

CONFORMAL CAPACITIES AND EXTREMAL METRICS

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For any non-compact Riemannian n -manifold M it is proved that the function $\lambda_M^{\frac{1}{1-n}} : M \times M \rightarrow \overline{\mathbb{R}}_+$ satisfies the triangular inequality. This result includes a Vuorinen conjecture. It follows from an extension to generalized condensers of a property of extremal functions.

Introduction.

For any non-compact Riemannian manifold M of dimension $n \geq 2$ we previously defined a function $\lambda_M : M \times M \rightarrow \overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$ only dependent on the conformal structure of M , and proved that for a class of manifolds containing all the proper subdomains of \mathbb{R}^n , $\lambda_M^{\frac{1}{1-n}}$ was a distance on M [F1, F2].

The case of a domain G of \mathbb{R}^n has been the object of several investigations leading to estimations of $\lambda_G[V_1, \dots, V_4]$ and to properties of the λ_G -Lipschitz mappings [FMV]. Then by considering the case of a ball M . Vuorinen has been led to conjecture that $\lambda_G^{\frac{1}{1-n}}$ is also a distance. In some way this improvement of my previous result is the best possible as it was proved in [AVV] that λ_G^{-p} cannot be a metric if $p > \frac{1}{n-1}$. Moreover its interest is reinforced by the fact that, if $n = 2$, it is relevant to the Teichmüller's theory of quadratic differentials and to the Jenkins's extremal metrics [J1]. This conjecture has been proven in [AVV] for $G = B^n$, by A.Y. Solynin [S] and J.A. Jenkins [J2] for $G = \mathbb{R}^2 \setminus \{0\}$, then extended by J.A. Jenkins [J2] to any plane domain with finite connectivity. But this extension is somewhat difficult to follow as it involves some families of homotopy groups which are not easy to define with precision when G is not simply-connected.

In the present paper we will prove the following general result, which includes the Vuorinen conjecture:

Theorem A. *With above notations, and for all $n \geq 2$, the function $m_M = \lambda_M^{\frac{1}{1-n}}$ satisfies the triangular inequality. Consequently if $\lambda_M(x, y)$ is finite when $x \neq y$, the function m_M is a distance on M .*

We must emphasize that when applying this result to a domain G of \mathbb{R}^n , we do not need any condition on the boundary of G , in opposition to Jenkins's theorem.

In fact this result will appear as a corollary of a general property of the level sets of extremal functions relevant to the theory of conformal capacities (Theorem B in Section 1).

This property is known in the case of classical condensers only (cf [H] 3.8). We will easily extend it to the case of condensers associated with non-compact continua by dividing it in two reverse inequalities. Although the first inequality (2.1) is sufficient for proving Theorem A in Section 3, it will be useful to set the complete equality in view of other applications [F4]. Then we will interpret our results in terms of moduli (cf. Section 4).

At last in Section 5 we will prove the existence of extremal functions associated with a pair of points, and in some way the present study may be considered as an extension to Riemannian manifolds of Jenkins's theory of extremal metrics [J1].

1. Preliminaries.

We will use the definitions and notations given in [F3]. Particularly M will always denote a non-compact Riemannian manifold of dimension $n \geq 2$ and $\hat{M} = M \cup \{\infty\}$ its Alexandrov compactification.

A relative continuum of the manifold M is a closed subset C of M without any compact connected component, or, in other terms: such that $C \cup \{\infty\}$ is connected in \hat{M} .

A function $u \in \mathcal{C}(M)$ is called monotone if for any relatively compact domain D of M , the supremum and infimum of u on ∂D are respectively equal to its supremum and infimum on D .

We denote $H(M) = \mathcal{C}(M) \cap L_n^1(M)$ the space of real-valued continuous functions u on M admitting a generalized gradient ∇u with

$$I(u, M) = \int_M |\nabla u|^n d\tau < +\infty,$$

in which $d\tau$ is the volume element of M .

Let $H^*(M)$ denote the set of monotone functions $u \in H(M)$.

For any pair (C_0, C_1) of closed subsets of M , the capacity of the condenser $\Gamma(C_0, C_1)$ is defined by

$$Cap(C_0, C_1) = \inf_{u \in A(C_0, C_1)} I(u, M)$$

in which $A(C_0, C_1)$ is the set of those functions $u \in H(M)$, called admissible for $\Gamma(C_0, C_1)$, satisfying $u = 0$ on C_0 , $u = 1$ on C_1 and $0 \leq u \leq 1$ everywhere. If $A(C_0, C_1) = \emptyset$ and, particularly, if $C_0 \cap C_1 \neq \emptyset$, we set $Cap(C_0, C_1) = +\infty$.

