PACIFIC JOURNAL OF MATHEMATICS
Vol. 180, No. 1, 1997

THE THETA DIVISOR OF BIDEGREE $(2,2)$ THREEFOLD IN $\mathbb{P}^2 \times \mathbb{P}^2$

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Recently A.Verra proved that the existence of two conic bundle structures (c.b.s.) on the bidegree $(2,2)$ divisor in the product of two projective planes implies a new counterexample to the Torelli theorem for Prym varieties. Let $J(T)$ be the jacobian of T. In this paper we prove that any of the two c.b.s. on T induces a parametrization of the theta divisor of $J(T)$ by the Abel-Jacobi image of a special family of elliptic curves of degree 10 (minimal sections of the given c.b.s.) on T. This result is an analog of the well-known Riemann theorem for curves. Further we use once again the geometry of curves on T, in order to prove the Torelli theorem for the bidegree $(2,2)$ threefolds. In the end, we study the bidegree $(2,2)$ threefold T with one node. It is shown that in this case the classical Dixon correspondence, between the two discriminant pairs defined by T, can be represented as a composition of two 4-gonal correspondences of Donagi.

0. Introduction.

0.1. In this paper we parametrize the theta divisor Θ and the special subvariety of stable singularities of Θ, of the intermediate jacobian of the Verra threefold T - the divisor of bidegree $(2,2)$ in $\mathbb{P}^2 \times \mathbb{P}^2$. As an application we prove the Torelli theorem for Verra threefolds. The threefold T deserves special attention because of the recent observation of A.Verra that the existence of two conic bundle structures on T implies a new counterexample to the Torelli theorem for Prym varieties. Moreover, this counterexample is not related to the 4- gonal correspondence of Donagi which had covbered all the known non-trivial counterexamples.

Let X be a smooth projective threefold for which the canonical bundle K_X has no sections. Then its intermediate jacobian $J(X) = H^{2,1}(X)^* / H^3(X, \boldsymbol{Z})$ *mod. torsion* is a principally polarized abelian variety (p.p.a.v.), with a naturally defined principal polarization Θ - the theta divisor of $J(X)$. In particular $J(T)$ is p.p.a.v., since $-K_T$ is ample. Let $\mathcal{C}$ be a family of algebraically equivalent curves C on X. The Abel-Jacobi map Φ sends $\mathcal{C}$ onto a subvariety $Z = \Phi(\mathcal{C}) \subset J(X)$; in other words, the family $\mathcal{C}$ *parametrizes* the subvariety

$Z \subset J(X)$. On the contrary, let Z be a subvariety of $J(X)$. One can state the question to find a family $\mathcal{C}_Z$ of curves on X which parametrizes Z.

In this paper we prove that if $X = T$ is the Verra threefold then: **1.** The 10-dimensional family $\mathcal{C}_\theta$ of elliptic curves on T of bidegree $(3,7)$ parametrizes the 8-dimensional theta divisor $\Theta \subset J(T)$ - see Theorem 4.1. **2.** The 6-dimensional family $\mathcal{D}$ of elliptic sextics on T of bidegree $(3,3)$ parametrizes a 3-dimensional component Z of stable singularities of Θ. This makes it possible to describe the tangent cones to Θ at the points of Z, which is used to prove the Torelli theorem for the Verra threefold - see Theorem 5.6.

0.2. In the preliminaries, we state in brief the Verra's counterexample, and remember some known facts about Prym varieties and minimal sections of ruled surfaces over elliptic curves.

In Section 2 and 3, we start from the representation of the intermediate Jacobian of T as a Prym variety to translate the Wirtinger description of $(J(T), \Theta)$ in the terms of curves on T. In particular, This implies the existence of a special family $\mathcal{C}_\theta$ of curves on T, which is mapped - via the Abel-Jacobi map - onto a copy of the theta divisor $\Theta(T)$ (see Theorem 4.1). The general curve $C \in \mathcal{C}_\theta$ is an elliptic curve of bidegree $(3,7)$. By construction, the curves of $\mathcal{C}_\theta$ parametrize the minimal sections (resp. the maximal subbundles of rank 1) of a special family of ruled surfaces (resp. rank 2 vector bundles) over the space of plane cubics - see Section 2, (3.1), [16], [20]. It turns out that $\mathcal{C}_\theta$ is generically an elliptic fibration over the family of effective divisors $\operatorname{Supp}\Theta$ related to the Wirtinger description of the theta divisor - see Sect. 2, 3, 4, and (1.2.2). The last is used in the proof of Theorem 4.1.

In Section 5, we separate a 3-dimensional component $Z \subset \operatorname{Sing}\Theta$ as an Abel-Jacobi image of the 6-dimensional family $\mathcal{D}$ of eliptic sextics of bidegree $(3,3)$ on T. The connection between the geometry of T and $\operatorname{Sing}\Theta$, based on the study of the family $\mathcal{D}$, is used to prove the Torelli theorem for the Verra 3-fold T: The threefold $T \subset \mathbb{P}^8$ coincides with the intersection of the tangent cones of Θ at the points of Z - see Theorem (5.6). Note that the Torelli theorem, stated in this form, is not a direct consequence of general facts about Prym varieties: one has also to see that the projective tangent cone Cone_z (z - a general point of Z) does not belong to the "trivial" component $D_6(W)$ of the determnantal locus $D_6(T)$ - see (5.3.3), (5.5.1), and the proof of Theorem (5.6).

In his paper [24], A.Verra proves that the correspondence between the two discriminant pairs, defined by two conic bundle structures on the same bidegree $(2,2)$ divisor T, is the same as the classical *Dixon correspondence* for plane sextics. For the original description of this correspondence - see [24], or [9]. In Section 6 we show that if T has a node then the Dixon correspondence can be represented as a composition of two 4-gonal correspondences of Donagi - see Corollary (6.3).

1. Preliminaries.

1.1. *The bidegree* $(2,2)$ *divisor* T (see [24]).

(1.1.1). Let $\mathrm{seg}: \mathbb{P}^2 \times \mathbb{P}^2 \longrightarrow \mathbb{P}^8$ be the Segre embedding, and let $W \subset \mathbb{P}^8$ be the image of seg. Let $p_1 : W \to \mathbb{P}^2$ and $p_2 : W \to \mathbb{P}^2$ be the canonical projections. Denote by $\mathcal{O}_W(m,n)$ the sheaf $p_1^*\mathcal{O}_{\mathbb{P}^2}(m) \otimes p_2^*\mathcal{O}_{\mathbb{P}^2}(n)$, m, n-integers.

(1.1.2). By definition, the elements T of the linear system $|\mathcal{O}_W(2,2)|$, are called *bidegree* $(2,2)$ *divisors* on W; we let $p_i : T \to \mathbb{P}^2$ be the restrictions of the projections p_i $(i = 1,2)$, on T.

(1.1.3). Let, moreover, T be general (esp. T is smooth). Then p_i defines a standard conic bundle structure on T, $i = 1,2$. Let $i \in \{1,2\}$ be fixed, and let

$$\begin{aligned} \Delta_i &= \{x \in \mathbb{P}^2 : \mathrm{Sing}\, p_i^{-1}(x) \neq \emptyset\} \\ &= \left\{x \in \mathbb{P}^2 : p_i^{-1}(x) = l + \bar{l} \text{ is a plane conic of rank } 2\right\} \end{aligned}$$

be the *discriminant curve* of p_i. For the general T, the curves Δ_i are smooth plane sextics (see [24]).

Denote by $\tilde{\Delta}_i = \{l - \text{a line in}\, T : \exists x \in \Delta_i, \text{ s.t.}\, l \text{ is a component of}\, p_i^{-1}(x)\}$ the induced double covering of the discriminant curve Δ_i, $i = 1,2$; i.e. $\tilde{\Delta}_i$ is the *curve of components of degenerate fibers* of p_i. With a probable abuse of the notation, we denote by $p_i : \tilde{\Delta}_i \to \Delta_i$ also the corresponding covering map. Since Δ_i is smooth, the map $p_i : \tilde{\Delta}_i \to \Delta_i$ is an unbranched double covering - defined by the sheaf $\eta_i \in Pic^0_{[2]}\Delta_i - \{\mathcal{O}_{\Delta_i}\}$. (Here $Pic^0_{[2]}\Delta_i = \left\{\eta \in Pic^0\Delta_i : \eta^{\otimes 2} = \mathcal{O}_{\Delta_i}\right\}$.)

(1.1.4). Let $J(T)$ be the intermediate jacobian (see e.g. [13]) of T, and let $P_i = P(\Delta_i, \eta_i)$ be the Prym variety (see e.g. [2], [21], or [1, Ch. VI, App. C]) of (Δ_i, η_i). It is well-known that $J(T)$ and P_i are isomorphic as principally polarized abelian varieties - see [2].

In particular, $\dim J(T) = \dim P_i = 9$. It follows immediately that $P(\Delta_1, \eta_1)$ and $P(\Delta_2, \eta_2)$ are isomorphic as p.p.a.v..

(1.1.5). In [24], A.Verra proves that the discriminant pairs (Δ_1, η_1) and (Δ_2, η_2) corresponds to each other by the classical Dixon construction. Moreover, let

$$\mathcal{P}_6 = \left\{(\Delta, \eta) : \Delta \text{ is a smooth plane curve of degree 6, and } \eta \in Pic^0_{[2]}\Delta - \{\mathcal{O}_\Delta\}\right\},$$

and let $p_6 : \mathcal{P}_6 \longrightarrow \mathcal{A}_9$, $(\Delta, \eta) \mapsto$ the principally polarized Prym variety $P(\Delta, \eta)$, be the *Prym map* for $\mathcal{P}_6$ (see e.g. [10], or [24]). Then $\deg p_6 = 2$, and p_6 is branched along the locus of intermediate jacobians of *nodal quartic double solids* (:= double coverings of $\mathbb{P}^3$ branched along nodal quartics) - see [24]. The general fiber of p_6 equals to a couple of pairs $((\Delta_1, \eta_1), (\Delta_2, \eta_2))$, which arises from a bidegree $(2,2)$ divisor T.

(1.1.6). We call the smooth bidegree $(2,2)$ divisor $T \in W$ *the Verra threefold.*

1.2. The intermediate jacobian $J(T)$ as a Prym variety.

(1.2.1). Let Θ be the divisor of the principal polarization (the theta divisor) of $J(T)$. It follows from the preceeding that we can identify Θ and the theta divisors of the Prym varieties P_1 and P_2. Then, because of the Wirtinger description of Prym varieties (see [19], or [21]), we can describe $J = J(T)$, Θ, etc., only in terms of $\tilde{\Delta}_i$ and Δ_i. As a direct corollary we obtain:

(1.2.2). Let p_i be fixed. Then:

(1). The jacobian $J(T)$ is isomorphic to

$$P\left(\tilde{\Delta}_i, \Delta_i\right) = \left\{\mathcal{L} \in Pic^{18}\tilde{\Delta}_i : \operatorname{Nm}\mathcal{L} = \omega_{\Delta_i}, \text{ and } h^0(\mathcal{L}) \text{ even}\right\}.$$

(2). $\Theta(T) \cong \Theta_i = \{\mathcal{L} \in J(T) : h^0(\mathcal{L}) \geq 2\}$.

(3). There exists a subset in $\operatorname{Sing}\Theta$, which is isomorphic to the set of stable singularities of Θ, with respect to p_i, i.e. $\operatorname{Sing}_i^{st}\Theta = \{\mathcal{L} \in \Theta : h^0(\mathcal{L}) \geq 4\}$; similarily - for the exceptional singularities of Θ, w.r.t. to p_i.

In Section 5 we shall describe a component Z of $\operatorname{Sing}\Theta$, the points of which are stable w.r.t. to both p_1 and p_2.

1.3. Minimal sections of ruled surfaces over elliptic curves.

(1.3.0). Here we collect some facts about ruled surfaces (esp. - on elliptic curves), which will be used in Section 2 - see [14, Ch.V, Sect. 2], [16], [20].

(1.3.1). Let C be a smooth curve. By definition, a ruled surface $p : S \to C$ is a surface which can be represented in the form $S = \mathbb{P}_C(\mathcal{E})$, where $\mathcal{E}$ is a locally free sheaf of rank 2 (a rank 2 vector bundle) over C. The representation $S = \mathbb{P}_C(\mathcal{E})$ is unique up to multiplication by an invertible sheaf: $\mathbb{P}_C(\mathcal{E}) \cong \mathbb{P}_C(\mathcal{E} \otimes \mathcal{L})$, $\mathcal{L} \in Pic\, C$.

The bundle $\mathcal{E}$ is called *normalized* if $h^0(\mathcal{E}) \neq 0$, but $h^0(\mathcal{E} \otimes \mathcal{L}) = 0$ for any invertible $\mathcal{L}$ such that $\deg \mathcal{L} < 0$.

(1.3.2). Any ruled surface S has a representation $S = \mathbb{P}_C(\mathcal{E})$, for some normalized $\mathcal{E}$. Such a representation is, in general, far from unique (see e.g. [16, Cor. 3.2], or (1.3.4) - in case $g(C) = 1$).

Let $\mathcal{E}$ be normalized, and let $C_0 \in \left|\mathcal{O}_{\mathbb{P}(\mathcal{E})/C}(1)\right|$ be a tautological section for $\mathcal{E}$. The invariant property of C_0 is that it is section of $p : S \to C$, for which the number

$$-e(C) = (C \cdot C)_S, \ C - \text{a section of } S$$

is minimal, i.e. C_0 is a *minimal section* of S. The number $e = e(S) = -(C_0 \cdot C_0)$ is called the *invariant* of S.

