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THE HELGASON FOURIER TRANSFORM FOR HOMOGENEOUS VECTOR BUNDLES OVER RIEMANNIAN SYMMETRIC SPACES

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The Helgason Fourier transform on a noncompact Riemannian symmetric space G/K is generalized to the homogeneous vector bundles E^τ ($\tau \in \hat{K}$) over G/K. The corresponding inversion formula is obtained by using the Plancherel formula on G and the Subrepresentation Theorem. For *radial systems of sections* of E^τ, the Helgason Fourier transform reduces to the (operator valued) spherical transform, defined with respect to the (operator valued) τ-spherical functions on G.

1. Introduction.

Let G be a connected noncompact semisimple Lie group with finite center, $K \subset G$ a maximal compact subgroup, and G/K the corresponding Riemannian symmetric space of the noncompact type. Let $G = KAN$ be an Iwasawa decomposition of G, and write

$$x = \mathbf{k}(x)\exp(H(x))n(x), \qquad x \in G, \tag{1.1}$$

where $\mathbf{k}(x) \in K$, $H(x) \in \mathbf{a}$ (the Lie algebra of A), and $n(x) \in N$. Let M be the centralizer of A in K, let $B = K/M$, and let $db = d(kM)$ be a K-invariant measure on B, normalized by $\int_B db = 1$.

We regard $C_0^\infty(G/K)$ as the set of compactly supported smooth functions on G which are right-invariant under K. For $b = kM \in B$, $\lambda \in \mathbf{a}^*$ (the real dual of $\mathbf{a}$), and $f \in C_0^\infty(G/K)$, define the Fourier transform by

$$\tilde{f}(\lambda, b) = \int_G e^{-(i\lambda+\rho)(H(g^{-1}k))} f(g)dg, \tag{1.2}$$

where ρ is half the sum of the positive restricted roots of G/K. [$\tilde{f}$ is well defined since $H(gm) = H(g)$, $\forall m \in M$.] Then, for a suitable normalization of the relevant Haar measures, the following inversion formula holds $\forall x \in G$ (see, e.g., [8], Lemma 9.2.1.6.):

$$f(x) = w^{-1} \int_{\mathbf{a}^*} \int_B \tilde{f}(\lambda, b) e^{(i\lambda-\rho)(H(x^{-1}k))} |c(\lambda)|^{-2} d\lambda\, db. \tag{1.3}$$

Here w is the order of the Weyl group of G/K, $d\lambda$ is a suitably normalized Euclidean measure on $\mathbf{a}^*$, and $c(\lambda)$ is the Harish Chandra function. Let ϕ_λ be the zonal spherical function on G corresponding to $\lambda \in \mathbf{a}^*_{\mathbf{C}}$ (the complexification of $\mathbf{a}^*$), given by

$$\phi_\lambda(x) = \int_K e^{(i\lambda-\rho)(H(xk))} dk, \qquad x \in G.$$

(The invariant measure dk on K is normalized by $\int_K dk = 1$.) Then $c(\lambda)$ is related to the asymptotic form of ϕ_λ at infinity in $A^+ = \exp(\mathbf{a}^+)$ ($\mathbf{a}^+$ the positive Weyl chamber in $\mathbf{a}$) by

$$c(\lambda) = \lim_{t\to+\infty} a_t^{(-i\lambda+\rho)} \phi_\lambda(a_t), \qquad \Re(i\lambda) \in \mathbf{a}^*_+, \tag{1.4}$$

where $a_t = \exp(tH)$ (H fixed in $\mathbf{a}^+$), and $\mathbf{a}^*_+$ is the positive Weyl chamber in $\mathbf{a}^*$. From (1.4) one obtains the integral representation

$$c(\lambda) = \int_{\bar{N}} e^{-(i\lambda+\rho)(H(\bar{n}))} d\bar{n}, \qquad \Re(i\lambda) \in \mathbf{a}^*_+, \tag{1.5}$$

where $\bar{N} = \theta N$ (θ the Cartan involution) (see [3], Theorem 6.14, p. 447). The integral in (1.5) converges absolutely as long as $\Re(i\lambda) \in \mathbf{a}^*_+$, and it is defined by meromorphic continuation for the other values of λ. The Haar measure on $\bar{N}$ is normalized so that

$$\int_{\bar{N}} e^{-2\rho(H(\bar{n}))} d\bar{n} = 1.$$

To prove (1.3) one uses the direct integral decomposition of $L^2(G/K)$ given by Harish Chandra, to write for all $f \in C_0^\infty(G/K)$ (see [4], Theorem 2.1)

$$f(x) = w^{-1} \int_{\mathbf{a}^*} (f * \phi_\lambda)(x) |c(\lambda)|^{-2} d\lambda, \qquad x \in G, \tag{1.6}$$

together with the following result for the translated spherical functions $\phi_\lambda(x^{-1}y)$ (see [3], Lemma 4.4, p. 418)

$$\phi_\lambda(x^{-1}y) = \int_K e^{(i\lambda-\rho)(H(x^{-1}k))} e^{-(i\lambda+\rho)(H(y^{-1}k))} dk, \qquad x, y \in G. \tag{1.7}$$

From (1.7) we find, using $\int_K g(k)dk = \int_{K/M} \left(\int_M g(km)dm\right) d(kM)$ ($g \in C(K)$),

$$(f * \phi_\lambda)(x) = \int_B \tilde{f}(-\lambda, b) e^{-(i\lambda+\rho)H(x^{-1}k)} db. \tag{1.8}$$

Using this in (1.6) and letting $\lambda \to -\lambda$ in the integral over $\mathbf{a}^*$ gives (1.3).

The purpose of this paper is to generalize the above construction to the homogeneous vector bundle E^τ over G/K associated with a given irreducible unitary representation τ of K. It is well known that a cross section $f \in \Gamma(E^\tau)$ may be identified with a vector-valued function $f : G \to V_\tau$ (V_τ the representation space of τ) which is right-K-covariant of type τ, i.e.,

$$f(gk) = \tau(k^{-1})f(g), \qquad \forall g \in G, \ \forall k \in K. \tag{1.9}$$

We denote by $C_0^\infty(G,\tau)$ the space of compactly supported smooth functions on G that are right-K-covariant of type τ, and by $L^2(G,\tau)$ the Hilbert space of square integrable such functions, with scalar product

$$\langle f_1, f_2\rangle = \int_G \langle f_1(x), f_2(x)\rangle dx. \tag{1.10}$$

The first step is to obtain the direct integral decomposition of $L^2(G,\tau)$ analogous to (1.6), with the $\mathrm{End}(V_\tau)$-valued spherical functions $\varphi_\tau^U = \psi_\tau^U * d_\tau\tau$ in place of ϕ_λ, where U is in the tempered spectrum of G, and ψ_τ^U is the spherical trace function of type τ relative to U (d_τ is the dimension of τ). This is done in Section 2 using the Plancherel Theorem on G.

The second step is to find the analog of (1.7) for the translated spherical functions $\varphi_\tau^U(x^{-1}y)$ when U is in the minimal principal series. This is similar to the scalar-case computation, and is done in Section 3 using the Eisenstein integral representation of $\varphi_\tau^U(x)$, together with a well-known change-of-variables formula for integrals over K (due to Harish-Chandra). The convolution $\varphi_\tau^U * f$ ($f \in C_0^\infty(G,\tau)$) can then be worked out in terms of the (naturally defined) Helgason Fourier transform $\tilde{f}$ of f, in analogy with (1.8) in the scalar case.

If G has only one conjugacy class of Cartan subalgebra (i.e., when G/K is split-rank), the results of Section 3 are actually enough to obtain the inversion formula on $C_0^\infty(G,\tau)$ ($\forall\tau \in \hat{K}$), since only the minimal principal series occur in the Plancherel formula. Similarly, if G/K has rank one and τ does not occur in any discrete series of G, then the inversion formula of Section 3 applies.

In Section 4 we use the Subrepresentation Theorem to reduce the general case to the minimal one. Given U in the tempered spectrum, we can find $U^{\sigma\mu}$ in the nonunitary (minimal) principal series ($\sigma \in \hat{M}$, $\mu \in \mathbf{a}_{\mathbf{C}}^*$), such that U is infinitesimally equivalent to a subrepresentation of $U^{\sigma\mu}$. For example if G has discrete series (i.e., $\hat{G}_d \neq \emptyset$), then an explicit embedding of each $U \in \hat{G}_d$ as a subrepresentation of a nonunitary principal series has been given by Knapp and Wallach in [6]. By a double induction argument, one obtains an embedding of any generalized principal series into a nonunitary minimal one.

Let m and n be the multiplicities of τ in $U^{\sigma\mu}|_K$ and $U|_K$, respectively. In general we have $m \geq n$, and ψ_τ^U is different from $\psi_\tau^{U^{\sigma\mu}}$, unless $m = n$. However, it is always possible to represent $\varphi_\tau^U(x)$ as a minimal Eisenstein integral, namely

$$\varphi_\tau^U(x) = \int_K \tau\left(\mathbf{k}(xk)\right) T_U\, \tau(k^{-1})\, e^{\mu(H(xk))} dk, \tag{1.11}$$

where T_U is a suitable element of $\mathrm{End}_M(V_\tau)$ (see Section 4 for details). The analog of (1.7) for $\varphi_\tau^U(x^{-1}y)$ follows then immediately from (1.11), and one can work out the contribution of U to the inversion formula.

Our main result can now be described as follows. Let $P = MAN$ be a minimal parabolic subgroup of G. Let P' be a cuspidal parabolic subgroup of G such that $P' \supseteq P$ and $A' \subseteq A$, where $P' = M'A'N'$ is a Langlands decomposition of P'. Let $K' = K \bigcap M'$ be maximal compact in M', and let $M' = K'A_1N_1$ be an Iwasawa decomposition of M' so that $A = A'A_1$ and $N = N'N_1$. Given σ' in the discrete series of M', choose parameters $\tilde{\sigma}' \in \hat{M}$ and $\mu_1 \in \mathbf{a}_1{}^*$ by the Subrepresentation Theorem ($\mathbf{a}_1$ the Lie algebra of A_1), so that σ' is infinitesimally equivalent with a subrepresentation of $\mathrm{ind}_{MA_1N_1}^{M'}(\tilde{\sigma}' \otimes e^{\mu_1} \otimes \mathbf{1})$ (see [6]). For $\nu' \in \mathbf{a}'^*$ (the real dual of the Lie algebra of A'), let $U^{\sigma'\nu'} = \mathrm{ind}_{P'}^G(\sigma' \otimes e^{i\nu'} \otimes \mathbf{1})$ be the representation in the (generalized) principal P'-series with parameters σ' and ν'. Then $U^{\sigma'\nu'}$ can be regarded as a subrepresentation of the (nonunitary) minimal principal series of G given by $\mathrm{ind}_P^G(\tilde{\sigma}' \otimes e^{i\nu'+\mu_1} \otimes \mathbf{1})$ and denoted $U^{\tilde{\sigma}', i\nu'+\mu_1-\rho}$ in Section 3. This follows by a double induction formula, see Proposition 4.1.

If $\tau \subset U^{\sigma'\nu'}|_K$, let $T_{\sigma'}$ be the element of $\mathrm{End}_M(V_\tau)$ such that

$$\varphi_\tau^{U^{\sigma'\nu'}}(x) = \int_K \tau\left(\mathbf{k}(xk)\right) T_{\sigma'}\, \tau(k^{-1})\, e^{(i\nu'+\mu_1-\rho)(H(xk))} dk.$$

For $\mu \in \mathbf{a}^*_{\mathbf{C}}$ we put

$$F^\mu(x) = e^{\mu(H(x))}\, \tau(\mathbf{k}(x)), \qquad x \in G. \tag{1.12}$$

Theorem 1.1. *Define the Helgason Fourier transform of* $f \in C_0^\infty(G,\tau)$ *by*

$$\tilde{f}(\lambda, k) = \int_G F^{i\bar{\lambda}-\rho}(x^{-1}k)^* f(x) dx, \qquad \lambda \in \mathbf{a}^*_{\mathbf{C}}, k \in K,$$

where $F^\mu(x)$ *is given by* (1.12), *and* $*$ *denotes adjoint.*

Let $d\mu(U^{\sigma'\nu'}) = p_{\sigma'}(\nu')d\nu'$ *be the Plancherel measure associated with the principal* P'*-series* $U^{\sigma'\nu'}$, *where* $d\nu'$ *is a properly normalized Euclidean measure on* $\mathbf{a}'^*$. *Then, for a suitable normalization of the relevant Haar measures* *and for suitable constants $c_{P'} > 0$, the following inversion formula holds*

$$f(x) = \frac{1}{d_\tau} \sum_{P'} c_{P'} \sum_{\sigma'} \int_{\mathbf{a}'^*} \int_K F^{i\nu'+\mu_1-\rho}(x^{-1}k) T_{\sigma'} \tilde{f}(\nu' - i\mu_1, k)\, p_{\sigma'}(\nu')\, d\nu' dk.$$

Here the sum $\sum_{P'}$ is over all cuspidal parabolic subgroups P' of G such that $P' \supseteq P$ and $A' \subseteq A$, and the sum $\sum_{\sigma'}$ is over all discrete series σ' of M' such that $U^{\sigma'\nu'}|_K \supset \tau$. (The parameter μ_1 can also be replaced by $-\mu_1$, and the integrations over K can also be written as integrations over K/M.)

The proof of this result will be done in several steps in Sections 2, 3 and 4.

In Section 5 we define radial systems of sections of E^τ and the associated convolution algebra $C_0^\infty(G, \tau, \tau)$. The radial systems of sections generalize the notion of K-invariant functions on G/K to the bundle case. Then we use the theory of spherical functions of type τ on G (see [2, 8]), to define a spherical transform for $F \in C_0^\infty(G, \tau, \tau)$. The corresponding inversion formula and Plancherel Theorem are obtained. Finally, we establish the relation between the Helgason Fourier transform of a radial section and its spherical transform.

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2. The direct integral decomposition of $L^2(G, \tau)$.

Fix an irreducible unitary representation τ of K on a vector space V_τ. Let $\hat{G}$ be the unitary dual of G, and let $\hat{G}(\tau) \subset \hat{G}$ denote the subset of those classes containing the class of τ upon restriction to K. Let (U, H_U) be a representative of the class $[U] \in \hat{G}(\tau)$. Since τ is fixed, we denote the multiplicity $m(\tau, U)$ simply by ξ_U. It is well known that ξ_U is bounded by d_τ, the dimension of τ. Let P_τ be the projection from H_U onto H_τ, the subspace of vectors which transform under K according to τ. Then P_τ is given by (see, e.g., [2])

$$P_\tau = d_\tau \int_K U(k) \chi_\tau(k^{-1}) dk, \tag{2.1}$$

where χ_τ is the character of τ. We have dim $H_\tau = d_\tau \xi_U$, and the representation of K on H_τ given by $k \to \tau_U(k) \equiv U(k)|_{H_\tau}$ is the direct sum of ξ_U

copies of τ. Therefore, we can regard H_τ as the K-module

$$H_\tau \simeq V_\tau \otimes \mathbf{C}^{\xi_U},$$

with $\tau_U(k) = \tau(k) \otimes \mathbf{1}$ ($\mathbf{1}$ the identity operator in $\mathbf{C}^{\xi_U}$).

