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THE LOCAL INDEX FORMULA FOR A HERMITIAN MANIFOLD

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Let M be a compact complex manifold of real dimension $m = 2\bar{m}$ with a Hermitian metric. Let $a_n(x, \Delta^{p,q})$ be the heat equation asymptotics of the complex Laplacian $\Delta^{p,q}$. Then $Tr_{L^2}(fe^{-t\Delta^{p,q}}) \sim \Sigma_{n=0}^{\infty} t^{(n-m)/2} \int_M f a_n(x, \Delta^{p,q})$ for any $f \in C^\infty(M)$; the a_n vanish for n odd. Let $ag(M)$ be the arithmetic genus and let $a_n(x, \bar{\partial}) := \Sigma_q (-1)^q a_n(x, \Delta^{0,q})$ be the supertrace of the heat equation asymptotics. Then $\int_M a_n(x, \bar{\partial}) dx = 0$ if $n \neq m$ while $\int_M a_m(x, \bar{\partial}) dx = ag(M)$. The Todd polynomial $Td_{\bar{m}}$ is the integrand of the Riemann Roch Hirzebruch formula. If the metric on M is Kaehler, then the local index theorem holds:

(1) $\qquad a_n(x, \bar{\partial}) = 0$ **for** $n < m$, **and** $a_m(x, \bar{\partial}) = Td_{\bar{m}}(x)$.

In this note, we show Equation (1) fails if the metric on M is not Kaehler.

2. Theorem. *Let x_0 be a point of a holomorphic manifold M of real dimension $m \geq 4$. There exists a Hermitian metric on M so that $a_m(x_0, \bar{\partial}) \neq Td_{\bar{m}}(x_0)$.*

3. Remark. The local index theorem, Equation (1), for the Dolbeault complex for a Kaehler manifold was first proved by Patodi [6]; other proofs have been given subsequently by other authors. Theorem 2 was first stated in the thesis of the first author [2], but the proof given there was never published. This theorem also follows from computations performed by the first author in [3]; however, this was never made explicit and the computations in [3] in any event contained a calculational error (subsequently fixed in [4]) that made the original paper somewhat difficult to use. It is a pleasant task to thank Professor H. Duistermaat for pointing out that the literature on this subject was rather incomplete; we hope the present note repairs the deficiency. We refer the interested reader to his recent work [1] on Lefschetz formulas in the holomorphic context which contains a more complete bibliography and history in this area than we present here.

4. Outline of the paper. We begin by reviewing the spectral geometry of operators of Laplace type and recalling well-known formulas for the invariants a_2 and a_4. We use these formulas to prove Theorem 2 in the special case $m = 4$. We then use product formulas to complete the proof of Theorem 2 in general.

5. Geometry of operators of Laplace type. Let $x = (x^i)$ be real coordinates on a real Riemannian manifold M of real dimension m. Let

$$D = -(g^{\nu\mu}\partial^2/\partial x^\nu\partial x^\mu + A^\sigma\partial/\partial x^\sigma + B)$$

be an operator of Laplace type on the space of smooth sections $C^\infty(V)$ of a vector bundle V over M; in this equation A and B are local sections of the bundles $TM \otimes \text{End}(V)$ and $\text{End}(V)$ respectively. We adopt the Einstein convention and sum over repeated indices.

There is a unique connection $\nabla = \nabla(D)$ and a unique endomorphism $E = E(D)$ so that $D = -(Tr(\nabla^2) + E)$; if ω_δ is the connection 1-form of ∇, then

$$\begin{aligned} \omega_\delta &= \frac{1}{2}g_{\nu\delta}(A^\nu + g^{\mu\sigma}\Gamma_{\mu\sigma}{}^\nu \cdot I_V), \\ E &= B - g^{\nu\mu}(\partial_\nu\omega_\mu + \omega_\nu\omega_\mu - \omega_\sigma\Gamma_{\nu\mu}{}^\sigma \cdot I_V); \end{aligned} \tag{6}$$

see [5, Lemma 4.1.1] for details. The asymptotics of the heat equation can be expressed in terms of this data. Let $\Delta = \delta d$ be the scalar Laplacian on $C^\infty(M)$, let τ be the scalar curvature of M, and let ρ^2 and R^2 be the norms of the Ricci and total curvature tensors. Let Ω be the curvature of the connection $\nabla(D)$. We refer to [5, Theorem 4.1.6] for the proof that:

(7)

$$\begin{aligned} a_2(x, D) &= (4\pi)^{-m/2}6^{-1}\,\text{Tr}(6E + \tau \cdot I_V), \quad \text{and} \\ a_4(x, D) &= (4\pi)^{-m/2}360^{-1}\{-\Delta\,\text{Tr}(60E + 12\tau \cdot I_V) \\ &\quad + \text{Tr}(60\tau E + 180E^2 + 30\Omega_{ij}\Omega_{ij} + (5\tau^2 - 2\rho^2 + 2|R|^2)I_V)\}. \end{aligned}$$

Decompose the exterior derivative $d = \partial + \bar\partial$ and the coderivative $\delta = \delta' + \delta''$, where $\bar\partial : C^\infty\Lambda^{p,q} \to C^\infty\Lambda^{p,q+1}$ and $\delta'' : C^\infty\Lambda^{p,q} \to C^\infty\Lambda^{p,q-1}$ are the operators of the Dolbeault complex. Then $\Delta^{p,q} := 2(\bar\partial\delta'' + \delta''\bar\partial)$; the normalizing factor of 2 is present to ensure that $\Delta^{p,q}$ is an operator of Laplace type and is inessential. Since $\tau\Sigma_q(-1)^q \dim\Lambda^{0,q} = 0$, Equation (7) implies there exists $\mathcal{E}$ involving only the first and second jets of the metric so that

$$a_2(x,\bar\partial)=(4\pi)^{-m/2}\Sigma_q(-1)^q\operatorname{Tr}(E(\Delta^{0,q})),\quad\text{and}$$
$$a_4(x,\bar\partial)=-6^{-1}\Delta a_2(x,\bar\partial)+\mathcal{E}.$$

Proof of Theorem 2 if $m=4$. The $a_n(\cdot,\bar\partial)$ and $Td_{\bar m}$ are local invariants. Thus we may take $M^4=\mathbb{C}^2/\mathbb{Z}^4$ to be the torus. Let $z^1=x^1+\sqrt{-1}y^1$ and $z^2=x^2+\sqrt{-1}y^2$ and let $ds^2=e^{2\phi}dz^1\circ d\bar z^1+dz^2\circ d\bar z^2$ for $\phi=\phi(x^2)$. Fix $P_0\in M^4$. We suppose $\phi(P_0)=0$ and $d\phi(P_0)=0$. We will show that the coefficient of $(\partial/\partial x^2)^2\phi(P_0)$ in $\Sigma_q(-1)^qTr(E(\Delta^{0,q}))(P_0)$ is non-trivial so $a_2(P_0,\bar\partial)\neq 0$. This will show that $a_4(\cdot,\bar\partial)$ involves the 4-jets of ϕ. Since the Todd polynomial Td_2 does not involve the 4-jets of ϕ, we can choose ϕ so that $a_4(P_0,\bar\partial)\neq Td_2(P_0)$. Since $d\phi(P_0)=0$, we use Equation (6) to see that $E(P_0)=\{B-\frac{1}{2}\partial_\nu(A^\nu+\Gamma_{\mu\mu}{}^\nu\cdot I_V)\}(P_0)$. Since we have that $\Gamma_{\mu\mu}{}^\nu\Sigma_q(-1)^q\dim\,\Lambda^{0,q}=0$, to complete the proof of Theorem 2 when $m=4$, we must show

$$\Sigma_q(-1)^q\operatorname{Tr}\left(B-\frac{1}{2}\partial_\nu A^\nu\right)(\Delta^{0,q})(P_0)\neq 0. \tag{8}$$

We evaluate the Hodge $\star$ operator:

$$\begin{aligned}
&\star(d\bar z^1)=-\frac{1}{2}d\bar z^1\wedge dz^2\wedge d\bar z^2,\quad \star(d\bar z^2)=-\frac{1}{2}e^{2\phi}d\bar z^2\wedge dz^1\wedge d\bar z^1,\\
&\star(dz^1\wedge d\bar z^1\wedge dz^2\wedge d\bar z^2)=-4e^{-2\phi},\quad \star(d\bar z^1\wedge d\bar z^2)=d\bar z^1\wedge d\bar z^2,\\
&\star(d\bar z^1\wedge dz^2\wedge d\bar z^2)=2d\bar z^1,\quad \star(d\bar z^2\wedge dz^1\wedge d\bar z^1)=2e^{-2\phi}d\bar z^2.
\end{aligned}$$