(1.3.3). The surface $S = \mathbb{P}(\mathcal{E})$ is called *decomposable* if $\mathcal{E}$ is decomposable - see e.g. [14, Ch.V, Sect. 2]. Otherwise, S is called *indecomposable* (ibid).

(1.3.4). The cardinality of the set of minimal sections of S closely depends on the decomposability of S, and on the parity of the invariant $e = e(S)$ (see [16, Cor. 3.2]).

In Section 3, we shall use the description of these sets only in case $g(C)$.

(*). *Minimal sections of a ruled surface over an elliptic curve.*

All of the next can be found in [14, Ch.V, Sect. 2]:

Let C be an elliptic curve, and let S be a ruled surface on C. Then one of the following alternatives is valid:

(1). S is decomposable, $e = e(S) > 0$. The normalized sheaf $\mathcal{E}$ for S is unique, and the minimal section C_0 is unique.

(2). S is decomposable, $e = e(S) = 0$. In this case $S = \mathbb{P}_C(\mathcal{O} \oplus \varepsilon)$, $\deg \varepsilon = 0$, and *either*

- **(a).** $\varepsilon = \mathcal{O}_C$, $S = C \times \mathbb{P}^1$, and the set of minimal sections of S is parametrized by the points of $\mathbb{P}^1$, *or*
- **(b).** $\varepsilon \neq \mathcal{O}_C$. Then the normalized sheaf $\mathcal{E}$ can be chosen in two ways: $\mathcal{E}^+ = \mathcal{E}$ and $\mathcal{E}^- = \mathbb{P}(\mathcal{O} \oplus (-\varepsilon))$, where $-\varepsilon = \varepsilon^{\otimes -1}$. Correspondingly, there are exactly two minimal sections of $S : C^+$ and C^- (each - the unique tautological section of the corresponding normalized bundle).

(3). S is unique indecomposable ruled surface, s.t. $e(S) = 0$. Then the normalized sheaf for S is unique, and the corresponding minimal section is unique.

(4). S is unique indecomposable ruled surface, s.t. $e(S) = -1$. Then the set $\{\mathcal{E} - \mathit{normalized} : S = \mathbb{P}(\mathcal{E})\}$ is parametrized by the points of the elliptic curve C.

2. Minimal sections of the canonical conic bundle surfaces.

2.0. Everywhere in this section the conic bundle structure $p : T \to \mathbb{P}^2$ is fixed; we let $p = p_1$, $\Delta = \Delta_1$, $\eta = \eta_1$, etc.

2.1. The sets Supp Θ and Supp P^-.

(2.1.1). Let $p = p_1 : T \to \mathbb{P}^2$, etc., be as above. Let $\mathrm{Nm} : Pic^{18}\tilde{\Delta} \to Pic^{18}\Delta$ be the norm map - see [1, Ch.V, App. C]. Then $\mathrm{Nm}^{-1}(\omega_\Delta)$ splits into two components:

$P^+ = \left\{\mathcal{L} \in \mathrm{Nm}^{-1}(\omega_\Delta) : h^0(\mathcal{L}) \text{ even}\right\}$ and $P^- =$ the same, but $h^0(\mathcal{L})$ odd.

(2.1.2). The general sheaf $\mathcal{L} \in P^+$ is non-effective, i.e. the linear system of effective divisors $|\mathcal{L}|$ is empty. However, the subset of the effective sheaves $\mathcal{L} \in P^+$ is exactly the theta divisor Θ - see (1.2.2). This gives a reason to define a set

$$\operatorname{Supp}\Theta := \{L \in |\mathcal{L}| : \mathcal{L} \in \Theta\},$$

i.e., $\operatorname{Supp}\Theta$ is the set of all effective divisors in the linear systems of the sheaves $\mathcal{L} \in \Theta$. Similarly

$$\operatorname{Supp} P^- := \left\{L \in |\mathcal{L}| : \mathcal{L} \in \ P^-\right\} \text{ (all the sheaves } \mathcal{L} \in P^- \text{ are effective).}$$

(2.1.3). *The maps* $p_*^+ : \operatorname{Supp}\Theta \to |\mathcal{O}_{\mathbb{P}^2}(3)|$ *and* $p_*^- : \operatorname{Supp} P^- \to |\mathcal{O}_{\mathbb{P}^2}(3)|$.

The set $\Theta \cup P^-$ coincides with the set of all "effective" sheaves in the preimage $\operatorname{Nm}^{-1}(\omega_\Delta)$. Moreover, on the level of effective divisors, the map Nm coincides with the usual projection $p_* : \operatorname{Symm}^{18}\tilde{\Delta} \to \operatorname{Symm}^{18}\Delta$. In particular, if $\mathcal{L} \in \operatorname{Nm}^{-1}\omega$ is effective and $L \in |\mathcal{L}|$, then $\mathcal{O}(p_*L) = \operatorname{Nm}\mathcal{L} = \omega_\Delta = \mathcal{O}_\Delta(3)$. Since $\deg\Delta = 6 > 3$, the linear system $|\mathcal{O}_\Delta(3)|$ is isomorphic to $|\mathcal{O}_{\mathbb{P}^2}(3)|$. In particular, the effective divisor $p_*L \in \operatorname{Symm}^{18}\Delta$ is an intersection of Δ with the unique plane cubic curve $C(L) := p_*(L)$. In particular, after composing with the corresponding restriction maps, the map p_* defines the maps

$$p_*^+ : \operatorname{Supp}\Theta \to |\mathcal{O}_{\mathbb{P}^2}(3)| \,(= \text{ the set of all the plane cubics}), \text{and}$$

$$p_*^- : \operatorname{Supp} P^- \to |\mathcal{O}_{\mathbb{P}^2}(3)|.$$

According to [21, Lemma 3.20]:

(1) the maps p_*^+ and p_*^- are surjective;

(2) all fibers of p_*^+ and p_*^- are finite.

2.2. The canonical conic bundle surface $S(C)$ and the preimage $p_*^{-1}(C)$.

(2.2.1). Let C be a sufficiently general plane cubic curve. In particular, C can be supposed to be smooth, and intersecting the discriminant sextic Δ in 18 distinct points $x_1, \ldots, x_{18}$.

The surface $S(C) = p^{-1}(C) \subset T$ is a standard conic bundle over the elliptic curve C; the degenerate fibers of $p : S(C) \to C$ are $f_i = p^{-1}(x_i) = l_i + \overline{l_i},\ i = 1, \ldots, 18$.

We call a surface $p^{-1}(C) \subset T$, C - any plane cubic (resp. C - a general plane cubic), a *canonical conic surface* on T (resp. - a *general c.c.b.s.*) - w.r. to $p = p_1$.

(2.2.2). *The set* $\Sigma(C)$

Let the cubic C be as in (2.2.1), and let

$$\Sigma(C) = \left\{\sigma : \cup\{x_i\} \to \cup\{l_i, \overline{l_i}\} : \sigma(x_i) \in \{l_i, \overline{l_i}\},\ i = 1, \ldots, 18\right\}$$

be the set of "choice" maps for C. We can define, in an obvious way, the two-argument signature function $\mathrm{sgn}: \Sigma(C) \times \Sigma(C) \to \{+1,-1\}$ as follows: $\mathrm{sgn}\,(\sigma', \sigma'') = +1$, if $\#\,(\mathrm{Image}\,\sigma' \cap \mathrm{Image}\,\sigma'') \in 2\boldsymbol{Z}$, otherwise $\mathrm{sgn}\,(\sigma', \sigma'') = -1$.

(2.2.3). *The map* $L : \Sigma(C) \to p_*^{-1}(C)$.

Let $\sigma \in \Sigma(C)$. The map σ defines the effective divisor $L(\sigma) = \sigma(x_1) + \cdots + \sigma(x_{18})$. Clearly $L(\sigma) \in p_*^{-1}(C) \in \mathrm{Supp}(\Theta) \cup \mathrm{Supp}(P^-)$.

(2.2.4). *The sets* $\Sigma_\Theta(C)$ *and* $\Sigma_{P^-}(C)$.

Lemma. *Let* C *be as above. Then*

(1) *The preimage* $p_*^{-1}(C)$ *coincides with the union of the disjoint sets - each of cardinality* 2^{17} :
$\Sigma_\Theta(C) = \{\sigma \in \Sigma(C) : L(\sigma) \in \mathrm{Supp}\,\Theta\}$ *and*
$\Sigma_{P^-}(C) = \{\sigma \in \Sigma(C) : L(\sigma) \in \mathrm{Supp}\,P^-\}$.

(2) $\mathrm{sgn}\,(\sigma', \sigma'') = +1$ *iff both* σ' *and* σ'' *belong to one of these two sets; otherwise* $\mathrm{sgn} = -1$.

Proof. Let $L\,(\sigma')$ and $L\,(\sigma'')$ be two elements of $p_*^{-1}(C)$. The divisors $L\,(\sigma')$ and $L\,(\sigma'')$ are obtained from each other by a finite number of replacements of the type: $L \mapsto L + l - \bar{l}$, where $l + \bar{l} = p^{-1}(x)$ for some $x \in \Delta$. In this case $x \in \{x_1, \ldots, x_{18}\}$, and $L\,(\sigma')$ and $L\,(\sigma'')$ are effective. Moreover, $L\,(\sigma')$ and $L\,(\sigma'')$ can be regarded as *general* elements of $\mathrm{Supp}\,\Theta \cup \mathrm{Supp}\,P^-$ (the cubic C is general). Therefore $h^0\,(L\,(\sigma'))$ and $h^0\,(L\,(\sigma''))$ can be only 1 or 2 (see e.g. [26]). Now, the lemma is a direct consequence of the following statement:

(2.2.5). (*) Let $\tilde{\Delta}$ be a smooth curve with the involution $l \leftrightarrow \bar{l}$ without fixed points. Let L be an effective (see (2.1)) invertible sheaf on $\tilde{\Delta}$, and let $l \in \tilde{\Delta}$. Then

$h^0(L) - h^0\left(L + l - \bar{l}\right) \in \{+1,-1\}$ - see e.g. [21, 3.14], where $(*)$ has been proved under more general conditions.

□

(2.2.6). Let $U = \{C$- a smooth plane cubic : $C \cap \Delta = 18$ distinct points$\}$, let $C \in U$, and let $\mathrm{Supp}\,\Theta(C) = \{L(\sigma) : \sigma \in \Sigma_\Theta(C)\}$. Let $\mathrm{Supp}\,\Theta^U = \cup\,\{\mathrm{Supp}\,\Theta(C) : C \in U\}$. Then the algebraic set $\mathrm{Supp}\,\Theta \subset \mathrm{Symm}^{18}\,\tilde{\Delta}$ is, in an obvious way, a closure of the open subset $\mathrm{Supp}\,\Theta^U$. The same (up to replacing of the notation) is true also for $\mathrm{Supp}\,P^-$ and $\mathrm{Supp}(P^-)^U$.

2.3. The global invariants $e(\Theta)$ and $e\,(P^-)$.

(2.3.1). *The maps* $\tilde{\sigma} : S(C) \to S(L(\sigma))$.

Let $C \in U$ be as in (2.2.5), and let $\sigma \in \Sigma(C)$ - see (2.2). Without any restriction we may assume that $L(\sigma) = l_1 + \cdots + l_{18}$. The lines l_i and $\overline{l_i}$ are

(-1)-curves on the surface $S(C) = p^{-1}(C)$. Therefore σ defines, in a unique way, a morphism $\tilde{\sigma} : S(C) \to S(L(\sigma))$, where $\tilde{\sigma}$ is the blow-down of the 18-tuple $\left\{\overline{l_1}, \dots, \overline{l_{18}}\right\}$.

(2.3.2). *The invariants $e(\Theta)$ and $e(P^-)$.*

Let $e(L(\sigma))$ be the invariant of the ruled surface $p(\sigma) : S(L(\sigma)) \to C$ (see (1.3.2)). This way, we define a map $\tilde{e} : \operatorname{Supp}\Theta^U \cup \operatorname{Supp}(P^-)^U \to \boldsymbol{Z}$, $\tilde{e} : L(\sigma) \mapsto e(L(\sigma))$ - see (2.2.5).

It is standard that the map $\tilde{e}$ must take a constant value on some open subset of each of the components of this domain. Therefore there exists a pair $(e(\Theta), e(P^-))$, and a pair of (possibly smaller) open subsets $\operatorname{Supp}\Theta^{op}$ and $\operatorname{Supp}(P^-)^{op}$, such that $\tilde{e}(l) = e(\Theta)$ for any $L \in \operatorname{Supp}\Theta^{op}$, and $\tilde{e}(L) = e(P^-)$ for any $L \in \operatorname{Supp}(P^-)^{op}$.

We shall find these two numbers.

2.4. $e(\Theta) = -1,\ e(P^-) = 0$.

(2.4.0). *Elementary transformations of ruled surfaces.*

Let z' be a point on the ruled surface S' over the curve C, and let $f' \subset S'$ be the fiber of S' through z'. Let $\sigma' : \tilde{S} \to S'$ be the blow-up of $z' \in S'$, let $e \subset \tilde{S}$ be the exceptional (-1)-curve of σ'. and let $f \subset \tilde{S}$ be the proper σ'-preimage of f. Then f is also a (-1)-curve on $\tilde{S}$, and one can blow-down $f \subset \tilde{S}$: $\sigma'' : \tilde{S} \to S''$. Clearly, S'' is also a ruled surface over C; moreover, S'' is birational to S' - via the birational isomorphism $\operatorname{elm}_{z'} = \sigma'' \circ (\sigma')^{-1} : S' \to S''$. We call $\operatorname{elm}_{z'}$ *the elementary transformation of S' centered at the point $z' \in S'$*. In particular, let $z'' = \sigma''(f) \in S''$ be the σ''-image of f. Then the inverse map of $\operatorname{elm}_{z'} : S' \to S''$ coincides with the elementary transformation $\operatorname{elm}_{z''} : S'' \to S'$ centered at the point $z'' \in S''$.

