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CLASSIFICATION OF CYCLIC GROUP ACTIONS ON NONCOMPACT SURFACES

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Classification of group actions on surfaces is a question arising from the Nielsen realization problem. Nielsen gives a classification theorem for cyclic group actions on a closed, oriented, connected surface which shows that the actions can be classified by their fixed point data. This paper considers certain group actions on *noncompact*, oriented, connected surfaces. The main difficulties are that a noncompact surface may have infinite genus, and the branch set of an action on a noncompact surface could be infinite. We introduce the *end data* and the *type of a cluster end*, and provide a complete classification of cyclic group actions on noncompact surfaces in terms of fixed point data, end data, and cluster end types.

1. Introduction.

We begin with some basic definitions. Throughout this paper, by a *surface* we mean a connected, oriented 2-dimensional manifold (without boundary unless specified); also, we always assume that actions and homeomorphisms are orientation-preserving.

Let M be a surface, and let G be a finite abelian group. A free G-action ϕ on M corresponds uniquely to a G-covering of the orbit space $N = M/\phi$, which is determined by an epimorphism $\phi\colon \pi_1(N, x_0) \to G$ from the fundamental group of N to the group G. Two free G-actions ϕ and ψ on M are equivalent (conjugate) if and only if they have the same orbit space N and their corresponding G-coverings are equivalent ([S]), that is, if and only if there is a homeomorphism $h\colon N, x_0 \to N, x_0$ such that $\psi = \phi \circ h_*$, where $h_*\colon \pi_1(N, x_0) \to \pi_1(N, x_0)$ is the automorphism induced by h. We may ignore the choice of the base point when G is abelian.

For an effective G-action ϕ on M, let $N = M/\phi$ be its orbit space and let $B \subset N$ be the set of all the branch points (B is called the *branch set*). The action ϕ can be presented by a G-branched covering of N which is determined by an epimorphism $\phi\colon \pi_1(N - B, x_0) \to G$. For each branch point $x \in B$, let D_x be a disk on N centered at x which contains no branch points other than x. Denote $\phi(x) = \phi([\partial D_x]) \in G$ where $[\partial D_x]$ is the homotopy class of

∂D_x in $\pi_1(N - B, x_0)$. Let $B^{\text{finite}} = \{x \in B \mid$ there are only finite number of elements $y \in B$ such that $\phi(y) = \phi(x)\}$. Note that $B = B^{\text{finite}}$ when the surface is compact.

Definition. The *fixed point data* of ϕ is a function $\mathcal{D}(\phi)\colon B^{\text{finite}} \to G$ given by $\mathcal{D}(\phi)(x) = \phi(x)$.

Theorem 1.1 [**N**]. *Let M be a closed surface.*

(a) *Any two free $\mathbb{Z}/n$-actions on M are equivalent.*

(b) *Two effective $\mathbb{Z}/n$-actions on M are equivalent if and only if they have the same fixed point data.*

As pointed out in [**E**], the classification problem of G-actions on a compact surface with boundary can be converted to the case of G-actions on a closed surface by gluing an invariant disk along each boundary component. Let ϕ be a free or effective action of a finite abelian group G on a compact surface M with boundary, and let $N = M/\phi$ be its orbit space. Then M is a G-covering or G-branched covering of N. The action ϕ is determined by an epimorphism $\phi\colon \pi_1(N - B, x_0) \to G$ (the branch set B is empty when ϕ is free).

Definition. Let $\gamma_1, \gamma_2, \dots, \gamma_k$ be the boundary components of the orbit space N, and let $[\gamma_i]$ be the homotopy class of γ_i in $\pi_1(N - B, x_0)$ ($i = 1, 2, \dots, k$). The *boundary data* $\mathcal{B}(\phi)$ is a function from the set of boundary components of N to the group G, defined by $\mathcal{B}(\phi)(\gamma_i) = \phi([\gamma_i])$ for $i = 1, 2, \dots, k$.

In the case $G = \mathbb{Z}/n$, we obtain a complete classification as a corollary of Theorem 1.1.

Corollary 1.2. *Let M be a compact surface with boundary.*

(a) *Two free $\mathbb{Z}/n$-actions on M are equivalent if and only if they have the same boundary data.*

(b) *Two effective $\mathbb{Z}/n$-actions on M are equivalent if and only if they have the same fixed point data and boundary data.*

Note that the sum of all the images of $\mathcal{D}(\phi)$ and $\mathcal{B}(\phi)$ is 0. This is the only condition for given fixed point data and boundary data to be realizable.

In this paper we consider group actions on *noncompact* surfaces. Theorem 1.1 and Corollary 1.2 do not extend to the noncompact case. The main difficulties are that a noncompact surface may have infinite genus, and the branch set of an action on a noncompact surface could be infinite. We introduce in §2 *decompositions* of noncompact surfaces and the *end data* for free actions to describe the behavior of an action around ends. §3 gives a classification theorem of free actions of a cyclic group on a noncompact surface. In

§4, we define the end data for effective actions, introduce *cluster ends* and the *type of a cluster end*, and give necessary and sufficient conditions for two cluster ends to be equivalent (Proposition 4.1). §5 discusses classification of effective actions of a cyclic group on noncompact surfaces. Another new concept, *stable equivalence*, is also introduced in §5.

The following are some of our main results. Let M be a noncompact surface with finitely many ends.

· If a $\mathbb{Z}/n$-action on M has at most finitely many branch points, it is determined up to conjugate equivalence by its fixed point data and end data (end data only, if the action is free). The related results are given in Theorems 3.1 and 5.1.

· If a $\mathbb{Z}/n$-action on M has infinitely many branch points, it is determined, up to stable equivalence, by its fixed point data, end data, cluster end types, and the genus of its orbit space (Theorem 5.2 and 5.2′).

Stronger results are also given for actions with only one cluster end (Corollary 5.7).

Theorem 5.4, Corollary 5.5 and Corollary 5.6 describe the relationship between conjugate equivalence and stable equivalence. More discussion on stable equivalence can be found in §5.

2. Decomposition of noncompact surfaces and end data.

In this section, we consider free actions of a finite abelian group on a noncompact surface with finitely many ends.

Definitions. Let M be a noncompact surface. An *end component of M* is a nested sequence $U_1 \supset U_2 \supset \cdots$ of connected open subsets of M which satisfy the following conditions:

(i) The boundary of U_n in M is compact for all n;

(ii) for any compact subset C of M, $U_n \cap C = \emptyset$ for n sufficiently large.

Two end components $U_1 \supset U_2 \supset \cdots$ and $U'_1 \supset U'_2 \supset \cdots$ are *equivalent* if and only if for any n, $U'_n \supset U_m$ and $U_n \supset U'_k$ for some integers m, k. An *end* of M is an equivalence class of end components. (See [**R**] where the term ideal boundary is used in place of end.)

