A LOEWNER-TYPE LEMMA FOR WEIGHTED BIHARMONIC OPERATORS

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In the present note we give a simpler proof of the recent result of Hedenmalm that the Green function for the weighted biharmonic operator $\Delta|z|^{2\alpha}\Delta$, $\alpha > -1$, on the unit disc **D** with the Dirichlet boundary conditions is positive. The main ingredient, which in the special case of the unweighted biharmonic operator Δ^2 is due to Loewner and which is of an independent interest, is a lemma characterizing, for a positive C^2 weight function w, the second-order linear differential operators which take any function u satisfying $\Delta w^{-1}\Delta u = 0$ into a harmonic function. Another application of this lemma concerning positivity of the Poisson kernels for the biharmonic operator Δ^2 is also given.

Let Ω be a bounded domain in the complex plane with smooth boundary, w a function which is C^2-continuous and positive on $\overline{\Omega}$ except for a finite set S of isolated singularities in Ω, where it can have a zero or become $+\infty$ (or even not be defined at all); such functions will be termed *weights*. We will be interested in the weighted biharmonic operators $\Delta w^{-1}\Delta$ on Ω, where Δ stands for the Laplacian. For brevity, let us call the above w-*biharmonic* operators, and similarly a function u satisfying $\Delta w^{-1}\Delta u = 0$ will be called a w-*biharmonic* function. The latter means, by definition, that u is a C^2 function on $\Omega \setminus S$ such that $w^{-1}\Delta u$ extends by continuity to a harmonic function on all of Ω. The w-biharmonic Green function $U(x,y)$ is the solution to the boundary value problem

$$\begin{cases} \Delta_x \dfrac{1}{w(x)} \Delta_x U(x,y) = \delta(x-y) & \text{(the delta function)}, \\ U(x,y) = \dfrac{\partial U}{\partial n_x}(x,y) = 0 & \text{for } x \in \partial\Omega, \end{cases}$$

where $\partial/\partial n$ denotes differentiation in the outward normal direction. The first equation must again be understood in the sense that, for fixed y, U is in $C^2(\Omega \setminus S \setminus \{y\})$ and $w^{-1}\Delta U = \Gamma(\,\cdot\,,y) + h_y$ where $\Gamma(x,y)$ is the ordinary Green function (for the Laplace operator) on Ω and h_y is harmonic on all of Ω. Quite recently, a lot of attention has been attracted by the problem of *positivity* of these Green functions, for various weights w. The motivation

for this comes mainly from applications in the theory, initiated by Heden-
malm, of contractive zero divisors and factorization of functions in Bergman
spaces, cf. [**DKSS1**], [**DKSS2**], [**H-jram**], [**H-OpP**]. For $w \equiv 1$, i.e. the
unweighted biharmonic operator, the Green function is related to bending
of a clamped plate and to the creeping flow on Ω, and the question concern-
ing its positivity comes back to Hadamard and Boggio (see the discussion
in [**H-Duke**] for more details and references). In that case, the answer is
known to be positive for a few domains (e.g. a disc, or Pascal's limaçon)
and negative for many others (sufficiently eccentric ellipses [**Gar**], [**ShT**],
sufficiently elongated rectangles [**NS**], all annuli [**E-Pee**]; also, granted we
relax the condition that Ω be bounded, the infinite strip [**Duff**]). For general
weight w and Ω the unit disc $\mathbf{D}$, various results were obtained by Heden-
malm [**H-Duke**], [**H-OpP**], Shimorin [**Shi1**], [**Shi2**], and Stessin [**Ste**]; see
also [**H-BVP**]. Observe that, if Ω is simply connected and ϕ is a conformal
map of $\mathbf{D}$ onto Ω, the pullback via ϕ of the Green function for Δ^2 on Ω is the
Green function for $\Delta|\phi'|^{-2}\Delta$ on $\mathbf{D}$; thus, some of the results just mentioned
(i.e., for general weight w on $\mathbf{D}$) can also be interpreted as results about
unweighted biharmonic operators on different domains (which are images of
$\mathbf{D}$ under a conformal map).