We choose an orthonormal basis $\{f_A\}_{A=1\cdots\infty}$ of H_U adapted to the decomposition of $U|_K$ into different K-types, $U|_K = \sum_{\delta\in\hat{K}} \delta$. We can always assume that for $A = 1, \dots, d_\tau \xi_U$ the vectors $\{f_A\}$ span H_τ, and we choose them as follows. Let $\mathrm{Hom}_K(H_U, V_\tau)$ be the space of all linear maps $T : H_U \to V_\tau$ such that

$$T\,U(k) = \tau(k)T, \qquad \forall k \in K.$$

This is a vector space of dimension ξ_U. Let $\{P_\xi\}_{\xi=1,\dots,\xi_U}$ be a basis of $\mathrm{Hom}_K(H_U, V_\tau)$, orthonormal with respect to the scalar product $\langle P, Q\rangle = (1/d_\tau)\,\mathrm{Tr}\,(P^* Q)$, where $*$ denotes adjoint. One has the relations $P_{\xi'} P_\xi^* = \delta_{\xi\xi'} \mathbf{1}_\tau$, and

$$P_\tau = \sum_\xi P_\xi^* P_\xi.$$

For each $\xi = 1, \dots, \xi_U$ and $\mathbf{v} \in V_\tau$, we put

$$f_{\mathbf{v}\xi} = P_\xi^* \mathbf{v}.$$

Let $\{\mathbf{v}_a\}_{a=1,\dots,d_\tau}$ be an orthonormal basis of V_τ, and let $f_{a\xi} \equiv f_{\mathbf{v}_a\xi}$. Then $\{f_{a\xi}\}$ is an orthonormal basis of H_τ.

Define the $\mathrm{End}(H_\tau)$-valued spherical function Ψ_τ^U by

$$\Psi_\tau^U(g) = P_\tau U(g) P_\tau, \qquad g \in G, \tag{2.2}$$

and let ψ_τ^U be the corresponding spherical trace function of type τ

$$\psi_\tau^U(g) = \mathrm{Tr}\ [P_\tau U(g) P_\tau] = \mathrm{Tr}\ \Psi_\tau^U(g).$$

For $\mathcal{O} \in \mathrm{End}(V_\tau \otimes \mathbf{C}^{\xi_U})$, let $\mathrm{Tr}\ |_{\mathbf{C}^{\xi_U}} \mathcal{O} \in \mathrm{End}(V_\tau)$ denote the operator obtained by taking the partial trace of $\mathcal{O}$ with respect to $\mathbf{C}^{\xi_U}$. Define the $\mathrm{End}(V_\tau)$-valued function φ_τ^U on G by

$$\varphi_\tau^U(g) = \mathrm{Tr}\ |_{\mathbf{C}^{\xi_U}} \Psi_\tau^U(g). \tag{2.3}$$

In terms of the basis $\{P_\xi\}$ of $\mathrm{Hom}_K(H_U, V_\tau)$, we have

$$\varphi_\tau^U(g) = \sum_{\xi=1}^{\xi_U} P_\xi U(g) P_\xi^*,$$

i.e., the matrix coefficients of $\varphi_\tau^U(g)$ are

$$\langle \varphi_\tau^U(g)\mathbf{v}, \mathbf{v}' \rangle = \sum_{\xi=1}^{\xi_U} \langle U(g) f_{\mathbf{v}\xi}, f_{\mathbf{v}'\xi} \rangle, \qquad \mathbf{v}, \mathbf{v}' \in V_\tau.$$

It is easy to see, using the Schur orthogonality relations, that $\varphi_\tau^U = \psi_\tau^U * d_\tau \tau$, i.e.,

$$\varphi_\tau^U(x) = d_\tau \int_K \psi_\tau^U(xk^{-1})\tau(k)dk. \qquad x \in G. \tag{2.4}$$

We have $\varphi_\tau^U(kxk') = \tau(k)\varphi_\tau^U(x)\tau(k')$, $\forall k, k' \in K$. Moreover Tr $\varphi_\tau^U(x) = \psi_\tau^U(x)$ (since $\psi_\tau^U * d_\tau \chi_\tau = \psi_\tau^U$). If $\xi_U = 1$, then $\varphi_\tau^U = \Psi_\tau^U$.

Proposition 2.1. *Let $f \in C_0^\infty(G, \tau)$. Then $\forall x \in G$*

$$f(x) = \frac{1}{d_\tau} \int_{\hat{G}(\tau)} (\varphi_\tau^U * f)(x)\, d\mu(U), \tag{2.5}$$

where $d\mu(U)$ is the Plancherel measure on $\hat{G}$ (suitably normalized), and the convolution is defined by

$$(\varphi_\tau^U * f)(x) \equiv \int_G \varphi_\tau^U(x^{-1}y) f(y) dy. \tag{2.6}$$

Proof. We write the Plancherel (inversion) formula on G as

$$f(x) = \int_{\hat{G}} \Theta_U(f \circ L_x) d\mu(U), \qquad f \in C_0^\infty(G), \tag{2.7}$$

where Θ_U is the global character of U, and L_x denotes left-translation on G, $L_x(y) = xy$.

Consider the *Fourier component* $\Theta_{U,\tau}$ of Θ_U, defined by the rule

$$\Theta_{U,\tau}(f) \equiv \Theta_U(f * d_\tau \bar{\chi}_\tau), \qquad f \in C_0^\infty(G),$$

the convolution being over K.

As proved in [8], Vol. II, p. 18 (the remark), the distribution $\Theta_{U,\tau}$ coincides with the spherical trace function ψ_τ^U, i.e.,

$$\Theta_{U,\tau}(f) = \int_G \psi_\tau^U(x) f(x) dx, \qquad f \in C_0^\infty(G).$$

Clearly if $[U]$ is not in $\hat{G}(\tau)$, then $\psi_\tau^U = 0$ and

$$\Theta_{U,\tau}(f) = 0, \qquad \forall f \in C_0^\infty(G).$$

Now let $f \in C_0^\infty(G,\tau)$. It follows from (1.9) that $f * d_\tau \bar{\chi}_\tau = f$, and similarly $(f \circ L_x) * d_\tau \bar{\chi}_\tau = f \circ L_x$. Applying Θ_U to each component of the vector-valued function $f \circ L_x$, gives

$$\Theta_U(f \circ L_x) = \Theta_U\left((f \circ L_x) * d_\tau \bar{\chi}_\tau\right) = \Theta_{U,\tau}(f \circ L_x),$$

and if $[U] \notin \hat{G}(\tau)$

$$\Theta_U(f \circ L_x) = \Theta_{U,\tau}(f \circ L_x) = 0.$$

We now apply the inversion formula (2.7) to each component of $f \in C_0^\infty(G,\tau)$, to find

$$\begin{aligned} f(x) &= \int_{\hat{G}(\tau)} \int_G \psi_\tau^U(y) f(xy)\, dy\, d\mu(U) \\ &= \int_{\hat{G}(\tau)} \int_G \psi_\tau^U(x^{-1}y) f(y)\, dy\, d\mu(U). \end{aligned}$$

Now

$$\begin{aligned} &\int_G \psi_\tau^U(x^{-1}y) f(y)\, dy = \int_G \int_K \psi_\tau^U(x^{-1}y) \tau(k) f(yk)\, dk\, dy \\ &= \int_G \int_K \psi_\tau^U(x^{-1}yk^{-1}) \tau(k) f(y)\, dk\, dy = \frac{1}{d_\tau} \int_G \varphi_\tau^U(x^{-1}y) f(y)\, dy. \end{aligned}$$

This proves (2.5). □

The space $\varphi_\tau^U * C_0^\infty(G,\tau)$ can be given the positive definite inner product

$$\langle \varphi_\tau^U * f_1, \varphi_\tau^U * f_2 \rangle = \frac{1}{d_\tau} \int_G \langle (\varphi_\tau^U * f_1)(x), f_2(x) \rangle\, dx \tag{2.8}$$

$$= \frac{1}{d_\tau} \int_G \langle f_1(x), (\varphi_\tau^U * f_2)(x) \rangle\, dx. \tag{2.9}$$

It is clear from (2.8)-(2.9) that this inner product is well-defined. We have the following corollary to Proposition 2.1.

Corollary 2.2. *Let $L_U^2(G,\tau)$ and $L^2(G,\tau)$ denote the Hilbert space completions of $\varphi_\tau^U * C_0^\infty(G,\tau)$ and $C_0^\infty(G,\tau)$, respectively. Let $\mathrm{ind}_K^G(\tau)$ denote the representation of G on $L^2(G,\tau)$ unitarily induced from $\tau \in \hat{K}$. Then we have the direct integral decompositions*

$$L^2(G,\tau) = \int_{\hat{G}(\tau)}^{\oplus} L_U^2(G,\tau)\, d\mu(U),$$

$$\operatorname{ind}_K^G(\tau) = \int_{\hat{G}(\tau)}^{\oplus} m(\tau, U)U\,d\mu(U),$$
$$\|f\|^2 = \int_{\hat{G}(\tau)} \|\varphi_\tau^U * f\|^2 d\mu(U), \qquad f \in C_0^\infty(G,\tau).$$

The theory above does not apparently involve any Fourier transform concept for general $f \in C_0^\infty(G,\tau)$. In order to investigate this point, we need to write down in a more precise way the convolution $\varphi_\tau^U * f$ for U in the tempered spectrum of a semisimple group G. We shall do this in two steps. First we consider the representations in the principal P-series, for P minimal parabolic. Then we discuss the representations in the generalized principal series and the discrete series.

3. τ-spherical functions for the minimal principal series.

Fix τ as before, and fix $\sigma \in \hat{M}$ so that σ occurs with multiplicity $m_\sigma > 0$ in $\tau|_M$. Hereafter we identify a class $\sigma \in \hat{M}$ with a representative (denoted (σ, V_σ)) in that class. Given a linear function $\mu : \mathbf{a} \to \mathbf{C}$, let $U^{\sigma\mu}$ be the representation of G in the nonunitary (minimal) principal series induced from the representation $\sigma \otimes \exp(\mu+\rho) \otimes \mathbf{1}$ of P:

(3.1) $$U^{\sigma\mu} = \operatorname{ind}_P^G(\sigma \otimes \exp(\mu+\rho)\otimes \mathbf{1}).$$

The representation space $E^{\sigma\mu}$ is the Hilbert space of those functions $f : G \to V_\sigma$ satisfying (see [8], Vol. I, p. 449)

$$f(xman) = a^{-(\mu+2\rho)}\sigma(m^{-1})f(x),$$

and $\|f\|^2 < +\infty$, where the scalar product is defined by

$$\langle f, g\rangle = \int_K \langle f(k), g(k)\rangle dk.$$

The action of G is just left translation of the argument:

$$[U^{\sigma\mu}(x)f](y) = f(x^{-1}y).$$

It is known that, independently of the continuous parameter μ, the restriction map $f \to f|_K$ is a bijection of $E^{\sigma\mu}$ onto $L^2(K,\sigma)$, the space of square integrable functions $f : K \to V_\sigma$ such that

$$f(km) = \sigma(m^{-1})f(k), \qquad m \in M, \;\; k \in K.$$

Thus $U^{\sigma\mu}|_K = \operatorname{ind}_M^K(\sigma)$ (see [8], Vol. I, p. 450), and by Frobenius Reciprocity τ occurs in $U^{\sigma\mu}|_K$ exactly m_σ times.

For each $\mu \in \mathbf{a}^*_{\mathbf{C}}$ we can define a representation of G in $L^2(K,\sigma)$ by

$$[\tilde{U}^{\sigma\mu}(x)f](k) = e^{-(\mu+2\rho)(H(x^{-1}k))} f(\mathbf{k}(x^{-1}k)), \qquad f \in L^2(K,\sigma).$$

The Hilbert space adjoint of $\tilde{U}^{\sigma\mu}(x)$ is then

$$\tilde{U}^{\sigma\mu}(x)^* = \tilde{U}^{\sigma,-\bar{\mu}-2\rho}(x^{-1}). \tag{3.2}$$

[This makes sense because the operators $\tilde{U}^{\sigma\mu}(x)$ ($\mu \in \mathbf{a}^*_{\mathbf{C}}$) all act on the same Hilbert space $L^2(K,\sigma)$.] Formula (3.2) follows immediately from the change-of-variables formula

$$\int_K f(k)dk = \int_K f(\mathbf{k}(yk'))e^{-2\rho(H(yk'))}dk', \qquad y \in G, \tag{3.3}$$

valid for all $f \in C(K)$ (see [8], Vol. II, p. 32). This result can be rewritten, with a little abuse of notation, as

$$dk = d\mathbf{k}(yk') = e^{-2\rho(H(yk'))}dk', \qquad \forall y \in G. \tag{3.4}$$

Eq. (3.3) is equivalent to the following one (see [3], Lemma 5.19, p. 197):

$$\int_K f(\mathbf{k}(y^{-1}k))dk = \int_K f(k')e^{-2\rho(H(yk'))}dk' \qquad y \in G.$$

[Replace f with $f \circ T_{y^{-1}}$ in (3.3), where $T_y(k) = \mathbf{k}(yk)$, and use $\mathbf{k}(y\mathbf{k}(y^{-1}k)) = k$.]

Conversely, given $f \in L^2(K,\sigma)$ we can extend it to a smooth function in $E^{\sigma\mu}$ by letting $f_\mu(kan) = a^{-(\mu+2\rho)}f(k)$, i.e.,

$$f_\mu(x) = e^{-(\mu+2\rho)(H(x))}f(\mathbf{k}(x)), \qquad x \in G. \tag{3.5}$$

Each $f \in E^{\sigma\mu}$ arises this way, since $(f|_K)_\mu = f$. We also have $(f_\mu)|_K = f$, and therefore $\langle f_\mu, f_\mu\rangle = \langle f, f\rangle$. In summary, we have the following (well-known) result.

Proposition 3.1. *The map $f \to A(f) \equiv f|_K$ is a bijection from $E^{\sigma\mu}$ onto $L^2(K,\sigma)$, for each $\mu \in \mathbf{a}^*_{\mathbf{C}}$, with inverse given by $A^{-1}(f) = f_\mu$. A is an intertwining operator for the representations $U^{\sigma\mu}$ and $\tilde{U}^{\sigma\mu}$, $A\,U^{\sigma\mu}(x) = \tilde{U}^{\sigma\mu}(x)A$, and $\langle A(f), A(f)\rangle = \langle f, f\rangle$. The Hilbert space adjoint of $U^{\sigma\mu}(x)$ is given by*

$$\begin{aligned}[U^{\sigma\mu}(x)^*f](y) &= e^{(\mu+\bar{\mu}+2\rho)(H(x\mathbf{k}(y)))}f(xy) \\ &= e^{(\mu+\bar{\mu}+2\rho)(H(x\mathbf{k}(y)))}[U^{\sigma\mu}(x^{-1})f](y).\end{aligned} \tag{3.6}$$

In particular, the representation $U^{\sigma\mu}$ is unitary if (and only if) $\mu = i\lambda - \rho$, with λ real-valued, and $i = \sqrt{-1}$.

Formula (3.6) follows easily from (3.2), and from the well-known properties of the Iwasawa functions: $\mathbf{k}(x\mathbf{k}(y)) = \mathbf{k}(xy)$, $H(x\mathbf{k}(y)) = H(xy) - H(y)$ (these are particular cases of the relations (3.13)-(3.14) below).

