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CLOSED COMMUTANTS OF THE BACKWARD SHIFT OPERATOR

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We characterize the closed operators with domain contained in the Hardy space H^2 that commute with the backward shift. Also, we give necessary and sufficient conditions for such an operator to be a Toeplitz operator with symbol the complex conjugate of a function in H^2. In particular, we show that this fact depends only on the domain.

Introduction.

Let F be a function in the Hardy space of the unit disk H^2. We can define the unbounded Toeplitz operator $T_{\overline{F}}$ operating on a suitable linear submanifold of H^2 (for instance H^∞). In recent years many questions have been raised about the behavior of these operators. Most of these problems appear naturally when studying the algebra of multipliers or the backward shift invariant subspaces of the so-called de Branges-Rovnyak spaces (see [11] and [17]). If S^* denotes the backward shift operator on H^2, it is not difficult to see that $T_{\overline{F}}$ commutes with S^*. More generally, if Q is a closed operator on some linear submanifold of H^2 that commutes with S^*, then the domain of Q, $D(Q)$ is dense in some (closed) S^* invariant subspace of H^2. Therefore Beurling's theorem assures that $D(Q)$ is dense in H^2 or in $(uH^2)^\perp = \mathcal{H}(u)$, for some inner function u.

If Q is a bounded operator on H^2 that commutes with S^*, it is easy to see that $Q = T_{\overline{\varphi}}$ with $\varphi \in H^\infty$. The analogous result for bounded operators on $\mathcal{H}(u)$ that commute with $S^*_u = S^*/_{\mathcal{H}(u)}$ is a well known theorem of Sarason [14]. Moreover, we can choose $\varphi \in H^\infty$ so that $Q = T_{\overline{\varphi}}/_{\mathcal{H}(u)}$ and $\|\varphi\|_\infty = \|Q\|$.

There are two natural questions appearing at this point. What are the closed operators that commute with S^* (or with S^*_u)? Do the above results for bounded operators have analogous versions for closed operators? The purpose of this paper is to answer these questions in both cases, when $D(Q)$ is dense in H^2 and in $\mathcal{H}(u)$, for some inner function u. In particular, we find necessary and sufficient conditions for such an operator Q to have the form $T_{\overline{F}}$ (or $T_{\overline{F}}/_{\mathcal{H}(u)}$) with $F \in H^2$. As a byproduct of this result we also obtain a short proof of Sarason's theorem.

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1. Closed operators with domain dense in H^2.

In what follows, a linear subspace of H^2 will be called a linear manifold; a 'closed' subspace will be simply called a subspace. Let $Q : D(Q) \subset H^2 \to H^2$ be a closed operator such that $D(Q)$ is S^* invariant and $S^*Q = QS^*$. It is easy to prove that if $\varphi \in H^\infty$ then $T_{\overline{\varphi}} D(Q) \subset D(Q)$ and $T_{\overline{\varphi}} Q = QT_{\overline{\varphi}}$ (see for instance [17], Lemma 6.3). It follows immediately that if $h \in D(Q) \cap \mathcal{H}(u)$ (for u an inner function) then $Qh \in \mathcal{H}(u)$, because $T_{\overline{u}} Qh = QT_{\overline{u}} h = 0$. Therefore the restriction of Q to $D(Q) \cap \mathcal{H}(u)$ is a closed operator that commutes with S_u^*, with domain and rank in $\mathcal{H}(u)$.

Definition. Let Q be a closed operator with domain $D(Q)$ dense in H^2 such that $S^*D(Q) \subset D(Q)$ and $S^*Q = QS^*$. We say that the operator Q is S^*-commuting . If u is an inner function and Q is a closed operator such that $D(Q) \subset \mathcal{H}(u)$ is dense, $S_u^* D(Q) \subset D(Q)$ and $S_u^* Q = QS_u^*$, we say that Q is S_u^*-commuting.

Consider the Hilbert space $H_2^2 = H^2 \times H^2$. For $\varphi \in L^\infty$, $(T_{\overline{\varphi}})_2$ denotes the matrix operator from H_2^2 into H_2^2 given by $(T_{\overline{\varphi}})_2 = T_{\overline{\varphi}}.I$, where I is the identity matrix. It is clear that if the operator Q is S^* or S_u^*-commuting, then its graph $G(Q)$ is a $(S^*)_2$ invariant subspace of H_2^2. So, a first approach to understand our operators is to classify all the $(S^*)_2$ invariant subspaces M of H_2^2. This was done by Lax in [9]. The strategy is as follows, beginning with Lax's characterization of the subspaces M, we study the extra conditions required to assure that M is a graph. Also, we distinguish two cases, when $M \subset \mathcal{H}(u) \times \mathcal{H}(u)$ for some inner function u, or when this inclusion does not hold for any inner function u.

Let $M_2(H^\infty)$ be the algebra of 2×2-matrices with entries in H^∞. We will think of $\sigma \in M_2(H^\infty)$ as a multiplication operator on H_2^2. So, even when we write $\sigma = (a_{ij})$, $1 \le i, j \le 2$, the adjoint operator is $\sigma^* = (T_{\overline{a}_{ji}})$. That is, we identify a_{ij} with $T_{a_{ij}}$ when $a_{ij} \in H^\infty$.

A matrix $\sigma \in M_2(H^\infty)$ is called an inner matrix if for almost every $e^{i\theta} \in \partial\mathbb{D}$ the complex matrix $\sigma(e^{i\theta})$ is a partial isometry of $\mathbb{C}^2$ with fixed initial space (i.e., not depending on $e^{i\theta}$). Let S be the forward shift operator on H^2. The theorem of Lax asserts that N is a $(S)_2$ invariant subspace of H_2^2 if and only if $N = \sigma H_2^2$, for some inner matrix σ. Since $(S)_2^* = (S^*)_2$, the $(S^*)_2$ invariant subspaces of H_2^2 are precisely the orthogonal complements of the $(S)_2$ invariant subspaces. Thus, $M \subset H_2^2$ is a $(S^*)_2$ invariant subspace if and only if there is an inner matrix σ such that $M = (\sigma H_2^2)^\perp = \text{Ker}\ \sigma^*$. Let

$$\sigma = \begin{bmatrix} a\ b \\ c\ d \end{bmatrix} \in M_2(H^\infty), \quad \text{and put} \quad \tau = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

A simple calculation shows that if $v = \det \sigma$, then $\sigma\tau = \tau\sigma = vI$. Therefore $\tau^*\sigma^* = \sigma^*\tau^* = (T_{\overline{v}})_2$, implying that $\text{Ker}\ \sigma^* \subset \text{Ker}\,(T_{\overline{v}})_2$. Hence, if $M = \text{Ker}\ \sigma^*$ (for σ an inner matrix) is a $(S^*)_2$ invariant subspace of H_2^2 not contained in $\mathcal{H}(u) \times \mathcal{H}(u)$ for any inner function u, then $\sigma(e^{i\theta})$ must be a partial isometry of $\mathbb{C}^2$ with nontrivial initial space for almost every $e^{i\theta} \in \partial\mathbb{D}$. Otherwise, $\sigma(e^{i\theta})$ is an isometry for a.e. $e^{i\theta} \in \partial\mathbb{D}$, and then $v = \det \sigma \neq 0$. Since v is an inner function, then $\text{Ker}\ \sigma^* \subset \mathcal{H}(v) \times \mathcal{H}(v)$. So, if Q is a nontrivial S^*-commuting operator, then $G(Q) = \text{Ker}\ \sigma^*$ for some inner matrix σ, where $\sigma(e^{i\theta})$ is a partial isometry with nontrivial initial space for a.e. $e^{i\theta} \in \partial\mathbb{D}$.

An elementary argument shows that if $(\alpha_1, \alpha_2) \in \mathbb{C}^2$ is normal, then the complex matrix of a partial isometry with initial space spanned by $(\overline{\alpha}_1, \overline{\alpha}_2)$ has the form

$$\nu = \begin{bmatrix} \lambda\alpha_1\ \lambda\alpha_2 \\ \beta\alpha_1\ \beta\alpha_2 \end{bmatrix},$$

where $\lambda, \beta \in \mathbb{C}$ and $|\lambda|^2 + |\beta|^2 = 1$. We are now ready to prove our first result.

Theorem 1.1. *The operator Q is S^*-commuting if and only if $G(Q) = \{(f, g) \in H_2^2 :\ T_{\overline{b}}g = -T_{\overline{a}}f\}$, where*
(i) *$a, b \in H^\infty$ satisfy $|a(e^{i\theta})|^2 + |b(e^{i\theta})|^2 = 1$ a.e. on $\partial\mathbb{D}$, and*
(ii) *b is an outer function.*
Moreover, $\mathcal{A} = \{(T_{\overline{b}}h, -T_{\overline{a}}h) :\ h \in H^2\}$ is dense in $G(Q)$.

Proof. The previous comments say that if Q is a S^*-commuting operator then $G(Q) = \text{Ker}\ \sigma^*$, where

$$\sigma = \begin{bmatrix} a\alpha_1\ a\alpha_2 \\ b\alpha_1\ b\alpha_2 \end{bmatrix} \in M_2(H^\infty), \tag{1.1}$$

for some $a,\ b$ satisfying (i) and some normal vector $(\alpha_1, \alpha_2) \in \mathbb{C}^2$. Then

$$(f, g) \in \text{Ker}\ \sigma^* \Leftrightarrow T_{\overline{a}}f + T_{\overline{b}}g = 0. \tag{1.2}$$

Additionally, $\text{Ker}\ \sigma^*$ is a graph if and only if $(0, g) \in \text{Ker}\ \sigma^*$ only when $g = 0$, or equivalently, if and only if $T_{\overline{b}}$ is one-to-one. So, the graph condition for $\text{Ker}\ \sigma^*$ is that b is an outer function. Conversely, for any matrix σ as before, Ker σ^* is the graph of some closed operator Q that commutes with S^*. So, we must show that

$$D(Q) = \{f \in H^2 : \ (f, g) \in \text{Ker } \sigma^* \quad \text{for some } \ g \in H^2\}$$

is dense in H^2. It is clear from (1.2) that $\mathcal{A}$ is contained in Ker σ^*. Since the set $\{T_{\overline{b}}h : \ h \in H^2\}$ of first components of elements in $\mathcal{A}$ is dense in H^2, so is $D(Q)$.

We use the symbol $\langle , \rangle$ to denote the inner product in H^2 or H^2_2. Let $(f, g) \in \text{Ker } \sigma^* \ominus \mathcal{A}$. Then for every $h \in H^2$,

$$\begin{aligned} 0 = \langle (f, g), (T_{\overline{b}}h, -T_{\overline{a}}h)\rangle &= \langle f, T_{\overline{b}}h\rangle + \langle g, -T_{\overline{a}}h\rangle \\ &= \langle bf - ag, h\rangle, \end{aligned} \tag{1.3}$$

implying that $bf = ag$. Multiplying this equality by $\overline{a}$ and using that $|a|^2 + |b|^2 = 1$, we obtain $b(\overline{a}f + \overline{b}g) = g \in H^2$. On the other hand, since $(f, g) \in$ Ker σ^*, $T_{\overline{a}}f + T_{\overline{b}}g = 0$, and consequently $\overline{a}f + \overline{b}g = \overline{F} \in \overline{H}_0^2$, the orthogonal complement of H^2 in L^2. Thus $\overline{z}F \in H^2$ and $\overline{b}(\overline{z}F) = \overline{zg} \in \overline{H}_0^2$ (here z denotes the function $z(e^{i\theta}) = e^{i\theta}$). Hence $T_{\overline{b}}(\overline{z}F) = 0$, and since b is outer, $F = 0$. Therefore $g = 0 = f$ and $\mathcal{A}$ is dense in Ker σ^*. $\square$

When Q, a and b satisfy the theorem, we say that the pair (a, b) defines the operator Q.