The main result of [**H-Duke**] is a proof of the positivity of the Green
function U for the operator $\Delta|z|^{-2\alpha}\Delta$, $\alpha > -1$, on $\mathbf{D}$. Although an explicit
formula for U is given for α an integer ([**H-Duke**], Theorem 4.6), the posi-
tivity is not completely obvious directly from it; instead, it is obtained for all
α as a consequence of the positivity of the Green function for the operator
$\Delta \exp(-2\alpha \mathrm{Im}\, z)\Delta$ on the upper half-plane, which in turn is reduced, by a
skillful and quite involved explicit computation using the above-mentioned
formula for integer α, to the positivity of a certain integral, and the last fi-
nally follows by an argument based on Pólya's lemma (see [**H-Duke**, p. 71]).
The purpose of this note is to present a simpler proof, which, essentially, uses
little beyond the ordinary maximum principle for harmonic functions. Its
basic ingredient is the following lemma, which may be of some indepen-
dent interest and whose special case $w \equiv 1$ goes essentially back to Charles
Loewner [**Loe**].

Lemma 1 (**Fundamental Lemma**). *Assume that w is C^2 and positive
on an open set $\mathcal{V}$. Let m be a function on $\mathcal{V}$ satisfying*

$$(1) \qquad\qquad \frac{1}{w}\Delta m = const.$$
$$\partial \frac{1}{w}\partial m = \overline{\partial}\frac{1}{w}\overline{\partial}m = 0$$

and form the second-order linear differential operator

$$(2) \qquad \mathcal{L}u := \frac{1}{w}[m\Delta u - \nabla m \cdot \nabla u + u\Delta m].$$

Then $\mathcal{L}$ transforms every w-biharmonic function on $\mathcal{V}$ into a harmonic function. Conversely, every second order linear differential operator sending an arbitrary w-biharmonic function on $\mathcal{V}$ into a harmonic one must be of the form (2) *with m satisfying* (1).

Here and elsewhere, $\nabla m \cdot \nabla u$ stands for the scalar product of the complex gradients

$$\nabla m \cdot \nabla u = \partial m \cdot \overline{\partial} u + \overline{\partial} m \cdot \partial u$$

where $\partial \equiv \partial/\partial z$, $\overline{\partial} \equiv \partial/\partial \overline{z}$ (the Wirtinger operators). Also we employ a slightly nonstandard definition of the Laplacian

$$\Delta := \partial\overline{\partial} = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \qquad\qquad (z = x + yi),$$

which differs from the usual one by a factor of 4.

Proof. We begin with the converse part. Assume that $\mathcal{L}$, which we choose to write in the form

$$(3) \qquad \mathcal{L}u = \frac{1}{w}[m\,\partial\overline{\partial}u + b\,\partial\partial u + c\,\overline{\partial}\overline{\partial}u] + A\,\partial u + B\,\overline{\partial}u + Cu,$$

sends an arbitrary w-biharmonic function into a harmonic one. Expanding we have

$$(4) \qquad \Delta\mathcal{L}u = m\Delta\frac{1}{w}\Delta u + b\Delta\frac{1}{w}\partial\partial u + c\Delta\frac{1}{w}\overline{\partial}\overline{\partial}u + \text{ terms of lower order.}$$

Since the only relation between the derivatives of u up to the fourth order is given by $\Delta w^{-1}\Delta u = 0$, we see already from (4) that

$$b = c = 0,$$

and we can thus write (3) in the form

$$(5) \qquad \mathcal{L}u = \frac{1}{w}m\Delta u + A\,\partial u + B\,\overline{\partial}u + Cu.$$