Now let P_τ denote the projection from $E^{\sigma\mu}$ onto H_τ, the subspace of vectors of $E^{\sigma\mu}$ which transform under K according to τ,

$$P_\tau = d_\tau \int_K U^{\sigma\mu}(k)\chi_\tau(k^{-1})dk$$

(cf. (2.1)). $U^{\sigma\mu}$ is neither irreducible nor unitary, in general. Nevertheless we can define, as before, $\Psi_\tau^{U^{\sigma\mu}}(x) = P_\tau U^{\sigma\mu}(x)P_\tau$, $\psi_\tau^{U^{\sigma\mu}}(x) = \mathrm{Tr}\ \Psi_\tau^{U^{\sigma\mu}}(x)$, and

$$\varphi_\tau^{U^{\sigma\mu}}(x) = d_\tau \int_K \tau(k)\psi_\tau^{U^{\sigma\mu}}(xk^{-1})dk \tag{3.7}$$

(cf. (2.4)). By [8], Corollary 6.2.2.3, we have

$$\psi_\tau^{U^{\sigma\mu}}(x) = d_\tau \int_K (\chi_\tau * \chi_\sigma)(\mathbf{k}(k^{-1}xk))\ e^{\mu(H(xk))}dk, \tag{3.8}$$

where χ_σ is the character of σ and the convolution is over M.

Lemma 3.2. *The function $\varphi_\tau^{U^{\sigma\mu}}$ admits the integral representation*

$$\varphi_\tau^{U^{\sigma\mu}}(x) = \frac{d_\tau}{d_\sigma}\int_K \tau\left(\mathbf{k}(xk)\right)P_\sigma\tau(k^{-1})\ e^{\mu(H(xk))}dk, \tag{3.9}$$

where

$$P_\sigma = d_\sigma \int_M \tau(m^{-1})\chi_\sigma(m)dm$$

is the projection from V_τ onto $H_\sigma \simeq V_\sigma \otimes \mathbf{C}^{m_\sigma} \subset V_\tau$ (the subspace of vectors of V_τ which transform under M according to σ), and d_σ is the dimension of σ.

Proof. This is a trivial calculation using (3.7)-(3.8), and the Schur orthogonality relations for K. □

The adjoint of $\varphi_\tau^{U^{\sigma\mu}}(x)$ can easily be calculated using (3.3), or (3.2),(3.6). We find

$$\varphi_\tau^{U^{\sigma\mu}}(x)^* = \varphi_\tau^{U^{\sigma,-\bar{\mu}-2\rho}}(x^{-1}),$$

with similar relations for the other functions $\Psi_\tau^{U^{\sigma\mu}}$ and $\psi_\tau^{U^{\sigma\mu}}$.

This is analogous to the scalar case, where the function

$$\psi_\mu(x) = \int_K e^{\mu(H(xk))} dk$$

satisfies (see, e.g., [8], Vol. II, p. 33)

$$\overline{\psi_{-\bar{\mu}-2\rho}(x^{-1})} = \psi_\mu(x).$$

In the unitary case ($\mu = i\lambda - \rho$, with λ real valued), the irreducibility of $U^{\sigma\mu}$ is discussed, e.g., in [8], Sect. 5.5.2, or in [5], p. 174. The result is that if $\lambda \in \mathbf{a}^*$ is *regular*, i.e., $\langle \lambda, \alpha \rangle \neq 0$ for all roots α of $(\mathbf{g}, \mathbf{a})$, then $U^{\sigma, i\lambda-\rho}$ is irreducible. Therefore for each $\sigma \in \hat{M}$, $U^{\sigma, i\lambda-\rho}$ is irreducible for an open dense subset of $\mathbf{a}^*$.

We now look for the analog of (1.7) for the translated spherical function $\varphi_\tau^{U^{\sigma\mu}}(x^{-1}y)$.

For $\mu \in \mathbf{a}^*_{\mathbf{C}}$, define $F^\mu : G \to \mathrm{End}(V_\tau)$ by

$$F^\mu(x) = e^{\mu(H(x))}\, \tau(\mathbf{k}(x)), \qquad x \in G. \tag{3.10}$$

We have

$$F^\mu(kxm) = \tau(k) F^\mu(x) \tau(m), \qquad \forall k \in K,\ \forall m \in M.$$

The following result generalizes Lemma 4.4, p. 418 of [3].

Proposition 3.3. *The translated spherical functions $\varphi_\tau^{U^{\sigma\mu}}(x^{-1}y)$ admit the integral representation*

$$\varphi_\tau^{U^{\sigma\mu}}(x^{-1}y) = \frac{d_\tau}{d_\sigma} \int_K F^\mu(x^{-1}k) P_\sigma F^{-\bar{\mu}-2\rho}(y^{-1}k)^* dk, \tag{3.11}$$

where $*$ denotes adjoint.

Proof. From (3.10) we have

$$\begin{aligned} &\int_K F^\mu(x^{-1}k) P_\sigma F^{-\bar{\mu}-2\rho}(y^{-1}k)^* dk \\ &= \int_K e^{\mu(H(x^{-1}k) - H(y^{-1}k)) - 2\rho(H(y^{-1}k))} \tau(\mathbf{k}(x^{-1}k)) P_\sigma \tau((\mathbf{k}(y^{-1}k))^{-1}) dk. \end{aligned} \tag{3.12}$$

Now notice that for all $x, y, z \in G$, we have the following cocycle relations for the Iwasawa functions $H(x)$ and $\mathbf{k}(x)$:

$$\mathbf{k}\left(x \mathbf{k}(yz)\right) = \mathbf{k}(xyz), \tag{3.13}$$

$$H\left(x\mathbf{k}(yz)\right) = H(xyz) - H(yz). \tag{3.14}$$

Indeed

$$\begin{aligned} xyz &= x\mathbf{k}(yz)e^{H(yz)}n(yz) \\ &= \mathbf{k}(x\mathbf{k}(yz))e^{H(x\mathbf{k}(yz))}n(x\mathbf{k}(yz))e^{H(yz)}n(yz) \\ &= \mathbf{k}(x\mathbf{k}(yz))e^{H(x\mathbf{k}(yz))+H(yz)}n'\, n(yz), \end{aligned} \tag{3.15}$$

where we have used the fact that A normalizes N, i.e., for all $a \in A$ and $n \in N$, $na = an'$, for some $n' \in N$. Equating (3.15) to $\mathbf{k}(xyz)e^{H(xyz)}n(xyz)$ gives (3.13)-(3.14), by the uniqueness of the Iwasawa decomposition.

Now make the change of variables $k' = \mathbf{k}(y^{-1}k)$ in the integral over K in (3.12). From (3.13) and (3.14), we obtain $\mathbf{k}(yk') = \mathbf{k}(y\mathbf{k}(y^{-1}k)) = k$, and

$$\begin{aligned} H(x^{-1}k) &= H(x^{-1}\mathbf{k}(yk')) = H(x^{-1}yk') - H(yk'), \\ H(y^{-1}k) &= H(y^{-1}\mathbf{k}(yk')) = -H(yk'), \\ \mathbf{k}(x^{-1}k) &= \mathbf{k}(x^{-1}\mathbf{k}(yk')) = \mathbf{k}(x^{-1}yk'). \end{aligned}$$

Using these relations and (3.4) in (3.12), we obtain, recalling (3.9),

$$\begin{aligned} &\int_K F^{\mu}(x^{-1}k)P_\sigma F^{-\bar{\mu}-2\rho}(y^{-1}k)^* dk \\ &= \int_K e^{\mu(H(x^{-1}yk'))}\tau(\mathbf{k}(x^{-1}yk'))P_\sigma \tau(k'^{-1})dk' = \frac{d_\sigma}{d_\tau}\varphi_\tau^{U^{\sigma\mu}}(x^{-1}y), \end{aligned}$$

which is (3.11). □

For $\mu = i\lambda - \rho$ ($\lambda \in \mathbf{a}^*_{\mathbf{C}}$), we obtain from (3.11)

$$\varphi_\tau^{U^{\sigma,i\lambda-\rho}}(x^{-1}y) = \frac{d_\tau}{d_\sigma}\int_K F^{i\lambda-\rho}(x^{-1}k)P_\sigma F^{i\bar{\lambda}-\rho}(y^{-1}k)^* dk. \tag{3.16}$$

Let $f \in C_0^\infty(G,\tau)$, and consider the convolution $\varphi_\tau^U * f$ (defined in (2.6)) for the unitary principal series $U = U^{\sigma,i\lambda-\rho}$ (λ real valued). Using (3.16) we find

$$\begin{aligned} (\varphi_\tau^{U^{\sigma,i\lambda-\rho}} * f)(x) &= \int_G \varphi_\tau^{U^{\sigma,i\lambda-\rho}}(x^{-1}y)f(y)dy \\ &= \frac{d_\tau}{d_\sigma}\int_K F^{i\lambda-\rho}(x^{-1}k)P_\sigma \tilde{f}(\lambda,k)dk, \end{aligned} \tag{3.17}$$

where we have defined, for $k \in K$, and $\lambda \in \mathbf{a}^*_{\mathbf{C}}$,

$$\tilde{f}(\lambda,k) \equiv \int_G F^{i\bar{\lambda}-\rho}(y^{-1}k)^* f(y)dy$$

$$(3.18) \qquad = \int_G e^{-(i\lambda+\rho)(H(y^{-1}k))}\tau(\mathbf{k}(y^{-1}k)^{-1})f(y)dy.$$

The function $\tilde{f} : \mathbf{a}^*_{\mathbf{C}} \times K \to V_\tau$ defined in (3.18) is called the Helgason Fourier transform of $f \in C_0^\infty(G,\tau)$. It reduces to the usual Helgason Fourier transform in the scalar case (cf. (1.2)).

The geometric interpretation of the Helgason Fourier transform is as follows. Let τ_M denote the restriction of τ to the subgroup M. For λ fixed in $\mathbf{a}^*_{\mathbf{C}}$, the function $k \to \tilde{f}(\lambda,k)$ satisfies $\tilde{f}(\lambda,km) = \tau(m^{-1})\tilde{f}(\lambda,k)$, $\forall m \in M$, and defines a cross section of the homogeneous vector bundle E^{τ_M} over K/M associated to τ_M. Let σ_τ be the restriction of τ_M to $H_\sigma = P_\sigma V_\tau$ (cf. Lemma 3.2). Then $\sigma_\tau \sim \sigma \otimes \mathbf{1}_{m_\sigma}$, and we have the direct sum decompositions $\tau_M = \sum_\sigma^\oplus \sigma_\tau$, $V_\tau = \sum_\sigma^\oplus H_\sigma = \sum_\sigma^\oplus V_\sigma \otimes \mathbf{C}^{m_\sigma}$. By writing $\tilde{f} = \sum_\sigma P_\sigma \tilde{f}$, we see that the function $\tilde{f}_\sigma(\lambda,\cdot) = P_\sigma\tilde{f}(\lambda,\cdot)$ defines a cross section of the homogeneous vector bundle

$$E^{\sigma_\tau} = \underbrace{E^\sigma \oplus \cdots \oplus E^\sigma}_{m_\sigma - times} \cong E^\sigma \otimes \mathbf{C}^{m_\sigma}$$

over K/M associated to σ_τ. Thus the Helgason Fourier transform maps cross sections of the bundle E^τ over G/K into cross sections of the bundle $E^{\tau_M} = \sum_\sigma^\oplus E^{\sigma_\tau}$ over K/M.

In some case this is enough to obtain the inversion formula on $C_0^\infty(G,\tau)$. Here are two examples.

Theorem 3.4. *Let G/K be a (noncompact) Riemannian symmetric space of split-rank type, i.e., such that* $\operatorname{rank} G = \operatorname{rank} K + \operatorname{rank} G/K$. *(Thus G has only one conjugacy class of Cartan subalgebra, or equivalently, all restricted roots have even multiplicity.) Let $d\mu(U^{\sigma,i\lambda-\rho}) = p_\sigma(\lambda)d\lambda$ be the Plancherel measure associated with the (minimal) unitary principal series $U^{\sigma,i\lambda-\rho}$, where $d\lambda$ is a suitably normalized Euclidean measure on $\mathbf{a}^*$. Then, for a suitable normalization of the relevant Haar measures, the following inversion formula holds ($\forall f \in C_0^\infty(G,\tau)$, $\forall \tau \in \hat{K}$)*

$$(3.19) \qquad f(x) = w^{-1}\sum_\sigma \frac{1}{d_\sigma}\int_{\mathbf{a}^*}\int_K F^{i\lambda-\rho}(x^{-1}k)P_\sigma\tilde{f}(\lambda,k)p_\sigma(\lambda)d\lambda\, dk,$$

where w is the order of the Weyl group of (G,A), and the sum is over all inequivalent M-types σ contained in $\tau|_M$. (The integrals over K can also be written as integrals over K/M.)

Theorem 3.5. *Let G/K be a (noncompact) Riemannian symmetric space of rank one. Fix $\tau \in \hat{K}$, and suppose that no discrete series of G (if there is* *any) contain τ. Then there is a normalization of Haar measures such that* (3.19) *holds,* $\forall f \in C_0^\infty(G,\tau)$.

Proof. If G has only one conjugacy class of Cartan subalgebra, then the set which supports the Plancherel measure in $\hat{G}$ consists precisely of the irreducible components of the unitary principal series $U^{\sigma,i\lambda-\rho}$, with $\sigma \in \hat{M}$ and $\lambda \in \mathbf{a}_+^*$. If G has real rank one and no discrete series of G contain τ, then the elements of $\hat{G}(\tau)$ with nonzero Plancherel measure are again the irreducible components of $U^{\sigma,i\lambda-\rho}$, with $\sigma \subset \tau|_M$ and $\lambda \in \mathbf{a}_+^*$. In both cases we obtain, from Proposition 2.1,

$$\begin{aligned} f(x) &= \frac{1}{d_\tau}\sum_\sigma \int_{\mathbf{a}_+^*} (\varphi_\tau^{U^{\sigma,i\lambda-\rho}} * f)(x)\,d\mu(U^{\sigma,i\lambda-\rho}) \\ &= \frac{1}{d_\tau}\sum_\sigma \frac{1}{w}\int_{\mathbf{a}^*} (\varphi_\tau^{U^{\sigma,i\lambda-\rho}} * f)(x)\,p_\sigma(\lambda)\,d\lambda. \end{aligned} \tag{3.20}$$

We have used the Weyl invariance of the Plancherel density, and the fact that $U^{\sigma,i\lambda-\rho}$ is unitarily equivalent with $U^{\sigma',i\lambda'-\rho}$ if and only if there exists w in the Weyl group such that $\sigma' = w\sigma$, and $\lambda' = w\lambda$ (see [8], Theorem 5.5.3.3). Using (3.17) in (3.20) proves (3.19). □

4. τ-spherical functions for the generalized principal series and the Helgason Fourier transform.

Let $\tau \in \hat{K}$ be fixed. The elements of $\hat{G}(\tau)$ with nonzero Plancherel measure consist not only of the irreducible components of the unitary principal series $U^{\sigma,i\lambda-\rho}$ $(\sigma \subset \tau|_M)$. In general τ will also be contained in some discrete series representations (when $\operatorname{rank} G = \operatorname{rank} K$), and if $\operatorname{rank} G/K > 1$, in some (generalized) principal series representations. These are the representations of G induced unitarily from certain representations (see below) of the nonminimal cuspidal parabolic (proper) subgroups of G.