Our study will be based on the following previous result (cf. [F1] Th. 6.5 or [F3] Prop. 3.4):

Theorem 1.1. *Let (C_0, C_1) be a pair of relative continua of M such that $A(C_0, C_1) \neq \emptyset$. Then there exists a unique function $u \in A(C_0, C_1)$ satisfying $I(u, M) = \text{Cap}(C_0, C_1)$.*

This function, denoted $\text{extr}(C_0, C_1)$, is monotone on M , hence belongs to $H^(M)$, and satisfies the variational condition*

$$(1.2) \quad \int_M |\nabla u|^{n-2} \nabla u \cdot \nabla v \, d\tau = 0$$

for any function $v \in H(M)$ vanishing on $C_0 \cup C_1$. In other terms, u is n -harmonic on $M \setminus (C_0 \cup C_1)$.

Remark. From ([F1] Prop. 4.2) or ([F3] Prop. 2.3) it follows that $u^{-1}(0)$ and $u^{-1}(1)$ are relative continua resp. containing C_0, C_1 . It must be observed that, if $M \setminus (C_0 \cup C_1)$ is not connected, C_0 [resp. C_1] can be a proper subset of $u^{-1}(0)$ [resp. $u^{-1}(1)$]. In fact, if there exists a component D of $M \setminus (C_0 \cup C_1)$ such that $\partial D \subset C_0$ [resp. $\partial D \subset C_1$] we have necessarily $u = 0$ [resp. $u = 1$] on D . For that reason we will need the following lemma:

Lemma 1.3. *Under the hypothesis of (1.1), the function $u = \text{extr}(C_0, C_1)$ satisfies*

$$\text{Cap}(u^{-1}(0), u^{-1}(1)) = \text{Cap}(C_0, C_1).$$

Proof. Obviously $A(u^{-1}(0), u^{-1}(1)) \subset A(C_0, C_1)$, hence $\text{Cap}(u^{-1}(0), u^{-1}(1)) \geq \text{Cap}(C_0, C_1)$. The reverse inequality follows from the fact that u is admissible for $\Gamma(u^{-1}(0), u^{-1}(1))$, hence the announced result. \square

Now our main result can be stated as follows:

Theorem B. *Let C_0, C_1 be two disjoint relative continua of a Riemannian manifold M such that $\gamma = \text{Cap}(C_0, C_1) < +\infty$, with $u = \text{extr}(C_0, C_1)$, $n = \dim(M)$. Then for any pair (α, β) of real numbers with $0 \leq \alpha \leq \beta \leq 1$, the sets $C_\alpha^- = \{x \in M | u(x) \leq \alpha\}$ and $C_\beta^+ = \{x \in M | u(x) \geq \beta\}$ are relative continua; the extremal function $u_{\alpha\beta} = \text{extr}(C_\alpha^-, C_\beta^+)$ is defined by $u_{\alpha\beta} = 0$ on C_α^- , $u_{\alpha\beta} = 1$ on C_β^+ and $u_{\alpha\beta}(x) = (u(x) - \alpha)/(\beta - \alpha)$ if $\alpha < u(x) < \beta$.*

At last the capacity of $\Gamma(C_\alpha^-, C_\beta^+)$ is $\gamma_{\alpha\beta} = \text{Cap}(C_\alpha^-, C_\beta^+) = (\beta - \alpha)^{1-n} \gamma$.

The fact that C_α^- and C_β^+ are relative continua comes from ([F1] 4.2) or ([F3] 2.3). The other assertions will directly follow from the inequalities (2.1) and (4.2). In Section 3. we will obtain Theorem A as a corollary of the first inequality (2.1).

2. First inequality (majorization of $\gamma_{\alpha\beta}$).

We observe that the variational equation (1.2) does not require any condition on v at infinity. When M is a domain G of \mathbb{R}^n we can interpret (1.2) by saying that u is a weak solution of $\operatorname{div}(|\nabla u|^{n-2}\nabla u) = 0$ on $G \setminus (C_0 \cup C_1)$ satisfying $u = 0$ on C_0 , $u = 1$ on C_1 and $\frac{du}{dn} = 0$ on ∂G .