Hence we have

$$
\begin{aligned}
\Delta\mathcal{L}u = {}& \Delta m \cdot \frac{1}{w}\Delta u + m \cdot \Delta\frac{1}{w}\Delta u + \partial m \cdot \overline{\partial}\frac{1}{w}\Delta u + \overline{\partial}m \cdot \partial\frac{1}{w}\Delta u \\
& + \Delta A \cdot \partial u + A \cdot \Delta\partial u + \partial A \cdot \Delta u + \overline{\partial}A \cdot \partial\partial u \\
& + \Delta B \cdot \overline{\partial}u + B \cdot \Delta\overline{\partial}u + \overline{\partial}B \cdot \Delta u + \partial B \cdot \overline{\partial}\overline{\partial}u \\
& + \Delta C \cdot u + C \cdot \Delta u + \partial C \cdot \overline{\partial}u + \overline{\partial}C \cdot \partial u.
\end{aligned}
$$

(6)

Here we have used the formula

$$\Delta(fg) = f\Delta g + g\Delta f + \partial f \cdot \overline{\partial}g + \overline{\partial}f \cdot \partial g \tag{7}$$

which will be of great service to us also later on.

Now, first of all, $\Delta\mathcal{L}u$ must vanish for harmonic u. For $\Delta u = 0$, the right-hand side of (6) reduces to

$$\Delta A \cdot \partial u + \overline{\partial}A \cdot \partial\partial u + \Delta B \cdot \overline{\partial}u + \partial B \cdot \overline{\partial}\partial u + \Delta C \cdot u + \partial C \cdot \overline{\partial}u + \overline{\partial}C \cdot \partial u.$$

Since the only relation between the derivatives of u up to the second order is $\Delta u = 0$, we must have

$$\Delta A + \overline{\partial}C = 0, \quad \Delta B + \partial C = 0, \quad \Delta C = 0, \quad \overline{\partial}A = 0, \quad \partial B = 0. \tag{8}$$

From the last two conditions we see that $\Delta A = \Delta B = 0$; the first two then give $\partial C = \overline{\partial}C = 0$, and we conclude that

$$C \equiv \text{const.}, \tag{9}$$

$$\overline{\partial}A = \partial B = 0. \tag{10}$$

Feeding this back into (6) and using again the formula (7), we obtain

$$\Delta\mathcal{L}u = \Delta m \cdot \frac{1}{w}\Delta u + m \cdot \overbrace{\Delta\frac{1}{w}\Delta u}^{=0} + \tag{11}$$

$$+ \partial m \cdot \overline{\partial}\Big(\frac{1}{w}\Big) \cdot \Delta u + \partial m \cdot \frac{1}{w}\overline{\partial}\Delta u + \overline{\partial}m \cdot \partial\Big(\frac{1}{w}\Big) \cdot \Delta u + \overline{\partial}m \cdot \frac{1}{w}\partial\Delta u +$$

$$+ A \cdot \Delta\partial u + B \cdot \Delta\overline{\partial}u + (\partial A + \overline{\partial}B) \cdot \Delta u + C \cdot \Delta u.$$

Comparing the coefficients at $\Delta\partial u$, $\Delta\overline{\partial}u$, and Δu, respectively, we see that for $\Delta\mathcal{L}u$ to vanish for arbitrary w-biharmonic function u it is necessary that

$$A + \frac{1}{w}\overline{\partial}m = 0, \qquad B + \frac{1}{w}\partial m = 0, \tag{12}$$

$$\frac{1}{w}\Delta m + \partial m \cdot \overline{\partial}\Big(\frac{1}{w}\Big) + \overline{\partial}m \cdot \partial\Big(\frac{1}{w}\Big) + \partial A + \overline{\partial}B + C = 0. \tag{13}$$

From (12) we have

$$\partial A = -\partial\Big(\frac{1}{w}\Big) \cdot \overline{\partial}m - \frac{1}{w}\Delta m, \qquad \overline{\partial}B = -\overline{\partial}\Big(\frac{1}{w}\Big) \cdot \partial m - \frac{1}{w}\Delta m,$$

$$\overline{\partial}A = -\overline{\partial}\frac{1}{w}\overline{\partial}m, \qquad\qquad\qquad \partial B = -\partial\frac{1}{w}\partial m,$$

so we may rewrite (13) as

$$0 = \frac{1}{w}\Delta m - \frac{1}{w}\Delta m - \frac{1}{w}\Delta m + C$$

or

$$(14) \qquad \frac{1}{w}\Delta m = C,$$

and (10) as

$$(15) \qquad 0 = \overline{\partial}\frac{1}{w}\overline{\partial}m = \partial\frac{1}{w}\partial m.$$

Our operator m must therefore be of the form

$$\mathcal{L}u = \frac{1}{w}m\Delta u - \frac{1}{w}\overline{\partial}m \cdot \partial u - \frac{1}{w}\partial m \cdot \overline{\partial}u + \frac{1}{w}\Delta m \cdot u,$$

which is (2), with m satisfying (9), (14), and (15), which is (1).