We say that an end e is *planar* if for any end component $U^e_1 \supset U^e_2 \supset \cdots$ at e, U^e_n is homeomorphic to a subset of the plane $\mathbb{R}^2$ for all sufficiently large n.

A noncompact surface M is of *finite genus* if there exists a compact surface with boundary $M' \subset M$ such that $M - M'$ is planar. In this case, the *genus* of M is defined to be the genus of M' (number of handles). In the contrary case, M is of *infinite genus*.

Let M be a noncompact surface with k ends $e_1, e_2, \ldots, e_k$. [**R**, Theorem 1] implies that two orientable noncompact surfaces with finitely many ends are homeomorphic if and only if they have the same genus and the same number of planar and nonplanar ends. The surface M is homeomorphic to a sphere with k punctures and at most countably many handles. We can choose an end component $U_1^{e_i} \supset U_2^{e_i} \supset \cdots$ at each e_i such that the boundary of each $U_n^{e_i}$ in M (denote as $\partial U_n^{e_i}$) is a simple closed curve and $U_1^{e_i} \cap U_1^{e_j} = \emptyset$ if $i \neq j$.

Definition. Let $U_n^{e_1}, U_n^{e_2}, \ldots, U_n^{e_k}$ be as above, and let $M_n = M - \bigcup_{i=1}^k U_n^{e_i}$ for $n = 1, 2, \ldots$. The sequence $M_1 \subset M_2 \subset \cdots$ is called a *decomposition of* M. Here each M_n is a compact surface with k boundary components $\partial M_n^{e_1}, \partial M_n^{e_2}, \ldots, \partial M_n^{e_k}$ $(\partial M_n^{e_i} = -\partial U_n^{e_i})$, and $\bigcup_{n=1}^\infty M_n = M$.

Given a decomposition $M_1 \subset M_2 \subset \cdots$ of M with $x_0 \in M_1$, note that the homomorphisms induced by the natural inclusions $i_*\colon \pi_1(M_n, x_0) \to \pi_1(M, x_0)$ and $j_*\colon \pi_1(M_n, x_0) \to \pi_1(M_m, x_0)$ $(n < m)$ are injective. Also

$$\pi_1(M, x_0) = \varinjlim_n \pi_1(M_n, x_0)$$

with above homomorphisms.

Now let G be a finite abelian group. We consider a free G-action ϕ on M with its orbit space $N = M/\phi$ which is a noncompact surface with finitely many ends. Assume that $\phi\colon \pi_1(N, x_0) \to G$. Let e be an end of N, and let $N_1 \subset N_2 \subset \cdots$ be a decomposition of N with $x_0 \in N_1$. We claim that $\phi([\partial N_n^e]) = \phi([\partial N_{n+1}^e])$ for $n = 1, 2, \ldots$. Let $a_1, b_1, a_2, b_2, \ldots, a_g, b_g$ be the canonical basis of the part of the surface N which is bounded by the curves ∂N_n^e and ∂N_{n+1}^e. Since $[\partial N_n^e]^{-1}[\partial N_{n+1}^e] = [a_1, b_1][a_2, b_2] \cdots [a_g, b_g]$ and G is abelian, we have that $\phi([\partial N_n^e]^{-1}[\partial N_{n+1}^e]) = 0$.

Define $\phi(e) = \phi([\partial N_n^e]) \in G$ and $G_\phi(e) = \bigcap_{n=1}^\infty \phi(\pi_1(U_n^e))$; here U_n^e is the connected component of $N - N_n$ which corresponds to the end e (so $U_1^e \supset U_2^e \supset \cdots$ is an end component of N at e), and $\phi(\pi_1(U_n^e))$ is the image of $\phi|\pi_1(U_n^e)$. One can check easily that $\phi(e)$ and $G_\phi(e)$ do not depend on the decomposition of N.

Definition. The *end data* $\mathcal{E}(\phi)$ of a free G-action ϕ is a function from the set of all ends (with planarities specified) of the orbit space N to $G \times$ {subgroups of G} given by $\mathcal{E}(\phi)(e) = (\phi(e),\ G_\phi(e))$.

Remark 2.1. From the definition, we obtain the following properties:

(a) The sum of $\phi(e)$, over all the ends e of N, is 0.

(b) If e is a planar end of N, then $G_\phi(e) = \langle \phi(e) \rangle$ is the subgroup of G generated by the element $\phi(e)$.

(c) If $G_\phi(e) = \bigcap_{n=1}^{\infty} \phi(\pi_1(U_n^e)) = \mathcal{H}$, then $\phi(\pi_1(U_n^e)) = \mathcal{H}$ for sufficiently large n. This is because $j_*\colon \pi_1(U_{n+1}^e) \to \pi_1(U_n^e)$ which is induced by the natural inclusion is injective, and G is finite. Furthermore, there is a subsequence of $\{U_n^e\}$, still denoted by $\{U_n^e\}$, such that $\phi(\pi_1(U_{n+1}^e - U_n^e)) = \mathcal{H}$.

(d) The number of the ends on M lifted from e is $n_e = |G|/|G_\phi(e)|$, *i.e.*, the preimage of U_n^e in M has exactly n_e connected components when n is sufficiently large.

3. A classification theorem of free $\mathbb{Z}/n$-actions.

Recall that in the compact case any two free $\mathbb{Z}/n$-actions on a closed surface are equivalent. This is no longer true in the noncompact case, however.

Theorem 3.1. *Let M be a noncompact surface with finitely many ends. Two free $\mathbb{Z}/n$-actions on M are equivalent if and only if they have the same end data.*

Note that if two free G-actions on M have the same end data, then their orbit spaces are homeomorphic. This can been shown by the Riemann-Hurwitz formula and [**R**, Theorem 1].

Proof of Theorem 3.1. Suppose that ϕ and ψ are two free $\mathbb{Z}/n$-actions on M which have the same end data. We may identify the two orbit spaces $M/\phi \cong N \cong M/\psi$. Then ϕ and ψ can be presented by epimorphisms $\phi,\ \psi\colon \pi_1(N, x_0) \to \mathbb{Z}/n$. Let $N_1 \subset N_2 \subset \cdots$ be a decomposition of N. We may assume that the base point x_0 is in N_1 and omit writing it. Let $\phi_n,\ \psi_n\colon \pi_1(N_n) \to \mathbb{Z}/n$ be the compositions of ϕ and ψ with $i_*\colon \pi_1(N_n) \to \pi_1(N)$ respectively, where i_* is induced by the natural inclusion. Without loss of generality, we may assume that all ϕ_n and ψ_n are surjective. Then the corresponding free $\mathbb{Z}/n$-actions ϕ_n and ψ_n act on the same compact surface with boundary, and their orbit spaces are N_n. Since ϕ and ψ have the same end data, we have that $\phi_n([\partial N_n^e]) = \psi_n([\partial N_n^e])$ for all ends e of N and sufficiently large n. Thus ϕ_n and ψ_n have the same boundary data, and they are equivalent by Corollary 1.2. Therefore, there is a homeomorphism h_n on N_n such that $\phi_n = \psi_n \circ h_{n*}$ where $h_{n*}\colon \pi_1(N_n) \to \pi_1(N_n)$ is induced by h_n.