Then by the same arguments as in [H] we obtain:

Proposition 2.1. *With the notation of Theorem B we have:*

$$\operatorname{Cap}(C_\alpha^-, C_\beta^+) \leq (\beta - \alpha)^{1-n} \operatorname{Cap}(C_0, C_1).$$

Proof. The function $u_{\alpha\beta}$ defined on M by $u_{\alpha\beta} = 0$ on C_α^- , $u_{\alpha\beta} = 1$ on C_β^+ and $u_{\alpha\beta} = (u - \alpha)/(\beta - \alpha)$ on $M_{\alpha\beta} = M \setminus (C_\alpha^- \cup C_\beta^+)$ is admissible for $\Gamma(C_\alpha^-, C_\beta^+)$ and also for $\Gamma(C_0, C_1)$. By using (1.2) with $v = u - u_{\alpha\beta}$ we get:

$$\begin{aligned} \operatorname{Cap}(C_0, C_1) &= I(u, M) = \int_M |\nabla u|^{n-2} \nabla u \cdot \nabla u_{\alpha\beta} d\tau = (\beta - \alpha)^{n-1} I(u_{\alpha\beta}, M) \\ &\geq (\beta - \alpha)^{n-1} \operatorname{Cap}(C_\alpha^-, C_\beta^+), \end{aligned}$$

hence the announced result. \square

Now we can observe that $u_{\alpha\beta}$ is n -harmonic on $M_{\alpha\beta}$ and satisfies the same boundary conditions as $\operatorname{extr}(C_\alpha^-, C_\beta^+)$. Theorem B would immediately follow if the unicity of such a function would be proved. However, as we are here dealing with functions which are not in $W_{n,0}^1$, it does not seem that this result can be considered as known for $n \geq 3$. We will achieve the proof in Section 4 by an elementary argument.

3. Proof of Theorem A.

We recall that the function λ_M has been defined by

$$(3.1) \quad \lambda_M(x, y) = \inf_{C_x, C_y} \operatorname{Cap}(C_x, C_y)$$

where C_x, C_y are relative continua resp. containing x and y , hence $\lambda_M(x, x) = +\infty$ for all $x \in M$.

As relative continua cannot reduce to single points, this function is strictly positive (cf. [F3] Prop. 3.6) but $\lambda_M(x, x)$ is not necessarily finite when $y \neq x$: for example, as $H^*(E^n)$ is reduced to constant functions, λ_{E^n} is identically $+\infty$ (cf. [F1] §7). We shall say that M is of class L if $\lambda_M(x, y)$ is always finite when $y \neq x$. For example, any proper subdomain of E^n is of class L .

More generally, an open submanifold M of a compact manifold N is of class L if, and only if, $N \setminus M$ contains at least two points (cf. [F1] Th. 9.1). For a still more general result see ([F3], Prop.7.6).

Then Theorem A will come from the following result in which M is not assumed to be of class L :

Proposition 3.2. *For any three points x, y, z in M such that $\lambda_M(x, y) < +\infty$, the function $m_M = \lambda_M^{\frac{1}{1-n}}$ satisfies the triangular inequality*

$$(3.3) \quad m_M(x, y) \leq m_M(x, z) + m_M(z, y).$$

Proof. $\varepsilon > 0$ given, let C_0, C_1 be two r -continua such that $x \in C_0$, $y \in C_1$ and $\text{Cap}(C_0, C_1) \leq \lambda_M(x, y) + \varepsilon$. From (1.3) we are allowed to assume that $u = \text{extr}(C_0, C_1)$ satisfies $u^{-1}(0) = C_0$, $u^{-1}(1) = C_1$. Then let us set $\alpha = u(z)$:

If $\alpha = 0$, $z \in C_0$ and $\lambda_M(z, y) \leq \text{Cap}(C_0, C_1)$.

If $\alpha = 1$, $z \in C_1$ and $\lambda_M(x, z) \leq \text{Cap}(C_0, C_1)$.

If $0 < \alpha < 1$, with the notations used in Theorem B, $z \in C_\alpha^- \cap C_\alpha^+$, hence from (2.1):

$$\begin{aligned} \lambda_M(x, z) &\leq \text{Cap}(C_0, C_\alpha^+) \leq \alpha^{1-n} \text{Cap}(C_0, C_1) \\ \lambda_M(z, y) &\leq \text{Cap}(C_\alpha^-, C_1) \leq (1-\alpha)^{1-n} \text{Cap}(C_0, C_1). \end{aligned}$$

By raising the first members of these inequalities to the negative power $1/(1-n)$ and adding them, we obtain

$$m_M(x, z) + m_M(z, y) \geq [\text{Cap}(C_0, C_1)]^{\frac{1}{1-n}} \geq [\lambda_M(x, y) + \varepsilon]^{\frac{1}{1-n}}$$

and the same inequality holds when $\alpha = 0$ or $\alpha = 1$.