2. The operators $T_{\overline{F}}$ with $F \in H^2$.

The Cauchy transform of a function $f \in L^1 = L^1(\partial\mathbb{D})$ is defined by the formula

$$K(f)(z) = \int_0^{2\pi} \frac{f(e^{i\theta})}{(1 - e^{-i\theta}z)} \frac{d\theta}{2\pi} \quad (z \in \mathbb{D}).$$

It is well known that this function is in H^p for every $0 < p < 1$. Thus $K(f)(z)$ has nontangential finite limit for almost every $e^{i\theta} \in \partial\mathbb{D}$ (see [4], pp. 17 and 39). Most of the time it will be convenient to think of $K(f)$ as its boundary function. When $f \in L^2$, $K(f)$ coincides with the projection of f on H^2, i.e., $K(f) = P_+(f)$, where $P_+ : L^2 \to H^2$ is the orthogonal projection. For $F \in H^2$ we define the operator (=linear transformation) $T_{\overline{F}}h = K(\overline{F}h)$ whenever $h \in H^2$ is such that $K(\overline{F}h) \in H^2$. Thus, $T_{\overline{F}}$ is an operator from $D(T_{\overline{F}}) = \{h \in H^2 : \ K(\overline{F}h) \in H^2\}$ into H^2, and we think of $D(T_{\overline{F}})$ as the domain of $T_{\overline{F}}$. We need several elementary facts about the operator $T_{\overline{F}}$.

Lemma 2.1. *Let $F \in H^2$ and $h \in D(T_{\overline{F}})$. Then for every integer $n \geq 0$,*

$$\langle T_{\overline{F}}h, e^{in\theta}\rangle = \int_0^{2\pi} \overline{F}he^{-in\theta} \frac{d\theta}{2\pi}.$$

Proof. The lemma is a direct consequence of the identities

$$\left\langle \left(\overline{F}h\right),\, e^{in\theta}\right\rangle = \widehat{K(\overline{F}h)}(n) = \frac{1}{n!}\left(\frac{d}{dz}\right)^n\Big|_{z=0} K\left(\overline{F}h\right)(z),$$

where

$$\left(\frac{d}{dz}\right)^n K\left(\overline{F}h\right)(z) = n!\int_0^{2\pi} \frac{\overline{F(e^{i\theta})}h(e^{i\theta})e^{-in\theta}}{(1-e^{-i\theta}z)^{n+1}}\,\frac{d\theta}{2\pi}.$$

□

Lemma 2.2. *The operator $T_{\overline{F}}$ is S^*-commuting.*

Proof. Since $H^\infty \subset D(T_{\overline{F}})$ then $D(T_{\overline{F}})$ is dense in H^2. Using the formula $(S^*f)(z) = (f(z) - f(0))/z$ $(f \in H^2)$ with $f(z) = K(\overline{F}h)(z)$, it is a simple calculation to show that $S^*D(T_{\overline{F}}) \subset D(T_{\overline{F}})$ and that S^* and $T_{\overline{F}}$ commute on $D(T_{\overline{F}})$. So, we only have to prove that $T_{\overline{F}}$ is closed.

Suppose that $\{h_k\} \subset D(T_{\overline{F}})$ is a sequence satisfying $h_k \xrightarrow{H^2} h$ and $T_{\overline{F}}h_k \xrightarrow{H^2} t$. By the Cauchy-Schwarz inequality, $\overline{F}h_k \longrightarrow \overline{F}h$ in L^1-norm. Since $e^{in\theta} \in L^\infty$, for $n \geq 0$ we have:

$$\begin{aligned}\langle \overline{F}h,\, e^{in\theta}\rangle &= \lim\langle \overline{F}h_k,\, e^{in\theta}\rangle = \lim\langle K(\overline{F}h_k),\, e^{in\theta}\rangle \\ &= \langle t,\, e^{in\theta}\rangle,\end{aligned}$$

where the second equality holds by Lemma 2.1. Then

$$\langle \overline{F}h - t,\, e^{in\theta}\rangle = 0 \quad \text{for all } \; n \geq 0,$$

and therefore $\overline{F}h - t \in \overline{H}^1_0$. Since $K(\overline{H}^1_0) = 0$ and K coincides with the identity on H^2, then $0 = K(\overline{F}h - t) = K(\overline{F}h) - t$ and $t = T_{\overline{F}}h$. □

Lemma 2.3. *Let $F \in H^2$, $\varphi \in H^\infty$ and $h \in D(T_{\overline{F}})$. Then $T_{\overline{F}}T_{\overline{\varphi}}h = T_{\overline{F\varphi}}h$.*

Proof. As stated at the beginning of Section 1 (see [17]) $T_{\overline{\varphi}}h \in D(T_{\overline{F}})$ and $T_{\overline{\varphi}}$ commutes with $T_{\overline{F}}$.

Let $P_- : L^2 \to \overline{H}^2_0$ be the orthogonal projection. Then

$$\begin{aligned}T_{\overline{F\varphi}}h = K(\overline{F\varphi}h) &= K[\overline{F}T_{\overline{\varphi}}h + \overline{F}P_-(\overline{\varphi}h)] \\ &= T_{\overline{F}}T_{\overline{\varphi}}h + K[\overline{F}P_-(\overline{\varphi}h)],\end{aligned}$$

where the last summand vanishes because $\overline{F}P_-(\overline{\varphi}h) \in \overline{H}^1_0$. □

Lemma 2.4. *Let $F = uF_0 \in H^2$, where u is inner and F_0 is outer ($F_0 \neq 0$). Then $h \in D(T_{\overline{F}})$ if and only if $T_{\overline{u}}h \in D(T_{\overline{F}_0})$, in which case $T_{\overline{F}}h = T_{\overline{F}_0}(T_{\overline{u}}h)$. Furthermore,* Ker $T_{\overline{F}} = \mathcal{H}(u)$.

Proof. For $h \in H^2$ we have:

$$\begin{aligned} K(\overline{F}h) &= K[\overline{F}_0(\overline{u}h)] = K\{\overline{F}_0[P_+(\overline{u}h) + P_-(\overline{u}h)]\} \\ &= K[\overline{F}_0(T_{\overline{u}}h)] + K[\overline{F}_0 P_-(\overline{u}h)], \end{aligned}$$

where the last summand vanishes regardless of h, because $\overline{F}_0 P_-(\overline{u}h) \in \overline{H}^1_0$. This proves the first part of the lemma. So, to prove that Ker $T_{\overline{F}} = \mathcal{H}(u)$ we only have to show that $T_{\overline{F}_0}$ is one-to-one. Let $k \in$ Ker $T_{\overline{F}_0}$, then for every polynomial p:

$$0 = \langle T_{\overline{F}_0}k,\, p\rangle = \langle \overline{F}_0 k,\, p\rangle = \langle k,\, F_0 p\rangle,$$

where the second equality is from Lemma 2.1. Since F_0 is outer, Beurling's theorem asserts that $\{pF_0 :\ p \text{ polynomial}\}$ is dense in H^2, and then $k = 0$. □

Now we study conditions for a S^*-commuting operator Q to be of the form $T_{\overline{F}}$ with $F \in H^2$. Although our first result lies near the surface, it will be fundamental in the sequel.

Proposition 2.5. *Let Q be a S^*-commuting operator defined by the pair of functions (a, b). Then there is $F \in H^2$ such that $D(Q) \subset D(T_{\overline{F}})$ and $Q = T_{\overline{F}}/_{D(Q)}$ if and only if $1/b \in H^2$. In that case, $F = -a/b$.*

Proof. Since for almost every point in $\partial\mathbb{D}$,

$$\frac{|a|^2}{|b|^2} = \frac{1 - |b|^2}{|b|^2} = \frac{1}{|b|^2} - 1,$$

then $a/b \in L^2$ if and only if $1/b \in L^2$. Moreover, since b is an outer function (Theorem 1.1), it is not difficult to see that the above conditions are equivalent to $a/b \in H^2$ and $1/b \in H^2$, respectively. So, actually the four conditions are equivalent.

By Theorem 1.1, $\mathcal{A} = \{(T_{\overline{b}}h,\, -T_{\overline{a}}h) :\ h \in H^2\}$ is dense in $G(Q)$. Therefore the set of pairs $(T_{\overline{b}}h,\, -T_{\overline{a}}h)$ with $h \in H^\infty$ is also dense in $G(Q)$. Consequently, $F \in H^2$ satisfies $D(Q) \subset D(T_{\overline{F}})$ and $Q = T_{\overline{F}}/_{D(Q)}$ if and only if

$$T_{\overline{F}}T_{\overline{b}}h + T_{\overline{a}}h = 0 \quad \text{for all} \quad h \in H^\infty. \tag{2.1}$$

Since by Lemma 2.3, $T_{\overline{F}}T_{\overline{b}}h + T_{\overline{a}}h = T_{\overline{Fb+a}}h$ for every $h \in H^\infty$, (2.1) is equivalent to the inclusion of H^∞ in Ker $T_{\overline{Fb+a}}$. By Lemma 2.4 this happens only when $Fb + a = 0$, that is, when $F = -a/b \in H^2$. □

It is not difficult at this point to show the existence of S^*-commuting operators others than the Toeplitz-like operators $T_{\overline{F}}$ for any $F \in H^2$. Let $\nu : \partial\mathbb{D} \to [0,1]$ be a function such that $\log \nu$ and $\log(1-\nu)$ belong to L^1 but $\nu^{-1} \notin L^1$. The first two conditions assure the existence of functions $a, b \in H^\infty$, b outer, so that $|b|^2 = \nu$ and $|a|^2 = 1 - \nu$ almost everywhere on $\partial\mathbb{D}$. The last condition on ν says that $1/b$ is not in L^2. So, if Q is the S^*-commuting operator defined by (a, b), Proposition 2.5 says that Q has not the form $T_{\overline{F}}$, with $F \in H^2$.

Theorem 2.6. *Let Q be a S^*-commuting operator. Then the following conditions are equivalent.*

(i) *There is $F \in H^2$ such that $D(Q) \subset D(T_{\overline{F}})$ and $T_{\overline{F}}/_{D(Q)} = Q$.*

(ii) *There is $F \in H^2$ such that $D(Q) \supset D(T_{\overline{F}})$ and $Q/_{D(T_{\overline{F}})} = T_{\overline{F}}$.*

(iii) *There is $F \in H^2$ such that $D(Q) = D(T_{\overline{F}})$ and $Q = T_{\overline{F}}$.*

The function F in (i), (ii) *and* (iii) *is the same.*

Proof. Clearly (iii) implies (i) and (ii). Suppose that (i) holds. Suppose that Q and $T_{\overline{F}}$ are defined by the two pairs of functions (a, b) and (a_F, b_F), respectively. By Proposition 2.5, $1/b$, $1/b_F \in H^2$ and $F = -a/b = -a_F/b_F$. Then, on $\partial\mathbb{D}$,

$$|F|^2 = \frac{1-|b|^2}{|b|^2} = \frac{1-|b_F|^2}{|b_F|^2},$$

implying that $|b| = |b_F|$ a.e. on $\partial\mathbb{D}$. Since b and b_F are outer functions, $b = \lambda b_F$ for some $\lambda \in \mathbb{C}$, $|\lambda| = 1$. Hence, $a/\lambda b_F = a/b = a_F/b_F$ and then $a = \lambda a_F$. Since clearly the pair (a_F, b_F) defines the same operator as the pair $(\lambda a_F, \lambda b_F) = (a, b)$, (iii) follows.