In order to get a homeomorphism $h\colon N \to N$ and

$$h_*\colon \pi_1(N) = \varinjlim_n \pi_1(N_n) \to \varinjlim_n \pi_1(N_n) = \pi_1(N)$$

such that $\phi = \psi \circ h_*$, we need the diagram

$$\begin{array}{ccc} \pi_1(N_n) & \xrightarrow{i_*} & \pi_1(N_{n+1}) \\ \Big\downarrow h_{n*} & & \Big\downarrow h_{n+1*} \\ \pi_1(N_n) & \xrightarrow{i_*} & \pi_1(N_{n+1}) \end{array}$$

to be commutative. In fact, we will show that $\{h_n\}$ can be chosen so that $h_{n+1}|N_n = h_n$.

Now let e be an end of N. Without loss of generality, we may assume that the base point is in ∂N_n^e. By Lemma 3.2 below we can choose each h_n satisfies $h_n|\partial N_n^e = \mathrm{Id}$.

Suppose now that we have the homeomorphism h_n on N_n with $h_n|\partial N_n^e = \mathrm{Id}$ and $\phi_n = \psi_n \circ h_{n*}$. Let N' be the part of the surface N which is bounded by ∂N_n^e and ∂N_{n+1}^e, and let $\phi' = \phi \circ i_* \colon \pi_1(N') \to \mathbb{Z}/n$ where $i_* \colon \pi_1(N') \to \pi_1(N)$ is induced by the inclusion $N' \hookrightarrow N$. Similarly, we define $\psi' = \psi \circ i_*$. We may also assume that $\phi'(\pi_1(N')) = G_\phi(e)$ and $\psi'(\pi_1(N')) = G_\psi(e)$ (see Remark 2.1 (c)). Since ϕ and ψ have the same end data, $G_\phi(e) = G_\psi(e)$ holds for all ends e of N. Denote $\mathcal{H} = G_\phi(e)$ $(= G_\psi(e))$.

If $|\mathcal{H}| = n$, then $\mathcal{H} = \mathbb{Z}/n$. Therefore, both ϕ' and ψ' are surjective. The corresponding free $\mathbb{Z}/n$-actions ϕ' and ψ' act on the same compact surface with boundary and have the orbit space N'. Since ϕ' and ψ' have the same boundary data, they are equivalent. Therefore, there is a homeomorphism h' on N' such that $\phi' = \psi' \circ h'_*$.

If $|\mathcal{H}| < n$, we define $\phi'', \psi'' \colon \pi_1(N') \to \mathcal{H}$ to be $\phi''(x) = \phi'(x)$ and $\psi''(x) = \psi'(x)$ respectively for any $x \in \pi_1(N')$. Then both ϕ'' and ψ'' are epimorphisms such that $\phi' = j \circ \phi''$ and $\psi' = j \circ \psi''$, where $j \colon \mathcal{H} \to \mathbb{Z}/n$ is the inclusion homomorphism. Using the same argument used in the case $|\mathcal{H}| = n$, we can show that there is a homeomorphism h' on N' such that $\phi'' = \psi'' \circ h'_*$. Therefore, $\phi' = \psi' \circ h'_*$.

Thus in both cases, there is a homeomorphism h' on N' such that $\phi' = \psi' \circ h'_*$. By Lemma 3.2, we may assume that $h'|\partial N' = \mathrm{Id}$. Let $h_{n+1} = h_n \cup h'$. Then h_{n+1} is a homeomorphism on N_{n+1} and $h_{n+1}|\partial N_{n+1}^e = \mathrm{Id}$. We also need to show that $\phi_{n+1} = \psi_{n+1} \circ h_{n+1*}$. This is true since $h_{n+1}|\partial N_n^e = \mathrm{Id}$ and since every element in $\pi_1(N_{n+1})$ can be written as a product of some elements presented by loops in N_n or N'.

Now define $h \colon N \to N$ by $h|N_n = h_n$. Then h is a homeomorphism on N, and $h_* = \varinjlim_n h_{n*}$. Therefore $\phi = \psi \circ h_*$. This shows that ϕ and ψ are equivalent.

The necessity can been seen easily from the definition of the equivalence and the end data. □

The following well-known result is used in the above proof.

Lemma 3.2. *Let N be a compact surface with boundary, and let h be a homeomorphism on N which keeps each boundary component invariant. Then there exists a homeomorphism h' on N such that $h'|\partial N = \mathrm{Id}$ and $h_* = h'_* \colon \pi_1(N) \to \pi_1(N)$.*

4. Equivalence of cluster ends.

If an action on a noncompact surface is effective, it may have infinitely many branch points. In this case, the end components of some ends intersect the branch set infinitely many times. We shall introduce the *type* and extend the definition of the end data for this situation.

Definitions. Let M be a noncompact surface with finitely many ends, and let G be a finite abelian group. If a G-action ϕ on M has infinitely many (countable) branch points, then there exist at least one end e of the orbit space $N = M/\phi$ such that: if $U_1^e \supset U_2^e \supset \cdots$ is an end component at e, then $U_n^e \cap B \neq \emptyset$ for all n. Such an end e is called a *cluster end of ϕ.*

The *type of a cluster end e of ϕ* is a subset of the group G which is defined by

$$ty_\phi(e) = \{g \in G \mid \forall n \ \exists x \in U_n^e \cap B \ \text{ such that } \ \phi(x) = g\}.$$

A cluster end is *planar* (or *nonplanar*) if it is planar (or nonplanar) as an end of the orbit space N.

Note that the preimages of cluster ends of N are always nonplanar ends of M. This is different from the case of finitely many branch points in which an end on the orbit space has the same planarity as its preimage in M.

Let $N_1 \subset N_2 \subset \cdots$ be a decomposition of the orbit space N. If an end e of N is not a cluster end, then $\phi([\partial N_n^e]) = \phi([\partial N_{n+1}^e])$ holds for all sufficiently large n. Therefore, we can still define $\phi(e)$ and $G_\phi(e)$ as in §2.