In order to obtain the inversion formula for the Helgason Fourier transform in the general case, we need to discuss the discrete series and the other continuous series in the tempered spectrum. The general situation is formally similar to the minimal one, and can be described as follows.

Let P' be a cuspidal parabolic subgroup of G, with a (fixed) Langlands decomposition $P' = M'A'N'$. Let $(\sigma', V_{\sigma'})$ be in the discrete series of M', and let ν' be an element of $\mathbf{a}'^*$, the real dual of the Lie algebra $\mathbf{a}'$ of A'. Let $U^{\sigma'\nu'}$ be the generalized principal series representation of G defined by

$$U^{\sigma'\nu'} = \operatorname{ind}_{P'}^G(\sigma' \otimes \exp(i\nu') \otimes \mathbf{1}).$$

Then $U^{\sigma'\nu'}$ is unitary, and depends only on the class of A', i.e., the set of all parabolic subgroups whose split component is A'. Moreover $U^{\sigma'\nu'}$ is irreducible provided that ν' is regular, i.e., $\langle \nu', \beta \rangle \neq 0$ for all roots β of $(\mathbf{g}, \mathbf{a}')$ (see [5], Theorem 14.15, p. 540). If P' is minimal, we may assume $P' = P = MAN$, where $G = KAN$ is an Iwasawa decomposition of G, and M is the centralizer of A in K. In this case $U^{\sigma\nu}$ is the (minimal) unitary principal series considered in Section 3, and denoted $U^{\sigma, i\nu - \rho}$ there. If G has discrete series, then G itself is cuspidal parabolic. If we take $P' = M' = G$, then the parameter ν' is trivial, and $U^{\sigma'} = \sigma'$ is in the discrete series of G. The irreducible components of the unitary representations $U^{\sigma'\nu'}$ constitute the so-called tempered spectrum of G. This is the set which supports the Plancherel measure in $\hat{G}$ [8].

In general, we may assume $A' \subset A$, $N' \subset N$, $M' \supset M$, and $P' \supset P$. As in the minimal case, A' and M' normalize N', and A' commutes with M', but M' is, in general, a noncompact reductive group. Since P' is cuspidal, M' has a compact Cartan subgroup contained in K', where

$$K' \equiv K \bigcap M' = K \bigcap P'$$

is a maximal compact subgroup of M'. Also $M \subset K' \subset K$. Every element of G can be written (nonuniquely) as kp', with $k \in K$ and $p' \in P'$. We write the decomposition $G = KM'A'N'$ as

$$x = \mathbf{k}'(x)\mu(x)e^{H'(x)}n'(x), \tag{4.1}$$

$\mathbf{k}'(x) \in K$, $\mu(x) \in M'$, $e^{H'(x)} \in A'$, and $n'(x) \in N'$. Here the A' and N' components are well defined, but there is an ambiguity in the definition of $\mathbf{k}'(x)$ and $\mu(x)$. (The ambiguity can be removed by requiring that $\theta(\mu(x)) = (\mu(x))^{-1}$.) As in [5], Proposition 7.14, p. 186, or Theorem 7.22, p. 196, we choose A_1 and N_1 as Iwasawa A and N components of M', so that $M' = K'A_1N_1$, $A = A'A_1$, and $N = N'N_1$. Then M is also the centralizer of A_1 in K', and $P_1 = MA_1N_1$ is a minimal parabolic subgroup of M' (see [5], p. 240). We call $\mathbf{k}_1$, H_1 and n_1 the corresponding Iwasawa functions on M',

$$m' = \mathbf{k}_1(m')e^{H_1(m')}n_1(m'), \qquad m' \in M', \tag{4.2}$$

$\mathbf{k}_1(m') \in K'$, $e^{H_1(m')} \in A_1$, $n_1(m') \in N_1$. Using (4.2) for $\mu(x)$ in (4.1), and comparing with (1.1), we find the following relations between the Iwasawa functions $\mathbf{k}$, $\mathbf{k}'$, $\mathbf{k}_1$, H, H', and H_1:

$$\mathbf{k}(x) = \mathbf{k}'(x)\mathbf{k}_1(\mu(x)), \tag{4.3}$$

$$H(x) = H'(x) + H_1(\mu(x)). \tag{4.4}$$

The product $\mathbf{k}'(x)\mu(x)$ is well defined, and satisfies

$$\mathbf{k}'(kx)\mu(kx) = k\mathbf{k}'(x)\mu(x), \qquad \forall k \in K,$$

(4.5)

$$\mathbf{k}'(xm'a'n')\mu(xm'a'n') = \mathbf{k}'(x)\mu(x)m', \quad \forall m' \in M',\ \forall a' \in A',\ \forall n' \in N'.$$

We also have

$$H'(xm'a'n') = H'(x) + \log a'.$$

By the Subrepresentation Theorem [5], we can identify $U^{\sigma'\nu'}$ with a subrepresentation of a nonunitary (minimal) principal series $U^{\tilde{\sigma}'\mu}$, for suitable parameters $\tilde{\sigma}' \in \hat{M}$, and $\mu \in \mathbf{a}^*_{\mathbf{C}}$. It is actually enough to employ the Subrepresentation Theorem for discrete series, in view of the following result.

Proposition 4.1. *Let σ' be in the discrete series of M', and choose parameters $\tilde{\sigma}' \in \hat{M}$ and $\mu_1 \in {\mathbf{a}_1}^*$ ($\mathbf{a}_1$ the Lie algebra of A_1) by the Subrepresentation Theorem, so that σ' is infinitesimally equivalent with a subrepresentation of the nonunitary (minimal) principal series of M' given by*

$$\omega_{\tilde{\sigma}'\mu_1} = \mathrm{ind}_{MA_1N_1}^{M'}(\tilde{\sigma}' \otimes e^{\mu_1} \otimes \mathbf{1}).$$

Then the generalized principal series $U^{\sigma'\nu'} = \mathrm{ind}_{M'A'N'}^{G}(\sigma' \otimes e^{i\nu'} \otimes \mathbf{1})$ is infinitesimally equivalent with a subrepresentation of the nonunitary (minimal) principal series of G

$$U^{\tilde{\sigma}',i\nu'+\mu_1-\rho} = \mathrm{ind}_{MAN}^{G}(\tilde{\sigma}' \otimes e^{i\nu'+\mu_1} \otimes \mathbf{1})$$

(*in the notations of Section 3*).

Proof. From a double induction formula (see [5], p. 171, 240), we obtain a canonical equivalence of representations

$$\begin{aligned}&\mathrm{ind}_{M'A'N'}^{G}\left(\mathrm{ind}_{MA_1N_1}^{M'}(\tilde{\sigma}' \otimes e^{\mu_1} \otimes \mathbf{1}) \otimes e^{i\nu'} \otimes \mathbf{1}\right)\\ &\cong \mathrm{ind}_{M(A'A_1)(N'N_1)}^{G}(\tilde{\sigma}' \otimes e^{i\nu'+\mu_1} \otimes \mathbf{1}).\end{aligned}$$

It follows that if σ' is a subrepresentation of $\omega_{\tilde{\sigma}'\mu_1}$, then $U^{\sigma'\nu'}$ may be regarded as a subrepresentation of $U^{\tilde{\sigma}',i\nu'+\mu_1-\rho}$. See [5], p. 240, for the precise identification of the representation spaces. □

Remark. An explicit embedding of a discrete series representation as a subrepresentation of a nonunitary principal series was given by Knapp and

Wallach in [6]. Using this work, one can give an explicit description of the parameters $\tilde{\sigma}'$ and μ_1 in Proposition 4.1.

Now fix σ' and ν' so that $\tau \subset U^{\sigma'\nu'}|_K$. For simplicity, denote the representation space of $U^{\sigma'\nu'}$ by $\mathcal{H}_{\sigma'}$, and put $\mathcal{H}_\tau = P_\tau \mathcal{H}_{\sigma'}$ (P_τ given by (2.1)), $\Psi_\tau^{U^{\sigma'\nu'}}(x) = P_\tau U^{\sigma'\nu'}(x) P_\tau$, $\psi_\tau^{U^{\sigma'\nu'}}(x) = \mathrm{Tr}\ \Psi_\tau^{U^{\sigma'\nu'}}(x)$, and define $\varphi_\tau^{U^{\sigma'\nu'}}(x)$ as in (2.4). Our aim is to find a (minimal) Eisenstein integral representation of $\varphi_\tau^{U^{\sigma'\nu'}}(x)$.

Let $\tilde{\sigma}'$ and μ_1 be determined as in Proposition 4.1. Put $\mu = i\nu' + \mu_1 - \rho$. Since μ_1 is real, the conjugate dual parameter to μ is $\mu' = -\bar{\mu} - 2\rho = i\nu' - \mu_1 - \rho$. We identify the representation spaces $E^{\tilde{\sigma}'\mu}$ and $E^{\tilde{\sigma}'\mu'}$ with $\mathcal{H}_{\tilde{\sigma}'} = L^2(K, \tilde{\sigma}')$ (cf. Section 3). By Proposition 4.1, there is a surjective linear map

$$S : \mathcal{H}_{\tilde{\sigma}'} \to \mathcal{H}_{\sigma'}$$

which carries the K-finite vectors of $U^{\tilde{\sigma}'\mu'}$ onto the K-finite vectors of $U^{\sigma'\nu'}$ in a $\mathbf{g}$-equivariant fashion. (If $U^{\sigma'\nu'}$ is in the discrete series of G, then S may be taken as the Szegö map of Knapp and Wallach, see [6].) Let $\ker S$ be the kernel of S. Since $U^{\sigma'\nu'}(x)S = SU^{\tilde{\sigma}'\mu'}(x)$, $\ker S$ is invariant under $U^{\tilde{\sigma}'\mu'}$, and $(\ker S)^\perp$ is invariant under $U^{\tilde{\sigma}'\mu}$. The adjoint of S gives the embedding of $U^{\sigma'\nu'}$ as a subrepresentation of $U^{\tilde{\sigma}'\mu}$ on $(\ker S)^\perp$.

Let m and n be the multiplicities of τ in $U^{\tilde{\sigma}'\mu}|_K$ and $U^{\sigma'\nu'}|_K$, respectively. In general we have $m \geq n$. Put $\tilde{\mathcal{H}}_\tau = P_\tau \mathcal{H}_{\tilde{\sigma}'}$, and write

$$\tilde{\mathcal{H}}_\tau = \mathcal{H}_1 \oplus \mathcal{H}_2,$$

where $\mathcal{H}_1 \subset (\ker S)^\perp$, and $\mathcal{H}_2 \subset \ker S$. Let P_1 and P_2 be the orthogonal projections onto $\mathcal{H}_1$ and $\mathcal{H}_2$, so that $P_\tau = P_1 + P_2$. We shall now prove that the spherical trace function $\psi_\tau^{U^{\sigma'\nu'}}$ equals the function $\psi_1 = \mathrm{Tr}[P_1 U^{\tilde{\sigma}'\mu'} P_1]$. Since $U^{\sigma'\nu'}$ is unitary, $\psi_\tau^{U^{\sigma'\nu'}}$ will also equal $\mathrm{Tr}[P_1 U^{\tilde{\sigma}'\mu} P_1]$.

Let $\{P_\xi\}_{\xi=1,\dots,m}$ be an orthonormal basis of $\mathrm{Hom}_K(\mathcal{H}_{\tilde{\sigma}'}, V_\tau)$ (cf. Section 2), such that

$$P_\xi^* V_\tau \subset \mathcal{H}_1, \quad \xi = 1, \dots, n,$$
$$P_\xi^* V_\tau \subset \mathcal{H}_2, \quad \xi = n+1, \dots, m.$$

We have $P_1 = \sum_{\xi=1}^n P_\xi^* P_\xi$, $P_2 = \sum_{\xi=n+1}^m P_\xi^* P_\xi$.

Let $\{\mathbf{v}_a\}_{a=1,\dots,d_\tau}$ be an orthonormal basis of V_τ, then the elements $f_{a\xi} = P_\xi^* \mathbf{v}_a$ give an orthonormal basis of $\mathcal{H}_1$ for $\xi = 1, \dots, n$, and of $\mathcal{H}_2$ for $\xi = n+1, \dots, m$. Clearly $\{g_{a\xi} = S(f_{a\xi})\}_{\xi=1,\dots,n}$ is a basis of $\mathcal{H}_\tau$ (not orthonormal, in general). Observing that $SP_\tau = SP_1 = P_\tau S$, we have for $\xi = 1, \dots, n$,

$$\Psi_\tau^{U^{\sigma'\nu'}}(x) g_{a\xi} = P_\tau U^{\sigma'\nu'}(x) P_\tau\, S f_{a\xi}$$

$$= P_\tau U^{\sigma'\nu'}(x) S\, P_1 f_{a\xi}$$
$$= P_\tau S\, U^{\tilde\sigma'\mu'}(x) P_1 f_{a\xi}$$
$$= S\, P_1 U^{\tilde\sigma'\mu'}(x) P_1 f_{a\xi}.$$

We see that the matrix representing $\Psi_\tau^{U^{\sigma'\nu'}}(x)$ in the basis $\{g_{a\xi}\}$ is the same as the matrix representing $\Psi_1(x) = P_1 U^{\tilde\sigma'\mu'}(x) P_1$ in the basis $\{f_{a\xi}\}$. Therefore $\psi_\tau^{U^{\sigma'\nu'}}$ equals $\psi_1 = \operatorname{Tr}\Psi_1$, as claimed. In turn this implies that $\varphi_\tau^{U^{\sigma'\nu'}}$ equals the $\operatorname{End}(V_\tau)$-valued function φ_1 on G given by

$$\varphi_1(x) = d_\tau \int_K \psi_1(xk^{-1})\tau(k)dk = \sum_{\xi=1}^{n} P_\xi U^{\tilde\sigma'\mu'}(x) P_\xi^*.$$

In order to find an explicit integral representation of $\varphi_\tau^{U^{\sigma'\nu'}}(x)$, we write down the basis of $\tilde{\mathcal{H}}_\tau$ more precisely as follows.