Conversely, let $\mathcal{L}$ be any operator of the form (5), with A and B given by (12), which satisfies (9), (14), and (15). Then it follows from the formulas (6) and (11) that

$$\Delta\mathcal{L}u = m \cdot \Delta\frac{1}{w}\Delta u,$$

and so $\mathcal{L}$ sends an arbitrary w-biharmonic function into a harmonic one. This completes the proof. $\qquad\square$

Corollary 2. *If m is a real-valued function satisfying*

$$\Delta m = w, \qquad \partial\frac{1}{w}\partial m = 0,$$

then the operator (2) *transforms w-biharmonic functions into harmonic ones.*

Indeed, for real-valued m, $\overline{\partial}w^{-1}\overline{\partial}m = \overline{\partial w^{-1}\partial m} = 0$, and so (1) is fulfilled.

As an application, we prove the promised positivity result for a weighted biharmonic Green function.

Theorem 3. *Let $\alpha > -1$. Then the Green function for the weighted biharmonic operator $\Delta|z|^{-2\alpha}\Delta$ on the unit disc $\mathbf{D}$ is positive.*

Proof. Since $w = |z|^{2\alpha}$, the hypotheses (1) of the Fundamental Lemma (or, rather, of its Corollary) are satisfied for

$$m = \frac{|z|^{2(\alpha+1)} - 1}{(\alpha + 1)^2}$$

on the open set $\mathcal{V} = \mathbf{D} \setminus \{0\}$ where w is C^2 and positive. Indeed, $\Delta m = (\alpha + 1)^{-2} \partial \overline{\partial} (z^{\alpha+1} \overline{z}^{\alpha+1} - 1) = w$, and

$$\partial \frac{1}{w} \partial m = \partial \left(\frac{1}{|z|^{2\alpha}} \cdot \frac{\overline{z}|z|^{2\alpha}}{\alpha + 1} \right) = \frac{1}{\alpha + 1} \partial \overline{z} = 0.$$

Hence, on $\mathbf{D} \setminus \{0\}$, the conclusion of the Lemma holds true for the operator

(16)
$$\begin{aligned} \mathcal{L}u &= \frac{m}{w} \Delta u - \left(\frac{\partial m}{w} \overline{\partial} u + \frac{\overline{\partial} m}{w} \partial u \right) + u \\ &= \frac{m}{w} \Delta u - \frac{1}{\alpha + 1} (\overline{z} \overline{\partial} u + z \partial u) + u. \end{aligned}$$

Suppose now to the contrary that the Green function $U(x, y) \leq 0$ for some $x, y \in \mathbf{D}$. Since the function $U(\,\cdot\,, y)$ vanishes on $\partial \mathbf{D}$, it must attain its minimum at some interior point $x_0 \in \mathbf{D}$, and $U(x_0, y) \leq 0$. Let us take $U(\,\cdot\,, y)$ as the function u in (16). The function $v := \mathcal{L}u$ is then harmonic on $\mathbf{D} \setminus \{0, y\}$. We claim that it is, in fact, harmonic on all of $\mathbf{D} \setminus \{y\}$. Indeed, in view of the definition of u,

$$\frac{1}{w(z)} \Delta u(z) = \log \left| \frac{z - y}{1 - \overline{y}z} \right|^2 + h_y(z)$$

with h_y harmonic on $\mathbf{D}$. Thus, if $y \neq 0$,

$$\Delta u(z) = |z|^{2\alpha} \cdot \text{a function real-analytic at the origin.}$$