For a cluster end e of ϕ with the type $ty_\phi(e)$, let $\langle ty_\phi(e)\rangle$ be a subgroup of G generated by the elements of $ty_\phi(e)$. Denote by ϕ^e the composition of the quotient homomorphism $G \to G/\langle ty_\phi(e)\rangle$ and $\phi\colon \pi_1(N-B) \to G$. Then $\phi^e \colon \pi_1(N-B) \to G/\langle ty_\phi(e)\rangle$ is an epimorphism, and it defines a $G/\langle ty_\phi(e)\rangle$-action ϕ^e on a noncompact surface with finitely many ends whose orbit space is N. The end e of N is not a cluster end of ϕ^e. Therefore, $\phi^e(e)$ and $G_{\phi^e}(e)$ are well-defined.

Definition. Let ϕ be an effective G-action on a noncompact surface with finitely many ends, and let N be its orbit space. The function $\mathcal{E}(\phi)(e) = (\phi^e(e),\ G_{\phi^e}(e))$ defined on the set of the ends of N (with planarities specified) is called the *end data of ϕ.*

Note that if e is not a cluster end, $ty_\phi(e) = \emptyset$; so $\phi^e(e) = \phi(e)$ and $G_{\phi^e}(e) = G_\phi(e)$.

Let ϕ be an effective G-action on a noncompact surface with the orbit space N, and let e be a cluster end of ϕ with the type $ty_\phi(e)$. An *admissible neighborhood* U^e of e is an element of $\{U_1^e, U_2^e, \cdots\}$, where $U_1^e \supset U_2^e \supset \cdots$ is an end component at e of N such that (i) the boundary of U^e in N is a simple closed curve, (ii) U^e is planar if e is planar, and (iii) if $x \in U^e \cap B$, then $\phi(x) \in ty_\phi(e)$.

Let $\phi\colon \pi_1(N - B) \to G$ and $\psi\colon \pi_1(N' - B') \to G$ be two effective G-actions on a noncompact surface M with finitely many ends and with N and N' as their orbit spaces respectively. Assume that e is an end of ϕ on N and e' an end of ψ on N'. We say that the end e of ϕ is *equivalent* to the end e' of ψ provided there exist admissible neighborhoods U^e of e in N, $U^{e'}$ of e' in N', and a homeomorphism $h\colon U^e, \partial U^e \to U^{e'}, \partial U^{e'}$ such that $\phi|\pi_1(U^e) = \psi|\pi_1(U^{e'})\circ h_*$.

Proposition 4.1. *Let M be a noncompact surface with finitely many ends. Let ϕ and ψ be effective $\mathbb{Z}/n$-actions on M with their orbit spaces N and N' respectively. A cluster end e of ϕ is equivalent to a cluster end e' of ψ if and only if*

(i) *$ty_\phi(e) = ty_\psi(e')$, and*

(ii) *$\mathcal{E}(\phi)(e) = \mathcal{E}(\psi)(e')$.*

Remark 4.2.

(a) The condition (ii) above is not a consequence of the condition (i).

(b) The conditions (i) and (ii) above imply that e and e' have the same planarity, and that there exist admissible neighborhoods U^e of e and $U^{e'}$ of e' such that

$$\phi([\partial U^e]) = \psi([\partial U^{e'}]) + \alpha_1 g_1 + \alpha_2 g_2 + \cdots \alpha_s g_s$$

for some $g_1, g_2, \ldots, g_s \in ty_\phi(e) = ty_\psi(e')$ and $\alpha_1, \alpha_2, \ldots, \alpha_s \in \mathbb{Z}$. This can be seen from the definition of the end data. If $\mathcal{E}(\phi)(e) = \mathcal{E}(\psi)(e')$, then $\phi^e(e) = \psi^{e'}(e')$ in $G/\langle ty_\phi(e)\rangle$. Therefore, there exist admissible neighborhoods U^e and $U^{e'}$ such that $\phi^e([\partial U^e]) = \psi^{e'}([\partial U^{e'}])$. Hence, $\phi([\partial U^e]) = \psi([\partial U^{e'}]) + g$ for some $g \in \langle ty_\phi(e)\rangle$.

(c) When $G = \mathbb{Z}/p$ (p is a prime), assume that the condition (i) in Proposition 4.1 holds. Then the condition (ii) is true if e and e' have the same planarity.

In order to prove Proposition 4.1, we introduce the following lemma.

Lemma 4.3. *Let D be a disk with the center O, and let $a_1, a_2, \ldots$ ($a_i \neq O$) be a sequence of distinct points in the disk D whose only limit point is O.*

Then there exist nonintersecting simple closed curves $\gamma_1, \gamma_2, \dots$ on D such that

(i) *each γ_n lies in the simple connected region bounded by γ_{n-1};*

(ii) *a_n is the only point from the sequence $\{a_i\}$ contained in the region bounded by γ_{n-1} and γ_n;*

(iii) $|\gamma_n| \to 0$ $(n \to \infty)$ *where* $|\gamma_n| = \sup_{x\in\gamma_n} |x - O|$.

The proof of the lemma is elementary and will be omitted.

Lemma 4.3 insures that the construction around a cluster end e of ϕ is based on $ty_\phi(e)$ but not on the positions of the branch points. Suppose that $ty_\phi(e) = \{\ g_1, g_2, \dots, g_r\ \}$. Then the situation near e looks as in Figure 4.1. Here, D is a plane disk with one puncture if e is planar, and D is an once punctured disk with infinitely many handles if e is nonplanar.

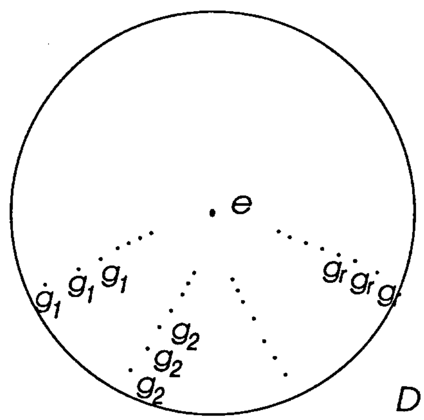

Figure 4.1.

Proof of Proposition 4.1. Necessity follows easily from the definition of the equivalence of two ends.

For sufficiency, assume that $ty_\phi(e) = \{g_1, g_2, \dots, g_r\}$. The conditions (i) and (ii) imply that $\phi([\partial U^e]) = \psi([\partial U^{e'}]) + \alpha_1 g_1 + \alpha_2 g_2 + \cdots + \alpha_s g_s$ for any admissible neighborhoods U^e of e and $U^{e'}$ of e' (for some g_i's and α_i's — see Remark 4.2); in particular, there exist admissible neighborhoods U^e of e and $U^{e'}$ of e' such that $\phi([\partial U^e]) = \psi([\partial U^{e'}])$. Here U^e and $U^{e'}$ are both planar or nonplanar.