(2.4.1).

Lemma. *Let $e = e(\Theta)$ and $\overline{e} = e(P^-)$ be as in (2.3.2). Then $|e - \overline{e}| = 1$.*

Proof. Let $C \in U$ (see (2.2.5)) be such that $L(\sigma) \in \operatorname{Supp}\Theta^{op} \cup \operatorname{Supp}(P^-)^{op}$, for any $\sigma \in \Sigma(C)$ (see (2.2), (2.3)). In particular, $e(S(L(\sigma))) = e$ for any $L(\sigma) \in \operatorname{Supp}\Theta(C)$, and $e(S(L(\sigma))) = \overline{e}$ for any $L(\sigma) \in \operatorname{Supp} P^-(C)$. Let $x_i, l_i, \overline{l_i}$, etc. be as in (2.2), (2.3), and let σ' and σ'' be such that $\sigma'(x_i) = \sigma''(x_i)$ for any i, except $i = j$, j - *fixed*. Let $z_j = l_j \cap \overline{l_j}$. Since the natural maps $S(C) \to S(L(\sigma'))$ and $S(C) \to S(L(\sigma''))$ are regular, the point $z_j \in S(C)$ has unique images z_j' on $S(L(\sigma'))$ and z_j'' on $S(L(\sigma''))$. It follows from the definition of $S(L(\sigma'))$ and $S(L(\sigma''))$ that $S(L(\sigma''))$ is obtained from $S(L(\sigma'))$ by the elementary transformation $\operatorname{elm}_{z_j'} : S(L(\sigma')) \to S(L(\sigma''))$ centered at the point $z_j' \in S(L(\sigma'))$. Similarily, $S(L(\sigma'')) = \operatorname{elm}_{z_j''}(S(L(\sigma'')))$.

Now, (2.4.1) is a consequence from the following:

Sublemma (*). (*See* [16, Lemma 4.3], *or* [20, Lemma 7]). *Let* $S' \to C$ *and* $S'' \to C$ *be two ruled surfaces over the smooth base curve* C, *and let* $S'' = \mathrm{elm}_P(S')$, *where* elm_P *denotes the elementary transformation of* S' *- centered at the point* $P \in S'$. *Then:*

(*i*) *If no minimal sections of* S' (*see* (1.3.4)) *passes through* P, *then* $e(S'') = e(S') - 1$;

(*ii*) *If the minimal section of* S' *passes through* P, *then* $e(S'') = e(S') + 1$. □

(2.4.2).

Lemma. $e = e(\Theta) = -1$.

Proof. Let $\mathcal{L} \in \Theta$ be general, and let $|\mathcal{L}| = \{L(t), t \in \mathbb{P}^1\}$ be the linear system of $\mathcal{L}$ - see (1.2.2). Just as in (2.1.3), the pencil $\{L(t)\}$ defines the family of plane cubics $\{C(t) = C(L(t))\}_{t\in\mathbb{P}^1}$. Since $\mathcal{L} \in \Theta$ is general, the general curve $C(t)$ of the 1-dimensional family $\{C(t)\}$ is a smooth plane cubic, and the only degenerations of $\{C(t)\}$ are a finite number of nodal plane cubics. Moreover, there are a finite number of plane cubics $C(t) \in \{C(t)\}$, which are simply tangent to Δ. There is an open subset $V_{\mathcal{L}} \subset \mathbb{P}^1$ such that $L(\sigma) \in \mathrm{Supp}\,\Theta^{op} \cup \mathrm{Supp}\,(P^-)^{op}$ for any $t \in V_{\mathcal{L}}$ and for any $\sigma \in \Sigma\,(C(t))$ (see (2.2.2) $-$ (2.2.4)).

Let $L(t) = l_1(t) + \cdots + l_{18}(t),\ t \in V_{\mathcal{L}}$. We shall prove the following:

(*) Sublemma. *If the bidegree* (2.2) *divisor* T *is general, then the general* $S(L)$, $L \in \mathrm{Supp}\,\Theta \cup \mathrm{Supp}\,P^-$ *cannot be of type* (1.3.4)(1), (1.3.4)(2.*a*) *or of type* (1.3.4)(3).

Proof. On the one hand, the choice of $C(t)$ implies that $C(t)$ can be identified with the *general* plane cubic; in particular, it is in general position w.r. to the discriminant sextic Δ. The various minimal ruled models of the surface $S(C)$ are obtained from each other by elementary transformations, related to the 18-tuple of degenerate fibers $\{l_i + \bar{l}_i = p^{-1}(x_i),\ i = \ldots, 18\}$. On the other hand, we can fix for a moment the smooth plane cubic C. Since the general plane sextic can be represented as a discriminant of (one of the conic bundle structures of) some bidegree $(2,2)$ divisor T (see [24]), we can choose the 18-tuple $\{x_1, \ldots, x_{18}\}$ without any closed restrictions. The rest repeats the proof of [20, Lemma 12]. □

It follows from $(*)$, and from description (1.3.4)$(*)$, that $e(\Theta)$ can be either 0 or -1. It remains to be seen that the assumption $e(\Theta) = 0$ leads to a contradiction. Indeed, let $e(\Theta) = 0$. Then the general surface $S(L(t))$ is of type (1.3.4)(2.b). In particular, the general $S(L), L \in \operatorname{Supp}\Theta$ has exactly two minimal sections.

One can define $\mathcal{C}$ to be the set of all these minimal sections (related to the component $\operatorname{Supp}\Theta$). Clearly, the set $\mathcal{C}$ is 9-dimensional. The general curve $C \in \mathcal{C}$ is mapped, via $p = p_1$, isomorphically onto a smooth plane cubic. Therefore $\deg C = (3, d)$, where deg is the bidegree map. A straightforward check, based on the normal bundle sequence for the triple $C \subset p^{-1}(p(C)) \subset T$, implies that the total degree of $C \in$ (the 9-dimensional family)$\mathcal{C}$ is 9 (see also [27]). Therefore $d = 6$.

The general pencil $\{L(t)\} \in \Theta$ defines a pencil of pairs $(C^+(t), C^-(t)) \subset \bar{\mathcal{C}}$; the curves $C^+(t)$ and $C^-(t)$ are the minimal sections of the surface $S(L(t))$, $t \in V_{\mathcal{L}}$ - see above.

Let B be the base of the 1-dimensional family of minimal sections

$$\mathcal{C}_{\mathcal{L}} = \left\{C : \exists t \in \mathbb{P}^1 \text{ s.t. } C = C^+(t), \text{ or } C = C^-(t)\right\},$$

defined by the pencil $|\mathcal{L}| = \{L(t)\}$. The curve B is a 2-sheeted covering of the base $\mathbb{P}^1$ of the pencil $|\mathcal{L}|$. We shall see that the curve B cannot be irreducible.

Let $S_{\mathcal{L}}$ be the union of the curves $C \in \mathcal{C}_{\mathcal{L}}$. Clearly, $S_{\mathcal{L}}$ is an effective divisor on T (the curves, which sweep $S_{\mathcal{L}}$ out, form a 1-dimensional algebraic family parametrized by the algebraic curve B).

The surface $S_{\mathcal{L}}$ can be reducible, or not. The irreducible components of this surface correspond to the irreducible components of the base $B = B_{\mathcal{L}}$.

Let B_0 be one of these irreducible components, let $\mathcal{C}_0 \to B_0$ be the corresponding irreducible family, and let S_0 be the corresponding irreducible components of $S_{\mathcal{L}}$. The surface S_0 represents the element

$$\operatorname{cl}(S_0) \in Pic\,T = \boldsymbol{Z}\cdot l + \boldsymbol{Z}\cdot h, \text{ where } l = \operatorname{cl}\left(p_1^*\mathcal{O}_{\mathbb{P}^2}(1)\right),\ h = \operatorname{cl}\left(p_2^*\mathcal{O}_{\mathbb{P}^2}(1)\right),\ p_1 = p.$$

Therefore $\operatorname{cl}(S_0) = al + bh$ for some integers a, b.

(**) Sublemma. *The integers a and b are non-negative.*

Proof. Let, for example, $b < 0$. Let f be the general fiber of $p = p_1 : T \to \mathbb{P}^2$. Since $S_0 \in |al+bh|$ is effective and f is not a fixed curve on T, the intersection number $(f \cdot S_0)_T$ must be non-negative. Therefore $0 \le (f \cdot (al + bh))_T = a(f \cdot l)_T + b(f \cdot h)_T = b(l^2 \cdot h)_T = 2b < 0$ - contradiction. ∎

Let t_0 be a sufficiently general fixed value of the rational parameter t. Let $\{C^+(t_0), C^-(t_0)\}$ be the pair of minimal sections of $S(L(t_0))$, and let e.g. $C^+(t_0) \in B_0$. Then, one of the following two alternatives is possible:

(1). $C^-(t_0) \in B_0$, and then $B_0 = B$;

(2). $C^-(t_0) \notin B_0$, and then $B = B_0 \cup B_1$, where $B_0 \cong B_1 \cong \mathbb{P}^1$.

Let $l \in \tilde{\Delta}$ be general. Since $\mathcal{L} \in \Theta$ is general, then $h^0\left(\tilde{\Delta}, \mathcal{L}\right) = 2$, and $h^0\left(\tilde{\Delta}, \mathcal{L} \otimes \mathcal{O}_{\tilde{\Delta}}(-l)\right) = 1$, i.e. there exists exactly one t such that l is a fiber of $S(L(t))$. In other words, $C^+(t)$ and $C^-(t)$ are the only two curves, of the corresponding family of minimal sections, which intersect l. In particular, $l \cdot S_0 = 2$ if $B_0 = B$ (Case (1)), and $l \cdot B_0 = 1$ if $B_0 \neq B$ (Case (2)).

Consider the Case (1). Then $\mathrm{cl}(S_0) = al + 2h$ for some $a \geq 0$. Since $C^+(t_0)$ and $C^-(t_0)$ do not intersect each other, and belong to the same algebraic family, $(C(\xi), C(\xi)) = 0$ for $\xi \in B$. Therefore $0 = \deg K_{C(\xi)} = (K_{S_0} + C(\xi)) \cdot C(\xi) = ((a-1)l + h) \cdot C(\xi) + 0 = 3(a-1) + 6 \geq 3$ - contradiction.

Therefore $B = B_0 \cup B_1$, $B_0 \cong B_1 \cong \mathbb{P}^1$, and the local splitting $(C^+(t_0), C^-(t_0))$ induces the global splitting $\{C^+(t)\} \cup \{C^-(t)\}$.

The assumption $C^+(t_0) \in B_0$ implies that $B_0 = \{C^+(t)\}$. Therefore $S_0 = \cup \{C^+(t) : t \in B_0 \cong \mathbb{P}^1\}$ is a pencil of elliptic curves, and $\mathrm{cl}(S_0) = al + h$, for some nonnegative integer a.

Let $a = 0$, i.e. $C^+(t) \subset S_0 = S_h = p_2^{-1}(h)$ for some line $h \subset \mathbb{P}^2$. Moreover, p_1 maps $C^+(t)$ isomorphically onto a plane cubic $C_3(t) = p_1(C^+(t))$, i.e. $C^+(t)$ lies on the surface $S_{3l}(t) = p_1^{-1}(C_3(t))$ of class $\mathrm{cl}(S_{3l}(t)) = 3l$. Therefore $C^+(t)$ must be a component of the effective 1-cycle $C_{3l \cdot h}(t) = S_{3l}(t) \cdot S_h$. Let $\bar{C}(t) = C_{3l \cdot h} - C^+(t)$ be the residue component. We shall compute the bidegree $\deg \bar{C}(t)$. On the one hand, $\deg C_{3l \cdot h}(t) = ((3l \cdot h) \cdot l, (3l \cdot h) \cdot h) = (3l^2 \cdot h \cdot 2(l+h), 3l \cdot h^2 \cdot 2(l+h))_{\mathbb{P}^2 \times \mathbb{P}^2} = (6, 6)$. On the other hand, $\deg C^+(t) = (3, 6)$ (see e.g. the proof of Sublemma (*)). Therefore $\deg \bar{C}(t) = (6,6) - (3,6) = (3,0)$, and $\bar{C}(t)$ must be a sum of fibers and components of fibers of p_2. Since $p_2 : T \to \mathbb{P}^2$ is a conic bundle, the irreducible components of $\bar{C}(t)$ are $(2,0)$-conics and $(1,0)$-lines. Therefore $p_1\left(\bar{C}(t)\right) \subset \mathbb{P}^2$ must be the sum of conics and lines, of total degree 3. Moreover, $\bar{C}(t) \subset p^{-1}\left(C_3(t)\right)$, i.e. all these conics and lines must lie on the cubic $C_3(t)$ - for any $t \in \mathbb{P}^1$. However, for general $t \in \mathbb{P}^1$ the cubic curve $C_3(t) = p_1\left(C^+(t)\right)$ is smooth (since $\mathcal{L} \in \Theta$ is supposed to be general). Therefore the general $C^+(t)$ cannot contain conics or lines - contradiction. Therefore $a > 0$.

Let $k = (C^+(t), C^+(t))_{S_0}$. Obviously $k \geq 0$, and (just as above): $0 = \deg K_{C^+(t)} = (a-1)l \cdot C^+(t) + k \geq k \Rightarrow k = 0,\ a = 1$. Therefore $\mathrm{cl}(S_0) = l + h$, i.e. $S_0 \subset T$ is a hyperplane section of T.