For $\mathbf{v} \in V_\tau$ and $T \in \operatorname{Hom}_M(V_\tau, V_{\tilde\sigma'})$, consider the function from K to $V_{\tilde\sigma'}$ given by

$$f_{\mathbf{v}T}(k) = \left(\frac{d_\tau}{d_{\tilde\sigma'}}\right)^{1/2} T\tau(k^{-1})\mathbf{v}.$$

This function is in $L^2(K, \tilde\sigma')$, and an easy computation shows that $f_{\mathbf{v}T} \in \tilde{\mathcal{H}}_\tau$. Moreover we have

$$\langle f_{\mathbf{v}T}, f_{\mathbf{v}'T'}\rangle = \langle \mathbf{v}, \mathbf{v}'\rangle \frac{1}{d_{\tilde\sigma'}} \operatorname{Tr}(T^*T').$$

Let $\{T_\xi\}_{\xi=1,\dots,m}$ be a basis of $\operatorname{Hom}_M(V_\tau, V_{\tilde\sigma'})$, orthonormal with respect to the scalar product $\langle P, Q\rangle = \frac{1}{d_{\tilde\sigma'}} \operatorname{Tr}(P^*Q)$. Then the functions $f_{\mathbf{v}\xi} \equiv f_{\mathbf{v},T_\xi}$ ($\mathbf{v} \in V_\tau$, $\xi = 1, \dots, m$) span $\tilde{\mathcal{H}}_\tau$. We assume (as we may) that they span $\mathcal{H}_1$ for $\xi = 1, \dots, n$, and $\mathcal{H}_2$ for $\xi = n+1, \dots, m$. The basis $\{P_\xi\}$ of $\operatorname{Hom}_K(\mathcal{H}_{\tilde\sigma'}, V_\tau)$ is then chosen so that

$$P_\xi^* \mathbf{v} = f_{\mathbf{v}\xi}.$$

We extend $f_{\mathbf{v}\xi}$ to $E^{\tilde\sigma'\mu'}$ by means of (3.5), and denote it by the same symbol. The matrix coefficients of $\varphi_\tau^{U^{\sigma'\nu'}}(x)$ are calculated as follows:

$$\begin{aligned}
&\langle \varphi_\tau^{U^{\sigma'\nu'}}(x)\mathbf{v}, \mathbf{v}'\rangle \\
&= \langle \varphi_1(x)\mathbf{v}, \mathbf{v}'\rangle \\
&= \sum_{\xi=1}^{n} \langle P_\xi U^{\tilde\sigma'\mu'}(x) P_\xi^* \mathbf{v}, \mathbf{v}'\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\xi=1}^{n} \langle U^{\tilde\sigma'\mu'}(x) f_{\mathbf{v}\xi}, f_{\mathbf{v}'\xi}\rangle \\
&= \sum_{\xi=1}^{n} \int_K \langle f_{\mathbf{v}\xi}(x^{-1}k), f_{\mathbf{v}'\xi}(k)\rangle dk \\
&= \frac{d_\tau}{d_{\tilde\sigma'}} \sum_{\xi=1}^{n} \int_K \langle T_\xi \tau(\mathbf{k}(x^{-1}k)^{-1})\mathbf{v}, T_\xi \tau(k^{-1})\mathbf{v}'\rangle e^{-(\mu'+2\rho)(H(x^{-1}k))} dk \\
&= \frac{d_\tau}{d_{\tilde\sigma'}} \sum_{\xi=1}^{n} \int_K \langle \tau(k) T_\xi^* T_\xi \tau(\mathbf{k}(x^{-1}k)^{-1})\mathbf{v}, \mathbf{v}'\rangle e^{-(\mu'+2\rho)(H(x^{-1}k))} dk \\
&= \frac{d_\tau}{d_{\tilde\sigma'}} \sum_{\xi=1}^{n} \int_K \langle \tau\left(\mathbf{k}(xk)\right) T_\xi^* T_\xi \tau(k^{-1})\mathbf{v}, \mathbf{v}'\rangle e^{\mu'(H(xk))} dk, \qquad (4.6)
\end{aligned}
$$

where we have used (3.4) in the last step. This proves the following result.

Proposition 4.2. *The τ-spherical function $\varphi_\tau^{U^{\sigma'\nu'}}$ is given by*

$$
\varphi_\tau^{U^{\sigma'\nu'}}(x) = \frac{d_\tau}{d_{\tilde\sigma'}} \int_K \tau(\mathbf{k}(xk)) T_{\tilde\sigma'} \tau(k^{-1}) e^{(i\nu'-\mu_1-\rho)(H(xk))} dk, \qquad (4.7)
$$

where

$$
T_{\tilde\sigma'} = \sum_{\xi=1}^{n} T_\xi^* T_\xi \in \mathrm{End}_M(V_\tau).
$$

Remark. The integral representation (4.7) is invariant under $\mu_1 \to -\mu_1$. This follows from the fact that $U^{\sigma'\nu'}$ is unitary, so that $\varphi_\tau^{U^{\sigma'\nu'}}(x)^* = \varphi_\tau^{U^{\sigma'\nu'}}(x^{-1})$.

Remark. Note that the operator $P_{\tilde\sigma'} = \sum_{\xi=1}^{m} T_\xi^* T_\xi$ is precisely the projector of V_τ onto $H_{\tilde\sigma'} \simeq V_{\tilde\sigma'} \otimes \mathbf{C}^m \subset V_\tau$ (the subspace of vectors of V_τ which transform under M according to $\tilde\sigma'$, see Lemma 3.2). We have $P_{\tilde\sigma'} = T_{\tilde\sigma'} + Q_{\tilde\sigma'}$, where $Q_{\tilde\sigma'} = \sum_{\xi=n+1}^{m} T_\xi^* T_\xi$. $T_{\tilde\sigma'}$ and $Q_{\tilde\sigma'}$ are projection operators, namely they project V_τ onto the subspaces $V_{\tilde\sigma'} \otimes \mathbf{C}^n$ and $V_{\tilde\sigma'} \otimes \mathbf{C}^{m-n}$ of $H_{\tilde\sigma'}$. If $m = n$ then $\tilde{\mathcal{H}}_\tau \subset (\ker S)^\perp$ and $\mathcal{H}_2 = \{0\}$. In this case $P_{\tilde\sigma'} = T_{\tilde\sigma'}$, and $\varphi_\tau^{U^{\sigma'\nu'}}$ equals the minimal spherical function $\varphi_\tau^{U^{\tilde\sigma'\mu}}$. The simplest example of this is when τ is multiplicity free, i.e., τ occurs at most once in every irreducible unitary representation of G. This is equivalent to the condition that all M-types contained in $\tau|_M$ occur with multiplicity one. In this case every spherical trace function of type τ on G equals a minimal one $\psi_\tau^{U^{\sigma\mu}}$, for suitable $\sigma \subset \tau|_M$, and $\mu \in \mathbf{a}_{\mathbf{C}}^*$.

Eq. (4.7) can also be proved by using the (nonminimal) Eisenstein integral representation of $\varphi_\tau^{U^{\sigma'\nu'}}$. According to this, $\varphi_\tau^{U^{\sigma'\nu'}}$ can be written as

$$\varphi_\tau^{U^{\sigma'\nu'}}(x) = E(P', F^{\sigma'}, \nu')(x), \tag{4.8}$$

where $E(P', f, \nu')$ is the Eisenstein integral[1]

$$E(P', f, \nu')(x) = \int_K e^{(i\nu'-\rho')(H'(xk))} f(xk)\tau(k^{-1})dk. \tag{4.9}$$

Here ρ' is the ρ-function for the roots of $(\mathbf{g}, \mathbf{a}')$, and the function $F^{\sigma'}$: $G \to \mathrm{End}(V_\tau)$ is defined as follows. Write $\tau|_{K'} = \sum_{\omega\in\hat{K}'} p_\omega \omega$, and $\sigma'|_{K'} = \sum_{\omega\in\hat{K}'} n_\omega \omega$. Frobenius Reciprocity implies that the multiplicity m_τ of τ in $U^{\sigma'\nu'}|_K$ is $m_\tau = \sum_{\omega\in\hat{K}'} p_\omega n_\omega$. (The sum here is finite, being only over those classes for which $p_\omega > 0$ and $n_\omega > 0$.) We identify a class $\omega \in \hat{K}'$ with a representative (denoted (ω, V_ω)) in that class. Let $\{P_j^{(\omega)}\}$ be a basis of $\mathrm{Hom}_{K'}(V_\tau, V_\omega)$, orthonormal with respect to $\langle P, P'\rangle = \frac{1}{d_\omega}\mathrm{Tr}\,(P^*P')$. Let $\psi_\omega^{\sigma'}$ be the spherical trace function of type ω on M' relative to σ'. Let $\varphi_\omega^{\sigma'} : M' \to \mathrm{End}(V_\omega)$ be the ω-spherical function on M' defined in the usual way, i.e.,

$$\varphi_\omega^{\sigma'}(m') = d_\omega \int_{K'} \omega(k')\psi_\omega^{\sigma'}(m'k'^{-1})dk'$$

(cf. (2.4)). Define $\hat{F}^{\sigma'} : M' \to \mathrm{End}(V_\tau)$ by

$$\hat{F}^{\sigma'}(m') = \sum_{\omega\in\hat{K}'} \frac{d_\tau}{d_\omega} \sum_{j=1}^{p_\omega} P_j^{(\omega)*}\varphi_\omega^{\sigma'}(m')P_j^{(\omega)}.$$

Then the function $F^{\sigma'}$ is obtained by extending $\hat{F}^{\sigma'}$ to all of G by letting $F^{\sigma'}(km'a'n') = \tau(k)\hat{F}^{\sigma'}(m')$, or equivalently (since $\mathbf{k}'(km'a'n')\mu(km'a'n') = km'$)

$$F^{\sigma'}(x) = \tau(\mathbf{k}'(x))\hat{F}^{\sigma'}(\mu(x)).$$

The function $F^{\sigma'}$ satisfies $F^{\sigma'}(kxk') = \tau(k)F^{\sigma'}(x)\tau(k')$, $\forall k \in K$, $\forall k' \in K'$.

---

[1] The relation between Eisenstein integrals and matrix coefficients of principal series is of course well known, see, e.g., [5], Prop. 14.3, p. 525 or [7], Vol. II, Section 13.1. (4.9) is only a special instance of Eisenstein integral. It corresponds to the case when the double representation of K is the canonical representation $\tau(k_1, k_2)T = \tau(k_1)T\tau(k_2)$ on $W = \mathrm{End}(V_\tau)$, see, e.g., [7], Vol. II, p. 216.

Applying the Subrepresentation Theorem to σ' (cf. Proposition 4.1), we can write $\varphi_\omega^{\sigma'}(m')$ as the following Eisenstein integral (relative to M'):

$$\varphi_\omega^{\sigma'}(m') = \frac{d_\omega}{d_{\tilde{\sigma}'}} \int_{K'} \omega(\mathbf{k}_1(m'k'))T_{\tilde{\sigma}'}^{(\omega)}\omega(k'^{-1})e^{(-\mu_1-\rho_1)(H_1(m'k'))}dk',$$

where ρ_1 is the ρ-function for M', and $T_{\tilde{\sigma}'}^{(\omega)} = \sum_{j=1}^{n_\omega} T_j^{(\omega)*}T_j^{(\omega)}$. [Here $\{T_j^{(\omega)}\}_{j=1,\dots,q_\omega}$ is an orthonormal basis of $\mathrm{Hom}_M(V_\omega, V_{\tilde{\sigma}'})$, q_ω is the multiplicity of $\tilde{\sigma}'$ in $\omega|_M$, and $q_\omega \geq n_\omega$, in general.]

Substituting this in (4.8) gives

$$\begin{aligned} \varphi_\tau^{U^{\sigma'\nu'}}(x) = \frac{d_\tau}{d_{\tilde{\sigma}'}} \int_{K\times K'} &\tau(\mathbf{k}'(xk)\mathbf{k}_1(\mu(xk)k'))T_{\tilde{\sigma}'}\tau((kk')^{-1}) \\ &\times e^{(-\mu_1-\rho_1)(H_1(\mu(xk)k'))+(i\nu'-\rho')(H'(xk))}dk\,dk', \end{aligned} \tag{4.10}$$

where we have used

$$T_{\tilde{\sigma}'} = \sum_{\omega\in\hat{K}'}\sum_{j=1}^{p_\omega} P_j^{(\omega)*}T_{\tilde{\sigma}'}^{(\omega)}P_j^{(\omega)},$$

which is easy to prove by Frobenius Reciprocity.

Using (4.5) and (4.3)-(4.4), one can show that $\forall x \in G$, $\forall k \in K$, and $\forall k' \in K'$,

$$\begin{aligned} \mathbf{k}'(xk)\mathbf{k}_1(\mu(xk)k') &= \mathbf{k}(xkk'), \\ H_1(\mu(xk)k') &= H_1(\mu(xkk')). \end{aligned}$$

Making the change of variables $kk' = \tilde{k}$ in the integral over K in (4.10), we see that the integral over K' drops out, and we obtain (4.7), since $\mathbf{a} = \mathbf{a}_1 \oplus \mathbf{a}'$, and (by (4.4)) $(-\mu_1-\rho_1)(H_1(\mu(xk))) = (-\mu_1-\rho_1)(H(xk))$, $(i\nu'-\rho')(H'(xk)) = (i\nu'-\rho')(H(xk))$, while the ρ-functions satisfy $\rho = \rho_1 + \rho'$.

We can now prove our main result.

Theorem 4.3. *Let G/K be a Riemannian symmetric space of the noncompact type. For $\tau \in \hat{K}$, let E^τ be the homogeneous vector bundle over G/K associated with τ, and define $C_0^\infty(G,\tau)$ and $L^2(G,\tau)$ in the usual way. Let $G = KAN$ be an Iwasawa decomposition of G, let M be the centralizer of A in K, and let $P = MAN$ be the corresponding minimal parabolic subgroup of G. Define the Helgason Fourier transform of $f \in C_0^\infty(G,\tau)$ as the map $\tilde{f} : \mathbf{a}_{\mathbf{C}}^* \times K \to V_\tau$ given by*

$$\tilde{f}(\lambda, k) = \int_G F^{i\bar{\lambda}-\rho}(x^{-1}k)^* f(x)dx, \qquad \lambda \in \mathbf{a}_{\mathbf{C}}^*, k \in K,$$

where $F^{\mu}(x)$ is given by (3.10), and $*$ denotes adjoint.

Let P' be a cuspidal parabolic subgroup of G such that $P' \supseteq P$ and $A' \subseteq A$, where $P' = M'A'N'$ is a Langlands decomposition of P'. Let $K' = K \bigcap M'$ be maximal compact in M', and let $M' = K'A_1N_1$ be an Iwasawa decomposition of M' so that $A = A'A_1$ and $N = N'N_1$. Given σ' in the discrete series of M', choose parameters $\tilde{\sigma}' \in \hat{M}$ and $\mu_1 \in \mathbf{a}_1{}^*$ by the Subrepresentation Theorem, so that σ' is infinitesimally equivalent with a subrepresentation of $\mathrm{ind}_{MA_1N_1}^{M'}(\tilde{\sigma}' \otimes e^{\mu_1} \otimes \mathbf{1})$ (see [6]).

Let $U^{\sigma'\nu'}$ denote the representation $\mathrm{ind}_{P'}^{G}(\sigma' \otimes e^{i\nu'} \otimes \mathbf{1})$ in the unitary principal P'-series with parameters σ' and $\nu' \in \mathbf{a}'^*$ (the real dual of the Lie algebra of A'). If $\tau \subset U^{\sigma'\nu'}|_K$, define $T_{\tilde{\sigma}'}$ as in Proposition 4.2. Let $d\mu(U^{\sigma'\nu'}) = p_{\sigma'}(\nu')d\nu'$ be the Plancherel measure associated with $U^{\sigma'\nu'}$, where $d\nu'$ is a properly normalized Euclidean measure on $\mathbf{a}'^*$. Then, for a suitable normalization of the relevant Haar measures and for suitable constants $c_{P'} > 0$, the following inversion formula holds

(4.11)
$$f(x) = \sum_{P'} c_{P'} \sum_{\sigma'} \frac{1}{d_{\tilde{\sigma}'}} \int_{\mathbf{a}'^*} \int_K F^{i\nu'+\mu_1-\rho}(x^{-1}k) T_{\tilde{\sigma}'} \tilde{f}(\nu' - i\mu_1, k)\, p_{\sigma'}(\nu') d\nu' dk.$$

Here the sum $\sum_{P'}$ is over all cuspidal parabolic subgroups P' of G such that $P' \supseteq P$ and $A' \subseteq A$, and the sum $\sum_{\sigma'}$ is over all discrete series σ' of M' such that $U^{\sigma'\nu'}|_K \supset \tau$. [These are in finite number only, since each ω in $\tau|_{K'}$ can only be contained in finitely many σ', see [5], Corollary 12.22, p. 455.]