Next we construct a homeomorphism h between U^e and $U^{e'}$ such that $\phi|\pi_1(U^e) = \psi|\pi_1(U^{e'})\circ h_*$. By Lemma 4.3, we may choose an end component $U^e = U^e_1 \supset U^e_2 \supset \cdots$ at e such that each U^e_n is an admissible neighborhood and each $U^e_n - U^e_{n+1}$ contains at least r branch points $x_1, x_2, \dots, x_r$ with $\phi(x_i) = g_i$, for $i = 1, 2, \dots, r$. Similarly, we choose an end component

$U^{e'} = U_1^{e'} \supset U_2^{e'} \supset \cdots$ at e' such that each $U_n^{e'} - U_{n+1}^{e'}$ is homeomorphic to $U_n^e - U_{n+1}^e$ and the fixed point data of ψ on $U_n^{e'} - U_{n+1}^{e'}$ is the same as the fixed point data of ϕ on $U_n^e - U_{n+1}^e$. Let $V_n^e = U_n^e - U_{n+1}^e$ and $V_n^{e'} = U_n^{e'} - U_{n+1}^{e'}$.

Now consider the homomorphisms $\phi_n = \phi \circ i_* \colon \pi_1(V_n^e) \xrightarrow{i_*} \pi_1(N) \xrightarrow{\phi} \mathbb{Z}/n$ and $\psi_n = \psi \circ i'_* \colon \pi_1(V_n^{e'}) \xrightarrow{i'_*} \pi_1(N) \xrightarrow{\psi} \mathbb{Z}/n$ where i_* and i'_* are induced by the inclusions. We want to show that there is a homeomorphism h_n between each V_n^e and $V_n^{e'}$ such that $\phi_n = \psi_n \circ h_{n*}$. Denote $ty(e) = ty_\phi(e)$ $(= ty_\psi(e'))$, and let p be the quotient homomorphism $p \colon \mathbb{Z}/n \to (\mathbb{Z}/n)/\langle ty(e) \rangle$. The condition (ii) of Proposition 4.1 implies that $\{U_n^e\}$ and $\{U_n^{e'}\}$ can be chosen such that $p \circ \phi_n(\pi_1(V_n^e)) = p \circ \psi_n(\pi_1(V_n^{e'})) = G_{\phi^e}(e)$. Then $\phi_n(\pi_1(V_n^e)) = \psi_n(\pi_1(V_n^{e'}))$ since $ty(e) \subset \phi_n(\pi_1(V_n^e)) \cap \psi_n(\pi_1(V_n^{e'}))$. Also by the constructions of U_n^e and $U_n^{e'}$, we have that $\phi_n([\partial U_n^e]) = \psi_n([\partial U_n^{e'}])$ and $\phi_n([\partial U_{n+1}^e]) = \psi_n([\partial U_{n+1}^{e'}])$. So a homeomorphism h_n between V_n^e and $V_n^{e'}$ such that $\phi_n = \psi_n \circ h_{n*}$ can be obtained in the same way as in the proof of Theorem 3.1. Using the method of Theorem 3.1 again and passing with h_{n*} to the direct limit, we get a homeomorphism h between U^e and $U^{e'}$ such that $\phi = \psi \circ h_*$. This completes the proof. □

From the proof of Proposition 4.1, we also get the following result.

Corollary 4.4. *If a cluster end e of ϕ is equivalent to a cluster end e' of ψ, and if admissible neighborhoods U^e of e and $U^{e'}$ of e' satisfy $\phi([\partial U^e]) = \psi([\partial U^{e'}])$, then $\phi|\pi_1(U^e) \sim \psi|\pi_1(U^{e'})$, that is, there exist a homeomorphism $h \colon U^e, \partial U^e \to U^{e'}, \partial U^{e'}$ such that $\phi|\pi_1(U^e) = \psi|\pi_1(U^{e'}) \circ h_*$.*

5. Classification of Effective $\mathbb{Z}/n$-actions.

For our convenience, all the sets (subsets) in this section are sets of elements with multiplicities. For any two sets A and B, let $A \sqcup B$ denote the union of elements with added multiplicities.

Let M be a noncompact surface with finitely many ends, and let $\phi \colon \pi_1(N - B, x_0) \to \mathbb{Z}/n$ be an effective $\mathbb{Z}/n$-action on M with the orbit space $N = M/\phi$.

If the branch set B is finite, that is, all the ends of N are not cluster, then we can choose a decomposition $N_1 \subset N_2 \subset \cdots$ of N with $B \subset N_1$. With the method used in the proof of Theorem 3.1, one can show that the $\mathbb{Z}/n$-action is determined by its fixed point data $\mathcal{D}(\phi)$ and end data $\mathcal{E}(\phi)$ (note that the sum of all the images of $\mathcal{D}(\phi)$ and all the first elements of the images of $\mathcal{E}(\phi)$ is 0). If B is finite, then two $\mathbb{Z}/n$-actions on M with the same fixed point data and end data have homeomorphic orbit spaces.

Theorem 5.1. *Let M be a noncompact surface with finitely many ends.*

Two effective $\mathbb{Z}/n$-actions on M with finitely many branch points are equivalent if and only if they have the same fixed point data and end data.

Now we consider $\mathbb{Z}/n$-actions with infinitely many branch points. Our observation shows that if two effective $\mathbb{Z}/n$-actions have the same orbit space, same fixed point data, same end data, and same corresponding end types, then they are *stably equivalent* (see the definition below).

Let ϕ be an effective $\mathbb{Z}/n$-action on M with its orbit space N (M and N are both noncompact surfaces with finitely many ends), and suppose that ϕ has the cluster ends $e_1, e_2, \dots, e_r$ on N with the types $ty(e_1),\ ty(e_2), \dots,\ ty(e_r)$ ($r \geq 1$).

Definitions. Let S be a finite set of nontrivial elements of $\mathbb{Z}/n$. It is called a *stabilizing data of* ϕ if

(i) each element of S is in $\bigcup_{i=1}^{r} ty(e_i)$, and

(ii) the sum of all the elements of S is 0.

Let S^2 be a sphere, and let $S = \{ g_1, g_2, \dots, g_k \}$ be a stabilizing data of ϕ. Construct a $\mathbb{Z}/n$-action $\chi(S)$ with S^2 as its orbit space in the following way: Pick $x_1, x_2, \dots, x_k \in S^2$ as the branch points, and define $\mathcal{D}(\chi(S))(x_i) = g_i$ for $i = 1, 2, \dots, k$. Note that $\chi(S)$ is unique up to (conjugate) equivalence. The *sum of* ϕ *and* $\chi(S)$, denoted by $\phi\#\chi(S)$, is obtained by taking the connected sum of their orbit spaces ($N\#S^2$) along a free orbit. Then $\phi\#\chi(S)$ is a well-defined effective $\mathbb{Z}/n$-action which still acts on M. Let ψ be another $\mathbb{Z}/n$-action on M. We say that ϕ is *stably equivalent* to ψ if $\psi \sim \phi\#\chi(S)$ for some stabilizing data S of ϕ (S is allowed to be empty).