Now suppose that (ii) holds and let (a, b) be a pair that defines Q. By Theorem 1.1 the set $\mathcal{B} = \{(T_{\overline{b}}h, -T_{\overline{a}}h) : h \in H^\infty\}$ is dense in $G(Q)$. On the other hand, since b and h are in H^∞, then $T_{\overline{b}}h \in D(T_{\overline{F}})$. Thus $\mathcal{B}$ is contained in $G(T_{\overline{F}})$ and therefore $G(Q) \subset G(T_{\overline{F}})$, which is precisely (i). □

It is worth noticing that Theorem 2.6 simply says that if $F \in H^2$, then no S^*-commuting operator is a proper extension or restriction of $T_{\overline{F}}$.

3. Closed operators on $\mathcal{H}(u)$.

Let u be a nonconstant inner function. As in Section 1, Lax's theorem will be the main tool to characterize the graph of S_u^*-commuting operators.

First notice that if $\sigma \in M_2(H^\infty)$ is an inner matrix so that Ker σ^* is the graph of a S_u^*-commuting operator, then $\sigma(e^{i\theta})$ must be an isometry of $\mathbb{C}^2$ for almost every $e^{i\theta} \in \partial\mathbb{D}$. Otherwise, we are in the situation described by

Theorem 1.1. Elementary calculations show that if $\tau \in M_2(\mathbb{C})$ is the matrix of an isometry of $\mathbb{C}^2$, then

$$\tau = \begin{bmatrix} \alpha_1 & \beta_2 \\ \beta_3 & \alpha_4 \end{bmatrix}, \qquad \text{where} \tag{3.1}$$

(i) $|\alpha_1|^2 + |\beta_2|^2 = 1$,

(ii) $|\alpha_1| = |\alpha_4|$ and $|\beta_2| = |\beta_3|$, and

(iii) $\alpha_1 \overline{\beta}_3 + \beta_2 \overline{\alpha}_4 = 0$.

Hence, if $\sigma \in M_2(H^\infty)$ and Ker σ^* is the graph of a S_u^*-commuting operator, then the boundary values of σ satisfy the above conditions for almost every point in $\partial\mathbb{D}$.

We need some general facts about the spaces $\mathcal{H}(u)$ (u inner). The function $h \in H^2$ is in $\mathcal{H}(u)$ if and only if $\overline{u}h \in \overline{H}_0^2$, or equivalently, if and only if $h_1 = \overline{z}u\overline{h} \in H^2$ (as before z denotes the function $z(e^{i\theta}) = e^{i\theta}$). Then, $\overline{u}h_1 = \overline{z}\overline{h} \in \overline{H}_0^2$, and consequently $h_1 \in \mathcal{H}(u)$. This means that the transformation $C_u h = \overline{z}u\overline{h}$ is a conjugation on $\mathcal{H}(u)$.

Let $u = wv$, where w and v are inner functions. Then we have the decomposition $\mathcal{H}(u) = v\mathcal{H}(w) \oplus \mathcal{H}(v)$, where the sum is orthogonal. Take $h \in \mathcal{H}(v)$, then $C_u h = \overline{z}u\overline{h} = w(\overline{z}v\overline{h}) = wC_v(h)$, and since $C_v : \mathcal{H}(v) \to \mathcal{H}(v)$ is onto, we obtain $C_u(\mathcal{H}(v)) = w\mathcal{H}(v)$. Since C_u is a conjugation, also $C_u(w\mathcal{H}(v)) = \mathcal{H}(v)$.

If we consider the inner functions normalized by the condition that the first nontrivial Taylor coefficient at $z = 0$ is positive, then the maximum common divisor between two inner functions is well defined. Therefore, for $f, g \in H^\infty$ such that $f \neq 0 \neq g$ we denote by $(f : g)$ the maximum common divisor between the inner factors of f and g. Also, we say that f and g are coprime when $(f : g) = 1$.

Theorem 3.1. *Let u be a nonconstant inner function and Q be a nontrivial S_u^*-commuting operator. Then $G(Q) = \mathrm{Ker}\ \sigma^*$, where*

$$\sigma = \begin{bmatrix} wa & -v\overline{b} \\ wb & v\overline{a} \end{bmatrix}, \tag{3.2}$$

(1) *w, v are inner functions and $u = wv$,*

(2) *$a, b \in \mathcal{H}(zv)$ and $|a|^2 + |b|^2 = 1$ a.e. on $\partial\mathbb{D}$, and*

(3) *$(a : b) = (wb : v\overline{a}) = 1$.*

Reciprocally, if σ is a matrix as before, then Ker σ^* *is the graph of a nontrivial S_u^*-commuting operator.*

Moreover, $\mathcal{A} = \{(T_{\overline{b}}h, -T_{\overline{a}}h) : \ h \in \mathcal{H}(v)\} + \{(ak, bk) : \ k \in \mathcal{H}(w)\}$ *is dense in* Ker σ^*.

Proof. Let $\sigma \in M_2(H^\infty)$ be an inner matrix such that for some nonconstant inner function u, Ker σ^* is the graph of a S_u^*-commuting operator. Then $\sigma(e^{i\theta})$ has the form (3.1) for a.e. $e^{i\theta} \in \partial\mathbb{D}$, where its entries satisfy (i), (ii) and (iii). Hence

$$\sigma = \begin{bmatrix} wa_1 & b_2 \\ wb_3 & a_4 \end{bmatrix} \quad \text{with} \ \ w,\, a_1,\, a_4,\, b_2,\, b_3 \in H^\infty,$$

where almost everywhere on $\partial\mathbb{D}$:

(i) $|a_1|^2 + |b_2|^2 = 1$,

(ii) $|a_1| = |a_4|$ and $|b_2| = |b_3|$,

(iii) $a_1\overline{b}_3 + b_2\overline{a}_4 = 0$, and

(iv) w is an inner function and $(a_1 : b_3) = 1$.

Actually, condition (iv) says that w is the maximum common divisor between the inner factors of the first and third entries of σ. We distinguish w when writing the matrix σ only because it will simplify further notation. Observe that condition (iv) also assumes that $a_1 \neq 0 \neq b_3$. Otherwise, it is easy to see that Ker σ^* must be the graph of the trivial operator on $\mathcal{H}(u)$. Multiplying (iii) by b_3a_4 we obtain

$$a_1a_4|b_3|^2 + b_2b_3|a_4|^2 = 0,$$

and since $|b_3|^2 + |a_4|^2 = 1$, then

$$a_1a_4 - b_2b_3 = \frac{-b_2b_3}{|b_3|^2} = \frac{a_1a_4}{|a_4|^2} = v \quad \text{inner}.$$

Therefore, $b_2 = -vb_3\overline{b}_3/b_3 = -v\overline{b}_3$ and $a_4 = va_1\overline{a}_1/a_1 = v\overline{a}_1$. So, $v\overline{b}_3$ and $v\overline{a}_1$ belong to $H^\infty \subset H^2$, which means that $(\overline{zv})b_3,\, (\overline{zv})a_1 \in \overline{H}_0^2$, or equivalently, $b_3,\, a_1 \in \mathcal{H}(zv)$. Thus, if we put $a = a_1$ and $b = b_3$, then

$$\sigma = \begin{bmatrix} wa & -v\overline{b} \\ wb & v\overline{a} \end{bmatrix},$$

where v is inner, $a,\, b \in \mathcal{H}(zv)$, $|a|^2 + |b|^2 = 1$, w is inner and $(a : b) = 1$. Hence, Ker σ^* is formed by the pairs $(f,\, g) \in H_2^2$ such that

$$T_{\overline{wa}}f + T_{\overline{wb}}g = 0 \tag{3.3}$$

and

$$T_{\overline{v}b}f - T_{\overline{v}a}g = 0. \tag{3.4}$$

Since Ker σ^* is a graph, then $(0,\, g) \in$ Ker σ^* only when $g = 0$. Looking at this condition in terms of the equalities (3.3) and (3.4), we see that Ker σ^* is a graph if and only if whenever $g \in H^2$ satisfies $T_{\overline{wb}}g = 0 = T_{\overline{v}a}g$ then $g = 0$, or equivalently, Ker $T_{\overline{wb}} \cap$ Ker $T_{\overline{v}a} = \{0\}$. Since Ker $T_{\overline{wb}} = \mathcal{H}(u_1)$, where u_1 is the inner factor of wb, and Ker $T_{\overline{v}a} = \mathcal{H}(u_2)$, where u_2 is the inner factor of $v\overline{a}$, then $\mathcal{H}(u_1) \cap \mathcal{H}(u_2) = 0$, meaning that $(wb : v\overline{a}) = (u_1 : u_2) = 1$.

Since $\det \sigma = wv$, the remarks of Section 1 say that Ker $\sigma^* \subset \mathcal{H}(wv) \times \mathcal{H}(wv)$; therefore Ker σ^* is formed by the pairs $(f,\, g) \in \mathcal{H}(wv) \times \mathcal{H}(wv)$ that satisfy equalities (3.3) and (3.4). We claim that the linear manifold

$$\mathcal{L} = \{f \in \mathcal{H}(wv) : \ \text{ there is } \ g \in \mathcal{H}(wv) \ \text{ with } \ (f,\, g) \in \text{Ker } \sigma^*\}$$

is dense in $\mathcal{H}(wv)$. Let $h \in \mathcal{H}(wv)$, then $(f,\, g) = (T_{\overline{wb}}h,\, -T_{\overline{wa}}h)$ clearly satisfies (3.3). Besides

$$\begin{aligned} T_{\overline{v}b}f - T_{\overline{v}a}g &= T_{\overline{v}b}T_{\overline{wb}}h + T_{\overline{v}a}T_{\overline{wa}}h \\ &= T_{\overline{wv}|b|^2}h + T_{\overline{wv}|a|^2}h \\ &= P_+[\overline{wv}(|b|^2 + |a|^2)h] \\ &= T_{\overline{wv}}h = 0. \end{aligned}$$

Consequently $(f,\, g)$ also satisfies (3.4), and then $(f,\, g) \in$ Ker σ^*. In a completely analogous way, $(T_{\overline{v}a}h,\, T_{\overline{v}b}h) \in$ Ker σ^* for $h \in \mathcal{H}(wv)$. So, the linear manifold

$$\{(T_{\overline{wb}}h + T_{\overline{v}a}k,\, -T_{\overline{wa}}h + T_{\overline{v}b}k) : \ h,\, k \in \mathcal{H}(wv)\} \tag{3.5}$$

is contained in Ker σ^*. Then

$$T_{\overline{wb}}\mathcal{H}(wv) + T_{\overline{v}a}\mathcal{H}(wv) \subset \mathcal{L}. \tag{3.6}$$