Proof. We write the Plancherel (inversion) formula on G as (see, e.g., [7], Theorem 13.4.1 or [5], Theorem 13.11)

$$f(x) = \sum_{P'} c_{P'} \sum_{\sigma'} \int_{\mathbf{a}'^*} \Theta_{\sigma'\nu'}(f \circ L_x) p_{\sigma'}(\nu') d\nu', \qquad f \in C_0^\infty(G), \tag{4.12}$$

where $\Theta_{\sigma'\nu'}$ is the global character of $U^{\sigma'\nu'}$, and the sum $\sum_{\sigma'}$ is over all discrete series of M'. For $f \in C_0^\infty(G, \tau)$ we find from (4.12) and Proposition 2.1

$$f(x) = \frac{1}{d_\tau} \sum_{P'} c_{P'} \sum_{\sigma'} \int_{\mathbf{a}'^*} (\varphi_\tau^{U^{\sigma'\nu'}} * f)(x)\, p_{\sigma'}(\nu') d\nu'.$$

From (4.7) we find (by the same proof as in Proposition 3.3)

$$\begin{aligned}\varphi_\tau^{U^{\sigma'\nu'}}(x^{-1}y) &= \frac{d_\tau}{d_{\tilde{\sigma}'}} \int_K F^{i\nu'-\mu_1-\rho}(x^{-1}k) T_{\tilde{\sigma}'} F^{i\nu'+\mu_1-\rho}(y^{-1}k)^* dk \\ &= \frac{d_\tau}{d_{\tilde{\sigma}'}} \int_K F^{i\nu'+\mu_1-\rho}(x^{-1}k) T_{\tilde{\sigma}'} F^{i\nu'-\mu_1-\rho}(y^{-1}k)^* dk.\end{aligned}$$

It follows that

$$(4.13)\qquad \begin{aligned}(\varphi_\tau^{U^{\sigma'\nu'}} * f)(x) &= \int_G \varphi_\tau^{U^{\sigma'\nu'}}(x^{-1}y)f(y)dy \\ &= \frac{d_\tau}{d_{\tilde{\sigma}'}}\int_K F^{i\nu'+\mu_1-\rho}(x^{-1}k)T_{\tilde{\sigma}'}\tilde{f}(\nu'-i\mu_1,k)dk.\end{aligned}$$

This proves (4.11). □

A standard computation, using (4.11), gives the following formula for the scalar product (1.10) of $f_1, f_2 \in C_0^\infty(G,\tau)$:

$$\begin{aligned}&\langle f_1, f_2\rangle \\ &= \sum_{P'} c_{P'} \sum_{\sigma'} \frac{1}{d_{\tilde{\sigma}'}} \int_{\mathbf{a}'^*}\int_K \langle T_{\tilde{\sigma}'}\tilde{f}_1(\nu'+i\mu_1,k), T_{\tilde{\sigma}'}\tilde{f}_2(\nu'-i\mu_1,k)\rangle\, p_{\sigma'}(\nu')d\nu' dk.\end{aligned}$$

Remark. If $\operatorname{rank} G = \operatorname{rank} K$, the contribution of $P' = G$ in (4.12) is the discrete series contribution. In this case the parameter ν' is trivial and the integral over $\mathbf{a}'^*$ drops out. The corresponding contribution in (4.11) is

$$f_{\text{discrete}}(x) = c_G \sum_\sigma \frac{1}{d_{\tilde{\sigma}}}\int_K F^{\mu_1-\rho}(x^{-1}k)T_{\tilde{\sigma}}\tilde{f}(-i\mu_1,k)p_\sigma dk,$$

where the sum is over all discrete series σ of G containing τ, and p_σ is the formal degree of σ. The parameters $\tilde{\sigma} \in \hat{M}$ and $\mu_1 \in \mathbf{a}^*$ are chosen by the Subrepresentation Theorem, so that σ is infinitesimally equivalent with a subrepresentation of $\operatorname{ind}_{MAN}^G(\tilde{\sigma}\otimes e^{\mu_1}\otimes \mathbf{1})$, see [6].

Remark. The Plancherel density $p_{\sigma'}(\nu')$ is determined by the asymptotic form of $\varphi_\tau^{U^{\sigma'\nu'}}(a)$ $(a \in A)$ at infinity in the positive Weyl chamber. By proceeding as in the scalar case, we find that

$$p_{\sigma'}(\nu') \propto (\operatorname{Tr}[C_{\sigma'}^\tau(\nu')C_{\sigma'}^\tau(\nu')^*])^{-1},$$

where (cf. (1.4)-(1.5))

$$\begin{aligned}C_{\sigma'}^\tau(\nu') &= \lim_{t\to+\infty} a_t^{(-i\nu'-\mu_1+\rho)}\varphi_\tau^{U^{\sigma'\nu'}}(a_t) \\ &= \frac{d_\tau}{d_{\tilde{\sigma}'}}T_{\tilde{\sigma}'}\int_{\bar{N}} e^{-(i\nu'+\mu_1+\rho)(H(\bar{n}))}\tau(\mathbf{k}(\bar{n})^{-1})d\bar{n}.\end{aligned}$$

The integral here converges absolutely as long as $\Re(i\nu'+\mu_1) \in \mathbf{a}_+^*$, and it is defined by meromorphic continuation for the other values of ν'.

5. The spherical transform.

Let f be a compactly supported smooth function on G which is biinvariant under K, $f(kxk') = f(x)$, $\forall k, k' \in K$. It is well known that the Fourier transform of f reduces to the spherical transform $\hat{f}(\lambda)$ in this case. Indeed, using the integral formula for the Cartan decomposition and the identity (cf. [3], Eq. (7), p. 419)

$$\phi_\lambda(x^{-1}) = \phi_{-\lambda}(x)$$

in (1.2), we find for f K-biinvariant

$$\tilde{f}(\lambda, b) = \hat{f}(\lambda) \equiv \int_G \phi_\lambda(x) f(x) dx. \tag{5.1}$$

From (1.8) we obtain

$$(f * \phi_\lambda)(x) = \hat{f}(-\lambda)\phi_{-\lambda}(x^{-1}) = \hat{f}(-\lambda)\phi_\lambda(x). \tag{5.2}$$

Using (5.1) in (1.3), or (5.2) in (1.6), gives the inversion formula for the spherical transform

$$\begin{aligned} f(x) &= w^{-1} \int_{\mathbf{a}^*} \hat{f}(\lambda)\phi_\lambda(x^{-1})|c(\lambda)|^{-2} d\lambda \\ &= w^{-1} \int_{\mathbf{a}^*} \hat{f}(-\lambda)\phi_\lambda(x)|c(\lambda)|^{-2} d\lambda. \end{aligned}$$

We want to generalize this to vector bundles. First we need to find the analog of K-biinvariant functions.

5.1. Radial systems of sections.

By a *radial system of sections* of E^τ we mean a map $F : G \to \mathrm{End}(V_\tau)$ such that

$$F(k_1 g k_2) = \tau(k_2^{-1}) F(g) \tau(k_1^{-1}), \qquad \forall g \in G, \ \forall k_1, k_2 \in K. \tag{5.3}$$

For any $\mathbf{v} \in V_\tau$, the vector valued function $f(g) = F(g)\mathbf{v}$ satisfies (1.9), and defines a *radial section* of E^τ. [We follow here Badertscher and Reimann [1], who studied vector fields over the real hyperbolic spaces.]

The radial systems of sections generalize the notion of K-biinvariant functions on G. They are called radial because they are determined by their restriction to the vector subgroup A, due to the Cartan decomposition $G = KAK$.

We denote by $C_0^\infty(G, \tau, \tau)$ and by $L^2(G, \tau, \tau)$ the obviously defined spaces of radial systems of sections, with scalar product

$$\langle F_1, F_2 \rangle = \int_G \mathrm{Tr}\, [F_1(x) F_2(x)^*]\, dx,$$

where $*$ denotes adjoint. For $F_1, F_2 \in C_0^\infty(G,\tau,\tau)$, define the convolution by

$$(F_1 * F_2)(x) = \int_G F_1(y^{-1}x)F_2(y)dy.$$

This definition is arranged so that $F_1 * F_2 \in C_0^\infty(G,\tau,\tau)$.

The convolution algebra $C_0^\infty(G,\tau,\tau)$ may be identified with a certain subalgebra of $C_0^\infty(G)$, which we now define. Let $I_{0,\tau}(G)$ denote the set of those $f \in C_0^\infty(G)$ which satisfy

$$f(kxk^{-1}) = f(x), \qquad x \in G, \;\; k \in K$$

(i.e., f is K-central), and

$$d_\tau \bar{\chi}_\tau * f = f = f * d_\tau \bar{\chi}_\tau. \tag{5.4}$$

Then $I_{0,\tau}(G)$ is a subalgebra of $C_0^\infty(G)$, and it is (anti)-isomorphic to $C_0^\infty(G,\tau,\tau)$. Indeed given $F \in C_0^\infty(G,\tau,\tau)$ define

$$f_F(x) \equiv d_\tau \operatorname{Tr} F(x).$$

It follows from (5.3) that f_F is K-central, and moreover it satisfies (5.4). Thus $f_F \in I_{0,\tau}(G)$. Viceversa, given $f \in I_{0,\tau}(G)$ put

$$F_f(x) \equiv \int_K \tau(k) f(kx)dk. \tag{5.5}$$

Then F_f is in $C_0^\infty(G,\tau,\tau)$. We have the following result (see [8], Vol. II, p. 3, Example 1).

Proposition 5.1. *The map $f \to F_f$ is a linear bijection of $I_{0,\tau}(G)$ onto $C_0^\infty(G,\tau,\tau)$. Its inverse is the map $F \to f_F$. These maps satisfy*

$$F_{f_1 * f_2} = F_{f_2} * F_{f_1},$$
$$f_{F_1 * F_2} = f_{F_2} * f_{F_1}.$$

5.2. Spherical functions of type τ on G.

Our objective is to find the analog of Eq. (5.2) for the convolution $\varphi_\tau^U * f$, when f is a radial section of E^τ, and U is in the tempered spectrum of G. In order to do this, we go back to the notations of Section 2.

Fix a representative (U, H_U) of $[U] \in \hat{G}(\tau)$. Let $H_\tau = P_\tau H_U$ (P_τ given by 2.1). We identify H_τ with $V_\tau \otimes \mathbf{C}^{\xi_U}$, by means of

$$f_{\mathbf{v}\xi} \leftrightarrow \mathbf{v} \otimes \mathbf{e}_\xi, \qquad \mathbf{v} \in V_\tau, \;\; \xi = 1, \ldots, \xi_U,$$

where $\{\mathbf{e}_\xi\}_{\xi=1,\dots,\xi_U}$ is a (fixed) orthonormal basis of $\mathbf{C}^{\xi_U}$. (Recall that $\xi_U = m(\tau, U)$.)

For $\mathcal{O} \in \mathrm{End}(V_\tau \otimes \mathbf{C}^{\xi_U})$, let $\mathrm{Tr}\,|_{V_\tau}\mathcal{O} \in \mathrm{End}(\mathbf{C}^{\xi_U})$ denote the partial trace of $\mathcal{O}$ with respect to V_τ. Put

$$\Phi_\tau^U(g) \equiv \frac{1}{d_\tau}\,\mathrm{Tr}\,|_{V_\tau}\Psi_\tau^U(g).$$

For each $g \in G$, $\Phi_\tau^U(g)$ is the element of $\mathrm{End}(\mathbf{C}^{\xi_U})$ with matrix coefficients

$$\begin{aligned}\langle \Phi_\tau^U(g)\mathbf{e}_\xi, \mathbf{e}_{\xi'}\rangle &= \frac{1}{d_\tau}\sum_{a=1}^{d_\tau}\langle U(g) f_{a\xi}, f_{a\xi'}\rangle \\ &= \frac{1}{d_\tau}\,\mathrm{Tr}\,[P_{\xi'}U(g)P_\xi^*].\end{aligned}$$

Notice that $\Phi_\tau^U(e) = \mathbf{1}$, and

$$\psi_\tau^U(g) = d_\tau\,\mathrm{Tr}\;\Phi_\tau^U(g). \tag{5.6}$$

The functions Ψ_τ^U and Φ_τ^U are related as follows.

Lemma 5.2. *Let the Haar measure on K be normalized by $\int_K dk = 1$. Then $\forall x \in G$*

$$\int_K \Psi_\tau^U(kxk^{-1})dk = \mathbf{1} \otimes \Phi_\tau^U(x), \tag{5.7}$$

$$d_\tau^2 \int_K \tau(k) \otimes \Phi_\tau^U(xk^{-1})dk = \Psi_\tau^U(x), \tag{5.8}$$

where $\mathbf{1}$ in (5.7) denotes the identity operator in V_τ.

Proof. Let $\Psi_{\tau,K}^U(x)$ denote the left-hand side of (5.7). We have

$$\Psi_{\tau,K}^U(x) = \int_K \tau_U(k)\Psi_\tau^U(x)\tau_U(k^{-1})dk. \tag{5.9}$$

Since $\Psi_{\tau,K}^U(x)$ commutes with all the $\tau_U(k)$, $k \in K$, it must be of the form

$$\Psi_{\tau,K}^U(x) = \mathbf{1} \otimes \Phi(x), \tag{5.10}$$

where Φ is a function on G with values in $\mathrm{End}(\mathbf{C}^{\xi_U})$. Taking the trace of both sides of (5.10) with respect to V_τ, and using (5.9), gives

$$\mathrm{Tr}\,|_{V_\tau}\Psi_{\tau,K}^U(x) = \mathrm{Tr}\,|_{V_\tau}\Psi_\tau^U(x) = d_\tau\Phi(x).$$

Thus $\Phi(x) = \Phi_\tau^U(x)$, and (5.7) is proved. In order to prove (5.8), we calculate the matrix coefficients of the left-hand side of (5.8). For $\mathbf{v}, \mathbf{v}' \in V_\tau$, we have

$$\begin{aligned}
&d_\tau^2 \int_K \langle \tau(k) \otimes \Phi_\tau^U(xk^{-1})\, \mathbf{v} \otimes \mathbf{e}_\xi, \mathbf{v}' \otimes \mathbf{e}_{\xi'} \rangle dk \\
&= d_\tau^2 \int_K \langle \tau(k)\mathbf{v}, \mathbf{v}' \rangle \langle \Phi_\tau^U(xk^{-1})\mathbf{e}_\xi, \mathbf{e}_{\xi'} \rangle\, dk \\
&= d_\tau \int_K \langle \tau(k)\mathbf{v}, \mathbf{v}' \rangle \sum_{a=1}^{d_\tau} \langle U(xk^{-1}) f_{a\xi}, f_{a\xi'} \rangle dk \\
&= d_\tau \sum_{a,b=1}^{d_\tau} \langle U(x) f_{b\xi}, f_{a\xi'} \rangle \int_K \langle \tau(k)\mathbf{v}, \mathbf{v}' \rangle \langle \tau(k^{-1})\mathbf{v}_a, \mathbf{v}_b \rangle dk \\
&= \sum_{a,b} \langle U(x) f_{b\xi}, f_{a\xi'} \rangle \langle \mathbf{v}, \mathbf{v}_b \rangle \overline{\langle \mathbf{v}', \mathbf{v}_a \rangle} \\
&= \langle U(x) f_{\mathbf{v}\xi}, f_{\mathbf{v}'\xi'} \rangle.
\end{aligned}$$

We have used the Schur orthogonality relations for the matrix coefficients of $\tau(k)$. This proves (5.8). □

It is clear from (5.7) that Φ_τ^U is K-central. Moreover, taking the partial trace in (5.8) with respect to V_τ, gives

$$d_\tau \chi_\tau * \Phi_\tau^U = \Phi_\tau^U = \Phi_\tau^U * d_\tau \chi_\tau.$$

For $f \in C_0^\infty(G)$, let $U(f)$ denote the operator

$$U(f) = \int_G f(x) U(x) dx.$$

Then, as is well known, $U(f)P_\tau = U(f * d_\tau \bar{\chi}_\tau)$, $P_\tau U(f) = U(d_\tau \bar{\chi}_\tau * f)$, and

$$P_\tau U(f) P_\tau = U(d_\tau \bar{\chi}_\tau * f * d_\tau \bar{\chi}_\tau).$$

In particular, for $f \in I_{0,\tau}(G)$ we have the following result.