Let $\partial_1=\partial/\partial z^1$ and $\partial_2=\partial/\partial z^2$. Since $\delta=-\star d\star$, we have that

$$\begin{aligned}
\delta(f_1d\bar z^1+f_2d\bar z^2)&=\star d\left(\frac{1}{2}f_1d\bar z^1\wedge dz^2\wedge d\bar z^2+\frac{1}{2}f_2e^{2\phi}d\bar z^2\wedge dz^1\wedge d\bar z^1\right)\\
&=\frac{1}{2}\star\{\partial_1f_1+\partial_2(f_2e^{2\phi})\}dz^1\wedge d\bar z^1\wedge dz^2\wedge d\bar z^2\\
&=-2e^{-2\phi}\{\partial_1f_1+\partial_2(f_2e^{2\phi})\},\\
\delta(fd\bar z^1\wedge d\bar z^2)&=-\star d(fd\bar z^1\wedge d\bar z^2)\\
&=-\star(\partial_1fdz^1\wedge d\bar z^1\wedge d\bar z^2+\partial_2fdz^2\wedge d\bar z^1\wedge d\bar z^2)\\
&=-2e^{-2\phi}\partial_1fd\bar z^2+2\partial_2fd\bar z^1.
\end{aligned}$$

Since $\delta''=\delta$ on $C^\infty\Lambda^{0,q}M$, we may compute:

$$\begin{aligned}
&\Delta^{0,0}f=2\delta\bar\partial f=2\delta(\bar\partial_1fd\bar z^1+\bar\partial_2fd\bar z^2)=-4e^{-2\phi}\{\partial_1\bar\partial_1f+\partial_2(e^{2\phi}\bar\partial_2f)\}\\
&=-4(e^{-2\phi}\partial_1\bar\partial_1+\partial_2\bar\partial_2+2\partial_2\phi\cdot\bar\partial_2)f,
\end{aligned}$$

$$\begin{aligned}
&\Delta^{0,1}(f_1 d\bar z^1 + f_2 d\bar z^2) = 2(\delta\bar\partial + \bar\partial\delta)(f_1 d\bar z^1 + f_2 d\bar z^2)\\
&= 2\{\delta(\bar\partial_1 f_2 - \bar\partial_2 f_1) d\bar z^1 \wedge d\bar z^2\} - 4\bar\partial\{e^{-2\phi}(\partial_1 f_1 + \partial_2(f_2 e^{2\phi}))\}\\
&= -4e^{-2\phi}\partial_1(\bar\partial_1 f_2 - \bar\partial_2 f_1) d\bar z^2 + 4\partial_2(\bar\partial_1 f_2 - \bar\partial_2 f_1) d\bar z^1\\
&\quad - 4\bar\partial_1\{e^{-2\phi}(\partial_1 f_1 + \partial_2(f_2 e^{2\phi}))\} d\bar z^1 - 4\bar\partial_2\{e^{-2\phi}(\partial_1 f_1 + \partial_2(f_2 e^{2\phi}))\} d\bar z^2,\\
&\Delta^{0,2}(f d\bar z^1 \wedge d\bar z^2) = 4\bar\partial(-e^{-2\phi}\partial_1 f d\bar z^2 + \partial_2 f d\bar z^1)\\
&= -4\{e^{-2\phi}\partial_1\bar\partial_1 - 2e^{-2\phi}\bar\partial_1\phi\cdot\partial_1 + \bar\partial_2\partial_2\} f d\bar z^1 \wedge d\bar z^2.
\end{aligned}$$

Note that $4\partial_2\bar\partial_2(e^{2\phi})(P_0) = 2(\partial/\partial x^2)^2\phi(P_0)$. We establish that Equation (8) holds and complete the proof by computing:

$$\begin{aligned}
&\mathrm{Tr}(\partial_\nu A^\nu(\Delta^{0,0}))(P_0) = 2(\partial/\partial x^2)^2\phi(P_0), \mathrm{Tr}(B(\Delta^{0,0}))(P_0) = 0,\\
&\mathrm{Tr}(\partial_\nu A^\nu(\Delta^{0,1}))(P_0) = 2(\partial/\partial x^2)^2\phi(P_0), \mathrm{Tr}(B(\Delta^{0,1}))(P_0) = 2(\partial/\partial x^2)^2\phi(P_0),\\
&\mathrm{Tr}(\partial_\nu A^\nu(\Delta^{0,2}))(P_0) = 0, \qquad \mathrm{Tr}(B(\Delta^{0,2}))(P_0) = 0.
\end{aligned}$$