Hence (3.3) when ε tends to zero. \square

4. End of the proof of Theorem B.

With the notations used in the statement of Theorem B let us set:

$$v_1 = \text{extr}(C_0, C_\alpha^+), v_2 = \text{extr}(C_\alpha^-, C_\beta^+), v_3 = \text{extr}(C_\beta^-, C_1).$$

For any positive numbers t_1, t_2, t_3 with $t_1 + t_2 + t_3 = 1$ the function $v = \sum_{i=1}^3 t_i v_i$ is admissible for $\Gamma(C_0, C_1)$.

As the open sets $M_{0\alpha}$, $M_{\alpha\beta}$ and $M_{\beta 1}$ are mutually disjoint we have:

$$\gamma = \text{Cap}(C_0, C_1) \leq I(v, M) = \sum_{k=1}^3 t_k^n I(v_k, M)$$

hence by choosing $t_1 = \alpha$, $t_2 = \beta - \alpha$, $t_3 = 1 - \beta$:

$$(4.1) \quad \gamma \leq \alpha^n I(v_1, M) + (\beta - \alpha)^n I(v_2, M) + (1 - \beta)^n I(v_3, M).$$

Now by applying (2.1) three times we have:

$$(4.2) \quad I(v_1, M) \leq \alpha^{1-n} \gamma, I(v_2, M) \leq (\beta - \alpha)^{1-n} \gamma, I(v_3, M) \leq (1 - \beta)^{1-n} \gamma.$$

The second member of (4.1) is therefore not greater than γ , and all the inequalities (4.2) are reduced to equalities. Particularly we have

$$Cap(C_\alpha^-, C_\beta^+) = I(v_2, M) = (\beta - \alpha)^{1-n} \gamma = I(u_{\alpha\beta}, M)$$

in which $u_{\alpha\beta}$ is the function defined in the proof of (2.1). Then it is obvious that $u_{\alpha\beta} = extr(C_\alpha^-, C_\beta^+)$ which completes the proof of Theorem B.

Interpretation in terms of moduli. With any condenser $\Gamma(C_0, C_1)$ one associates the family $\Sigma(C_0, C_1)$ of hypersurfaces contained in M and separating C_0 from C_1 . Then a non-negative Borel function f is called admissible for $\Sigma(C_0, C_1)$ if, for any $\sigma \in \Sigma(C_0, C_1)$, $\int_\sigma f dH^{n-1} \geq 1$ where dH^{n-1} is the $(n-1)$ -dimensional Hausdorff measure.

The conformal modulus of $\Sigma(C_0, C_1)$ is defined by:

$$(4.3) \quad \text{Mod}(C_0, C_1) = \inf_f \int_M f^m d\tau \quad \left(m = \frac{n}{n-1} \right)$$

where f is admissible for $\Sigma(C_0, C_1)$.

The formula of duality

$$(4.4) \quad \text{Mod}(C_0, C_1) = \left(Cap(C_0, C_1) \right)^{\frac{1}{1-n}}$$

has been proved by Gehring [G1] for smooth separating surfaces, then by Ziemer [Z] for arbitrary sets by using the Fuglede's notation of modulus of a system of measures [Fu]. In those theories M is a domain of \mathbb{R}^n and C_0, C_1 are two compact sets contained in \bar{M} (see also [C]). In order to avoid all difficulty for extending (4.4) to the present case, we can use the functional definition of admissible functions, not very far from Ziemer's one, given in [BLF] with suitable justifications:

Definition 4.5. Let C_0, C_1 be two relative continua of M . A non-negative function f is called admissible for $\Sigma(C_0, C_1)$ if it belongs to $L^m(M)$ with $m = n/(n-1)$ and satisfies $\int_M f |\nabla u| d\tau \geq 1$ for any $u \in A(C_0, C_1)$.

With that definition it is easy to check that the infimum in (4.3) is obtained with $f = |\nabla u|^{n-1}/I(u, M)$ in which $u = \text{extr}(C_0, C_1)$, hence (4.4).

Then we have in the most general case

$$(4.6) \quad m_M(x, y) = \lambda_M^{\frac{1}{1-n}}(x, y) = \sup_{C_x, C_y} \text{Mod}(C_x, C_y)$$

where C_x, C_y are relative continua resp. containing x, y .

The extremal distance m_M is thus defined in terms of moduli and we can observe that the second part of the proof of Theorem B is similar to the argument used in ([M] 7.2) for proving the general inequality

$$(4.7) \quad \text{Mod}(D) \geq \sum_{k=1}^m \text{Mod}(D_k)$$

in which the D_k are mutually disjoint shells separating the boundaries of the shell D . But we emphasize the inefficiency of (4.7) for proving the Vuorinen conjecture, as this inequality is not in the good direction.