It follows from the preceding that if $e(\Theta) = 0$, $L(t) \in \mathrm{Supp}\,\Theta$ is general, and $C^+(t)$ is any of the two minimal sections defined by L, then $C^+(t)$ is a smooth elliptic curve of bidegree $(3, 6)$, such that:

(a). $C^+(t)$ lies in a hyperplane section S_0 of T.

Let $\mathcal{C}_{3,6}(T)$ be the family of effective 1-cycles on T, of bidegree $(3, 6)$. Let $\bar{\mathcal{C}}$ be the closure, in $\mathcal{C}_{3,6}(T)$, of the family $\mathcal{C}$ of "general" minimal sections

$C^+(t)$ - see above, and also - the proof of Sublemma $(*)$. This way, we have seen that the general element $C^+(t)$ of $\bar{\mathcal{C}}$ lies in a hyperplane section of T. Since (a) is a closed condition on the families of 1-cycles on T, the last implies:

(b). Any $C \in \bar{\mathcal{C}}$ lies in a hyperplane section of T.

We shall prove that (b) is impossible. In order to see this, it is enough to find an element of $\bar{\mathcal{C}}$ which does not lie in a hyperplane $\mathbb{P}^7 \subset \mathbb{P}^8 = \operatorname{Span} T$.

Let $C_{3,3}$ be a general bidegree $(3,3)$ curve of genus 1 on T - see (5.1).

In particular, $C_{3,3}$ is a smooth elliptic curve on T, $\operatorname{Span} C = \mathbb{P}^5 \subset \mathbb{P}^8$ (see (5.3.1) or the proof of (5.2)$(*)$), and $p = p_1$ project $C_{3,3}$ isomorphically onto a smooth plane cubic C_3, which intersects the discriminant Δ in 18 distinct points x_i, $i = 1, \ldots, 18$. Let $l_i \subset p^{-1}(x_i)$, $i = 1, \ldots, 18$, be the eighteen $(0,1)$-lines which intersect $C_{3,3}$. One can always choose one of these 18 lines, l_1, such that $\operatorname{Span}(C_{3,3} + l_1) = \mathbb{P}^6$. In fact, by the general choice of $C_{3,3}$, the general $\mathbb{P}^6 \supset C_{3,3}$, intersects T in the 1-cycle $C_{3,3} + \bar{C}_{3,3}$, where $\bar{C}_{3,3}$ is also a smooth elliptic curve of bidegree $(3,3)$. Therefore the general $\mathbb{P}^6 \supset C_{3,3}$ does not contain lines $l \subset T$; in particular $\mathbb{P}^5 = \operatorname{Span} C_{3,3}$ does not contain lines $l \subset T$.

Let $\mathbb{P}^6 := \operatorname{Span}(C_{3,3} + l_1)$. Clearly, the intersection $C(\mathbb{P}^6) = T \cap \mathbb{P}^6$ is an effective 1-cycle on T of bidegree $(6,6)$, containing the cycle $C_{3,3} + l_1$. Since $\deg(C_{3,3} + l_1) = (3,4)$, $C(\mathbb{P}^6)$ cannot contain more than 2 lines of bidegree $(0,1)$ - different from l_1. Therefore, one can always choose a line, say l_2, $\in \{l_2, \ldots, l_{18}\}$ (=a set of cardinality > 2) such that $\operatorname{Span}(C_{3,3} + l_1 + l_2) = \mathbb{P}^7$. Let $S = T \cap \mathbb{P}^7$ be the hyperplane section of T defined by $\mathbb{P}^7$.

Let $\mathbb{P}^6$ be a general hyperplane in $\mathbb{P}^7$; in particular, we may assume that the 1-cycle $C(\mathbb{P}^6) = T \cap \mathbb{P}^6 = S \cap \mathbb{P}^6$ is reduced. Remember that $C(\mathbb{P}^6)$ is a (singular) canonical curve of $p_a = 7$; moreover the curve $C_{3,3}$ is an elliptic component of $C(\mathbb{P}^6)$, and of degree 6. Therefore the effective 1-cycle $\bar{C} = C(\mathbb{P}^6) - C_{3,3}$, being a component of the canonical curve $C(\mathbb{P}^6)$, is a (possibly singular) curve of $p_a = 1$, of degree 6, and $\dim \operatorname{Span} \bar{C} = 5$. Denote by $\bar{\mathbb{P}}^5$ the 5-space $\operatorname{Span} \bar{C}$, and let $\{\mathbb{P}^6(t) : t \in \mathbb{P}^1\}$ be the pencil of codimension 1 subspaces of $\mathbb{P}^7$ through $\bar{\mathbb{P}}^5$. If $t \in \mathbb{P}^1$ is general then - just as above - the effective 1-cycle $C_{3,3}(t) := S \cap \mathbb{P}^6(t) - \bar{C}$ is of $p_a = 1$ and of bidegree $(3,3)$. Since $\mathbb{P}^6 \supset \bar{\mathbb{P}}^5$, $\mathbb{P}^6 = \mathbb{P}^6(t_0)$ for some $t_0 \in \mathbb{P}^1$. Therefore the curve $C_{3,3}$ moves in the elliptic pencil $\{C_{3,3}(t)\}$ on S, and $C_{3,3} = C_{3,3}(t_0)$. The rationally equivalent curves $C_{3,3}(t) \subset S$ are numerically equivalent to each other; in particular, the general element $C_{3,3}(t)$ of $\{C_{3,3}(t)\}$ is - just like $C_{3,3} = C_{3,3}(t_0)$ - an elliptic curve of bidegree $(3,3)$ which intersects the lines l_1 and l_2.

Clearly, the general $(0,1)$-line $l \subset T$ does not lie on S. Let l be such a line. Then l intersects the hyperplane section S in a single point $x = l \cap S$. Since the curves $C_{3,3}(t)$ sweep S out, there exists $C_{3,3}(t')$ such that $x \in C_{3,3}(t')$.

This way, we obtained the connected effective 1-cycle $C = C_{3,3}(t') + l_1 + l_2 + l$ such that:

(1). $C_{3,3}(t') \in \mathcal{D} =$ the closure of the family of elliptic curves of bidegree $(3,3)$ on T - see (5.1);

(2). The $(0,1)$-lines l_1, l_2 and l intersect $C_{3,3}(t_0)$;

(3). $\mathrm{Span}\,(C_{3,3}(t') + l_1 + l_2 + l) = \mathbb{P}^8$.

Since (3) is an open condition on the family

$$\Sigma_{3,6} = \{C_{3,3}+l_1+l_2+l : C_{3,3} \in \mathcal{D},\ l_1, l_2, l \text{ are} (0,1)-\text{lines on } T \text{ intersecting } C_{3,3}\},$$

then (3) remains valid for the general $C_{3,3} \in \mathcal{D}$, and for the general (hence - any) triple (l_1, l_2, l_3) of $(0,1)$-lines intersecting $C_{3,3}$.

Turning back to our initial (general) $C_{3,3}$, we conclude that $\mathrm{Span}\,(C_{3,3} + l_1 + l_2 + l_3) = \mathbb{P}^8$ - independently of the choice of the triple (l_1, l_2, l_3) of distinct $(0,1)$-lines intersecting $C_{3,3}$.

It remains to see that $C = C_{3,3} + l_1 + l_2 + l_3 \in \bar{\mathcal{C}}$.

Let, as above, $l_1, \dots, l_{18}$ be the $(0,1)$-lines which intersect $C_{3,3}$, let $\bar{l}_i = p^{-1}\,(p\,(l_i)) - l_i$, $i = 1, \dots, 18$ be their residue $(0,1)$-lines, let $L = l_1 + \dots + l_{18}$, and let $S(L)$ be the ruled surface defined by blowing down $\bar{l}_i$, $i = 1, \dots, 18$.

Let $L' = \bar{l}_1 + \bar{l}_2 + \bar{l}_3 + l_4 + \dots + l_{18}$, let $C_3 = p(C) = p\,(C_{3,3})$, let $S\,(C_3) = p^{-1}(C)$.

Let $\sigma : S\,(C_3) \to S(L)$ and $\sigma' : S\,(C_3) \to S(L')$ be the natural maps defined by L and L' (see (2.3.1)). The map σ sends the component $C_{3,3}$ of C isomorphically onto a section $C_0 := \sigma\,(C_{3,3})$ of $S(L)$. By the general choice of $C_{3,3}$, $(C_0, C_0)_{S(L)} = (C_{3,3}, C_{3,3})_{S(C_3)}$. Now, the obvious equality $\deg N_{C_{3,3}|T} = \deg C_{3,3} = 6$, and the normal bundle sequence for the embedding $C_{3,3} \subset S\,(C_3) \subset T$, imply $(C_0, C_0)_{S(L)} = (C_{3,3}, C_{3,3})_{S(C_3)} = \deg N_{C_{3,3}|S(C_3)} = -3$. Therefore $e(S(L)) = 3$, and C_0 is the unique minimal section of $S(L)$ - see (1.3.4)(*).

The map $\sigma' : S\,(C_3) \to S(L')$ sends $C_{3,3}$ isomorphically onto a section C_0' of $S(L')$, blows-down the lines l_1, l_2 and l_3 (and blows-down no other line which intersects $C_{3,3}$). Therefore $\sigma'(C) = \sigma'(C_{3,3}) = C_0'$, and $(C_0', C_0')_{S(L')} = (C, C)_{S(C_3)} = (C_{3,3}, C_{3,3})_{S(C_3)} + 3 = 0$, i.e. $C_0' = \sigma(C)$ is a minimal section of $S(L')$ and $e(S(L')) = 0$ - see (1.3.4)(*). Therefore $L' \in \mathrm{Supp}\,\Theta$, and $C \in \bar{\mathcal{C}}$ - contradiction ($\Leftarrow C = C_{3,3} + l_1 + l_2 + l_3 \in \bar{\mathcal{C}}$ fulfills the conditions (3) and (b)).

It follows that $e(\Theta)$ cannot be 0, and Lemma (2.4.2) is proved.

□

(2.4.3).

Corollary. $e = e(\Theta) = -1,\ \bar{e} = e(P^-) = 0.$

3. The family of minimal sections $\mathcal{C}_\Theta$.

3.1. Definition of $\mathcal{C}_\Theta$.

(3.1.1). Let $L \in \operatorname{Supp}\Theta$ be general, and let $S(L)$ be the corresponding minimal model of $S(C(L))$ - see (2.3). It follows from (2.4.1) – (2.4.3) that $e(S(L)) = e = -1$, $S(L)$ is isomorphic to an indecomposable surface of type (1.3.4)(4), and the family of minimal sections $\mathcal{C}_\Theta(L)$ of $S(L)$ is parametrized by the base curve C - see (1.3.4)(4), or [14, Ch. 5, Exer. (2.7)].

Let L be as above, and let C be a general element of $\mathcal{C}_\Theta(L)$. Since the projection $p = p_1$ maps C isomorphically onto a plane cubic, $\deg C = (3, d)$ for some integer d; and a straightforward check shows that $d = 7$ (see e.g. (2.4.1) – (2.4.3)). In particular,

$$C \in \mathcal{C}_{3,7}(T) := \{\text{the family of effective connected 1-cycles on } T, \text{of bidegree } (3,7)\}.$$

On the base of the preceding, we define:

$$\mathcal{C}_\Theta := (\text{ the closure in } \mathcal{C}_{3,7}(T) \text{ of}) \{C : C \in \mathcal{C}_\Theta(L),\ L \in \operatorname{Supp}\Theta \text{ is general }\},$$

where the term “general” can be defined in an obvious way.

Clearly, $\dim \mathcal{C}_\Theta = 10$ (the family $\mathcal{C}_\Theta$ is generically a finite covering of an elliptic fibration over the 9-dimensional projective space of all the plane cubics).

(3.1.2).

Proposition. *Let $\mathcal{C}_\Theta$ be the family of minimal sections defined in (3.1.1), and let C be a general element of $\mathcal{C}_\Theta$. Then $\deg C = (3,7)$, and $p = p_1$ maps C isomorphically onto the smooth plane cubic $p(C) \subset \mathbb{P}^2$.*

4. The Abel-Jacobi image of the family $\mathcal{C}_\Theta$.

Theorem 4.1. *Let $\Phi : \mathcal{C}_\Theta \to J(T)$ be the Abel-Jacobi map for the family $\mathcal{C}_\Theta$ (see e.g. [5]). Then the image $\Phi(\mathcal{C}_\Theta)$ is a copy of the theta divisor $\Theta(T)$.*

Proof. It follows from (3.1.1), (3.1.2) that the family $\mathcal{C}_\Theta$ is generically an elliptic fibration over the family $\operatorname{Supp}\Theta$. Let $L \in \operatorname{Supp}\Theta$ be general. Then the fiber $\mathcal{C}_\Theta(L) = \xi^{-1}(L)$ of the natural map $\xi : \mathcal{C}_\Theta \to \operatorname{Supp}\Theta$ is isomorphic to the plane cubic C defined by the image of L under the map $p_* : \operatorname{Supp}\Theta \to |\mathcal{O}_{\mathbb{P}^2}(3)|$. The general element $C(x)$, $x \in C$ of $\xi^{-1}(L)$ is mapped, by p, isomorphically onto the plane cubic C. The only degenerations of $C(x)$ are caused by the 18 intersection points of C and Δ. They can be described as follows: Let $L = l_1 + \cdots + l_{18}$, and let $\bar{l}_i$ be the complementary line of

l_i, i.e. $\bar{l}_i = p^{-1}(p(l_i)) - l_i$. Let $L_i = L + \bar{l}_i - l_i$. The ruled model $S(L_i)$ is of type (1.3.4)(2.b), and the degenerate elements of the fiber $\xi^{-1}(L)$ are $C_i^+ + \bar{l}_i$, $C_i^- + \bar{l}_i$, where C_i^+ and C_i^- are the minimal sections of $S(L_i)$.

