r)| &\leq \frac{-G_1'(r)}{1 - G_1^2(r)}(1 - |g(r)|^2) \\ &= \frac{\sqrt{3}(1 - |g(r)|^2)}{2r(\sqrt{3} - 2r)}\end{aligned}$$

for $r \in (0, 1/\sqrt{3})$. Therefore,

$$\operatorname{Re}\left\{\frac{rg'(r)}{g(r)}\right\} \geq -\left|\frac{rg'(r)}{g(r)}\right| \geq -\frac{\sqrt{3}(1 - |g(r)|^2)}{2(\sqrt{3} - 2r)|g(r)|}$$

which gives

$$\kappa_e(f(r), f \circ \gamma) \geq \frac{\left(1 - \frac{r}{\sqrt{3}}\right)^2}{r} \left[\frac{\sqrt{3}}{2(\sqrt{3} - 2r)} + \frac{\frac{\sqrt{3}+r}{\sqrt{3}-r}|g(r)| - \frac{\sqrt{3}}{2(\sqrt{3}-2r)}}{|g(r)|^2}\right]$$

$$= \frac{\left(1 - \frac{r}{\sqrt{3}}\right)^2}{r} \left[\frac{\sqrt{3}}{2(\sqrt{3} - 2r)} + h(|g(r)|) \right].$$

Here h is the function of the last lemma. For $r \in (0, 1/\sqrt{3})$ we have

$$0 \leq \frac{1 - \sqrt{3}r}{1 - \frac{r}{\sqrt{3}}} = G_1(r) \leq |g(r)| \leq 1$$

with strict inequality unless $g = G_1$ (see the proof of Theorem 2(i)). Since h is increasing on $\left[\frac{1-\sqrt{3}r}{1-\frac{r}{\sqrt{3}}}, 1\right]$, the preceding lemma gives $h(|g(r)|) \geq h(G_1(r))$ with strict inequality unless $g = G_1$. By making use of this inequality we obtain

$$\begin{aligned} \kappa_e(f(r), f \circ \gamma) &\geq \frac{\left(1 - \frac{r}{\sqrt{3}}\right)^2}{r} \left[\frac{\sqrt{3}}{2(\sqrt{3} - 2r)} + h(G_1(r)) \right] \\ &= \kappa_e(F_1(r), F_1 \circ \gamma) = \frac{\left(1 - \frac{r}{\sqrt{3}}\right)^2 \left(1 - \frac{4}{\sqrt{3}} r - r^2\right)}{r(1 - \sqrt{3}r)^2} \end{aligned}$$

with strict inequality unless $g = G_1$; that is, $f = F_1$. □

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