Proposition 5.3. *Let $f \in I_{0,\tau}(G)$, and let $[U] \in \hat{G}(\tau)$. Then*

$$\begin{aligned}
&P_\tau U(f) P_\tau = U(f), \\
&U(f)U(k) = U(k)U(f), \qquad k \in K.
\end{aligned}$$

Let $U_\tau(f)$ denote the restriction of $U(f)$ to H_τ. Then the set of operators $U_\tau(f)$, $f \in I_{0,\tau}(G)$, is the centralizer of the representation $k \to \tau_U(k)$ of K on H_τ.

Proof. See [8], Vol. I, p. 307 and Prop. 4.5.1.7, p. 310. □

It follows from this proposition and from (5.7), that for $f \in I_{0,\tau}(G)$

$$\begin{aligned} U_\tau(f) &= \int_G f(x)\Psi_\tau^U(x)dx = \int_G \int_K f(kxk^{-1})\Psi_\tau^U(x)dkdx \\ &= \int_G \int_K f(y)\Psi_\tau^U(k^{-1}yk)dkdy = \mathbf{1} \otimes \int_G f(x)\Phi_\tau^U(x)dx, \end{aligned}$$

and that

$$\Theta_U(f) \equiv \mathrm{Tr}\ U(f) = \mathrm{Tr}\ U_\tau(f) = \int_G f(x)\psi_\tau^U(x)dx.$$

Since

$$U(f_1 * f_2) = U(f_1)U(f_2),$$

we see that the map $f \rightarrow \hat{f}(U) \in \mathrm{End}(\mathbf{C}^{\xi_U})$, where

$$\hat{f}(U) \equiv \int_G f(x)\Phi_\tau^U(x)dx, \tag{5.11}$$

is a representation of the algebra $I_{0,\tau}(G)$ on $\mathbf{C}^{\xi_U}$,

$$(\widehat{f_1 * f_2})(U) = \hat{f}_1(U)\hat{f}_2(U), \qquad f_1, f_2 \in I_{0,\tau}(G).$$

It is not difficult to show that this representation is irreducible (see [8], Vol. I, p. 310, the remark).

In summary, the function $x \rightarrow \Phi(x) \equiv \Phi_\tau^U(x) \in \mathrm{End}(\mathbf{C}^{\xi_U})$ satisfies the following three conditions:

$$\text{(i)} \quad \Phi(kxk^{-1}) = \Phi(x), \qquad x \in G, \ \ k \in K; \tag{5.12}$$

$$\text{(ii)} \quad d_\tau\chi_\tau * \Phi = \Phi\ (= \Phi * d_\tau\chi_\tau); \tag{5.13}$$

(iii) The map $f \rightarrow \int_G f(x)\Phi(x)dx$ is an irreducible representation of $I_{0,\tau}(G)$.

Definition 5.4. (See [8], Vol. II, p. 14.) Let $\tau \in \hat{K}$ be fixed. A quasi-bounded (cf. [8], Vol. II, p. 6) continuous function Φ on G with values in $\mathrm{End}(E)$ (E a finite-dimensional vector space) is called a spherical function of type τ on G, if it satisfies (i)-(iii) above. We say that Φ is of positive type if E admits the structure of a Hilbert space such that for all positive integers n, all $x_1, \dots, x_n$ in G, and all $c_1, \dots, c_n$ in $\mathbf{C}$, $\sum_{i,j} c_i\bar{c}_j\Phi(x_i^{-1}x_j)$ is a positive operator.

The connection between spherical functions of type τ and representations of G is as follows. Let U be a topologically completely irreducible (TCI) Banach representation of G (see [8], Vol. I, p. 228 for the definition of TCI). Suppose that τ occurs in $U|_K$, and define Ψ_τ^U, Φ_τ^U by means of Eqs. (2.2), (5.7). Then Φ_τ^U is a spherical function of type τ on G. Conversely, if Φ is a spherical function of type τ on G, there exists a TCI Banach representation U of G such that $\Phi = \Phi_\tau^U$. [See [8], Vol. II, p. 15.] The spherical functions of positive type may be characterized as follows.

Proposition 5.5. *Let Φ be a spherical function of type τ on G. Then:*

(1) *Φ is of positive type if and only if the scalar-valued function $x \to \operatorname{Tr} \Phi(x)$ is positive definite on G;*

(2) *If Φ is of positive type, there exists $[U] \in \hat{G}(\tau)$ such that $\Phi = \Phi_\tau^U$. Conversely, if $[U] \in \hat{G}(\tau)$ then Φ_τ^U is of positive type.*

Proof. See [8], Vol. II, p. 15, the remark. □

Let $\mathcal{H}$ denote the set of all spherical functions of type τ on G, and let $\mathcal{H}_P$ denote the subset of spherical functions of positive type.

Definition 5.6. The spherical transform $\hat{f}$ of $f \in I_{0,\tau}(G)$ is the (operator-valued) function on $\mathcal{H}$ defined by

$$\hat{f}(\Phi) = \int_G f(x)\Phi(x)dx, \qquad \Phi \in \mathcal{H}. \tag{5.14}$$

For $\Phi = \Phi_\tau^U \in \mathcal{H}_P$ ($[U] \in \hat{G}(\tau)$), the function $\hat{f}$ on $\mathcal{H}_P$ defined by (5.14) is called the spherical Fourier transform of f, and we use the notation $\hat{f}(\Phi_\tau^U) = \hat{f}(U)$ (cf. (5.11)). For $\Phi \in \mathcal{H}$ ($\Phi : G \to \operatorname{End}(E)$), let $\Psi(x) \in \operatorname{End}(V_\tau \otimes E)$ be given by the left hand side of (5.8) (with $\Phi_\tau^U \to \Phi$), and define the spherical transform $\hat{F}$ of $F \in C_0^\infty(G, \tau, \tau)$ by

$$\hat{F}(\Phi) = \frac{1}{d_\tau} \int_G \operatorname{Tr}|_{V_\tau} [\Psi(x)(F(x) \otimes \mathbf{1}_E)]\, dx \in \operatorname{End}(E), \tag{5.15}$$

where $\mathbf{1}_E$ is the identity operator in E, and $\operatorname{Tr}|_{V_\tau}$ means the partial trace in $V_\tau \otimes E$ with respect to V_τ. For $[U] \in \hat{G}(\tau)$, write $\hat{F}(U)$ in place of $\hat{F}(\Phi_\tau^U) \in \operatorname{End}(\mathbf{C}^{\xi_U})$.

Lemma 5.7. *In the notations of Proposition 5.1 we have, for all $\Phi \in \mathcal{H}$,*

$$\hat{F}(\Phi) = \hat{f}_F(\Phi), \qquad \hat{f}(\Phi) = \hat{F}_f(\Phi). \tag{5.16}$$

Proof. Using (5.8) in (5.15) we obtain

$$\begin{aligned}\hat{F}(\Phi) &= d_\tau \int_G \int_K \mathrm{Tr}\,|_{V_\tau}\left[(\tau(k)\otimes\Phi(xk^{-1}))(F(x)\otimes \mathbf{1}_E)\right] dk\,dx \\ &= d_\tau \int_K \int_G \mathrm{Tr}\,[F(xk^{-1})]\Phi(xk^{-1})dx\,dk \\ &= d_\tau \int_G \mathrm{Tr}\,[F(y)]\Phi(y)dy \\ &= \int_G f_F(y)\Phi(y)dy = \hat{f}_F(\Phi).\end{aligned}$$

Reading the argument backwards, proves the second equality in (5.16). ☐

5.3. The functional equation.

As in the scalar case, spherical functions of type τ on G satisfy a certain functional equation.

Theorem 5.8. *Let Φ be a spherical function of type τ on G. Then Φ satisfies the functional equation*

$$\int_K \Phi(xkyk^{-1})dk = \Phi(x)\Phi(y), \qquad x,y\in G. \tag{5.17}$$

Conversely, let Φ be a nonzero quasi-bounded continuous function on G with values in $\mathrm{End}(E)$ *(E a finite-dimensional vector space), which satisfies* (5.17) *together with the conditions* (i) *and* (ii) *in the definition of spherical function of type τ. Then the map $f \to \int_G f(x)\Phi(x)dx$ is a representation of the algebra $I_{0,\tau}(G)$ on E. If this representation is irreducible, then Φ is a spherical function of type τ on G.*

Proof. For $f\in C_0^\infty(G)$ put

$$f_K(x) = \int_K f(kxk^{-1})dk,$$

and $f^\tau = d_\tau\bar{\chi}_\tau * f * d_\tau\bar{\chi}_\tau$, i.e., explicitly,

$$f^\tau(x) = d_\tau^2 \int_{K\times K} f(k_1xk_2)\chi_\tau(k_1)\chi_\tau(k_2)dk_1dk_2.$$

It is easy to see that $(f_K)^\tau = (f^\tau)_K$. Put $f^\# = (f^\tau)_K$. Then the map $f\to f^\#$ is a projection of $C_0^\infty(G)$ onto $I_{0,\tau}(G)$.

Let Φ be a spherical function of type τ on G, and put $\Phi(f) = \int_G f(x)\Phi(x)dx$ ($f \in C_0^\infty(G)$). Let f_1, f_2 be in $C_0^\infty(G)$. Since the map $f \to \Phi(f)$ is a representation of $I_{0,\tau}(G)$, we have

$$\begin{aligned} 0 &= \Phi(f_1^\# * f_2^\#) - \Phi(f_1^\#)\Phi(f_2^\#) \\ &= \int_G (f_1^\# * f_2^\#)(z)\Phi(z)dz - \int_G f_1^\#(x)\Phi(x)dx \int_G f_2^\#(y)\Phi(y)dy \\ &= \int_{G\times G} f_1^\#(zy^{-1})f_2^\#(y)\Phi(z)dydz - \int_{G\times G} f_1^\#(x)f_2^\#(y)\Phi(x)\Phi(y)dxdy \\ &= \int_{G\times G} [\Phi(xy) - \Phi(x)\Phi(y)]\, f_1^\#(x)f_2^\#(y)dxdy. \end{aligned}$$

A straightforward calculation, using the definition of $f^\#$ and (5.12), (5.13), shows that the latter expression equals

$$\int_{G\times G} \left[\int_K \Phi(xkyk^{-1})dk - \Phi(x)\Phi(y)\right] f_1(x)f_2(y)dxdy.$$

This proves (5.17), since f_1, f_2 are arbitrary in $C_0^\infty(G)$.

Conversely, suppose that $\Phi : G \to \mathrm{End}(E)$ satisfies (5.12), (5.13), and (5.17). It is easy to show that the map $f \to \int_G f(x)\Phi(x)dx$ is a representation of $I_{0,\tau}(G)$. Therefore if this representation is irreducible, then Φ is a spherical function of type τ on G. For a different proof see [8], Vol. II, p. 16. □

Theorem 5.9. *Let Φ be a spherical function of type τ on G. Then $\Phi(e) = \mathbf{1}$, and for all $f \in I_{0,\tau}(G)$*

$$\int_G \Phi(xy)f(y)dy = \Phi(x)\hat{f}(\Phi). \tag{5.18}$$

Conversely, let Φ be a quasi-bounded continuous function on G with values in $\mathrm{End}(E)$ *(E a finite-dimensional vector space), satisfying (5.12), (5.13), and $\Phi(e) = \mathbf{1}$. Suppose that for any $f \in I_{0,\tau}(G)$ there exists an element $\hat{f}(\Phi)$ of* $\mathrm{End}(E)$*, such that (5.18) holds. Then $\hat{f}(\Phi) = \int_G f(x)\Phi(x)dx$, and the map $f \to \hat{f}(\Phi)$ is a representation of $I_{0,\tau}(G)$ on E. If this representation is irreducible, then Φ is a spherical function of type τ on G.*

Proof. Let Φ be a spherical function of type τ on G. Then (5.17), together with the irreducibility of the representation $f \to \int_G f(x)\Phi(x)dx$ of $I_{0,\tau}(G)$, implies $\Phi(e) = \mathbf{1}$. Therefore (5.18) holds at $x = e$. For $x \neq e$ we have

$$\int_G \Phi(xy)f(y)dy = \int_G \int_K \Phi(xy)f(k^{-1}yk)dkdy$$

$$= \int_G \int_K \Phi(xkzk^{-1}) f(z) dk dz$$
$$= \Phi(x) \int_G \Phi(z) f(z) dz = \Phi(x) \hat{f}(\Phi),$$

where we have used the functional Equation (5.17).

Conversely, let Φ satisfy (5.12), (5.13), $\Phi(e) = \mathbf{1}$, and (5.18). Setting $x = e$ in (5.18) gives $\hat{f}(\Phi) = \int_G f(x)\Phi(x)dx$. Let $f_1, f_2 \in I_{0,\tau}(G)$. Then

$$\begin{aligned}(\widehat{f_1 * f_2})(\Phi) &= \int_G \Phi(z)(f_1 * f_2)(z)dz = \int_G \int_G \Phi(z) f_1(zy^{-1}) f_2(y) dy dz \\ &= \int_G \int_G \Phi(xy) f_1(x) f_2(y) dy dx = \int_G \Phi(x) \hat{f}_2(\Phi) f_1(x) dx \\ &= \hat{f}_1(\Phi) \hat{f}_2(\Phi),\end{aligned}$$

where we have used (5.18). Therefore the map $f \to \hat{f}(\Phi)$ is a representation of $I_{0,\tau}(G)$, and Φ is a spherical function of type τ provided that this representation is irreducible. □

5.4. The inversion formula.

The inversion formula for the spherical transform on $I_{0,\tau}(G)$ follows easily from the Plancherel formula on the group G.