We shall prove the following.

Lemma 4.2. *Let $L \in \operatorname{Supp}\Theta$, etc., be as above, and let C' and C'' be two general elements of the fiber $\xi^{-1}(L)$. Then $\Phi(C') = \Phi(C'')$.*

Proof. The ruled model $S(L)$ is of type (1.3.4)(4). we fix $C' = C(x)$ to be the tautological section of an indecomposable rank 2 bundle $E(x)$, defined by the extension $0 \to \mathcal{O}_C \to E(x) \to \mathcal{O}_C(x) \to 0$ (here $C \cong C'$ is the base of the ruled surface $S(L)$). In particular, C' is a minimal section of $S(L)$, i.e. $C' \in \xi^{-1}(L)$. Let C'' be a section of $S(L) = \mathbb{P}(E(x))$. Let f be the "fiber" of $S(L)$, and let $C' = C(x)$ be the "section". Then, in $Pic\, S(L) = \boldsymbol{Z} \cdot C' + Pic(C) \cdot f \;\Rightarrow\; C'' = C' + \delta \cdot f$, for some $\delta \in Pic\, C$; and one can always choose the divisor δ such that $\operatorname{Supp}\delta$ does not intersect $C \cap \Delta$. By assumption, $C'' \in \xi^{-1}(L)$. Therefore the section C'' is minimal and $\deg\delta = 0$. Therefore the sections C' and C'', regarded as 1-cycles on the threefold T, differ from each other by the cycle $\delta \cdot f = p_1^{-1}(\delta)$ which is mapped to zero by the cycle-class map - since all the fibers of $p : T \to \mathbb{P}^2$ are rationally equivalent. In particular $\Phi(C') = \Phi(C'')$. $\square$

Let $\Psi : \operatorname{Supp}\Theta \to \Theta$ be the Prym-Abel-Jacobi map, defined by sending the elements $L(t)$ of the linear system $|\mathcal{L}|$ to the sheaf $\mathcal{L}$, and let $\xi : \mathcal{C}_\Theta \to \operatorname{Supp}\Theta$ be as above. It follows from (4.2) that the map ξ factors through Φ and Ψ, i.e., there exists a correctly defined map $i : \Phi(\mathcal{C}_\Theta) \to \Theta$ such that $i \circ \Phi = \Psi \circ \xi$. It follows immediately from the definition of i that the map i is an isomorphism, and Theorem (4.1) is proved.

$\square$

Corollary 4.3. *Let $C \in \mathcal{C}_\Theta$ be general. Let $\mathcal{L} = \Psi \circ \xi(C)$, let $|\mathcal{L}| = \{L(t) : t \in \mathbb{P}^1\}$, and let $\{C(t) : t \in \mathbb{P}^1\}$ be the corresponding 1-dimensional rational family of plane cubics. Then the connected component of the fiber $\xi^{-1} \circ \Psi^{-1}(\mathcal{L}) = i^{-1} \circ \Phi^{-1}(\mathcal{L})$, which passes through C, is isomorphic to an elliptic fibration $f : \mathcal{C}_\mathcal{L} \to \mathbb{P}^1$; the fiber $f^{-1}(t)$ is isomorphic to the plane cubic curve $C(t)$.*

5. The family $\mathcal{D}$ of elliptic sextics of bidegree (3,3) on T.

5.1. In this section we prove that the Abel-Jacobi map sends the 6-dimensional family $\mathcal{D}$ of elliptic sextics on T of bidegree (3,3) onto a 3-dimensional subvariety $Z \subset J(T)$ - isomorphic to a component of $\operatorname{Sing}\Theta$.

Definition of $\mathcal{D}$.

Let $C \subset T$ be a reduced and connected curve of arithmetic genus $p_a(C) = 1$, and of total degree $|\deg C| = 6$. Here deg is the usual bidegree map $\deg : \{\text{1-cycles on}\, T\} \to \boldsymbol{Z} \oplus \boldsymbol{Z}$. If such a curve $C \subset T$ does exists, the bidegree $\deg C$ can be one of the pairs $(2,4), (3,3), (4,2)$. We shall consider the middle case: $\deg C = (3,3)$. By definition

$$\mathcal{D} = (\text{the closure of})\{C - \text{ a reduced and connected curve on}\, T : \\ p_a(C) = 1, \deg(C) = (3,3)\}.$$

We call $\mathcal{D}$ the *family of elliptic sextics of bidegree* $(3,3)$ *on* T.

Lemma 5.2. *Let T be general, and let $\mathcal{D}(T)$ be the family of elliptic sextics on T of bidegree $(3,3)$. Then*

(i) $\dim \mathcal{D}(T) = 6$.

(ii) *The general curve $C \in \mathcal{D}(T)$ is smooth, and the projections $p_1 = p$ and p_2 map C isomorphically onto plane cubics.*

(iii) *($p = p_1$): Let $L(C) \in \mathrm{Symm}^{18}\,\tilde{\Delta}$ be, as usual, the effective divisor on $\tilde{\Delta}$ defined by the 18-tuple of lines of bidegree $(0,1)$ which intersect the curve $C \in \mathcal{D}$. Then $L(C) \in \mathrm{Supp}\,\Theta \cup \mathrm{Supp}\,P^-$.*

Proof. By definition, any bidegree $(2,2)$ threefold T belongs to the linear system $|\mathcal{O}_W(2)| \cong \mathbb{P}^{35}$ (see e.g. the notation of (5.3.2.)). Denote by $\mathcal{D}(W)$ the family of elliptic sextics $C \subset W$, of bidegree $\deg C = (3,3)$. Then, for any $T \in |\mathcal{O}_W(2)|$, and for any $C \in \mathcal{D}(T)$, the curve C belongs to the family $\mathcal{D}(W)$. Let $\Pi \subset \mathcal{D}(W) \times |\mathcal{O}_W(2)|$ be the incidence set $\Pi = \{(C,T) : C \subset T\}$, and let $q_1 : \Pi \to \mathcal{D}(W)$ and $q_2 : \Pi \to |\mathcal{O}_W(2)|$ be the canonical projections.

(*) Sublemma. *q_2 is surjective.*

Proof. Let $q_2(\Pi) \subset |\mathcal{O}_W(2)|$ be the image of q_2. Since $|\mathcal{O}_W(2)| \cong \mathbb{P}^{35}$ is irreducible, the surjectivity of q_2 will follow from the inequality $\dim q_2(\Pi) \geq 35$. We shall prove this inequality.

(1). First of all, we shall evaluate $\dim \Pi$.

Let $G = G(5 : \mathrm{Span}\, W) = G(6,9)$ be the Grassmanian of the 5-dimensional projective subspaces $\mathbb{P}^5 \subset \mathrm{Span}\, W = \mathbb{P}^8$, let $\mathbb{P}^5$ be a general element of G, and let $C = C\,(\mathbb{P}^5) = W \cap \mathbb{P}^5$. It is not hard to see that C is an elliptic sextic of bidegree $(3,3)$ in W, s.t. $\mathrm{Span}\, C\,(\mathbb{P}^5) = \mathbb{P}^5$ - see (2). In other words, the set

$$G_0 = \{\mathbb{P}^5 : C\,(\mathbb{P}^5) = W \cap \mathbb{P}^5 \,\text{is a smooth elliptic sextic of bidegree } (3,3), \\ \text{s.t. } \mathrm{Span}\, C\,(\mathbb{P}^5) = \mathbb{P}^5\}$$

is an open subset of G, and the map $\phi_0 : G_0 \to \mathcal{D}(W)$, $\phi_0 : \mathbb{P}^5 \to C\left(\mathbb{P}^5\right)$ is a regular embedding. Let $\mathcal{D}_0 = \phi\left(G_0\right)$ be the isomorphic image of G_0, and let $\mathcal{D}' \subset \mathcal{D}(W)$ be the closure (in $\mathcal{D}(W)$) of $\mathcal{D}_0$.

It follows from the preceding that the biregular isomorphism $\phi_0 : G_0 \to \mathcal{D}_0$ defines a birational map $\phi : G \to \mathcal{D}'$. In particular, $\dim \mathcal{D}(W) \geq \dim \mathcal{D}' = \dim G = 18$.

Let $C \in \mathcal{D}(W)$ be general, and let $\mathcal{J}^C \subset \mathcal{O}_W$ be the sheaf of ideals of $C \subset W$ (see [14, Ch. 2, Sect. 5]). After tensoring the exact sequence $0 \to \mathcal{J}^C \to \mathcal{O}_W \to \mathcal{O}_C \to 0$ by $\mathcal{O}(2)$, and passing to cohomologies, we obtain the exact sequence

$$0 \to H^0\left(W, \mathcal{J}^C(2)\right) \to H^0\left(W, \mathcal{O}_W(2)\right) \to H^0\left(C, \mathcal{O}_C(2)\right) \to \dots$$

Therefore $\dim H^0\left(W, \mathcal{J}^C(2)\right) \geq h^0\left(W, \mathcal{O}_W(2)\right) - h^0\left(C, \mathcal{O}_C(2)\right) = 36 - h^0\left(C, \mathcal{O}_C(2)\right)$.

Since C is an elliptic sextic $\deg \mathcal{O}_C(2) = 12$. Then the Riemann-Roch formula implies $h^0\left(\mathcal{O}_C(2)\right) = 12$. Therefore $h^0\left(W, \mathcal{J}^C(2)\right) \geq 24$.

Let $q_1^{-1}(C) \cong \{T \in |\mathcal{O}_W(2)| : C \in \mathcal{D}(T)\}$ be the fiber of q_1 at C. It follows from the definition of $\mathcal{J}^C$ that $\{T \in |\mathcal{O}_W(2)| : C \subset T\} \cong \mathbb{P}\left(H^0\left(W, \mathcal{J}^C(2)\right)\right)$, i.e. the dimension of the general fiber of $q_1 = \dim q_1^{-1}(C) = \dim \mathbb{P}\left(H^0\left(W, \mathcal{J}^C(2)\right)\right) \geq 23$.

Therefore $\dim \Pi = \dim \mathcal{D}(W) + \dim($ the general fiber of $q_1) \geq 18 + 23 = 41$.

(2). In order to evaluate $\dim q_2(\Pi)$ we shall find the dimension of the general fiber of q_2.

First of all, we shall prove (ii) for the general element $T \in q_2(\Pi)$.

Clearly, (ii) is an open condition on Π. Therefore, in order to prove (ii) for the general element T of $q_2(\Pi)$, it is enough to find a smooth bidegree $(2,2)$ threefold T and a smooth elliptic sextic $C \subset T$ which fulfills (ii). In order to find a pair $(C \subset T) \in \Pi$ which fulfills (ii), one proceeds as follows:

Let $p_i : W \to \mathbb{P}^2$, $i = 1,2$ be the canonical projections on $W \cong \mathbb{P}^2 \times \mathbb{P}^2$, and let $\mathbb{P}^6$ be a general codimension 2 subspace in $\mathbb{P}^8$. Then $S = S\left(\mathbb{P}^6\right) = W \cap \mathbb{P}^6$ is a smooth Del Pezzo surface of degree 6 embedded anticanonically in $\mathbb{P}^6$. The hyperplane sections of S are, in fact, the same as the elliptic curves of bidegree $(3,3)$ on S. Indeed, by the Theorem of Bertini, the general element C of the system $|\mathcal{O}_S(1)|$ is smooth. Moreover, the projections p_1 and p_2 map C isomorphically onto plane cubics. Indeed, the composition $p_2 \circ p_1^{-1} : \mathbb{P}^2 \to S \to \mathbb{P}^2$ is an elementary Cremona transformation, defined by triples of points $(x_1, x_2, x_3) \subset \mathbb{P}^2 = p_1(S)$ and $(y_1, y_2, y_3) \subset \mathbb{P}^2 = p_2(S)$. In particular, the curve $C \subset S$ must be a proper p_1-preimage of a cubic $C_1 \supset \{x_1, x_2, x_3\}$, as well - a proper p_2-preimage of a cubic $C_2 \supset \{y_1, y_2, y_3\}$. Therefore p_1 and p_2 map the elliptic curve C isomorphoically onto plane cubics; in particular C is an elliptic sextic on W of bidegree $(3,3)$.

In order to find $T \supset C$, we choose a general quadric $Q \subset \mathbb{P}^8$ which passes through C, and let $T = Q \cap W$. Then the pair $(C \subset T) \in \Pi$ fulfills (ii). This proves (ii) for the general $T \in q_2(\Pi)$.

We are ready to compute $\dim q_2^{-1}(T)$.

Let $T \in q_2(\Pi)$ be general, and let $C \in \mathcal{D}(T)$ be a general elliptic sextic of bidegree $(3,3)$ on T. Let $S(C_1) = p_1^{-1}(C_1)$ and $S(C_2) = p_2^{-1}(C_2)$, where $C_1 = p_1(C)$ and $C_2 = p_2(C)$ be the isomorphic projections of C. Just as in Sections 2, 3, 4, the curve C defines, via intersection, the effective divisors $L_i(C)$ on $\tilde{\Delta}_i$, which belong to the linear system of some $\mathcal{L}_i \in \mathrm{Nm}^{-1}(\omega_{\Delta_i})$, $i = 1,2$.

According to the agreement we fix $p = p_1$. Let $\Delta = \Delta_1$, $L(C) = L_1(C)$, $S(C) = S(C_1)$, etc., be the corresponding objects. Obviously, C is a minimal section of the ruled surface $S(L(C))$ (see (2.3.1), (2.3.2)), and the same arguments as in Sect. 2, 3, 4 imply that $(C,C)_{S(C)} = (C,C)_{S(L(C))} = e(S(L(C))) = 3$ (see also the proof of Lemma (2.4.2)). Let

$$0 \to N_{C|S(C)} \to N_{C|T} \to N_{S(C)|T} \otimes \mathcal{O}_C \to 0$$

be the normal bundle sequence for $C \subset S(C) \subset T$. It follows from the preceding that:

(a). $\deg N_{C|S(C)} = (C,C)_{S(C)} = e(S(C)) = 3$; therefore $h^0\left(N_{C|S(C)}\right) = 3$, $h^1\left(N_{C|S(C)}\right) = 0$.