Consequently,

$$u(z) = |z|^{2(\alpha+1)} f_1(z) + h(z)$$

and

$$\partial u(z) = \overline{z}|z|^{2\alpha} f_2(z) + f_3(z)$$

with $f_1(z), f_2(z), f_3(z)$ real-analytic at the origin and $h(z)$ harmonic on $\mathbf{D}$. In particular, as $\alpha > -1$, it follows that $u(z)$, $z \, \partial u(z)$ and $\overline{z} \overline{\partial} u(z) = \overline{z \, \partial u(z)}$ are continuous at the origin, and, consequently, so is v. By the removable singularities theorem for harmonic functions, the harmonicity of v on $\mathbf{D} \setminus \{0, y\}$ therefore implies that v is harmonic on all of $\mathbf{D} \setminus \{y\}$.

Now on the unit circle, both u and ∇u vanish (by the definition of the Green function) and so does m; hence, $v = 0$ on $\partial \mathbf{D}$. At y, u and $\overline{z} \overline{\partial} u + z \partial u$ are continuous, while $w(z)^{-1} \Delta u(z) \sim \log |z - y|^2$ tends to $-\infty$; it follows

that $v(y) = +\infty$. By the maximum principle, $v \geq 0$ in $\mathbf{D} \setminus \{y\}$, and actually $v > 0$ since an equality at any point would imply that $v \equiv 0$, contradicting the fact that v have a logarithmic singularity at y. [1] Thus, in particular, $v(x_0) > 0$; since $u(x_0) \leq 0$ by assumption and, x_0 being a local minimum of u, $\nabla u(x_0)$ vanishes, we get

$$0 < v(x_0) = u(x_0) + \frac{m}{w}\Delta u(x_0) \leq \frac{m}{w}\Delta u(x_0).$$

If $\underline{x_0 \neq 0}$, this implies $\Delta u(x_0) < 0$ ($\Delta u(x_0) = -\infty$ if $x_0 = y$), which is contradictory to x_0 being a local minimum of u. So $\underline{x_0 = 0}$. That is, the only place where $U(\,\cdot\,, y)$, $y \in \mathbf{D}$, can have a local minimum is the origin. Owing to the symmetry of the Green function ($U(z,y) = U(y,z)$), we can apply the same reasoning to the function $u(y) = U(y,0) = U(0,y)$, which leads to the conclusion that *if $U(x,y) \leq 0$ for some $x, y \in \mathbf{D}$, then U must have its global minimum at the point $(0,0)$.*

On the other hand, by the definition of the Green function we have

$$\Delta_z U(z,0) = |z|^{2\alpha}(\log|z|^2 + h_0(z))$$

for some harmonic function h_0. In view of the rotational symmetry, $h_0(z)$ depends only on the modulus $|z|$, and so must be a constant function. An easy integration therefore gives

$$U(z,0) = \tfrac{1}{(\alpha+1)^2}|z|^{2(\alpha+1)}(\log|z|^2 - \tfrac{2}{\alpha+1} + h_0) + h_1(z)$$

with another harmonic function h_1 (which again reduces to a constant, for the same reason). Thus

$$U(r,0) - U(0,0) = (\alpha+1)^{-2}r^{2(\alpha+1)}(\log r^2 + h_0 - \tfrac{2}{\alpha+1}) < 0$$

for sufficiently small $r > 0$, which however shows that $U(0,0)$ cannot be a local — not to say global — minimum for $U(\,\cdot\,, 0)$. Thus we have arrived at a contradiction, which completes the proof. □

If ϕ is a biholomorphic mapping of Ω_1 onto Ω_2 and G is the Green function for $\Delta w^{-1}\Delta$ on Ω_2, it is easily seen that $G \circ \phi$ must be the Green function for $\Delta[|\phi'|^2(w \circ \phi)]^{-1}\Delta$ on Ω_1. Using conformal mapping, we therefore immediately obtain the following result.