5. Appendix. Some more results relative to the function λ_M . Extremal functions relative to a pair of points.

It is noteworthy that we have set Theorem A without using the existence of a pair (γ_x, γ_y) of relative continua satisfying $\text{Cap}(\gamma_x, \gamma_y) = \lambda_M(x, y)$. We recall that the Jenkins's proof for plane domains was based on the existence of such continua which, in this special case, are trajectories of a quadratic defferential $Q(z)dz^2$ with poles at x, y . The following general result is therefore of some interest:

Theorem 5.1. *For any pair (x, y) of points in M such that $\lambda_M(x, y) < +\infty$ there exists at least a pair (γ_x, γ_y) of relative continua resp. containing x, y with $\text{Cap}(\gamma_x, \gamma_y) = \lambda_M(x, y)$. Hence the function $u = \text{extr}(\gamma_x, \gamma_y)$ belongs to $H^*(M)$ and satisfies $I(u, M) = \lambda_M(x, y)$.*

Proof. From [F1] it is known that

$$\lambda_M(x, y) = \inf_{u \in \alpha(x, y)} I(u, M)$$

in which $\alpha(x, y)$ is the set of functions $u \in H^*(M)$ satisfying $u(x) = 0$, $u(y) = 1$.

If $\lambda_M(x, y) < +\infty$ there exists a sequence (u_k) in $\alpha(x, y)$ such that $\lambda_M(x, y) = \lim I(u_k, M)$.

From classical properties of $H^*(M)$ (cf. [F3] Prop. 1.6) there exists a c -convergent subsequence of (u_k) whose limit u belongs to $\alpha(x, y)$ and satisfies

$I(u, M) \leq \lim I(u_k, M) = \lambda_M(x, y)$; hence necessarily $I(u, M) = \lambda_M(x, y)$. Then $\gamma_x = u^{-1}(0)$ and $\gamma_y = u^{-1}(1)$ satisfy the required condition. \square

Remark. The following example shows that such a pair (γ_x, γ_y) is not necessarily unique: if $M = \mathbb{R}^n \setminus \{0\}$, x and y being symmetrical with respect to 0, there are two pairs of relative continua satisfying the required condition, symmetrical with respect to 0 : in one pair (γ_x, γ_y) γ_x joins x to 0 and γ_y joins y to ∞ ; in the other one γ_x joins x to ∞ and γ_y joins y to 0.

Alternative definitions for λ_M . It is easy to see that we do not change the value of $\lambda_M(x, y)$ if in (3.1) we restrict C_x and C_y to be connected (or, in other terms, to be non-compact continua, see [F3] §5). However another definition of λ_G has been used in [V3] for a domain G of \mathbb{R}^n : this definition restricts C_x and C_y to be paths with endpoints on ∂G .

This alternative definition is perhaps not equivalent to our definition in the case of arbitrary domains. However, for domains G with ∂G of class C^1 the two definitions agree as we now show.

Sketch of the proof. Suppose $\lambda_G(x, y) < \infty$ and $\varepsilon > 0$ given. Let C_0, C_1 be non-compact continua such that $Cap(C_0, C_1) < \lambda_G(x, y) + \varepsilon$. If ∂G has no double point, \bar{C}_0 and \bar{C}_1 cannot have a common point a in ∂G : otherwise, as ∂G is of class C^1 , it is easy to prove that any function $u \in A(C_0, C_1)$ would satisfy an inequality of the type

$$\int_0^R \frac{\omega^n(r) dr}{r} \leq C_n I(u, G)$$

in which $\omega(r)$ is the oscillation of u on $G \cap S(a, r)$ and C_n a constant. Hence a contradiction with the fact that for r sufficiently small $\omega(r) = 1$.

Then we can construct two polygonal lines $\gamma_0 = [x, x_1, \dots, x_h]$ and $\gamma_1 = [y, y_1, \dots, y_k]$ with $x_h \in \partial G$, $y_k \in \partial G$, x_{h-1} and y_{k-1} being sufficiently near ∂G , $u(x_i)$ ($i = 1, 2, \dots, h-1$) and $1 - u(y_j)$ ($j = 1, 2, \dots, k-1$) being sufficiently small, in such a way that $Cap(\gamma_0, \gamma_1) \leq Cap(C_0, C_1) + \varepsilon \leq \lambda_G(x, y) + 2\varepsilon$. The same result holds when ∂G has double points, but the proof is slightly more elaborated since we have to distinguish the two sides of ∂G .

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Received April 5, 1994 and revised February 14, 1995.

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