(b). $N_{S(C)|T} = p^*\left(\mathcal{O}_{\mathbb{P}^2}(3)\right)$; therefore $h^0\left(N_{S(C)|T} \otimes \mathcal{O}_C\right) = 9$, $h^1\left(N_{S(C)|T} \otimes \mathcal{O}_C\right) = 0$.

Now (a) and (b) imply: $h^1\left(N_{C|T}\right) = 0$, and $\dim \mathcal{D}(T) = h^0\left(N_{C|T}\right) = 6$.

Since $q_2^{-1}(T)$ is canonically isomorphic to $\mathcal{D}(T)$, this implies the equality $\dim q_2^{-1}(T) = 6$, for the general $T \in q_2(\Pi)$.

(3). It follows from (1) and (2) that

$$\dim q_2(\Pi) = \dim \Pi - \dim(\text{ the general fiber of } q_2) \geq 41 - 6 = 35.$$

This proves $(*)$ = the surjectivity of q_2. □

It follows from the surjectivity of q_2 that (i) and (ii) are true for the general element of $|\mathcal{O}_W(2)|$ (see (2)). As for (iii) - it is a direct consequence from the arguments stated in the end of (2). □

5.3. The family $\mathcal{D}$ and the quadrics of rank 6 through T.

(5.3.1). *The elements of $\mathcal{D}$ as components of canonical curves on T.*

Let $\mathbb{P}^6$ be a subspace of $\mathbb{P}^8 = \operatorname{Span} T$ such that $\dim\left(T \cap \mathbb{P}^6\right) = 1$, and let $C\left(\mathbb{P}^6\right) = T \cap \mathbb{P}^6$. The curve $C\left(\mathbb{P}^6\right) \subset T$ is a canonical curve of degree 12, and of arithmetic genus 7. Call such a cure C a *canonical curve on T*.

Obviously, all the canonical curves on T are rationally equivalent, and the family of canonical curves on T can be represented by the 14-dimensional Grassmann variety $G(7,9) = G(6 : \mathbb{P}^8)$.

The curves of the family $\mathcal{D}$ are closely connected with the degenerations of the family of canonical curves on T. More precisely, let $C \in \mathcal{D}$ be general, let $\mathbb{P}^5(C) = \operatorname{Span} C$, and let $\mathbb{P}^6 \in \mathbb{P}^2(C) := \mathbb{P}^8/\mathbb{P}^5(C)$ be a 6-space through $\mathbb{P}^5(C)$. Then the canonical curve $C(\mathbb{P}^6)$ splits into two component: $C(\mathbb{P}^6) = C + \tilde{C}$, where $\tilde{C} \in \mathcal{D}$, and $\delta\left(C, \tilde{C}\right) = \#\left(C \cap \tilde{C}\right) = 6$. In particular, C and $\tilde{C}$ are projectively normal, since C and $\tilde{C}$ are components of a canonical curve. (In other words, $\operatorname{Span} C$ and $\operatorname{Span} \tilde{C}$ must be projective spaces of dimension 5.)

(5.3.2). *The determinantal subvarieties of* $\mathbb{I}_2(W)$.

Let $W \subset \mathbb{P}^8$ bethe Segre image of $\mathbb{P}^2 \times \mathbb{P}^2$, and let $\mathbb{P}^2 \times \mathbb{P}^2 = \mathbb{P}(E) \times \mathbb{P}(F)$, where E and F are complex 3-spaces. Let $\mathbb{P}^8 = \mathbb{P}(E \otimes F)$. Then, as it follows from the definition of the Segre map, the elements of $W \subset \mathbb{P}^8$ are in 1:1 correspondence with the $\mathbb{C}^*$ -classes of tensor products $u \otimes v : u \in E, v \in F$.

Let $(e_i,\, i = 1,2,3)$ and $(f_j,\, j = 1,2,3)$ be bases for E and F, with dual bases $(x_i,\, i = 1,2,3)$ and $(y_j,\, j = 1,2,3)$. Then $(g_{ij} = e_i \otimes f_j,\, i,j = 1,2,3)$ is a basis for $E \otimes F$ with dual basis $(z_{ij} = x_i \otimes y_j,\, i,j = 1,2,3)$. Let $[z_{ij}]_{i,j=1,2,3}$ be the coordinate matrix, and let $\mathbb{I}_2(W)$ be the projective space of quadrics in $\mathbb{P}^8$ which pass through W. Then $\mathbb{I}_2(W)$ is spanned by the two-by-two minors of $[z_{ij}]$, i.e. $\mathbb{I}_2(W)$ is projective 8-space. Clearly, the choice of the coordinates z_{ij} defines a linear isomorphism

$$\psi(z) : \mathbb{P}^8 = \mathbb{P}(E \otimes F) \cong \mathbb{I}_2(W).$$

The linear map $\psi(z)$ sends the $\mathbb{C}^*$ -classes of the unit tensor products (i.e. - the elements of W) to quadrics of rank 4. Moreover, any quadric of rank 4, which contains W, can be represented in this way. It is well known (see e.g. [17]) that the set

$$\operatorname{Sec} W \;:=\; \text{the closure of the union of all the secant lines of } W,$$

is a cubic hypersurface in $\mathbb{P}^8$, i.e. W is one of the four Severi varieties (ibid.). It follows from the definition of $\psi(z)$ that $\psi(z)$ sends the points of $\operatorname{Sec} W$ to quadrics of rank 6. Moreover, any quadric of rank 6 which contains W can be represented (in a non-unique way) as an image of a point of $\operatorname{Sec} W$. We collect these observations in the following

(*) Lemma. *Let* $\mathbb{I}_2(W)$ *be the projective space of quadrics in* $\mathbb{P}^8$ *which contain the fourfold* W*, and let* $D_k(W) = D_k(\mathbb{I}_2(W)) =$ *(the closure of)* $\{Q \in \mathbb{I}_2(W) : \operatorname{rank}(Q) = k\}$ *be the k-th determinantal subvariety of* $\mathbb{I}_2(W)$*.*

Then $D_k(W) \neq \emptyset \Leftrightarrow k \in \{4,6,9\}$. Moreover, there exists a linear isomorphism $\psi : \mathbb{P}^8 \to \mathbb{I}_2(W)$ such that

$$\psi(W) = D_4(W),$$

$$\psi(\operatorname{Sec} W) = D_6(W).$$

Proof. Let $[z_{ij}]$ be as above, and let $\psi = \psi(z)$. Then the natural action of $PGL(E) \times PGL(F)$ on $\mathbb{P}^8$, which does not change the rank of the 3×3 matrix $[z_{ij}]$, splits $\mathbb{P}^8$ into 3 orbits - the fourfold W, and the quasi-projective varieties $(\operatorname{Sec} W) - W$ and $\mathbb{P}^8 - \operatorname{Sec} W$. The linear map ψ sends the closure of any of these orbits onto a determinantal subvariety of $\mathbb{I}_2(W)$. Now, it can be easily seen that these three determinantal subvarieties are D_4, D_6 and D_9. □

(5.3.3). *Quadrics of rank* 6 *related to the incidence correspondence $\Sigma \subset \mathcal{D} \times \mathcal{D}$.*

Let T be a general bidegree $(2,2)$ divisor, and let $Q \subset \mathbb{P}^8$ be any quadric such that $Q \cap W = T$. It follows from (5.3.2) that such a quadric Q is not unique - Q can be replaced by any quadric in $\operatorname{Span}(Q, \mathbb{I}_2(W)) - \mathbb{I}_2(W) = \mathbb{P}^9 - \mathbb{P}^8$; however Q is unique $\operatorname{mod}. \mathbb{I}_2(W)$.

Let $\Sigma \in G(7,9)$ (see (5.3.1)) be the incidence correspondence

$$\Sigma = (\,\text{the closure of}\,) \left\{ \mathbb{P}^6 : C(\mathbb{P}^6) = C + \tilde{C}, \text{ where } C, \tilde{C} \in \mathcal{D} \right\}.$$

The set Σ can be regarded (up-to closed subsets of codim. > 1) also as an incidence correspondence $\Sigma \subset \mathcal{D} \times \mathcal{D}$.

Let $\mathbb{P}^6 \in \Sigma$ be general. the codim. 2 subspace $\mathbb{P}^6 \subset \mathbb{P}^8$ intersects the fourfold W in an anticanonically embedded del Pezzo surface $S(\mathbb{P}^6)$ of degree 6, and the quadric Q intersects $S(\mathbb{P}^6)$ in a pair of elliptic sextics $C + \tilde{C}$. Let $\mathbb{P}^5(C)$ and $\mathbb{P}^5\left(\tilde{C}\right)$ be, as in (5.3.1), the spans of C and $\tilde{C}$, and let H and $\tilde{H}$ be linear forms on $\mathbb{P}^8$ such that $(H)_0 \cap \mathbb{P}^6 = \mathbb{P}^5(C)$, $\left(\tilde{H}\right)_0 \cap \mathbb{P}^6 = \mathbb{P}^5\left(\tilde{C}\right)$. The splitting $Q \cap S(\mathbb{P}^6) = C + \tilde{C}$ implies:

$$Q|_{\mathbb{P}^6} = H \cdot \tilde{H}|_{\mathbb{P}^6} \,(\operatorname{mod}. \mathbb{I}_2\left(S(\mathbb{P}^6)\right) = \text{ the ``restriction'' of } \mathbb{I}_2(W) \text{ on } \mathbb{P}^6).$$

(Here Q is the quadratic form of Q, and we disregard the multiplication by a non-zero constant.) Let $\mathbb{P}^6 = (H_1 = H_2 = 0)$ be any pair of linear equations which define the subspace $\mathbb{P}^6 \subset \mathbb{P}^8$. It follows from the preceding that Q can be represented in the form

$$Q = H \cdot \tilde{H} + H_1 \cdot \tilde{H}_1 + H_2 \cdot \tilde{H}_2, \ \operatorname{mod}. \mathbb{I}_2(W),$$

where $\tilde{H}_1$ and $\tilde{H}_2$ are some linear forms. Clearly, the quadric of equation $H \cdot \tilde{H} + H_1 \cdot \tilde{H}_1 + H_2 \cdot \tilde{H}_2$ does not belong to the set $D_6(W)$ (the restriction of this quadric to $\mathbb{P}^6$ does not contain the surface $S\left(\mathbb{P}^6\right)$). In particular, the quadric Q in the definition of T can be replaced by this quadric of rank 6.

5.4. The Abel-Jacobi image $Z = \Phi\left(\mathcal{D}\right)$.

Let $T \subset W$, $\mathcal{D}$, etc., be as above, and let $\Phi : \mathcal{D} \to J = J(T)$ be the Abel-Jacobi map for the family $\mathcal{D}$. Let $C \in \mathcal{D}$ be general, and let $\Phi^* : H^1\left(T, \Omega^2\right) \to H^1\left(N_{C/T} \otimes \omega_T\right)$ be the codifferential of Φ at the point $C \in \mathcal{D}$ - see e.g. [5].

The space $H^1\left(T, \Omega^2\right)$ is naturally isomorphic to $H^0\left(\Omega_{J(T)}\right)$ - the cotangent space of $J(T)$ at a fixed point. The normal bundle sequence for the embedding $T \subset W$, and the formulae of Bott and Künneth imply the isomorphism

$$\alpha : \, H^0\left(\mathcal{O}_{\mathbb{P}^8}(1)\right) \cong H^0\left(T, \mathcal{O}(1,1)\right) \to H^1\left(T, \Omega^2\right)$$

(see also [24]). In particular, the elements of $H^1\left(T, \Omega^2\right)$ can be regarded as linear forms on $\mathbb{P}^8$.

The following proposition is an analog of Lemma 4.6 in [25].

Proposition 5.4.1. *Let $C \in \mathcal{D}$ be general, let $\mathbb{P}^5(C) = \operatorname{Span} C$ be as in (5.3.1), let $\mathbb{P}^6$ be any 6-space through $\mathbb{P}^5(C)$, and let $Q = H \cdot \tilde{H} + H_1 \cdot \tilde{H}_1 + H_2 \cdot \tilde{H}_2, \operatorname{mod} . \mathbb{I}_2(W)$ be any of the representations of the quadric Q defined by the element $\mathbb{P}^6 \in \Sigma$ - see (5.3.3).*

Let Φ^* and α be as above, and let $\Phi \cdot \alpha$ be their composition. Then the subspace $\ker\left(\Phi \cdot \alpha\right) \subset H^0\left(\mathcal{O}_{\mathbb{P}^8}(1)\right)$ is spanned by the forms $H, \tilde{H}, H_1, \tilde{H}_1, H_2, \tilde{H}_2$.

Proof. see [25], the proof of Lemma 4.6. □

Note that the analog of the family $\mathcal{D}$, studied in [25], is the family of "halves" of canonical curves (= the elliptic quadrics) on the *quartic double solid* (= the Fano threefold which is a double covering of $\mathbb{P}^3$ branched along a quartic surface); see also (5.3.1).