Clearly, if ϕ and ψ are stably equivalent, they have the same orbit space, same fixed point data, same end data, and same corresponding cluster end types. Stable equivalence is an equivalence relation among effective $\mathbb{Z}/n$-actions on a noncompact surface M. In particular, if $\psi \sim \phi\#\chi(S)$, then $\phi \sim \psi\#\chi(S')$, here S' is the set which satisfies that $g \in S$ with the multiplicity k if and only if $g \in S'$ with the multiplicity $n-k$ ($n = |\mathbb{Z}/n|$).

Theorem 5.2. *Let M be a noncompact surface with finitely many ends. Two effective $\mathbb{Z}/n$-actions on M are stably equivalent if and only if they have the same orbit space, same fixed point data, same end data, and their corresponding cluster ends are equivalent.*

Note that orbit spaces of two effective $\mathbb{Z}/n$-actions on M may not have the same genus even if they have the same fixed point data, same end data, and their corresponding cluster ends are equivalent (in the case when all the ends of their orbit spaces are planar).

By Proposition 4.1, we can rephrase the above theorem as follows.

Theorem 5.2′. *Let M be a noncompact surface with finitely many ends. Two effective $\mathbb{Z}/n$-actions on M are stably equivalent if and only if they have the same orbit space, same fixed point data, same end data, and same corresponding cluster end types.*

Proof of Theorem 5.2. Suppose that $\phi,\ \psi\colon \pi_1(N-B)\to\mathbb{Z}/n$ are effective $\mathbb{Z}/n$-actions on M with the same orbit space N, same fixed point data, same end data, and corresponding cluster end types $ty(e_1),\ ty(e_2),\dots,ty(e_r)$ $(r\geq 1)$. Assume that the cluster end e_i of ϕ is equivalent to e_i of ψ, then there exist admissible neighborhoods $U_\phi^{e_i}$ of e_i with respect to ϕ and $U_\psi^{e_i}$ of e_i with respect to ψ such that

$$\phi([\partial U_\phi^{e_i}])=\psi([\partial U_\psi^{e_i}])+\alpha_{i,1}g_{i,1}+\alpha_{i,2}g_{i,2}+\cdots+\alpha_{i,i_s}g_{i,i_s}$$

where $g_{i,j}\in ty(e_i),\ \alpha_{i,j}\in\mathbb{Z}$, and $|\alpha_{i,j}|<n$ (Remark 4.2). The above formula actually holds for any admissible neighborhoods $U_\phi^{e_i}$ and $U_\psi^{e_i}$.

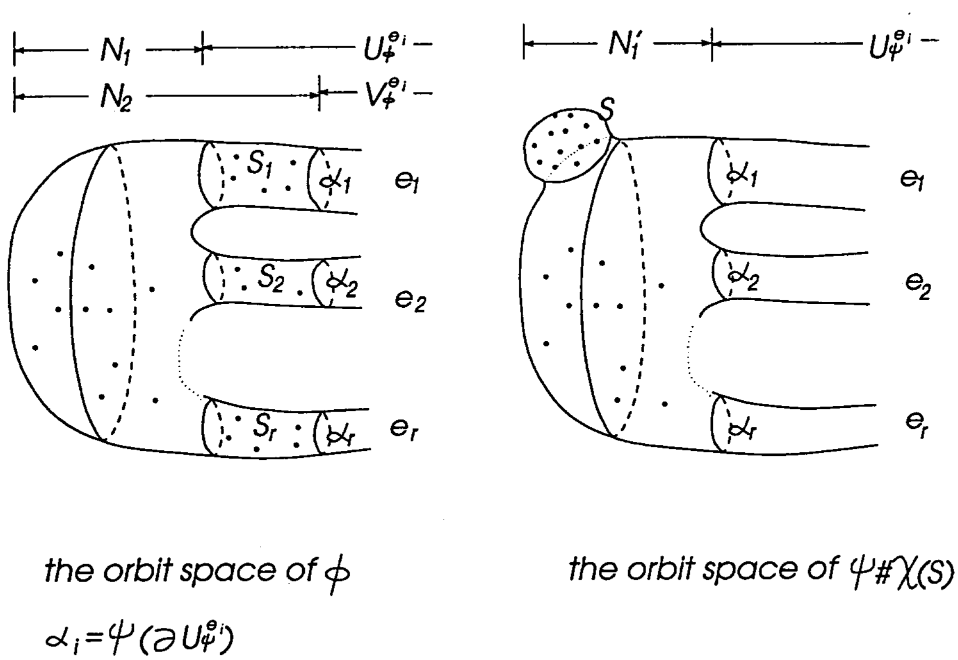

Figure 5.1.

Now we choose admissible neighborhoods $U_\phi^{e_1},U_\phi^{e_2},\dots,U_\phi^{e_r}$ and $U_\psi^{e_1},\ U_\psi^{e_2},\dots,\ U_\psi^{e_r}$ (see Figure 5.1) which satisfy the following conditions.

(i) $\overline{U_\phi^{e_i}}\cap\overline{U_\phi^{e_j}}=\emptyset$ and $\overline{U_\psi^{e_i}}\cap\overline{U_\psi^{e_j}}=\emptyset$ when $i\neq j$;

(ii) $N_1 = N - \bigcup_{i=1}^r U_\phi^{e_i}$ and $N_1' = N - \bigcup_{i=1}^r U_\psi^{e_i}$ have the same genus;

(iii) $N_1 \cap B \supseteq B^{\text{finite}}$ and $N_1' \cap B \supseteq B^{\text{finite}}$;

(iv) $\phi|\pi_1(N_1)\colon \pi_1(N_1) \to \mathbb{Z}/n$ and $\psi|\pi_1(N_1')\colon \pi_1(N_1') \to \mathbb{Z}/n$ are surjective and have the same fixed point data.

By the condition (iv) above, $\sum_{i=1}^r \phi([\partial U_\phi^{e_i}]) = \sum_{i=1}^r \psi([\partial U_\psi^{e_i}])$. Therefore, $\sum_{i=1}^r \sum_{j=1}^{i_s} \alpha_{i,j} g_{i,j} = 0$.