Theorem 5.10. *The spherical transform* (5.14) *is inverted by*

$$f(x) = d_\tau \int_{\hat{G}(\tau)} \operatorname{Tr} \left[\Phi_\tau^U(x^{-1}) \hat{f}(U) \right] d\mu(U), \qquad f \in I_{0,\tau}(G), \tag{5.19}$$

where $d\mu(U)$ is the Plancherel measure on $\hat{G}$ (suitably normalized).

Proof. By reasoning as in the proof of Proposition 2.1, we find for any $f \in I_{0,\tau}(G)$

$$\begin{aligned}f(x) &= \int_{\hat{G}} \Theta_U(f \circ L_x) d\mu(U) \\ &= \int_{\hat{G}(\tau)} \Theta_{U,\tau}(f \circ L_x) d\mu(U) \\ &= \int_{\hat{G}(\tau)} \int_G \psi_\tau^U(y) f(xy) dy d\mu(U) \\ &= \int_{\hat{G}(\tau)} \int_G \psi_\tau^U(x^{-1}y) f(y) dy d\mu(U).\end{aligned} \tag{5.20}$$

Taking the trace in Eq. (5.18) and using (5.6), we find for all $f \in I_{0,\tau}(G)$

$$\int_G \psi_\tau^U(x^{-1}y) f(y) dy = d_\tau \operatorname{Tr} \left[\Phi_\tau^U(x^{-1}) \hat{f}(U) \right]. \tag{5.21}$$

Using (5.21) in (5.20) gives (5.19). □

From Theorem 5.10, we obtain the following inversion formula for radial systems of sections of E^τ.

Corollary 5.11. *Let $F \in C_0^\infty(G,\tau,\tau)$. Then the spherical transform $F \to \hat{F}$, defined in (5.15), is inverted by*

$$F(x) = \frac{1}{d_\tau} \int_{\hat{G}(\tau)} \mathrm{Tr}\, |_{\mathbf{C}^{\xi_U}} \left[\Psi_\tau^U(x^{-1})(\mathbf{1} \otimes \hat{F}(U)) \right] d\mu(U), \tag{5.22}$$

where $\mathbf{1}$ is the identity operator in V_τ, and $\mathrm{Tr}\, |_{\mathbf{C}^{\xi_U}}$ means the partial trace in $V_\tau \otimes \mathbf{C}^{\xi_U}$ with respect to $\mathbf{C}^{\xi_U}$.

Proof. We apply the map $f \to F_f$ (see Prop. 5.1) to $f(x)$ given by (5.19). Using Eq. (5.16), we have

$$\begin{aligned}
F_f(x) &= \int_K \tau(k) f(kx) dk \\
&= d_\tau \int_K \int_{\hat{G}(\tau)} \tau(k)\, \mathrm{Tr} \left[\Phi_\tau^U(x^{-1}k^{-1}) \hat{F}_f(U) \right] d\mu(U)\, dk \\
&= d_\tau \int_{\hat{G}(\tau)} \int_K \mathrm{Tr}\, |_{\mathbf{C}^{\xi_U}} \left[\tau(k) \otimes \Phi_\tau^U(x^{-1}k^{-1}) \hat{F}_f(U) \right] dk\, d\mu(U) \\
&= d_\tau \int_{\hat{G}(\tau)} \mathrm{Tr}\, |_{\mathbf{C}^{\xi_U}} \left[\left(\int_K \tau(k) \otimes \Phi_\tau^U(x^{-1}k^{-1}) dk \right) (\mathbf{1} \otimes \hat{F}_f(U)) \right] d\mu(U) \\
&= \frac{1}{d_\tau} \int_{\hat{G}(\tau)} \mathrm{Tr}\, |_{\mathbf{C}^{\xi_U}} \left[\Psi_\tau^U(x^{-1})(\mathbf{1} \otimes \hat{F}_f(U)) \right] d\mu(U),
\end{aligned}$$

where we have used (5.8) in the last step. The result now follows from Prop. 5.1 (the map $f \to F_f$ being onto). □

The Plancherel theorem for the spherical transform follows now from the inversion formula by well known standard arguments.

Corollary 5.12. *Let $L^2(\hat{G}(\tau), d\mu(U))$ denote the set of all functions $\hat{F}$ on $\hat{G}(\tau)$ with values in the set $\bigcup_{n=1}^\infty \mathrm{End}(\mathbf{C}^n)$ satisfying*

$$\begin{aligned}
&1)\ \ \hat{F}(U) \in \mathrm{End}(\mathbf{C}^{\xi_U}), \qquad \textit{for all } [U] \in \hat{G}(\tau), \\
&2)\ \ \|\hat{F}\|^2 \equiv \int_{\hat{G}(\tau)} \mathrm{Tr} \left[\hat{F}(U) \hat{F}(U)^* \right] d\mu(U) < +\infty.
\end{aligned}$$

$L^2(\hat{G}(\tau), d\mu(U))$ is a Hilbert space, with the inner product

$$\langle \hat{F}_1, \hat{F}_2 \rangle = \int_{\hat{G}(\tau)} \mathrm{Tr}\left[\hat{F}_1(U)\hat{F}_2(U)^*\right] d\mu(U).$$

Then the spherical Fourier transform $F \to \hat{F}$ extends to an isometry of the Hilbert space $L^2(G,\tau,\tau)$ of radial systems of sections of E^τ onto $L^2(\hat{G}(\tau), d\mu(U))$.

The relation between the spherical transform and the Helgason Fourier transform of a radial section can now be made clear. First we have:

Proposition 5.13. *Let $F \in C_0^\infty(G,\tau,\tau)$, and let φ_τ^U ($[U] \in \hat{G}(\tau)$) be the τ-spherical function given by (2.4) or (2.3). Then*

$$(\varphi_\tau^U * F)(x) = \mathrm{Tr}\,|_{\mathbf{C}^{\xi_U}}\left[\Psi_\tau^U(x^{-1})(\mathbf{1}\otimes\hat{F}(U))\right], \tag{5.23}$$

the convolution being defined in (2.6). Therefore (2.5) reduces to (5.22) in this case.

Proof. Multiplying in (5.8) by $\mathbf{1}\otimes\hat{f}_F(U)$, taking the partial trace with respect to $\mathbf{C}^{\xi_U}$, and using (5.21) and (2.4), we obtain

$$\begin{aligned}\mathrm{Tr}\,|_{\mathbf{C}^{\xi_U}}\left[\Psi_\tau^U(x)(\mathbf{1}\otimes\hat{f}_F(U))\right] &= d_\tau^2\int_K \tau(k)\,\mathrm{Tr}\,[\Phi_\tau^U(k^{-1}x)\hat{f}_F(U)]dk\\ &= d_\tau\int_{K\times G}\tau(k)\psi_\tau^U(k^{-1}xy)f_F(y)dy\,dk\\ &= \int_G \varphi_\tau^U(xy)f_F(y)dy = \int_G \varphi_\tau^U(xy)F(y)dy.\end{aligned}$$

The last step is a trivial computation (let $y \to yk$, integrate in dk, and use (5.5)). This proves (5.23). □

Eq. (5.23) generalizes (5.2) to the case of bundles. Notice that we have not used, so far, the structure theory of semisimple Lie groups, and we have kept the notations as general as possible. In fact the theory of spherical functions of type τ on G and the corresponding spherical transform can be formulated for any pair (G,K) of a locally compact unimodular Lie group G and a compact subgroup $K \subset G$, provided that every $U \in \hat{G}$ is K-finite (see [2], Section 3). If G admits a uniformly large compact subgroup K (see the definition in [8], Vol. I, p. 305), then the inversion formula (5.19) holds (by the same proof given here). This includes all reductive pairs and all motion groups [8].

We now specialize the inversion formula (5.22) to the semisimple case. For P' cuspidal parabolic, consider the generalized unitary principal P'-series $U^{\sigma'\nu'}$ (see Section 4), and suppose that ν' is regular, so that $U^{\sigma'\nu'}$ is irreducible. If $\tau \subset U^{\sigma'\nu'}|_K$ put, as usual, $\Psi_\tau^{U^{\sigma'\nu'}}(x) = P_\tau U^{\sigma'\nu'}(x) P_\tau$, and define $\Phi_\tau^{U^{\sigma'\nu'}}(x)$ as in (5.7). Then $\Phi_\tau^{U^{\sigma'\nu'}}$ is a spherical function of type τ on G of positive type. Let $\hat{F}(U^{\sigma'\nu'})$ be the spherical Fourier transform of $F \in C_0^\infty(G,\tau,\tau)$ relative to $U^{\sigma'\nu'}$, given by (see (5.14), (5.15), and (5.16))

$$\hat{F}(U^{\sigma'\nu'}) = \int_G f_F(x) \Phi_\tau^{U^{\sigma'\nu'}}(x) dx. \tag{5.24}$$

Then the inversion formula (5.22) takes the form

$$F(x) = \frac{1}{d_\tau} \sum_{P'} c_{P'} \sum_{\sigma'} \int_{\mathbf{a}'^*} \operatorname{Tr}|_{\mathbf{C}^{\xi_{U^{\sigma'\nu'}}}} \left[\Psi_\tau^{U^{\sigma'\nu'}}(x^{-1}) (\mathbf{1}_\tau \otimes \hat{F}(U^{\sigma'\nu'})) \right] p_{\sigma'}(\nu') d\nu' \tag{5.25}$$

(same notations as in Theorem 4.3). This formula holds for any K-type τ. It can also be obtained from (4.11), by specializing to a radial section.

In order to see this, we compute the Helgason Fourier transform of a radial section. Going back to the notations of Section 3, consider the (nonunitary) minimal principal series $U^{\sigma\mu}$ of G defined in (3.1). If $\sigma \subset \tau|_M$ (with multiplicity m_σ), define $\Psi_\tau^{U^{\sigma\mu}}$ and $\Phi_\tau^{U^{\sigma\mu}}$ in the usual way (cf. (2.2) and (5.7)). These functions admit integral representations similar to (3.9). For example $\Phi_\tau^{U^{\sigma\mu}}$ is given as follows. Let $\Phi_\sigma^\tau(k)$ be the spherical function of type σ on K associated with τ. This is the function from K to $\operatorname{End}(\mathbf{C}^{m_\sigma})$ defined by

$$\int_M \Psi_\sigma^\tau(mkm^{-1}) dm = \mathbf{1}_\sigma \otimes \Phi_\sigma^\tau(k),$$

where $\Psi_\sigma^\tau(k) = P_\sigma \tau(k) P_\sigma$, and P_σ is the projection from V_τ onto $H_\sigma \simeq V_\sigma \otimes \mathbf{C}^{m_\sigma}$ (the subspace of vectors of V_τ which transform under M according to σ). The function $\Phi_\tau^{U^{\sigma\mu}}$ is then found to be

$$\Phi_\tau^{U^{\sigma\mu}}(x) = \int_K {}^t\Phi_\sigma^\tau(\mathbf{k}(k^{-1}xk)) e^{\mu(H(xk))} dk \in \operatorname{End}(\mathbf{C}^{m_\sigma}), \tag{5.26}$$

where ${}^t\mathcal{O}$ denotes the transpose of $\mathcal{O} \in \operatorname{End}(\mathbf{C}^n)$.

By the Subquotient Theorem, every TCI Banach representation U of G is infinitesimally equivalent to a subquotient representation of a principal series $U^{\sigma\mu}$, for suitable $\sigma \in \hat{M}$, and $\mu \in \mathbf{a}_{\mathbf{C}}^*$. By a similar argument as in the proof of Proposition 4.2, we find that every (nonzero) spherical function of type τ on G can be written as $T\Phi_\tau^{U^{\sigma\mu}}T$, where $\sigma \subset \tau|_M$, $\mu \in \mathbf{a}^*_{\mathbf{C}}$, and T is a suitable projection operator in $\mathbf{C}^{m_\sigma}$ (see, e.g., Eq. (5.29) below).

Given $F \in C_0^\infty(G,\tau,\tau)$ define $\hat{F}_\sigma : \mathbf{a}^*_{\mathbf{C}} \to \mathrm{End}(\mathbf{C}^{m_\sigma})$ $(\sigma \subset \tau|_M)$ by

$$\hat{F}_\sigma(\lambda) = \hat{F}(\Phi_\tau^{U^{\sigma,i\lambda-\rho}}) = \int_G f_F(x)\Phi_\tau^{U^{\sigma,i\lambda-\rho}}(x)dx, \qquad \lambda \in \mathbf{a}^*_{\mathbf{C}}. \tag{5.27}$$

Let $f(g) = F(g)\mathbf{v}$ ($\mathbf{v} \in V_\tau$) be a radial section of E^τ. An easy calculation shows that the Helgason Fourier transform of f is given by

$$\tilde{f}(\lambda,k) = \sum_\sigma [\mathbf{1}_\sigma \otimes {}^t\hat{F}_\sigma(\lambda)]P_\sigma\tau(k^{-1})\mathbf{v}, \qquad \lambda \in \mathbf{a}^*_{\mathbf{C}},\ k \in K, \tag{5.28}$$

where the sum is over all inequivalent M-types contained in $\tau|_M$. Eq. (5.28) generalizes (5.1) to vector bundles.

In particular, consider the principal P'-series $U^{\sigma'\nu'}$, and suppose that $\tau \subset U^{\sigma'\nu'}|_K$. Using the same notations as in Proposition 4.2, we find that the spherical function $\Phi_\tau^{U^{\sigma'\nu'}}$ is given by

$$\Phi_\tau^{U^{\sigma'\nu'}}(x) = T_n\, \Phi_\tau^{U^{\tilde{\sigma}',i\nu'+\mu_1-\rho}}(x)\, T_n, \tag{5.29}$$

where T_n projects onto the first n coordinates in $\mathbf{C}^m$. [Recall that $m = m(\tilde{\sigma}',\tau) \geq n = m(\tau, U^{\sigma'\nu'})$.] The spherical Fourier transform $\hat{F}(U^{\sigma'\nu'})$ of $F \in C_0^\infty(G,\tau,\tau)$ is then found to be

$$\hat{F}(U^{\sigma'\nu'}) = T_n\, \hat{F}_{\tilde{\sigma}'}(\nu' - i\mu_1)\, T_n. \tag{5.30}$$

Calculating the term $T_{\tilde{\sigma}'}\tilde{f}(\nu' - i\mu_1, k)$ in (4.11) for a radial section $f(g) = F(g)\mathbf{v}$ as above, we find from (5.28) and (5.30)

$$T_{\tilde{\sigma}'}\tilde{f}(\nu' - i\mu_1, k) = [\mathbf{1}_{\tilde{\sigma}'} \otimes {}^t\hat{F}(U^{\sigma'\nu'})]T_{\tilde{\sigma}'}\tau(k^{-1})\mathbf{v}.$$

Using this in (4.11) gives the inversion formula (5.25) (by an easy computation).

Conversely, the inversion formula (4.11) for the Helgason Fourier transform can be deduced from the spherical inversion formula (5.25). The proof of this is an adaptation of [8], Lemma 9.2.1.6. to the bundle case. [One uses Eq. (5.25) for the function $F_g \in C_0^\infty(G,\tau,\tau)$ given by

$$F_g(x) = \int_K (f(gkx) \otimes v)\, \tau(k)dk,$$

where $g \in G$, $f \in C_0^\infty(G,\tau)$, $v \in V_\tau$, together with the formula $\mathrm{Tr}\, F_g(e) = \langle f(g), \bar{v}\rangle$.]

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