Corollary 5.4.2. *Let $Z = \Phi\left(\mathcal{D}\right)$ be the Abel-Jacobi image of the family of elliptic sextics on T of bidegree $(3,3)$. Then $\dim Z = 3$.*

(5.4.3). *The fiber $\Phi^{-1}(z)$.*

(A). Let $C \in \mathcal{D}$ be general, let $z = \Phi(C)$, and let $\Phi^{-1}(z)_0$ be the irreducible component of $\Phi^{-1}(z)$ such that $C \in \Phi^{-1}(z)_0$. Let $\mathbb{P}^5(C) = \operatorname{Span} C$, and let u, v be the local parameters at the (general) point $\mathbb{P}^7 =$

$\mathbb{P}^7(0,0) \in \mathbb{P}^2(C)^* = (\mathbb{P}^8/\mathbb{P}^5(C))^*$. Let $S(u,v) = T \cap \mathbb{P}^7(u,v)$. The surface $S(u,v)$ is a hyperplane section of T which contains (the general) $C \in \mathcal{D}$. Such a surface $S(u,v)$ cannot be reducible. Otherwise $l_i = p_i(C) = p_i(S(u,v))$, $i = 1,2$ will be lines. Therefore $S(u,v)$ is a surface of type $K3$, and the elliptic curve $C = C(u,v;0) \subset S(u,v)$ moves in a pencil $\{C(u,v;t) \subset S(u,v) : t \in \mathbb{P}^1\}$. Obviously, $C(u,v;t) \in \mathcal{D}$ and $\Phi(C(u,v;t)) = \Phi(C)$, since the curves $C(u,v;t)$ and $C = C(u,v;0)$ are rationally equivalent on T, i.e. $C(u,v;t) \in \Phi^{-1}(z)$, $\forall u,v,t$.

Denote by $\{C(u,v;t)\}$ the family of all these curves. By the construction of $C(u,v;t)$, the family $\{C(u,v;t)\}$ is a 3-dimensional and irreducible family of rationally equivalent curves, and $C = C(u,v;0) \in \{C(u,v;t)\}$. Moreover, $\dim \Phi^{-1}(z)_0 = \dim \Phi^{-1}(z) = \dim \mathcal{D} - \dim Z = 6 - 3 = 3$. Therefore $\{C(u,v;t)\} = \Phi^{-1}(z)_0$.

(B). Let $\mathbb{P}^6 \supset \mathbb{P}^5(C)$, and let $Q_0 = H \cdot \tilde{H} + H_1 \cdot \tilde{H}_1 + H_2 \cdot \tilde{H}_2$ be defined as in (5.3.3), (5.4); let Q_0 be also the quadric defined by the equation $Q_0 = 0$.

Let Λ be the ruling of Q_0 defined by the condition $\mathbb{P}^5(C) \in \Lambda$, and let $\mathbb{P}^5_0 \in \Lambda$, $\mathbb{P}^5_0 \neq \mathbb{P}^5(C)$. Let $l = \mathrm{Span}\,\{\mathbb{P}^5_0, \mathbb{P}^5(C)\}$ be the line spanned by $\mathbb{P}^5_0$, $\mathbb{P}^5(C) \in \Lambda = \mathbb{P}^3$. Being a subset of Λ, the line l can be regarded as a family of spaces $\{\mathbb{P}^5(x) : x \in \mathbb{P}^1 = \mathbb{C} \cup \infty,\ \mathbb{P}^5(0) = \mathbb{P}^5_0,\ \mathbb{P}^5(\infty) = \mathbb{P}^5(C)\}$. Let $C(x) = \mathbb{P}^5(x) \cap T$, $x \in \mathbb{C} \cup \infty$.

Clearly $C(x) \in \mathcal{D}$, $\forall x$; moreover - by construction - $C(x)$ is rationally equivalent to $C(\infty) = C$, $\forall x$. Therefore $\Phi(C(x)) = \Phi(C(\infty)) = \Phi(C) = z$, $\forall x$. In particular, if $C_0 := \mathbb{P}^5_0 \cap T = \mathbb{P}^5(0) \cap T = C(0)$ then $\Phi\,(C_0) = z$.

By assumption, $\mathbb{P}^5_0$ is chosen to be an arbitrary element of $\Lambda - \{\mathbb{P}^5(C)\}$, $C_0 = \mathbb{P}^5_0 \cap C$, and $C = \mathbb{P}^5(C) \cap T$. Therefore, for any $\mathbb{P}^5 \in \Lambda$, the curve $C\,(\mathbb{P}^5) := \mathbb{P}^5 \cap T$ is an element of the fiber $\Phi^{-1}(z)$. This way, we define the map

$$\lambda : \mathbb{P}^3 \cong \Lambda \to \Phi^{-1}(z),\ \mathbb{P}^5 \mapsto \lambda\left(\mathbb{P}^5\right) = C\left(\mathbb{P}^5\right) = \mathbb{P}^5 \cap T.$$

Since the different $\mathbb{P}^5 \in \Lambda$ intersect different curves $C\,(\mathbb{P}^5) = \mathbb{P}^5 \cap T$, the map λ is an embedding of Λ in the fiber $\Phi^{-1}(z)$, i.e. $\Lambda \cong \lambda(\Lambda) \subset \Phi^{-1}(z)$ (where $z = \Phi(C)$). Moreover $\Lambda \cong \mathbb{P}^3$, $\dim \Phi^{-1}(z) = \dim \Phi^{-1}(z)_0 = 3$, and $\mathbb{P}^5(C) \in \Lambda$ (i.e. $C = \mathbb{P}^5(C) \cap T = \lambda\,(\mathbb{P}^5(C)) \in \lambda(\Lambda)$). Therefore $\mathbb{P}^3 \cong \Lambda \cong \lambda(\Lambda) = \Phi^{-1}(z)_0$.

(C). It follows from the preceding that the quadric Q_0 does not depend on the element $C(u,v;t) \in \Phi^{-1}(z)_0$. We write $Q_0 = Q_0(C) = Q_0(C(u,v;t)) = Q_0(z)$.

The considerations from (A) and (B) imply that if $C \in \mathcal{D}$ is general and $z = \Phi(C)$, then the elements of the irreducible component $(C \in)\Phi^{-1}(z)_0$ of the fiber $\Phi^{-1}(z) \in \mathcal{D}$ can be described by two alternative ways:

(1) as element of the family $\{C(u,v;t)\}$ - see (A);

(2) as intersections $C\,(\mathbb{P}^5) = \mathbb{P}^5 \cap T,\ \mathbb{P}^5 \in \Lambda$, where Λ is the ruling of the

quadric $Q_0(z)$ defined by $\mathbb{P}^5(C) \in \Lambda \cong \mathbb{P}^3$.

In particular $\Phi^{-1}(z)_0 \cong \mathbb{P}^3$.

Proposition 5.5. *Let $T \subset W$ be a general bidegree $(2,2)$ divisor, let $p_i : T \to \mathbb{P}^2$, $i = 1,2$ be the projections, and let $\operatorname{Sing}_i^{st} \Theta$ be as in (1.2.2). Then:*

(i) *There exist canonically defined maps $\mathcal{L}_i : Z \to \mathcal{L}_i(Z) \subset P\left(\tilde{\Delta}_i, \Delta_i\right) \cong J(T)$, where $\mathcal{L}_i(Z)$ is a component of $\operatorname{Sing}_i^{st}\Theta$, $i = 1,2$.*

(ii) *Let $C \in \mathcal{D}$ be general, and let $z = \Phi(C)$. Then the quadric $Q_0(C) = Q_0(z)$ - see (5.4.3)$(*)$ - coincides with the projective tangent cone Cone_z of Θ at the point $z \in Z \subset \operatorname{Sing}\Theta$.*

Proof. Let $Q \in \mathbb{P}^8$ be any quadric such that $T = W \cap Q$, and let $\mathbb{I}_2(T) = \mathbb{P}\left(H^0\left(\mathbb{P}^8, \mathcal{O}(2-T)\right)\right)$ be the space of quadrics through T (see also (5.3.2)). Clearly, $\mathbb{I}_2(T) \cong \operatorname{Span}\{\mathbb{I}_2(W), Q\} \cong \mathbb{P}^9$.

Let $D_k(T) :=$ (the closure of) $\{P \in \mathbb{I}_2(T) : \operatorname{rank}(P) = k\}$ be the k-th determinantal locus in $\mathbb{I}_2(T)$.

Let $k = 6$. By (5.3.2)$(*)$, $D_6(T) \supset D_6(W) \cong \operatorname{Sec} W$, and $Q_0(z)$ does not belong to $\mathbb{I}_2(W) \supset D_6(W)$ - see (5.3.3). Therefore the rule $z \mapsto Q_0(z)$ defines a map $Q_0 : Z \to D_6(T)$, and the image $Q_0(Z)$ is not a subset of $D_6(W)$.

Lemma 5.5.1. *$Q_0(Z)$ is a component of $D_6(T)$, and $D_6(T) = Q_0(Z) \cup D_6(W)$.*

Proof. Let $Q_0 \in D_6(T) - D_6(W)$, and let Λ be one of the two rulings of the quadric Q_0. Let $\mathbb{P}^5 \in \Lambda$. Then the set $C\left(\mathbb{P}^5\right) = T \cap \mathbb{P}^5 = (W \cap Q) \cap \mathbb{P}^5 = (W \cap Q_0) \cap \mathbb{P}^5 = W \cap \mathbb{P}^5$ is a curve on T. Moreover, as it follows from the elementary projective properties of the fourfold W, $C\left(\mathbb{P}^5\right)$ is an elliptic curve of bidegree $(3,3)$, i.e. $C \in \mathcal{D}$. Clearly, $Q_0 = Q_0\left(C\left(\mathbb{P}^5\right)\right) = Q_0(z)$, where $z = \Phi\left(C\left(\mathbb{P}^5\right)\right) \in Z$. $\square$

(5.5.2). *The differential dQ_0 via the Gauss map of Z.*

It follows from the definition of $Q_0(Z)$ that $\dim Q_0(Z) \leq \dim Z = 3$. Moreover, $Q_0(Z)$ is a component of the determinantal locus $D_6(T) \subset \mathbb{I}_2(T) \cong \mathbb{P}^9$, and the general quadric $Q \in \mathbb{I}_2(T)$ has rank 9. It follows from the general properties of the determinantal varieties (see [12, Ch. 14]) that the components of $D_6(T) = D_{9-3}(T)$ cannot be of codimension greater than $3 \cdot (3+1)/2 = 6$. Therefore $\dim Q_0(Z) = \dim Z = 3$, and the map $Q_0 : Z \to Q_0(Z)$ is generically finite. In particular, the differential $dQ_0 : T_Z \to T_{Q_0(Z)}$ is generically an isomorphism. Let $\mathbb{P}\left(dQ_0\right) : \mathbb{P}\left(T_z\right) \to \mathbb{P}\left(T_{Q_0(Z)}\right)$ be the projectivization of dQ_0.

By definition, Z is a subvariety of the abelian variety $J(T)$, where $\dim Z = 3$, $\dim J(T) = 9$. Let $Z_0 \subset Z$ be the open subset of smooth points of Z. One can define the regular map

$$\mathrm{Gauss}_0 : \; Z_0 \to G(3,9) = G\left(2 : \mathbb{P}^9\right),$$

$$z \mapsto [(\text{ the translate in } 0 \in J(T), \text{ of}) \text{ the tangent space of } Z \subset J(T) \\ \text{at the point } z \in Z].$$

By definition, the rational *Gauss map* for $Z \subset J(T)$ is the rational map $\mathrm{Gauss} : \; Z \to G(3,9)$, defined by Gauss_0 - see e.g. [13, Ch. 2, Sect. 7].

Let $z \in Z$ be general. (In particular, z is a smooth point of Z.) It follows from Proposition (5.4.1) that the projective 2-space $\mathbb{P}^2(z) = \mathrm{vertex}\,(Q_0(z))$ can be identified with $\mathbb{P}(\mathrm{Gauss}(z)) \subset \mathbb{P}^8 = \mathbb{P}\left(T_{J(T)}|_0\right)$. Since $\mathbb{P}\left(dQ_0\right)$ is a local isomorphism, we can identify the spaces $\mathbb{P}\left(T_{Q_0(Z)}|_z\right)$ and $\mathbb{P}^2(z) = \mathrm{vertex}\,(Q_z)$ (see also [24]).

(5.5.3). *Proof of* (5.5)(i). Let $i \in \{1,2\}$ be fixed, and let $L_i : \mathcal{D} \to \mathrm{Symm}^{18}\,\tilde{\Delta}_i$ be the map defined in the proof of Lemma (5.2). (In (5.2), $p = p_1$ and $L(C) = L_1(C)$; the definition of $L_2(C)$ is evident.) Let $C_0 \in \mathcal{D}$ be general, and let $z = \Phi\,(C_0)$. Let Λ be the ruling of the quadric $Q_0(z)$ defined by the condition $\mathbb{P}^5\,(C_0) \in \Lambda$.

On the one hand, the element $\mathbb{P}^5 \in \Lambda$ determines uniquely the curve $C\,(\mathbb{P}^5) = T \cap \mathbb{P}^5 \in \mathcal{D}$.

On the other hand, if $C \in \mathcal{D}$, and $\mathrm{Span}\, C = \mathbb{P}^d$, then (by Riemann-Roch) $d \leq \deg C - p_a(C) = 5$. Moreover if $\mathbb{P}^6 \supset \mathrm{Span}\, C$ is general, then, by adjunction, $C\,(\mathbb{P}^6) = T \cap \mathbb{P}^6$ must be a degenerate canonical curve of degree 12. Since C is a component of the canonical curve $C\,(\mathbb{P}^6)$, $\dim \mathrm{Span}\, C$ must be maximum, i.e. $d = \dim \mathrm{Span}\, C = \deg C - p_a(C) = 6 - 1 = 5$. In particular, if $C \in \mathcal{D}$ is such that $C = C\,(\mathbb{P}^5) = T \cap \mathbb{P}^5$ for some $\mathbb{P}^5 \in \Lambda$, then this $\mathbb{P}^5$ is unique. This way, one can regard Λ as a subvariety of $\mathcal{D}$.