□

Proof of Theorem 2 for $m > 4$. It suffices to construct a single example. Let $X^{4+2k} := M^4 \times (S^2)^k$ and let $ds_X^2 = ds_M^2 + (ds_{S^2}^2)^k$ where $ds_{S^2}^2$ is the standard homogeneous metric on the Riemann sphere S^2 and where we take the k fold product. The invariants a_n and the Todd polynomial are multiplicative with respect to such products. Let $X = M_1 \times \cdots \times M_\ell$ with a product metric. The multiplicative nature of the Dolbeault complex implies that

$$a_n(z_1, ..., z_\ell, \bar\partial) = \Sigma_{n=q_1+\cdots+q_\ell}\Pi_{1\le i\le\ell} a_{q_i}(z_i, \bar\partial). \tag{9}$$

Let $X = M^4 \times S^2 \times \cdots \times S^2$ have real dimension $4 + 2k$. Since the metric on S^2 is homogeneous, $a_q(z, \bar\partial)$ is independent of z. The arithmetic genus of S^2 is 1. Since the metric on S^2 is homogeneous and Kaehler, we use Equation (1) to see $a_2(z, \cdot) = Td_1(z)$ is a non-zero constant and $a_n(z, \cdot) = 0$ for $n \ne 2$. We use Equation (9) to see that $a_m(\vec z, \bar\partial) = a_4(z_1, \bar\partial)\cdot a_2(\cdot, \bar\partial)^k$. □

There are some partial results concerning the vanishing of the lower order terms in the heat equation trace:

10. Corollary. *Let M be a holomorphic manifold of real dimension $m \ge 4$.*
(a) *Fix $x_0 \in M$. If $m = 4k$, assume $2j \ge 2k$; if $m = 4k + 2$, assume $2j \ge 2k + 2$. Then there exists a Hermitian metric on M so that $a_{2j}(x_0, \bar\partial) \ne 0$.*

(b) *If* $3j < m$, *then* $a_{2j}(\cdot, \bar\partial) = 0$.

Proof. Suppose first $m = 4$. We use [5, Theorem 4.1.9(a)] to see

$$a_{2j}(z, \bar\partial) = (-1)^{j+1}(8j+4)/(2^{j+1}\cdot 1\cdot 3\cdot \ldots \cdot(2j+1))\Delta^{j-1}a_2(z, \bar\partial) + \mathcal{E}_{2j}, \tag{11}$$

where $\mathcal{E}_{2j}$ involves lower order jets of the metric. Since a_2 is not the zero polynomial in the jets of the metric, we can specify $a_{2j}(z_0, \bar\partial)$ arbitrarily for all $j \geq 1$; (a) now follows if $m = 4$. The general case now follows from Equation (9) by taking suitable products. Since the techniques used to prove (b) are tangential to the thrust of this note, we only sketch the proof of (b); the argument is similar to a counting argument given in [5, Theorem 2.5.2] to prove a vanishing theorem for invariants defined by the de Rham complex. Fix z_0 and normalize the metric so $ds^2(z_0) = dz^i \circ d\bar z^i$. Expand $a_{2j}(\cdot, \bar\partial)$ as a homogeneous polynomial of order $2j$ in the derivatives of the metric. Equation (9) shows $a_{2j}(\cdot, \bar\partial) \equiv 0$ on a product with a flat torus and thus $a_{2j}(\cdot, \bar\partial) \equiv 0$ if the metric is flat in one factor. Consequently if $a_{2j}(\cdot, \bar\partial) \not\equiv 0$, every holomorphic index must appear in every monomial of $a_{2j}(\cdot, \bar\partial)$. Since $a_{2j}(\cdot, \bar\partial)$ is $U(1)$ invariant, every anti-holomorphic index must appear as well. The total number of indices which can appear in a monomial of $a_{2j}(\cdot, \bar\partial)$ is $3j$. This yields the estimate $m \leq 3j$ so $a_{2j}(\cdot, \bar\partial) = 0$ if $3j < m$. □

Corollary 10 does not completely answer the question of when the invariants $a_{2j}(\cdot, \bar\partial)$ vanish; roughly speaking we have shown they vanish when $j < m/3$ and are non-zero when $j > m/2$; the range $[m/3, m/2]$ requires further study.

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