Corollary 4. *Let Ω be a simply connected domain and ϕ a Riemann mapping of Ω onto $\mathbf{D}$. Then the Green function for the operator $\Delta[|\phi'|^2 \cdot |\phi|^{2\alpha}]^{-1}\Delta$, $\alpha > -1$, on Ω is positive.*

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[1]In fact, one must have, quite explicitly, $v(z) = m(y)\log|(z-y)/(1-\bar{y}z)|^2$.

In particular, the Green function for the operator

$$\Delta\big|(z-a)^{\alpha}/(1-\overline{a}z)^{\alpha+2}\big|^{-2}\Delta \qquad (a \in \mathbf{D}, \alpha > -1)$$

on $\mathbf{D}$ *is positive.*

We conclude by another result which can be obtained along the same vein. Recall that the function

$$D(\xi, y) := \frac{1}{w(\xi)} \Delta_{\xi} U(\xi, y), \qquad \xi \in \partial\Omega, \ y \in \Omega,$$

plays the role of the *Poisson kernel* for the operator $\Delta w^{-1}\Delta$. That is, for f a function on $\partial\Omega$, the function

$$F(y) := -\int_{\partial\Omega} f(\xi) D(\xi, y) \, d\sigma(\xi)$$

is w-biharmonic in Ω and has boundary values $F = 0$, $\partial F/\partial n = f$. In other words, $D(\xi, y)$ solves the boundary value problem

$$\begin{cases} \Delta_y \dfrac{1}{w(y)} \Delta_y D(\xi, y) = 0, & y \in \Omega, \\ D(\xi, y) = 0, & y \in \partial\Omega, \\ \dfrac{\partial}{\partial n_y} D(\xi, y) = -\delta(y - \xi), & y \in \partial\Omega. \end{cases}$$

(See e.g. [**E-Pee**], Section 5, or [**H-Duke**], [**H-BVP**].) It is easy to see that if $U(x, y) \geq 0$ on $\Omega \times \Omega$, then $D(\xi, y) \geq 0$ on $\partial\Omega \times \Omega$. In fact, $D(\xi, y) < 0$ implies that $\dfrac{\partial^2}{\partial n_{\xi}^2} U(\xi, y) = \Delta_{\xi} U(\xi, y) = w(\xi) D(\xi, y) < 0$, and as $U(\xi, y) = \dfrac{\partial}{\partial n_{\xi}} U(\xi, y) = 0$, it would follow that $U(x, y) < 0$ for all x lying sufficiently close to ξ on the inward normal from ξ. It is an open question whether $D(\xi, y) > 0$ on $\partial\Omega \times \Omega$ does not actually imply that already $U(x, y) > 0$ on $\Omega \times \Omega$. The general opinion seems to be that more likely this is not the case; a small contribution in favour of the opposite is provided by the assertion below. Loosely speaking, its sense is that if $U(x, y)$ has a negative minimum for x "not far" from y, then $D(\,\cdot\,, y)$ has to take negative values somewhere on the boundary.

Proposition 5. *Let Ω be a bounded domain whose boundary consists of finitely many Jordan curves and let $U(x, y)$ be the Green function for the unweighted biharmonic operator Δ^2 on Ω. If $U(x, y) < 0$ is a local minimum*

for some x, y which lie in a disc $B(z_0, r)$ contained wholly in Ω, then there is $\xi \in \partial\Omega$ for which $D(\xi, y) < 0$.

Proof. Since $w \equiv 1$, we can take

$$m(z) = |z - z_0|^2 - r^2$$

as our function m, and $u(z) = U(z, y)$ as the function u in (2). This choice of m satisfies

$$m(y) < 0, \qquad m > 0 \ \text{ on } \ \partial\Omega, \qquad m(x) < 0.$$

From the Fundamental Lemma we conclude that

$$v(z) := m(z)\Delta_z U(z, y) - \nabla m \cdot \nabla_z U(z, y) + U(z, y)$$

is harmonic on $\Omega\backslash\{y\}$. For $z = x$, $v(x) = m(x)\Delta_x U(x, y) + U(x, y)$ is negative by assumption. Since $v(y) = +\infty$ (by a similar argument as in the proof of Theorem 3), we must have $v(\xi) < 0$ at some point ξ on $\partial\Omega$. But in view of the definition of the Green function, $v(z) = m(z)\Delta_z U(z, y) = m(z)D(z, y)$ on $\partial\Omega$, so $D(\xi, y) < 0$ as well. $\qquad\qquad\square$