Write $wb = u_1b_1$ and $v\overline{a} = u_2a_2$, where $u_1 = (wb : wv)$ and $u_2 = (v\overline{a} : wv)$. The graph condition for Ker σ^* (i.e.: $(wb : v\overline{a}) = 1$) implies that $(u_1 : u_2) = 1$. Since u_1 and u_2 divide wv, there are inner functions v_1 and v_2 such that $u_1v_1 = u_2v_2 = wv$. Henceforth, by the comments preceding the theorem,

$$T_{\overline{wb}}\mathcal{H}(wv) = T_{\overline{b}_1}T_{\overline{u}_1}[u_1\mathcal{H}(v_1) \oplus \mathcal{H}(u_1)] = T_{\overline{b}_1}\mathcal{H}(v_1).$$

Besides, since $u_1 = (wb : wv)$, then $(b_1 : v_1) = 1$. It is well known that if b_1 and v_1 are coprime, then $T_{\overline{b}_1}\mathcal{H}(v_1)$ is dense in $\mathcal{H}(v_1)$ (an easy argument involving the F. and M. Riesz theorem shows that the orthogonal complement of $T_{\overline{b}_1}\mathcal{H}(v_1)$ in $\mathcal{H}(v_1)$ is trivial). Analogously, $T_{\overline{v}a}\mathcal{H}(wv) = T_{\overline{a}_2}\mathcal{H}(v_2)$ is dense in $\mathcal{H}(v_2)$. So, by (3.6), $\mathcal{L} \cap (\mathcal{H}(v_1) + \mathcal{H}(v_2))$ is dense in $\mathcal{H}(v_1) + \mathcal{H}(v_2)$, and consequently our claim will follow if $\mathcal{H}(v_1) + \mathcal{H}(v_2)$ is dense in $\mathcal{H}(wv)$. In what follows, C denotes the conjugation of $\mathcal{H}(wv)$. Also, the orthogonal complements are taken with respect to $\mathcal{H}(wv)$. We have

$$(\mathcal{H}(v_1) + \mathcal{H}(v_2))^\perp = \mathcal{H}(v_1)^\perp \cap \mathcal{H}(v_2)^\perp = v_1\mathcal{H}(u_1) \cap v_2\mathcal{H}(u_2),$$

and

$$C[v_1\mathcal{H}(u_1) \cap v_2\mathcal{H}(u_2)] = C[v_1\mathcal{H}(u_1)] \cap C[v_2\mathcal{H}(u_2)] = \mathcal{H}(u_1) \cap \mathcal{H}(u_2).$$

Thus $(\mathcal{H}(v_1) + \mathcal{H}(v_2))^\perp = C(\mathcal{H}(u_1) \cap \mathcal{H}(u_2))$, and since $(u_1 : u_2) = 1$ then $\mathcal{H}(u_1) \cap \mathcal{H}(u_2) = 0$.

We have proved that if $\sigma \in M_2(H^\infty)$ satisfies the conditions of the theorem, then Ker σ^* is the graph of some closed operator Q that commutes with S^* (because σ is an inner matrix), and such that $D(Q) = \mathcal{L} \subset \mathcal{H}(wv)$ is dense. That is, Q is S_u^*-commuting for $u = wv$.

Summing up, if Q is a nontrivial S_u^*-commuting operator, then by Lax's theorem $G(Q) = \text{Ker } \sigma^*$ for some inner matrix $\sigma \in M_2(H^\infty)$. Moreover, we have shown that σ must satisfy conditions (2) and (3) of the theorem, and since $D(Q) = \mathcal{L}$ is dense in both $\mathcal{H}(u)$ and $\mathcal{H}(wv)$ (in $\mathcal{H}(u)$ by hypothesis), then $u = wv$.

Reciprocally, if $\sigma \in M_2(H^\infty)$ is a matrix as before, the above reasoning also shows that Ker σ^* is the graph of some S_u^*-commuting operator.

To finish our proof we must show that $\mathcal{A} \subset \text{Ker } \sigma^*$ is dense. Let $\mathcal{B} = \{(T_{\overline{wb}}h,\, -T_{\overline{wa}}h) :\ h \in \mathcal{H}(wv)\}$. We already saw that $\mathcal{B}$ is contained in Ker σ^*. The decomposition $\mathcal{H}(wv) = w\mathcal{H}(v) + \mathcal{H}(w)$ immediately leads to $\mathcal{B} = \{(T_{\overline{b}}h,\, -T_{\overline{a}}h) :\ h \in \mathcal{H}(v)\}$. The theorem will follow if we show that

$$\text{Ker } \sigma^* \ominus \mathcal{B} = \{(ak,\, bk) :\ k \in \mathcal{H}(w)\}.$$

Suppose that $(f,\, g) \in \text{Ker } \sigma^*$ is orthogonal to $\mathcal{B}$. Then for every $h \in \mathcal{H}(v)$,

$$\begin{aligned} 0 = \langle (f,\, g), (T_{\overline{b}}h,\, -T_{\overline{a}}h)\rangle &= \langle f, T_{\overline{b}}h\rangle - \langle g, T_{\overline{a}}h\rangle \\ &= \langle T_b f - T_a g, h\rangle, \end{aligned} \tag{3.7}$$

Thus, $bf - ag \in \mathcal{H}(v)^\perp = vH^2$, or equivalently,

$$\overline{v}(bf - ag) \in H^2. \tag{3.8}$$

Since $(f,\, g) \in \operatorname{Ker} \sigma^*$, (3.4) and (3.8) imply that $0 = T_{\overline{v}b} f - T_{\overline{v}a} g = \overline{v}(bf - ag)$. On the other hand, (3.3) says that $\overline{wa} f + \overline{wb} g \in \overline{H}^2_0$. Then

$$\begin{aligned} b\overline{H}^2_0 \ni b(\overline{wa} f + \overline{wb} g) &= \overline{wa} bf + \overline{w}|b|^2 g \\ &= \overline{w}(|a|^2 g + |b|^2 g) = \overline{w} g. \end{aligned}$$

Thus, $g = bw\overline{G}$ with $G \in H^2_0$. So,

$$0 = bf - ag = bf - abw\overline{G} = b(f - aw\overline{G}),$$

which implies that $f = aw\overline{G}$. Besides,

$$T_{\overline{zbw}} G = P_+(\overline{zbw} G) = P_+(\overline{zg}) = 0,$$

and consequently $G \in \mathcal{H}(wzu_b)$, where u_b is the inner factor of b. By the same reason $G \in \mathcal{H}(wzu_a)$, where u_a is the inner factor of a. Since $(u_a : u_b) = 1$,

$$G \in \mathcal{H}(wzu_b) \cap \mathcal{H}(wzu_a) = \mathcal{H}(wz),$$

and since also $G \in H^2_0$, then $G \in z\mathcal{H}(w)$. Thus $G = zF$ with $F \in \mathcal{H}(w)$ and $(f,\, g) = (aw\overline{z}\overline{F},\, bw\overline{z}\overline{F})$.

The map $F \mapsto w\overline{z}\overline{F}$ is the conjugation C_w of $\mathcal{H}(w)$, so $w\overline{z}\overline{F} = k \in \mathcal{H}(w)$, and we have that every $(f,\, g) \in \operatorname{Ker} \sigma^* \ominus \mathcal{B}$ has the form $(f,\, g) = (ak,\, bk)$ with $k \in \mathcal{H}(w)$, as claimed.

Now we can reverse this process. Take $k \in \mathcal{H}(w)$ and $(f,\, g) = (ak,\, bk) = (T_{\overline{v}a} vk,\, T_{\overline{v}b} vk)$. Since $vk \in v\mathcal{H}(w) \subset \mathcal{H}(wv)$, the functions f and g belong to $\mathcal{H}(wv)$. It is immediate to verify that $(f,\, g)$ satisfies (3.3) and (3.4), so $(f,\, g) \in \operatorname{Ker} \sigma^*$. Furthermore, $bf - ag = 0$ and then (3.7) implies that $(f,\, g)$ is orthogonal to $\mathcal{B}$. ∎

Definition. Let u be an inner function and Q be a nontrivial S^*_u-commuting operator. If $\sigma \in M_2(H^\infty)$ is a matrix as in Theorem 3.1 such that $G(Q) = \operatorname{Ker} \sigma^*$ we say that Q is defined by the 4-tuple $(a,\, b,\, w,\, v)$.

4. The operators $T_{\overline{F}} | D(T_{\overline{F}}) \cap \mathcal{H}(u)$.

Lemma 4.1. *Let u be an inner function and Q be a S^*_u-commuting operator defined by $(a,\, b,\, w,\, v)$. Then, $Q = T_{\overline{F}} /_{D(Q)}$ (with $F \in H^2$) if and only if*

(I) $Fb + a \in vH^2$ *and*

(II) $F\overline{a} - \overline{b} \in wH^2$.

Proof. It is clear from Theorem 3.1 that

$$\mathcal{A}_\infty = \{(T_{\overline{b}} h,\, -T_{\overline{a}} h) :\ h \in \mathcal{H}(v) \cap H^\infty\} + \{(ak,\, bk) :\ k \in \mathcal{H}(w) \cap H^\infty\}$$

is dense in $G(Q)$. Since $T_{\overline{F}}$ $(F \in H^2)$ is closed, $Q = T_{\overline{F}}/_{D(Q)}$ if and only if $\mathcal{A}_\infty \subset G(T_{\overline{F}})$, that is, if and only if

(i) $T_{\overline{F}} T_{\overline{b}} h = -T_{\overline{a}} h$ for all $h \in \mathcal{H}(v) \cap H^\infty$ and

(ii) $T_{\overline{F}} ak = bk$ for all $k \in \mathcal{H}(w) \cap H^\infty$.

By Lemma 2.3, $T_{\overline{F}} T_{\overline{b}} = T_{\overline{Fb}}$ on H^∞. So, (i) is equivalent to $T_{\overline{Fb+a}} h = 0$ for all $h \in \mathcal{H}(v) \cap H^\infty$, which according to Lemma 2.4, is equivalent to $Fb + a \in vH^2$. That is, (i) and (I) are equivalent.

Condition (ii) can be rewritten as $(\overline{F}a - b)k \in \overline{H}_0^2$ for all $k \in \mathcal{H}(w) \cap H^\infty$. Since the inner functions are normalized, then w is constant only when $w \equiv 1$. We consider two cases. First, suppose that $w \not\equiv 1$ and take $k = S^* w \in \mathcal{H}(w)$. Thus $(\overline{F}a - b)(w - w(0)) \in \overline{H}^2$. Therefore, $(\overline{F}a - b)(1 - w(0)\overline{w}) \in \overline{w}H^2$, and then $\overline{F}a - b \in \overline{w}(1 - w(0)\overline{w})^{-1}\overline{H}^2 \subset \overline{w}\overline{H}^2$. Henceforth, (ii) implies (II).