Let S_i be the set of $g_{i,j}$ such that the multiplicity of $g_{i,j}$ in S_i is $\alpha_{i,j}$ (if $\alpha_{i,j} > 0$), or is $n + \alpha_{i,j}$ (if $\alpha_{i,j} < 0$) for $i = 1, 2, \dots, r$. Now take admissible neighborhoods $V_\phi^{e_1}$, $V_\phi^{e_2}, \dots,$ $V_\phi^{e_r}$ of ϕ which satisfy for each $1 \le i \le r$,

(1) $V_\phi^{e_i} \subset U_\phi^{e_i}$,

(2) the genus of $U_\phi^{e_i} - V_\phi^{e_i}$ is 0, and

(3) the image set of the fixed point data of ϕ on $U_\phi^{e_i} - V_\phi^{e_i}$ is S_i.

Then we have $\phi([\partial V_\phi^{e_i}]) = \psi([\partial U_\psi^{e_i}])$. By Corollary 4.4, $\phi|\pi_1(V_\phi^{e_i}) \sim \psi|\pi_1(U_\psi^{e_i})$.

Note that $S = \bigsqcup_{i=1}^r S_i$ is a stabilizing data of ϕ. Let $N_2 = N - \bigcup_{i=1}^r V_\phi^{e_i}$. Then $\phi|\pi_1(N_2)$ and $\psi|\pi_1(N_1')$ have the same boundary data, and the fixed point data of $\phi|\pi_1(N_2)$ is the union of the fixed point data of $\psi|\pi_1(N_1')$ and $\chi(S)$. Thus $\phi|\pi_1(N_2) \sim \psi|\pi_1(N_1')\#\chi(S)$ by Corollary 1.2. Therefore, we have $\phi \sim \psi\#\chi(S)$.

Necessity of the theorem is trivial by the definition of the stable equivalence. $\square$

The following example shows that two effective $\mathbb{Z}/n$-actions on M which are stably equivalent are not necessarily conjugate equivalent.

Example. Let N be a noncompact surface with exactly two planar ends. Let $\{x_n\}$, $\{y_n\}$ be two sequences of distinct points of N, and let γ be a simple closed curve as shown in Figure 5.2. Denote $B = \{x_1, y_1, x_2, y_2, \dots\}$.

Define two $\mathbb{Z}/3$-actions $\phi,\ \psi\colon \pi_1(N - B) \to \mathbb{Z}/3$ by

$$\phi(x_n) = 2, \quad \phi(y_n) = 1, \quad \phi([\gamma]) = 2$$

and

$$\psi(x_n) = 2, \quad \psi(y_n) = 1, \quad \psi([\gamma]) = 0$$

for $n = 1, 2, \dots$. Both ϕ and ψ act on a noncompact surface with three ends, all of which are nonplanar. We have $ty_\phi(e_1) = ty_\psi(e_1) = \{2\}$ and $ty_\phi(e_2) = ty_\psi(e_2) = \{1\}$. Then $(\mathbb{Z}/3)/\langle ty_\phi(e_i)\rangle \cong \{0\}$, and $\mathcal{E}(\phi)(e_i) = (0,\ \{0\}) = \mathcal{E}(\psi)(e_i)$ for $i = 1, 2$. So ϕ and ψ has the same fixed point data, same end data, and same corresponding cluster end types. Hence, ϕ and ψ

are stably equivalent. In fact, $\phi \sim \psi\#\chi(S)$ for $S = \{2, 1\}$. But ϕ is not conjugate equivalent to ψ since $\phi([\gamma]) \neq \psi([\gamma])$.

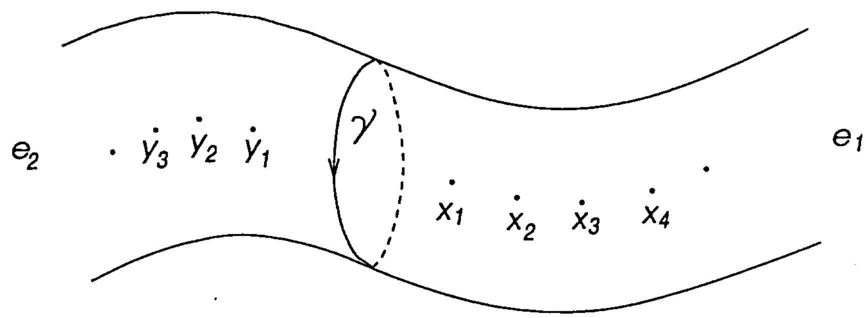

Figure 5.2.

Note that if $\phi \sim \psi\#\chi(S)$, S is not necessarily unique. It may even happen that $\phi \sim \phi\#\chi(S)$ holds for some nonempty S. In order to get a reduced form for stabilizing data and to see when two stably equivalent $\mathbb{Z}/n$-actions on M are actually conjugate, we need more information.

Let ϕ be an effective $\mathbb{Z}/n$-action on M with finitely many ends, and let $e_1, e_2, \dots, e_r$ be the cluster ends of ϕ with the types $ty(e_1),\ ty(e_2), \dots,\ ty(e_r)$.

Definitions. A stabilizing data E of ϕ is *trivial* if it can be written as $E = E_1 \sqcup E_2 \sqcup \cdots \sqcup E_k$ for some integer k, and if each E_i $(i = 1, 2, \dots, k)$ satisfies:

(i) All the elements of E_i are in the same $ty_\phi(e_j)$ for some $1 \leq j \leq r$, and

(ii) $\sum_{g \in E_i} g = 0$.

Two stabilizing data S and S' of ϕ are *equivalent* if there exist trivial stabilizing data E and E' of ϕ such that $S \sqcup E = S' \sqcup E'$. A stabilizing data S of ϕ is *irreducible* if no subset of S is equivalent to a trivial stabilizing data of ϕ.

We claim that the conjugate equivalency of an effective $\mathbb{Z}/n$-action will not be affected by adding a trivial stabilizing data. This can be seen from Corollary 4.4.

Lemma 5.3. *If E is a trivial stabilizing data of ϕ, then $\phi\#\chi(E) \sim \phi$.*

Theorem 5.4. *Let ϕ be an effective $\mathbb{Z}/n$-action on a noncompact surface with finitely many ends, and let S be a stabilizing data of ϕ. Then ϕ is conjugate equivalent to $\phi\#\chi(S)$ if and only if S is equivalent to a trivial stabilizing data of ϕ.*

Before proving the theorem, we establish the following corollary first.