We close with a few comments regarding the solvability of the system (1):

$$(17) \qquad \begin{aligned} \frac{1}{w}\Delta m &= C, \\ \partial\frac{1}{w}\partial m = \overline{\partial}\frac{1}{w}\overline{\partial}m &= 0, \end{aligned}$$

where C is a complex constant. Obviously, $m \equiv$ const. is a solution for any w. On the other hand, we have seen that for $w = |f'|^2$, with f a holomorphic function, one gets a nontrivial solution $m = |f|^2$. It turns out that these are essentially the only possibilities that can happen.

Proposition 6. *Let w be positive and C^2 on a simply connected open set $\mathcal{V}$. Then the system (17) has nontrivial solutions if and only if the function $\log w$ is harmonic. In that case, we can write $w = |H'|^2$ where H is a holomorphic function on $\mathcal{V}$ whose derivative does not vanish, and all solutions to (17) are given by*

$$m = C|H|^2 + \alpha H + \beta\overline{H} + \gamma,$$

where α, β, γ are arbitrary complex numbers.

Proof. Let m be a solution of (17). Owing to the second equation in (17), we must have

$$(18) \qquad\qquad \partial m = \overline{f}w, \qquad \overline{\partial}m = gw$$

for some functions f, g holomorphic on $\mathcal{V}$. Applying $\overline{\partial}$ to the first and ∂ to the second equation and using the first equation in (17), we obtain

$$(19) \qquad \overline{f}\,\frac{\overline{\partial} w}{w} + \overline{f'} = g\,\frac{\partial w}{w} + g' = C.$$

If $f = g \equiv 0$, (18) shows that m is a constant; so let us assume that, for instance, g is not identically zero. Then (19) gives

$$\frac{\partial w}{w} = \frac{C - g'}{g}.$$

The right-hand side is a meromorphic function on $\mathcal{V}$, whereas the left-hand side is continuous by hypothesis. Hence $\partial w/w$ must be a holomorphic function on $\mathcal{V}$, and so

$$0 = \overline{\partial}\,\frac{\partial w}{w} = \overline{\partial}\,(\partial \log w) = \Delta \log w,$$

as asserted.

Conversely, if $\log w$ is harmonic, then owing to the simple connectivity of $\mathcal{V}$ we can write $\log w = F + \overline{F}$ with F holomorphic on $\mathcal{V}$. Thus $w = |H'|^2$ where H is any primitive of the holomorphic function e^F. Since $\Delta |H|^2 = w$, any solution m to (17) must satisfy

$$m = C|H|^2 + f + \overline{g}$$

for some holomorphic functions f and g. Feeding this into the second equation in (17) gives

$$\frac{1}{H'}\overline{\left(\frac{g'}{H'}\right)'} = \frac{1}{\overline{H'}}\left(\frac{f'}{H'}\right)' = 0,$$

so both f'/H' and g'/H' must be constant functions. It follows that $f + \overline{g} = \alpha H + \beta \overline{H} + \gamma$ for some constants α, β, γ, which is what we needed to prove. $\qquad\square$

Corollary 7. *Let w be positive and C^2 on an open set $\mathcal{V}$. Then a nontrivial solution to* (17) *exists if and only if $w = |H'|^2$ and*

$$(20) \qquad m = C|H|^2 + \alpha H + \beta \overline{H} + \gamma$$

for some complex numbers α, β, γ and some (possibly multi-valued) analytic function H on $\mathcal{V}$ for which the expression (20) *is single-valued and the derivative H' does not vanish.*

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NOTE: THE REFERENCES TO SHIMORIN'S PAPERS ON PAGE 344 WERE GIVEN IN THE PRINTED VERSION AS [SH1] AND [SH2].