If $w \equiv 1$, $\mathcal{H}(w) = \{0\}$ and condition (ii) is trivial. We will see that in this case, (II) is a consequence of (i). Since (i) implies (I),

$$Fb + a = vR, \quad \text{with} \quad R \in H^2. \tag{4.1}$$

Multiplying by $\overline{a}$ we obtain $\overline{a}Fb + 1 - b\overline{b} = (\overline{a}v)R$. Then $(\overline{a}F - \overline{b})b = (\overline{a}v)R - 1 \in H^2$. Multiplying (4.1) by $\overline{b}$ we have $F(1 - \overline{a}a) + \overline{b}a = (\overline{b}v)R$, from which $a(\overline{a}F - \overline{b}) = -(\overline{b}v)R + F \in H^2$. Therefore, $\overline{az}P_+(a\overline{F} - b)$ and $\overline{bz}P_+(a\overline{F} - b)$ belong to $\overline{H}_0^2$. Henceforth,

$$P_+(a\overline{F} - b) \in \operatorname{Ker} T_{\overline{az}} \cap \operatorname{Ker} T_{\overline{bz}} = \mathcal{H}(zu_a) \cap \mathcal{H}(zu_b),$$

where u_a and u_b are the inner factors of a and b respectively. Since by Theorem 3.1 $(a : b) = 1$, the last intersection is $\mathcal{H}(z) = \mathbb{C}$. So $P_+(a\overline{F} - b) \in \mathbb{C}$, meaning that $a\overline{F} - b \in \overline{H}^2$, which proves (II) with $w \equiv 1$. Since (II) implies (ii) trivially, then (i) and (ii) are equivalent to (I) and (II) in any case, and the lemma follows. $\square$

Theorem 4.2. *Let u be an inner function and Q be a S_u^*-commuting operator defined by (a, b, w, v). Then there is $F \in H^p$ $(p \geq 2)$ such that $Q = T_{\overline{F}}/_{D(Q)}$ if and only if there are G, $J \in H^p$ such that*

$$(\overline{a}v)G + (wb)J = 1. \tag{4.2}$$

If (4.2) holds we can take $F = -awJ + (\overline{b}v)G$.

Proof. If $Q = T_{\overline{F}}|_{D(Q)}$, with $F \in H^p$ (for $p \geq 2$), then $F \in H^2$ and Lemma 4.1 implies that $A = bF + a \in vH^2 \cap L^p = vH^p$ and $B = F\overline{a} - \overline{b} \in wH^2 \cap L^p = wH^p$. Therefore

$$\overline{a}A - bB = \overline{a}bF + |a|^2 - bF\overline{a} + |b|^2 = 1 \in (\overline{a}v)H^p + (bw)H^p,$$

as claimed. Conversely, suppose that (4.2) holds. Then $\overline{a}(vG-a)=1-wbJ-|a|^2=b(\overline{b}-wJ)$, and multiplying by a, $(1-|b|^2)(vG-a)=ab(\overline{b}-wJ)$. Consequently

$$
\begin{aligned}
(vG-a) &= ab(\bar{b}-wJ)+b\bar{b}(vG-a) \\
&= a|b|^2-abwJ+b(\bar{b}v)G-a|b|^2 \\
&= b[-awJ+(\bar{b}v)G].
\end{aligned}
\tag{4.3}
$$

So, taking $F=-awJ+(\bar{b}v)G$ we see that $F\in H^p$ and satisfies condition (I) of Lemma 4.1.

Equality (4.3) allows us to rewrite Condition (4.2) in terms of F, as $\overline{a}(bF+a)+wbJ=1$. Then $b\overline{a}F+bwJ=1-|a|^2=b\bar{b}$, implying that $\overline{a}F+wJ=\bar{b}$. Consequently $\overline{a}F-\bar{b}=-wJ\in wH^2$, which is condition (II) of Lemma 4.1. So, $Q=T_{\overline{F}}/_{D(Q)}$ by Lemma 4.1. $\square$

Clearly, Theorem 4.2 is related to the problem of characterizing the pairs of functions h, $k\in H^\infty$ such that there exist G, $J\in H^p$ (for $p\geq 2$) so that

$$
hG+kJ=1. \tag{4.4}
$$

For $p=\infty$ the famous corona theorem of L. Carleson [2] asserts that (4.4) holds if and only if there is $\delta>0$ such that $|h(z)|+|k(z)|>\delta$ for all $z\in\mathbb{D}$. A pair $(h,\,k)$ that satisfies this condition is called a corona pair.

If Q is a *bounded* S_u^*-commuting operator defined by $(a,\,b,\,w,\,v)$, then we know from Sarason's theorem that Q has the form $T_{\overline{F}}$, with $F\in H^\infty$. Thus, by Theorem 4.2 and the corona theorem, the boundedness of Q is equivalent to the fact that $(\overline{a}v,\,wb)$ is a corona pair. On the other hand, if a, $b\in H^\infty$ are such that $|a|^2+|b|^2=1$, then $(a,\,b)$ can fail to be a corona pair even if a and b are outer functions (see [15]).

To the best of the author's knowledge, the problem for $2\leq p<\infty$ is still open, although many information have been obtained in recent years (see [10]). In order to show the existence of an inner function u and a S_u^*-commuting operator Q not having the form $T_{\overline{F}}/_{D(Q)}$ for any $F\in H^2$, we need some elementary facts about the above problem for $p=2$. For $\alpha\in\partial\mathbb{D}$, the cone with vertex α is

$$
\Lambda(\alpha)=\{z\in\mathbb{D}:\ |z-\alpha|<2(1-|z|)\}.
$$

The nontangential maximal function of a function R defined on $\mathbb{D}$ is

$$
R^*(\alpha)=\sup_{z\in\Lambda(\alpha)}|R(z)| \qquad (\alpha\in\partial\mathbb{D}).
$$

It is well known that an analytic function R on $\mathbb{D}$ is in H^p $(0 < p < \infty)$ if and only if R^* is in L^p of $\partial\mathbb{D}$ (see [6]). If h, $k \in H^\infty$ satisfy Equation (4.4) for some G, $J \in H^2$, then by the Cauchy-Schwarz inequality,

$$R = \frac{1}{|h| + |k|} \le \frac{1}{(|h|^2 + |k|^2)^{1/2}} \le (|G|^2 + |J|^2)^{1/2} \le |G| + |J|.$$

So,

$$R^* = \left[\frac{1}{|h| + |k|}\right]^* \le G^* + J^* \in L^2. \tag{4.5}$$

Thus, $R^* \in L^2$ is a necessary condition for h, k to satisfy (4.4) with $p = 2$. In [10] Lin proved that if $R^* \in L^{4+\varepsilon}$ for some $\varepsilon > 0$, then h, k satisfy (4.4).

Proposition 4.3. *There is a Blaschke product u and a S_u^*-commuting operator Q not of the form $T_{\overline{F}}/_{D(Q)}$, for any $F \in H^2$.*

Proof. Let v and w be the Blaschke products whose zeros are $z_n = 1 - 2^{-n}$ and $\omega_n = 1 - 2^{-n} + 4^{-n}$ for $n = 2, 3, \ldots$ respectively, where as always $v(0)$, $w(0) > 0$. For $u = wv$ let Q be the S_u^*-commuting operator defined by $(a, b, w, v) = (1/2, \sqrt{3}/2, w, v)$. Two Blaschke products with disjoint zero sequences are coprime, and since $z_n < \omega_n < z_{n+1}$, then $(wb : v\overline{a}) = (w\sqrt{3}/2 : v/2) = 1$. The other conditions of Theorem 3.1 are trivially fulfilled. Put $R(z) = (|(\sqrt{3}/2)w(z)| + |(1/2)v(z)|)^{-1}$ for $z \in \mathbb{D}$. According to the comments preceding the proposition, if R^* is not in L^2 then there are no functions G, $J \in H^2$ such that $(1/2)vG + (\sqrt{3}/2)wJ = 1$. In this case, Theorem 4.2 implies that Q is not of the form $T_{\overline{F}}/_{D(Q)}$, for any $F \in H^2$. A simple calculation shows that

$$\begin{aligned} |w(z_n)| &= \prod_{j \ge 2} \left|\frac{\omega_j - z_n}{1 - \overline{z}_n \omega_j}\right| < \left|\frac{\omega_n - z_n}{1 - \overline{z}_n \omega_n}\right| \\ &= \frac{1}{2(2^n - 1) + 2^{-n}} < \frac{1}{2^n} \end{aligned}$$

for all $n \ge 2$. Thus,

$$R(z_n) = \frac{2}{\sqrt{3}|w(z_n)|} > \frac{2^{n+1}}{\sqrt{3}} > 2^n. \tag{4.6}$$

For $n \ge 2$ let $\alpha_n = (1 - 4^{-n}) + i2^{-n}(2 - 4^{-n})^{1/2} \in \partial\mathbb{D}$. Then $z_n \in \Lambda(\alpha_n)$, because

$$\begin{aligned} |z_n - \alpha_n|^2 &= (1 - 2^{-n} - 1 + 4^{-n})^2 + 2^{-2n}(2 - 4^{-n}) \\ &= 4^{-n}(3 - 2.2^{-n}) \\ &< 4.4^{-n} = [2(1 - |z_n|)]^2. \end{aligned}$$

This means that α_n belongs to the ball centered at z_n and radius $2(1-|z_n|)$. Since the point $1 \in \partial\mathbb{D}$ is also contained in this ball, it is geometrically clear that every point α in the arc-interval $(1,\, \alpha_n) \subset \partial\mathbb{D}$ is also contained in this ball, and therefore $z_n \in \Lambda(\alpha)$. In particular, if $I_n \subset \partial\mathbb{D}$ is the open arc-interval $(\alpha_{n+1},\, \alpha_n)$, then $z_n \in \Lambda(\alpha)$ for every $\alpha \in I_n$ (for all $n \geq 2$). Hence, if $\alpha \in I_n$:

$$\begin{aligned} R^*(\alpha) &= \sup_{z\in\Lambda(\alpha)} \frac{1}{\left|\frac{\sqrt{3}}{2}w(z)\right| + \left|\frac{1}{2}v(z)\right|} \\ &\geq \frac{1}{\left|\frac{\sqrt{3}}{2}w(z_n)\right| + \left|\frac{1}{2}v(z_n)\right|} = R(z_n) > 2^n, \end{aligned} \tag{4.7}$$

by (4.6). Let us denote by m the normalized Lebesgue measure on $\partial\mathbb{D}$. If $I_n = (\alpha_{n+1},\, \alpha_n)$, then

$$\begin{aligned} 2\pi\, m(I_n) &\geq |\alpha_{n+1} - \alpha_n| \geq \operatorname{Re}\, \alpha_{n+1} - \operatorname{Re}\, \alpha_n \\ &= 4^{-n} - 4^{-n-1} = (3/4)4^{-n}. \end{aligned} \tag{4.8}$$

Therefore, $m(I_n) \geq C4^{-n}$ for some constant $C > 0$ independent of n. Finally, Inequalities (4.7) and (4.8) yield

$$\begin{aligned} \int_{\partial\mathbb{D}} |R^*|^2\, dm &\geq \sum_{n\geq 2} \int_{I_n} |R^*|^2\, dm \\ &\geq \sum_{n\geq 2} 4^n m(I_n) \geq \sum_{n\geq 2} C = \infty. \end{aligned}$$

□

5. Domain conditions for S_u^*-commuting operators.

In the last section we characterized the S_u^*-commuting operators Q of the form $Q = T_{\overline{F}}|_{D(Q)}$, for some $F \in H^2$, in terms of the 4-tuple of functions defining Q. Since the operator Q is not always given by its defining functions, sometimes the criterion of Theorem 4.2 is not very practical. Even more, if Q is given by its defining 4-tuple $\xi = (a,\, b,\, w,\, v)$, the test on ξ provided by Theorem 4.2 is not always easy to perform.