Corollary 5.5. *Let ϕ be an effective $\mathbb{Z}/n$-action on a noncompact surface with finitely many ends, and let S and S' be stabilizing data of ϕ. Then $\phi\#\chi(S) \sim \phi\#\chi(S')$ if and only if S is equivalent to S'.*

Proof. If S is equivalent to S', there are trivial stabilizing data E and E' of ϕ such that $S \sqcup E = S' \sqcup E'$. By Lemma 5.3, $\phi\#\chi(S) \sim (\phi\#\chi(S))\#\chi(E) = \phi\#\chi(S \sqcup E) = \phi\#\chi(S' \sqcup E') = (\phi\#\chi(S'))\#\chi(E') \sim \phi\#\chi(S')$.

Conversely, suppose that $\phi\#\chi(S) \sim \phi\#\chi(S')$. We want to show S is equivalent to S'. Choose a stabilizing data R of ϕ such that $S \sqcup R = E$ where E is trivial. Then we have $\phi \sim \phi\#\chi(E) = \phi\#\chi(S \sqcup R) = (\phi\#\chi(S))\#\chi(R) \sim (\phi\#\chi(S'))\#\chi(R) = \phi\#\chi(S' \sqcup R)$. Therefore, $S' \sqcup R$ is equivalent to a trivial stabilizing data by Theorem 5.4. That is, there are trivial stabilizing data F and F' such that $S' \sqcup R \sqcup F = F'$. Then $S \sqcup F' = S \sqcup S' \sqcup R \sqcup F = S' \sqcup (E \sqcup F)$. Since both F' and $E \sqcup F$ are trivial, S is equivalent to S'. $\square$

Proof of Theorem 5.4. Let N be the orbit space of ϕ with the cluster ends $e_1, e_2, \ldots, e_r$ $(r \geq 1)$. Choose nonintersecting admissible neighborhoods U_0^i of e_i on N $(i - 1, 2, \ldots, r)$ such that $(N - \bigcup_{i=1}^r U_0^i) \cap B = B^{\text{finite}}$. Let $\alpha_i = \phi([\partial U_0^i])$ for $i = 1, 2, \ldots, r$.

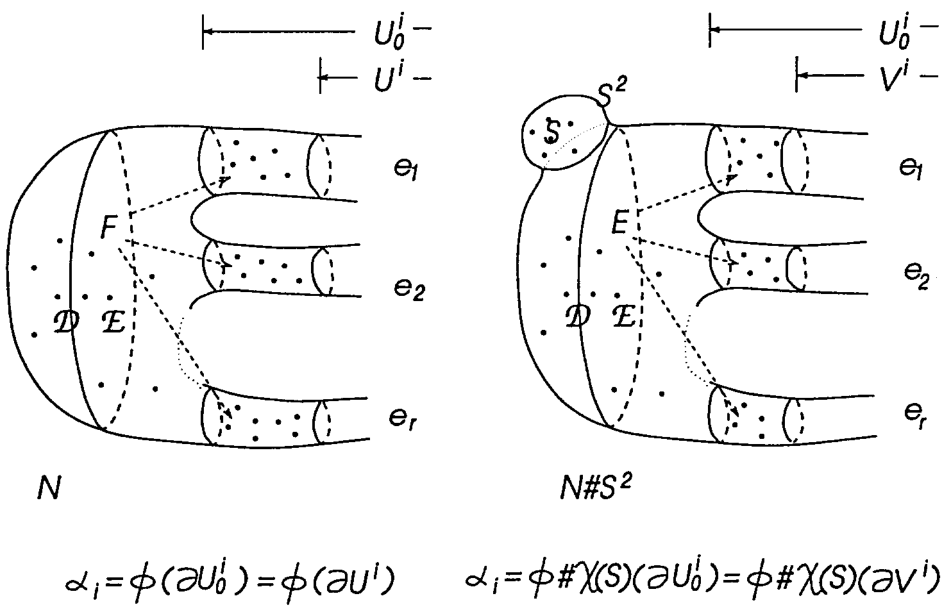

Figure 5.3.

Now choose another set of admissible neighborhoods $U^1, U^2, \ldots, U^r$ on N such that $U^i \subset U_0^i$, $\phi([\partial U^i]) = \alpha_i$ for $i = 1, 2, \ldots, r$, and the set $F =$

$\{\phi(x) \mid x$ is a branch point on $U_0^i - U^i$ for $i = 1, 2, \ldots, r\}$ contains S. Then F is a trivial stabilizing data of ϕ since $\phi([\partial U_0^i]) = \phi([\partial U^i])$ for $i = 1, 2, \ldots, r$.

Suppose $\phi \sim \phi \# \chi(S)$. We may assume that the connected sum $N \# S^2$ is performed on $N - \bigcup_{i=1}^r U_0^i$. Let $V^1, V^2, \ldots, V^r$ be the corresponding admissible neighborhoods of $U^1, U^2, \ldots, U^r$ on $N \# S^2$ under the conjugate equivalence (see Figure 5.3). We may choose $U^1, U^2, \ldots, U^r$ small enough so that each V^i is a subset of U_0^i. Therefore, $\phi \# \chi(S)([\partial V^i]) = \phi \# \chi(S)([\partial U_0^i]) = \alpha_i$ for $i = 1, 2, \ldots, r$. Let $E = \{\phi(x) \mid x$ is a branch point of $\phi \# \chi(S)$ on $U_0^i - V^i$, $i = 1, 2, \ldots, r\}$. Then E is a trivial stabilizing data of ϕ, and $S \sqcup E = F$. Thus S is equivalent to a trivial stabilizing data of ϕ.

On the other hand, if S is equivalent to a trivial stabilizing data of ϕ, there exist trivial stabilizing data E and E' such that $S \sqcup E = E'$. By Lemma 5.3, $\phi \# \chi(S) \sim (\phi \# \chi(S)) \# \chi(E) = \phi \# \chi(S \sqcup E) = \phi \# \chi(E') \sim \phi$. $\square$

Corollary 5.6. *Let ϕ be an effective $\mathbb{Z}/n$-action on a noncompact surface M with finitely many ends, and let S be irreducible stabilizing data of ϕ. Then $\phi \# \chi(S) \sim \phi$ if and only if $S = \emptyset$.*

In the following two cases, any stabilizing data of ϕ is trivial. Therefore, we have the following results by Theorem 5.2 and Lemma 5.3.

Corollary 5.7. *Let M be a noncompact surface with finitely many ends, and let ϕ and ψ be two effective $\mathbb{Z}/n$-actions on M with only one cluster end. Then ϕ and ψ are conjugate equivalent if and only if they have the same orbit space, same fixed point data, same end data, and same cluster end type.*

Corollary 5.8. *Let M be a noncompact surface with finitely many ends, and let ϕ and ψ be two effective $\mathbb{Z}/n$-actions on M such that at least one cluster end type is $\mathbb{Z}/n$. Then ϕ and ψ are conjugate equivalent if and only if they have the same orbit space, same fixed point data, same end data, and same corresponding cluster end types.*

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