In this section we study domain conditions for Q to be of the desired form. The fundamental tool is a strong notion of cyclicity.

Let $\mathbb{H}$ be a Hilbert space and $V : \mathbb{H} \to \mathbb{H}$ be a bounded operator. An element $x \in \mathbb{H}$ is called a cyclic vector for V if the span of $x, Vx, V^2x, \ldots$ is dense in $\mathbb{H}$. The cyclic vectors for S^* have been characterized by Douglas, Shapiro and Shields in terms of pseudocontinuation across $\partial\mathbb{D}$ [3], where many other characterizations were obtained as byproduct of this one. The noncyclic vectors for S^* are the functions that belong to $\mathcal{H}(u)$ for some inner function u.

It is well known that S^*u is a cyclic vector for the operator S_u^*. We introduce now a stronger notion of cyclicity.

Definition. The function $h \in \mathcal{H}(u)$ is an "exact cyclic" vector for S_u^* if for every $k \in \mathcal{H}(u)$ there is $F \in H^2$ such that $T_{\overline{F}}h = k$.

Let $h \in \mathcal{H}(u)$ be an exact cyclic vector for S_u^* and let $\mathcal{V} \subset \mathcal{H}(u)$ be the subspace generated by $h, S^*h, (S^*)^2h, \dots$ By Beurling's theorem, $\mathcal{V} = \mathcal{H}(v)$ for some inner function v that divides u. Thus, whenever $T_{\overline{F}}h$ is defined (i.e.: $K(\overline{F}h) \in H^2$), it belongs to $\mathcal{H}(v)$. Since h is an exact cyclic vector for S_u^* then $\mathcal{H}(v) = \mathcal{H}(u)$, and consequently h is a cyclic vector for S_u^*.

To show the existence of exact cyclic vectors for S_u^*, we simply observe that S^*u is one of them. Let $k \in \mathcal{H}(u)$, then $F = C_uk = \overline{z}u\overline{k} \in H^2$ and

$$\begin{aligned} T_{\overline{F}}(S^*u) &= P_+[\overline{u}kz(S^*u)] = P_+(\overline{u}ku) - P_+(\overline{u}ku(0)) \\ &= P_+(k) - u(0)P_+(\overline{u}k) = k. \end{aligned}$$

Let u be an inner function. In the next proposition we simply say cyclic or exact cyclic vector without mentioning the operator S_u^*.

Proposition 5.1. *Let u be an inner function and $\varphi \in H^\infty$.*

(1) *Let $h \in \mathcal{H}(u)$ be cyclic. Then $T_{\overline{\varphi}}h$ is cyclic if and only if $(\varphi : u) = 1$.*

(2) *Let $h \in \mathcal{H}(u) \cap H^\infty$ be exact cyclic. Then $T_{\overline{\varphi}}h$ is exact cyclic if and only if (φ, u) is a corona pair.*

Proof. (1) Since h is a cyclic vector and $(S_u^*)^nT_{\overline{\varphi}}h = T_{\overline{\varphi}}(S_u^*)^nh$ for $n \geq 0$, then $T_{\overline{\varphi}}h$ is a cyclic vector if and only if the range of $T_{\overline{\varphi}}|_{\mathcal{H}(u)}$ is dense in $\mathcal{H}(u)$. As pointed out in the proof of Theorem 3.1, the last condition is known to be equivalent to $(\varphi : u) = 1$.

(2) Suppose that there are $f, g \in H^\infty$ so that $f\varphi + gu = 1$, and let $t \in \mathcal{H}(u)$. Since h is an exact cyclic vector, there is $F \in H^2$ such that $T_{\overline{F}}h = t$. Then by Lemma 2.3

$$T_{\overline{Ff}}(T_{\overline{\varphi}}h) = T_{\overline{F}}(T_{\overline{f\varphi}}h) = T_{\overline{F}}(T_{1-\overline{gu}}h) = T_{\overline{F}}h = t,$$

so $T_{\overline{\varphi}}h$ is an exact cyclic vector.

On the other hand, if $T_{\overline{\varphi}}h$ is exact cyclic, for every $t \in \mathcal{H}(u)$ there is $F \in H^2$ such that $t = T_{\overline{F}}T_{\overline{\varphi}}h = T_{\overline{\varphi}}T_{\overline{F}}h$. The second equality holds because

$h \in H^\infty \subset D(T_{\overline{F}})$. Thus the range of $T_{\overline{\varphi}}|_{\mathcal{H}(u)}$ is $\mathcal{H}(u)$, so by part (1) of the proposition, $(\varphi : u) = 1$. Then $T_{\overline{\varphi}}|_{\mathcal{H}(u)}$ is one-to-one. That is, $T_{\overline{\varphi}}|_{\mathcal{H}(u)}$ is invertible. The inverse operator A necessarily commutes with S_u^*. So, by Sarason's theorem $A = T_{\overline{f}}|_{\mathcal{H}(u)}$ for some $f \in H^\infty$. Therefore $T_{\overline{\varphi f - 1}}|_{\mathcal{H}(u)} = 0$, and then $\varphi f - 1 \in uH^2 \cap H^\infty = uH^\infty$, as claimed. □

As a consequence of Proposition 5.1, we obtain that if u is an inner function other than a finite Blaschke product, then there are cyclic vectors for S_u^* that are not exact cyclic vectors. Just pick $\varphi \in H^\infty$ so that $(\varphi : u) = 1$ but (φ, u) is not a corona pair, and take $T_{\overline{\varphi}} S^* u$.

Obviously the notion of exact cyclic vector also makes sense for S^*. However, it is easy to see that S^* has not exact cyclic vectors at all. Let $h \in H^2$ such that $T_{\overline{F}} h = 1$ for some $F \in H^2$. Then $T_{\overline{zF}} h = S^* T_{\overline{F}} h = S^* 1 = 0$, so $h \in \mathcal{H}(v)$, where v is the inner factor of zF. Therefore h cannot be an exact cyclic vector for S^*. Roughly speaking, H^2 is too big to admit exact cyclic vectors for S^*.

Let u be an inner function and $F \in H^2$. Put $D_u(T_{\overline{F}}) = D(T_{\overline{F}}) \cap \mathcal{H}(u)$. Then $T_{\overline{F}}|_{D_u(T_{\overline{F}})}$ is a S_u^*-commuting operator. We proved in Theorem 2.6 that $T_{\overline{F}}$ ($F \in H^2$) cannot be properly extended or restricted to a S^*-commuting operator. The same holds for $T_{\overline{F}}|_{D_u(T_{\overline{F}})}$.

Theorem 5.2. *Let u be an inner function and $F \in H^2$.*

(1) *If $D \subset D_u(T_{\overline{F}})$ is a dense linear submanifold of $\mathcal{H}(u)$ such that $T_{\overline{F}}|_D$ is S_u^*-commuting, then $D = D_u(T_{\overline{F}})$.*

(2) *If Q is a S_u^*-commuting operator such that $D(Q) \supset D_u(T_{\overline{F}})$ and $Q|_{D_u(T_{\overline{F}})} = T_{\overline{F}}|_{D_u(T_{\overline{F}})}$, then $D(Q) = D_u(T_{\overline{F}})$.*

Proof. Let us write $D_u = D_u(T_{\overline{F}})$.
(1) We claim that

$$\mathcal{K} = \{(T_{\overline{\varphi}} S^* u,\, T_{\overline{F}} T_{\overline{\varphi}} S^* u) :\ \varphi \in H^\infty\}$$

is dense in $G(T_{\overline{F}}|_{D_u})$. Let $(a,\, b,\, w,\, v)$ be a 4-tuple of H^∞ functions defining $T_{\overline{F}}|_{D_u}$. By Theorem 3.1 and the decomposition $\mathcal{H}(u) = w\mathcal{H}(v) + \mathcal{H}(w)$, the pairs

$$(T_{\overline{wb}} h + T_{\overline{v}a} k,\, -T_{\overline{wa}} h + T_{\overline{v}b} k), \qquad h,\ k \in \mathcal{H}(u)$$

form a dense subset of $G(T_{\overline{F}}|_{D_u})$. Moreover, we can take h and k in any dense subset of $\mathcal{H}(u)$. In particular, we can take $h = T_{\overline{p}} S^* u$ and $k = T_{\overline{q}} S^* u$, where p and q are polynomials (because $S^* u$ is a cyclic vector for S_u^*). Henceforth, the pairs

$$\begin{aligned}&([T_{\overline{wb}} T_{\overline{p}} + T_{\overline{v}a} T_{\overline{q}}] S^* u,\, [-T_{\overline{wa}} T_{\overline{p}} + T_{\overline{v}b} T_{\overline{q}}] S^* u) \\ &= (T_{\overline{wbp} + \overline{v}a\overline{q}} S^* u,\, T_{-\overline{wap} + \overline{v}b\overline{q}} S^* u),\end{aligned}$$

with p and q polynomials, are dense in $G(T_{\overline{F}}|_{D_u})$. Taking $\varphi = wbp + v\overline{a}q \in H^\infty$, necessarily $T_{\overline{F}}T_{\overline{\varphi}}S^*u = T_{-\overline{wap}+\overline{vbq}}S^*u$, and the claim follows.

Now let $(a,\, b,\, w,\, v)$ be a 4-tuple of functions in H^∞ that defines $T_{\overline{F}}|_D$. By Theorem 4.2 there exist $G,\, J \in H^2$ so that $(v\overline{a})G + (wb)J = 1$. Since for $l \in \mathcal{H}(u) \cap H^\infty$ the functions $h = T_{\overline{J}}l$ and $k = T_{\overline{G}}l$ belong to $\mathcal{H}(u)$, then by Theorem 3.1 the following pairs belong to $G(T_{\overline{F}}|_D)$:

$$\begin{aligned}&(T_{\overline{wb}}(T_{\overline{J}}l) + T_{\overline{v}a}(T_{\overline{G}}l),\, -T_{\overline{wa}}(T_{\overline{J}}l) + T_{\overline{v}b}(T_{\overline{G}}l))\\ &= (l,\, [-T_{\overline{wa}}T_{\overline{J}} + T_{\overline{v}b}T_{\overline{G}}]l) = (l,\, T_{\overline{F}}l).\end{aligned}$$

Consequently, for every $l \in \mathcal{H}(u) \cap H^\infty$ and every $\varphi \in H^\infty$:

$$(T_{\overline{\varphi}}l,\, T_{\overline{\varphi}}T_{\overline{F}}l) = (T_{\overline{\varphi}}l,\, T_{\overline{F}}T_{\overline{\varphi}}l) \in G(T_{\overline{F}}|_D).$$

This means that $\mathcal{K} \subset G(T_{\overline{F}}|_D)$, and since $\mathcal{K}$ is dense in $G(T_{\overline{F}}|_{D_u})$ and $G(T_{\overline{F}}|_D) \subset G(T_{\overline{F}}|_{D_u})$ is closed, both graphs coincide, implying that $D = D_u$.

(2) The argument is similar. Let $(a,\, b,\, w,\, v)$ be a 4-tuple that defines Q. Then

$$\{(T_{\overline{wb}}h + T_{\overline{v}a}k,\, -T_{\overline{wa}}h + T_{\overline{v}b}k) :\ h,\, k \in \mathcal{H}(u) \cap H^\infty\}$$

is dense in $G(Q)$. Since for $h,\, k \in \mathcal{H}(u) \cap H^\infty$, $T_{\overline{wb}}h + T_{\overline{v}a}k$ is contained in D_u (because $wb,\ v\overline{a} \in H^\infty$) and Q coincides with $T_{\overline{F}}$ on D_u, it is immediate that $G(T_{\overline{F}}|_{D_u})$ is dense in $G(Q)$. Therefore they coincide. ☐

Theorem 5.2 can be rephrased by saying that if u is an inner function, and Q is a S_u^*-commuting operator, then no restriction or extension of Q has the form $T_{\overline{F}}|_{D_u(T_{\overline{F}})}$ for $F \in H^2$ unless $Q = T_{\overline{F}}|_{D_u(T_{\overline{F}})}$.

Theorem 5.3. *Let u be an inner function and Q be a S_u^*-commuting operator. Then the following conditions are equivalent.*

(1) *There is $F \in H^2$ such that $Q = T_{\overline{F}}|_{D_u(T_{\overline{F}})}$.*

(2) *$D(Q) \supset \mathcal{H}(u) \cap H^\infty$.*

(3) *$D(Q)$ contains some exact cyclic vector for S_u^*.*

Proof. Since $D_u(T_{\overline{F}}) \supset \mathcal{H}(u) \cap H^\infty$ then (1) $\Rightarrow$ (2), and since $S^*u \in \mathcal{H}(u) \cap H^\infty$ is exact cyclic for S_u^*, then (2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (1). Let $h \in D(Q)$ be an exact cyclic vector for S_u^*. Then there exists $F \in H^2$ so that $T_{\overline{F}}h = Qh$. Therefore for p polynomial:

$$(T_{\overline{p}}h,\, T_{\overline{F}}T_{\overline{p}}h) = (T_{\overline{p}}h,\, QT_{\overline{p}}h) \in G(Q). \tag{5.1}$$

Since h is cyclic for S_u^*, the set $\{T_{\overline{p}}h :\ p \text{ polynomial}\}$ is dense in $\mathcal{H}(u)$. Hence, the closure of

$$\mathcal{F} = \{(T_{\overline{p}}h,\, T_{\overline{F}}T_{\overline{p}}h) :\ p\ \text{ polynomial}\},$$

is contained in $G(T_{\overline{F}}|_{D_u})$ ($D_u = D_u(T_{\overline{F}})$), and it is the graph of some S_u^*-commuting operator R. So, $R = T_{\overline{F}}|_{D_u}$ by Theorem 5.2, and then (5.1) implies that Q is a S_u^*-commuting extension of $T_{\overline{F}}|_{D_u}$. Thus, $Q = T_{\overline{F}}|_{D_u}$ by Theorem 5.2. □

As a corollary of Theorem 5.3 we obtain a short proof of Sarason's theorem. For $\sigma \in L^2$ the Hankel operator with symbol σ is the densely defined operator on H^2 with codomain $\overline{H}_0^2$, given by $H_\sigma(f) = (I-K)(\sigma f)$, whenever this function is in $\overline{H}_0^2$ (K is the Cauchy transform and I is the identity). If for instance f is in H^∞, then $H_\sigma(f) = P_-(\sigma f)$. A famous theorem of Nehari [12] asserts that H_σ is bounded if and only if there is $G \in H^2$ such that $\sigma - G = \psi \in L^\infty$, in which case ψ can be choosen so that $\|H_\sigma\| = \|\psi\|_\infty$ (actually, $H_\sigma = H_\psi$).

Corollary 5.4 (Sarason). *Let u be an inner function and $Q : \mathcal{H}(u) \to \mathcal{H}(u)$ be a bounded operator that commutes with S_u^*. Then there is $\varphi \in H^\infty$ such that $Q = T_{\overline{\varphi}}|_{\mathcal{H}(u)}$ and $\|\varphi\|_\infty = \|Q\|$.*

Proof. By Theorem 5.3 there is $F \in H^2$ such that $Q = T_{\overline{F}}|_{D_u}$, where $D_u = D_u(T_{\overline{F}}) = D(Q) = \mathcal{H}(u)$. Consider the conjugation C ($= C_u$) of $\mathcal{H}(u)$, and put $z = e^{i\theta}$. If $g \in \mathcal{H}(u) \cap H^\infty$, we write $\overline{F}g = \sum_{-\infty<n<\infty} c_n z^n$ for the Fourier series of $\overline{F}g$. The next string of equalities is straightforward

$$\begin{aligned} \overline{zT_{\overline{F}}(g)} &= \overline{zP_+(\overline{F}g)} = \sum_{n\geq 0} \overline{c}_n \overline{z}^{n+1} \\ &= P_-(\overline{z}F\overline{g}) = H_{\overline{u}F}(u\overline{zg}) = H_{\overline{u}F}(Cg). \end{aligned}$$

So, for any $f \in \mathcal{H}(u) \cap H^\infty$,

$$H_{\overline{u}F}(f) = \overline{zT_{\overline{F}}(Cf)} = \overline{zQ(Cf)}. \tag{5.2}$$

Since C is a conjugation and Q is bounded, (5.2) implies that $H_{\overline{u}F}|_{\mathcal{H}(u)}$ is bounded, with norm $\|H_{\overline{u}F}|_{\mathcal{H}(u)}\| = \|Q\|$. Since H^2 decomposes into the orthogonal sum $H^2 = \mathcal{H}(u) \oplus uH^2$, and $H_{\overline{u}F}|_{uH^2} = 0$, then $H_{\overline{u}F}$ is bounded and $\|H_{\overline{u}F}\| = \|H_{\overline{u}F}|_{\mathcal{H}(u)}\|$. By Nehari's theorem there are $G \in H^2$ and $\psi \in L^\infty$ such that $\overline{u}F - G = \psi$, where $\|\psi\|_\infty = \|H_{\overline{u}F}\| = \|Q\|$. Therefore $\varphi = u\psi = F - uG \in L^\infty \cap H^2 = H^\infty$, $\|\varphi\|_\infty = \|Q\|$, and clearly $T_{\overline{\varphi}}|_{\mathcal{H}(u)} = T_{\overline{F}}|_{\mathcal{H}(u)} = Q$. □

6. S^*-commuting operators and the Smirnov class.

This final section is essentially devoted to prove that if Q is a S^*-commuting operator whose domain contains H^∞, then there is $F \in H^2$ such that

$Q = T_{\overline{F}}$. The analogous result for S_u^*-commuting operators was proved in Theorem 5.3. It is not possible to imitate the proof of Theorem 5.3 here, because of the lack of exact cyclic vectors for S^*. We will take a detour through the spaces $\mathcal{H}(u)$. First we establish some general background about analytic functions. The books of Duren [4] and Garnett [6] are excellent sources for this material.

Let us write $|I|$ for the normalized Lebesgue measure of an interval $I \subset \partial\mathbb{D}$. If $|I| < 1$, put

$$\Gamma(I) = \{z \in \mathbb{D} : \ |z| \geq 1 - |I| \ \text{ and } \ z/|z| \in I\}.$$

For completeness reasons we take $\Gamma(I) = \mathbb{D}$ if $|I| = 1$. A positive Borel finite measure μ on $\mathbb{D}$ is called a Carleson measure if $\mu(\Gamma(I)) \leq C|I|$ for every interval $I \subset \partial\mathbb{D}$, where C denotes a generic constant, not necessarily the same in each occurrence. It is well known that μ is a Carleson measure if and only if given $1 \leq p < \infty$,

$$\int_{\mathbb{D}} |f|^p \, d\mu \leq C\|f\|_p^p \quad \text{for every} \quad f \in H^p,$$

where C only depends on p. We say that a sequence $\{z_n\} \subset \mathbb{D}$ is of type **C** if $\mu = \sum(1 - |z_n|)\delta_{z_n}$ is a Carleson measure, where δ_z is the probability measure with mass at z. Therefore $\{z_n\} \subset \mathbb{D}$ is of type **C** if and only if for a given $1 \leq p < \infty$,

$$\sum (1 - |z_n|)|f(z_n)|^p \leq C\|f\|_p^p \qquad (f \in H^p). \tag{6.1}$$

The closed graph theorem tells us that (6.1) holds if and only if the linear transformation $f \mapsto \{(1 - |z_n|)^{1/p} f(z_n)\}_{n \geq 1}$ maps H^p into l^p.

The condition for a sequence $\{z_n\} \subset \mathbb{D}$ to be the zero sequence of a Blaschke product is $\sum(1 - |z_n|) < \infty$. Consequently, every sequence of type **C** is the zero sequence of a Blaschke product. Actually, a Blaschke product B factorizes as a finite product of interpolating and finite Blaschke products if and only if its zero sequence is of type **C**. We are not going to use this fact here. We write I° for the interior (respect to $\partial\mathbb{D}$) of an interval $I \subset \partial\mathbb{D}$.

Lemma 6.1. *Let $\{I_j : \ j = 1, 2, \dots\}$ be a collection of intervals in $\partial\mathbb{D}$ with non-void pairwise disjoint interiors, and let $z_j \in \Gamma(I_j)$. Then $\{z_j\}$ is a sequence of type* **C**.

Proof. We must prove that for every interval $I \subset \partial\mathbb{D}$,

$$\sum_{z_j \in \Gamma(I)} (1 - |z_j|) \leq C|I|$$

for some $C > 0$. Since I is contained in an open interval of measure at most $2|I|$, we can assume I open, say $I = (\alpha, \beta)$. If $z_j \in \Gamma(I)$ then the interior of I_j meets I. Therefore $I_j^\circ \subset I$ or I_j° contains at least one of the points α or β. Since the family $\{I_k^\circ : \ k \geq 1\}$ is pairwise disjoint, there are at most one interval I_{j_0} such that $\alpha \in I_{j_0}$ and one interval I_{j_1} such that $\beta \in I_{j_1}$. Consequently, whether there are j_0 and j_1 as before or not,

$$\sum_{z_j \in \Gamma(I)} (1 - |z_j|) \leq \sum_{I_j^\circ \subset I} (1 - |z_j|) + 2|I| \leq 3|I|.$$

□

The Smirnov class is formed by the functions $f = h/k$, where $h, k \in H^\infty$ and k is an outer function. Obviously f is analytic on $\mathbb{D}$ and has nontangential boundary values $h(e^{i\theta})/k(e^{i\theta})$ for almost every $e^{i\theta} \in \partial\mathbb{D}$. One of the fundamental features of the Smirnov class is that it contains the spaces H^p for all $p > 0$. Besides, a function in the Smirnov class is in H^p $(p > 0)$ if and only if its boundary function is in L^p of $\partial\mathbb{D}$.

Lemma 6.2. *Let f be a function in the Smirnov class and let $1 \leq p < \infty$. Then $f \in H^p$ if and only if for every sequence $\{z_n\}$ of type* $\mathbf{C}$,

$$\sum (1 - |z_n|)|f(z_n)|^p < \infty. \tag{6.2}$$

Proof. The 'if' part of the lemma comes from (6.1). So, suppose that $f \notin H^p$. We will construct a sequence of type $\mathbf{C}$ that fails to satisfy (6.2). There is no loss of generality if we write $f = a/b$, where $a, b \in H^\infty$, $\|a\|_\infty \leq 1$, b is outer and $b(0) > 0$. First we prove the lemma for $p = 1$.

Suppose that $f \notin L^1$ and let $I_1 \subset \partial\mathbb{D}$ be a closed interval such that $|I_1| < 1$ and $\int_{I_1} |f| \, dm = \infty$, where m is the normalized Lebesgue measure on $\partial\mathbb{D}$. For N a positive integer consider the outer function

$$|b_N(e^{i\theta})| = \begin{cases} |b(e^{i\theta})| & \text{if } \ |b(e^{i\theta})| > 1/N \\ 1/N & \text{if } \ |b(e^{i\theta})| \leq 1/N, \end{cases}$$

where $b_N(0) > 0$. Hence, there exists an integer $N = N(I_1)$ such that if $f_1 = a/b_N$, then

$$\int_{I_1} |f_1| \, dm > 5. \tag{6.3}$$

Partition I_1 in N closed intervals of measure $(1/N)|I_1|$. So, there is at least one of these intervals, say I_2, such that $\int_{I_2} |f| \, dm = \infty$. Since $|f_1| = |a/b_N| \leq$

N, then $\int_{I_2} |f_1|\,dm \le N|I_2| \le |I_1| < 1$. So, (6.3) yields

$$\int_{I_1\setminus I_2} |f_1|\,dm > 4. \tag{6.4}$$

The set $I_1 \setminus I_2^\circ$ is formed by one or the union of two disjoint closed intervals. In any case (6.4) implies that there is one of these intervals, J, so that $\int_J |f_1|\,dm > 2$. Consider the sequence of functions $g_n(e^{i\theta}) = f_1((1-\frac{|J|}{n})e^{i\theta})$, $n \ge 1$. It is immediate from Parseval's identity that $g_n \to f_1$ in L^2-norm, and a fortiori in L^1-norm. Consequently, there is an integer n_1 such that

$$\int_J |g_{n_1}|\,dm > 1. \tag{6.5}$$

Partition J into n_1 closed intervals of measure $|J|/n_1$, $J_1, \dots, J_{n_1}$. Since g_{n_1} is continuous on J_k, there exists $\omega_k \in J_k$ such that $\int_{J_k} |g_{n_1}|\,dm = |g_{n_1}(\omega_k)|.|J_k| = |g_{n_1}(\omega_k)|.(|J|/n_1)$ for every $1 \le k \le n_1$. Put $z_k = (1-\frac{|J|}{n_1})\omega_k$. Then $f_1(z_k) = g_{n_1}(\omega_k)$, $|z_k| = 1 - (|J|/n_1)$ and by (6.5),

$$\begin{aligned}\sum_{k=1}^{n_1}(1-|z_k|)|f_1(z_k)| &= \sum_{k=1}^{n_1}\frac{|J|}{n_1}|g_{n_1}(\omega_k)| = \sum_{k=1}^{n_1}\int_{J_k}|g_{n_1}|\,dm \\ &= \int_J |g_{n_1}|\,dm > 1.\end{aligned} \tag{6.6}$$

Observe that $z_k \in \Gamma(J_k)$ for every J_k. The function b_N is invertible in H^∞ and $|b_N^{-1}b| \le 1$ a.e. on $\partial\mathbb{D}$. So, this inequality also holds on $\mathbb{D}$. Then for $z \in \mathbb{D}$,

$$|f(z)| = \frac{|a(z)|}{|b(z)|} \ge \frac{|a(z)|}{|b_N(z)|} = |f_1(z)|.$$

Thus, (6.6) gives

$$\sum_{k=1}^{n_1}(1-|z_k|)|f(z_k)| > 1. \tag{6.7}$$

Since $\int_{I_2} |f|\,dm = \infty$, we can repeat the above process with I_2 instead of I_1, obtaining a closed interval $I_3 \subset I_2$ that plays with respect to I_2 the same role that I_2 plays with respect to I_1, and so forth...

In the $l-$step of this process we have

(I) a closed interval $I_{l+1} \subset I_l$ such that $\int_{I_{l+1}} |f|\,dm = \infty$,

(II) a family $J_1^l, \dots, J_{n_l}^l$ of closed subintervals of $I_l \setminus I_{l+1}^\circ$ whose interiors are pairwise disloint, and

(III) a family of points $z_k^l \in \Gamma(J_k^l)$ $(1 \le k \le n_l)$ such that

$$\sum_{k=1}^{n_l}(1-|z_k^l|)|f(z_k^l)|>1. \tag{6.8}$$

By (I) and (II) two different intervals of the family $\{J_k^l:\ 1\le k\le n_l,\ 1\le l\}$ have disjoint interiors. Then Lemma 6.1 says that the sequence $\{z_k^l\}$, $1\le k\le n_l$, $1\le l$, is of type **C**. Besides (6.8) implies that

$$\sum_{l\ge 1}\sum_{k=1}^{n_l}(1-|z_k^l|)|f(z_k^l)|=\infty.$$

This proves the lemma for $p=1$. Now suppose that $p\ge 1$ and $f\notin H^p$. Factorize $a=a_0v$, where a_0 is outer and v is inner. Since a_0 and b are outer functions, there are analytic p-powers a_0^p and b^p. For $(p)=$ the smallest integer greater or equal than p, consider $f_p=a_0^pv^{(p)}/b^p$. The function f_p is in the Smirnov class and by hypothesis $f_p\notin L^1$. By the case $p=1$ there exists a sequence $\{z_n\}$ of type **C** such that

$$\sum(1-|z_n|)\frac{|a_0^p(z_n)v^{(p)}(z_n)|}{|b^p(z_n)|}=\infty. \tag{6.9}$$

Since $|v(z)|\le 1$ for all $z\in\mathbb{D}$ and $(p)\ge p$, $|v(z)|^p\ge|v(z)|^{(p)}=|v(z)^{(p)}|$. So, by (6.9)

$$\sum(1-|z_n|)\frac{|a_0(z_n)v(z_n)|^p}{|b(z_n)|^p}=\infty,$$

and the lemma follows. □

Let Q be a S^*-commuting operator; if u is inner and $D_u(Q)=D(Q)\cap\mathcal{H}(u)$ then $Q|_{D_u(Q)}$ is a closed operator that commutes with S_u^*. Moreover, if $(a,\,b)$ is a pair of functions in H^∞ that defines Q, then by Theorem 1.1 $T_{\overline{b}}h$ is in $D_u(Q)$ for every $h\in\mathcal{H}(u)$. Since b is outer, $D_u(Q)$ is dense in $\mathcal{H}(u)$. That is, $Q|_{D_u(Q)}$ is S_u^*-commuting.

Theorem 6.3. *Let Q be a S^*-commuting operator and $2\le p\le\infty$. Then there is $F\in H^p$ such that $Q=T_{\overline{F}}$ if and only if for every inner function u there is $F_u\in H^p$ such that $D_u=D_u(Q)=D_u(T_{\overline{F}_u})$ and $Q|_{D_u}=T_{\overline{F}_u}|_{D_u}$.*

Proof. The necessity is trivial. Suppose that $(a,\,b)$ is a pair of functions defining Q. For u an inner function, Theorem 1.1 implies that

$$\{(T_{\overline{b}}h,\,-T_{\overline{a}}h):\ h\in\mathcal{H}(u)\}\subset G(Q)\cap[\mathcal{H}(u)\times\mathcal{H}(u)]\subset G(Q|_{D_u}).$$

By hypothesis there exists $F_u\in H^p$ such that $T_{\overline{F}_u}T_{\overline{b}}h=-T_{\overline{a}}h$ for every $h\in\mathcal{H}(u)$. In particular, if $h\in\mathcal{H}(u)\cap H^\infty$, $T_{\overline{F}_u\overline{b}+\overline{a}}h=0$. By Lemma 2.4 then there is $G\in H^2$ so that

$$bF_u+a=uG. \tag{6.10}$$

Let us first assume that $p = \infty$. Suppose that Q is not of the form $T_{\overline{F}}$ for any $F \in H^\infty$; then Proposition 2.5 says that a/b is not in H^∞. Hence, there is a sequence $\{z_n\} \subset \mathbb{D}$ such that the sequence $a(z_n)/b(z_n)$, $n = 1, 2, \ldots$ is unbounded. Substracting a suitable subsequence we can assume that $\{z_n\}$ is the zero sequence of a Blaschke product v (i.e., $\sum(1 - |z_n|) < \infty$). Thus, choosing $u = v$ in (6.10) and evaluating at z_n we have

$$b(z_n)F_v(z_n) + a(z_n) = v(z_n)G(z_n) = 0 \qquad \text{for } n \geq 1.$$

Consequently $F_v(z_n) = -a(z_n)/b(z_n)$ is unbounded, which contradicts our hypothesis $F_v \in H^\infty$.

Suppose now that $2 \leq p < \infty$. Again, if Q is not of the form $T_{\overline{F}}$ for any $F \in H^p$, Proposition 2.5 implies that $a/b \notin H^p$. Henceforth Lemma 6.2 says that there is a sequence $\{z_n\} \subset \mathbb{D}$ of type **C** such that the sequence

$$(1 - |z_n|)^{1/p} \frac{a(z_n)}{b(z_n)}, \quad n \geq 1$$

is not in l^p. Let u in (6.10) be the Blaschke product with zeros z_n. Then $F_u(z_n) = -a(z_n)/b(z_n)$ for all $n \geq 1$, and consequently

$$(1 - |z_n|)^{1/p} F_u(z_n) = -(1 - |z_n|)^{1/p} \frac{a(z_n)}{b(z_n)}$$

for all $n \geq 1$. So, $\{(1 - |z_n|)^{1/p} F_u(z_n)\}_{n \geq 1} \notin l^p$, which according to Lemma 6.2 contradicts the hypothesis $F_u \in H^p$. $\square$

The case $p = \infty$ of the above theorem is related to a question recently raised by Lotto and Sarason [11]. They ask if a closed operator R densely defined on H^2 such that $R|_{\mathcal{H}(u)}$ is bounded for every inner function u must be bounded. They were able to solve affirmatively this problem for Hankel operators with symbol in L^2. Theorem 6.3 for $p = \infty$ answers affirmatively this question when R commutes with S^*. Recently Michael Sand solved affirmatively the problem for a large class of closed operators [13]. Finally, an affirmative solution to the general problem was found by the author in [18]. We finish the paper with the following

Corollary 6.4. *Let Q be a S^*-commuting operator. The following condition are equivalent.*

(1) *There is $F \in H^2$ such that $Q = T_{\overline{F}}$.*

(2) *$H^\infty \subset D(Q)$.*

(3) *$D(Q)$ contains all the inner functions.*

Proof. Obviously (1) $\Rightarrow$ (2) $\Rightarrow$ (3). So, suppose that (3) holds. Then $S^*u \subset D(Q) \cap \mathcal{H}(u) = D_u(Q)$ for every inner function u. Since S^*u is an exact cyclic vector for S_u^*, Theorem 5.3 implies that there is $F_u \in H^2$ such that $D_u(T_{\overline{F}_u}) = D_u(Q)$ and $Q|_{D_u(Q)} = T_{\overline{F}_u}|_{D_u(Q)}$ for every u. Henceforth (1) follows from Theorem 6.3. □

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