Let $C \in \Lambda$. Being an element of $\mathcal{D}$, the curve C defines uniquely the effective divisors $L_i(C) \in \mathrm{Symm}^{18}\,\tilde{\Delta}_i$, $i = 1,2$ (see Lemma 5.2). Moreover, if $L = L_i(C)$ for some $C \in \mathcal{D}$ then this C must be unique.

Indeed, let (as usual) $i = 1$, $p = p_1$, etc., and let $C' \in \mathcal{D}$ be such that $L_1\,(C') = L = L_1(C)$. Then both C and C' have to be minimal sections of the ruled model $S(L)$ (see 2.3.1); moreover $e(S(L)) = 3$ - see the proof of Lemma (2.4.2). Therefore $C = C'$ (see (1.3.4)(1)), i.e. the curve $C \in \mathcal{D}$ defining the fixed $L = L_1(C)$ must be unique.

As it follows from the preceding, the rule $C \mapsto L_i(C)$, defines an embedding $\mathbb{P}^3 = \Lambda \to \mathrm{Symm}^{18}\,\tilde{\Delta}_i$. In other words, if $\Lambda_i = L_i(\Lambda) \subset \mathrm{Symm}^{18}\,\tilde{\Delta}_i$ is the L_i-image of Λ then the map $L_i : \Lambda \to \Lambda_i$ is an isomorphism. In particular $\Lambda_i \cong \mathbb{P}^3$, i.e. the set Λ_i is a rational subfamily of $\mathrm{Supp}\,\Theta_i \cup \mathrm{Supp}\,P_i^-$, see (5.2)(iii). Here Θ_i and P_i^- are the components of the effective part of $\mathrm{Nm}^{-1}(\omega_{\Delta_i})$, see e.g. [26]. Therefore all the divisors $L_i(C) \in \Lambda_i$, $C = C(\mathbb{P}^5)$, $\mathbb{P}^5 \in \Lambda$ belongs to the same linear system $|L_i(C_0)|$ on $\tilde{\Delta}_i$.

Let $\mathcal{L}_i = \mathcal{L}(L_i(C_0)) \in Pic^{18}\tilde{\Delta}_i$ be the invertible sheaf defined by the effective divisor $L_i(C_0) \in \mathrm{Symm}^{18}\tilde{\Delta}_i$. The rule $L_i(C_0) \mapsto \mathcal{L}(L_i(C_0))$ defines a map $\mathcal{L} : L_i(\mathcal{D}) \to \Theta \cup P^-$. It follows from Corollary (5.4.2), and from the definition of the map $\mathcal{L} \circ L_i$, that $\dim \mathcal{L} \circ L_i(\mathcal{D}) = \dim Z = 3$. This way, we obtained a 3-dimensional subset $\mathcal{L} \circ L_i(\mathcal{D}) \subset \mathrm{Nm}^{-1}(\omega_{\Delta_i})$ such that $h^0(\mathcal{L}) \geq 4$, for any $\mathcal{L} \in \mathcal{L} \circ L_i(\mathcal{D})$; here we use the same symbol $\mathcal{L}$ for the sheaf $\mathcal{L}$ and for the map $\mathcal{L}$.

The number $d = \min\{\dim|\mathcal{L}| : \mathcal{L} \in \mathcal{L} \circ L_i(\mathcal{D})\}$ is a constant through an open subset of $\mathcal{L} \circ L_i(\mathcal{D})$.

(*) Lemma 5.1. *Let d be as above. Then $d = 3$.*

Proof. Let, e.g., $d = 4$. (The case $d \geq 5$ can be treated in a similar way.) Let $^- : \tilde{\Delta}_i \to \tilde{\Delta}_i$ be the involution induced by the double covering $\tilde{\Delta}_i \to \Delta_i$, and let

$$W_i = \left\{\mathcal{M} = \mathcal{F} \otimes \mathcal{O}(x - \bar{x}) : \mathcal{F} \in \mathcal{L} \circ L_i(\mathcal{D}),\ x \in \tilde{\Delta}\right\}.$$

It follows from the definition of W_i that $W_i \subset \mathrm{Nm}^{-1}(\omega_{\Delta_i})$, and $h^0(\mathcal{M}) = 4$ for the general $\mathcal{M} \in W_i$ (see e.g. [21, Lemma 3.14]).

Therefore W_i is a 4-dimensional subset of $\mathrm{Sing}_i^{st}\Theta$ - see (1.2.2). However $\dim \mathrm{Sing}\,\Theta = 3$ (see [24]). Therefore d cannot be 4. $\square$

(**) Remark. The intermediate jacobian $J(T)$ is a Prym variety which arises from a double covering of a general plane sextic, in contrast to the intermediate jacobian $J(B)$ of the desingularized nodal quartic double solid B - in which case the plane sextic Δ has a totally tangent conic (i.e., there exists a conic q for which the intersection multiplicity is even at all points of $q \cap \Delta$, see e.g. [24]). The existence of a totally tangent conic is a closed condition of codimension one, on the 19-dimensional moduli space of the plane sextics. Moreover $\dim \mathrm{Sing}\,\Theta(B) = 4$ (see [8, Sect. 7]), in contrast to $\dim \mathrm{Sing}\,\Theta(T) = 3$. This, probably, once more explains why the Dixon correspondence, which can be identified with a bidegree $(2,2)$ divisor, cannot be applied for a discriminantal pair which comes from a nodal quartic double solid - see [24].

It follows from the preceding that the sheaf $\mathcal{L} \circ L_i(C)$ does not depend on the particular choice of the curve $C \in \Phi^{-1}(z)_0$, $z = \Phi(C) = \Phi(C_0)$. Therefore the map $\mathcal{L} \circ L_i : \mathcal{D} \to J(T)$ factors through the Abel-Jacobi map $\Phi : \mathcal{D} \to Z$. Denote by $\mathcal{L}_i : Z \to \mathcal{L}_i(Z) = \mathcal{L} \circ L_i(\mathcal{D})$ the quotient map. It follows from $(*)$ that $\mathcal{L}_i(Z)$ is a 3-dimensional component of $\operatorname{Sing}_i^{st}\Theta$. This proves (i).

(5.5.4). *Proof of* (5.5)(ii).

Let $\eta_i \in Pic^0_{[2]}\Delta_i$ be as usual, the sheaf which defines the unbranched double covering $\tilde{\Delta}_i \to \Delta_i$ (see (1.3.3)), and let ω_{Δ_i} be the canonical sheaf of Δ_i. The linear system $|\omega_{\Delta_i} \otimes \eta_i|$ defines the *Prym-canonical map* $\phi_i : \Delta_i \to \mathbb{P}^8$. It is known by [24] that if the smooth plane sextics Δ_i, $i = 1, 2$ are the discriminants of bidegree $(2, 2)$ divisor, then the Prym-canonical map is an embedding and the isomorphic images $\Delta_i^T = \phi_i(\Delta_i)$ are projectively normal.

Let $\mathcal{L}_i(z)$, $z = \Phi(C_o)$, etc., be as above. The sheaf $\mathcal{L}_i(z)$ is a stable singularity of Θ, with respect to p_i. Therefore the projective tangent cone $\mathrm{Cone}_{\mathcal{L}_i(z)}$ of Θ, at the point $\mathcal{L}_i(z)$, is a quadric which passes through the Prym-canonical image Δ_i^T of the discriminant curve Δ_i, $i = 1, 2$ - see e.g. [23].

Here we use the following results, due to Verra - see [24]:

(*). Let $s : \Delta_i \to T$ be the Steiner map, defined by the rule $s : x \mapsto \operatorname{Sing} p_i^{-1}(x)$. Then the image $s(\Delta_i)$ coincides with the Prym-canonical curve Δ_i^T.

(**). Let $Q \subset \mathbb{P}^8$ be a quadric which passes through the Steiner curves $s(\Delta_1)$ and $s(\Delta_2)$. Then $Q \supset T$.

It follows from the preceding, and from $(*)$ and $(**)$, that

(i). $\mathrm{Cone}_z := \mathrm{Cone}_{\mathcal{L}_1(z)} = \mathrm{Cone}_{\mathcal{L}_2(z)}$;

(ii). $\mathrm{Cone}_z \supset T$, i.e. $\mathrm{Cone}_z \in \mathbb{I}_2(T)$.

It is well-known that Cone_z is a quadric of rank 5 or 6 (see [15]), i.e. $\mathrm{Cone}_z \in D_5(T) \cup D_6(T)$. It is not hard to see that if T is general then $D_5(T) = \emptyset$. In fact, the general choice of the quadric Q, such that $W \cap Q = T$, implies that $\operatorname{codim}(D_5(T) \subset \mathbb{I}_2(T)) = 10$, outside the fixed determinantal $D_4(T) = D_4(W) \cong W$ - see (5.3.2) and [12, Ch. 14]. Therefore $\operatorname{rank}\mathrm{Cone}_z = 6$ - since $\dim \mathbb{I}_2(T) = 9 < 10$.

The maps $\mathcal{L}_1$ and $\mathcal{L}_2$ are local isomorphisms. Therefore the projective tangent spaces $\operatorname{vertex}(Q_0(z)) = \mathbb{P}^2(z) = \mathbb{P}(T_Z|_z)$ (see (5.4.1)), and $\mathbb{P}(T_{\mathcal{L}_i(Z)}|_{\mathcal{L}_i(z)}) = \operatorname{vertex}(\mathrm{Cone}_{\mathcal{L}_i(z)}) = \operatorname{vertex}(\mathrm{Cone}_z)$, $i = 1, 2$ (see [21], 2.7 and 3.20) can be identified.

It follows that the quadrics Cone_z and $Q_0(z)$ have the same vertex $\mathbb{P}^2(z)$, and $\mathrm{Cone}_z \supset T$, $Q_0(z) \supset T$. Moreover, Cone_z and $Q_0(z)$ belong to the determinantal locus $D_6(T)$. An elementary projective considerations implies that these two quadrics must coincide. This proves (ii).

$\square$

As a corollary we obtain:

Theorem 5.6 (The Torelli theorem for the Verra threefold). *Let T be a general smooth bidegree $(2,2)$ divisor in the Segre image W of $\mathbb{P}^2 \times \mathbb{P}^2$, and let $(J(T), \Theta)$ be the principally polarized intermediate jacobian of T. Then there exists a component Z of $\mathrm{Sing}\,\Theta$ such that $\dim Z = 3$, and T coincides with the intersection of all the projective tangent cones of Θ at the regular poins of Z.*

Proof. It follows from the preceding that the map Q_0 sends the set $Z = \Phi(\mathcal{D})$ onto the component $Q_0(Z)$ of $D_6(T)$ (see (5.3.3) and (5.5.1)), and the space $\mathbb{I}_2(T) \cong \mathbb{P}^9$ is spanned by the quadrics of the determinantal locus $D_6(T)$. Moreover, $D_6(T) = Q_0(Z) \cup D_6(W)$, and $\mathrm{Span}\, D_6(W)$ coincides with the proper subspace $\mathbb{I}_2(W) \cong \mathbb{P}^8$ of $\mathbb{I}_2(T)$ - see (5.3.2)$(*)$ and (5.5.1).

On the one hand, for the general T, the component $\mathbb{I}_2(T)$ is spanned by the quadrics from $Q_0(Z)$, or - equivalently:

(*). $\mathbb{I}_2(T) = \mathrm{Span}\, Q_0(Z)$, for T - general.

Clearly, $(*)$ is an open condition on the pace of bidegree $(2,2)$ threefolds. Therefore, in order to prove $(*)$ for the general T, it is enough to find a bidegree $(2,2)$ threefold $T(0)$ such that $\mathbb{I}_2(T(0)) = \mathrm{Span}\, Q_0(T(0))$.

Let $[z_{ij}]_{1\le i,j\le 3}$ be as in (5.3.2), let $Q(0) = z_{11}^2 + z_{22}^2 + z_{33}^2$, and let $T(0) = W \cap (Q(0) = 0)$. Let Δ_{ij} be the $(ij)^{th}$ minor of the 3×3 matrix $[z_{ij}]$; by (5.3.2), $\mathbb{I}_2(W) = \mathrm{Span}\{\Delta_{11}, \Delta_{12}, \dots, \Delta_{33}\}$. One has:

(1) $Q(0) \notin \mathbb{I}_2(W)$ & $\mathrm{rk}\, Q(0) = 3 \le 6 \Rightarrow Q(0) \in Q_0(Z) - (D_6(W) \cap Q_0(Z))$.

(2) $\mathrm{rk}\,(Q(0) + \Delta_{ij}) = 6$ & $(Q(0) + \Delta_{ij}) \notin D_6(W)$, $1 \le i,j \le 3$. In particular, $(Q(0) + \Delta_{ij}) \in D_6(T) - D_6(W) = Q_0(Z) - (D_6(W) \cap Q_0(Z))$, $1 \le i,j, \le 3$.

(3) $Q(0)$, $Q(0) + \Delta_{11}$, $Q(0) + \Delta_{12}$, $\dots$, $Q(0) + \Delta_{33}$ are linearly independent in $\mathbb{I}_2(T(0)) \subset H^0(\mathbb{P}^8, \mathcal{O}(2))$, and $\dim \mathbb{I}_2(T(0)) = 10$. In particular, $\mathrm{Span}\{Q(0), Q(0) + \Delta_{11}, Q(0) + \Delta_{12}, \dots, Q(0) + \Delta_{33}\} = \mathbb{I}_2(T(0))$.

It follows from (1), (2) and (3) that $\mathbb{I}_2(T(0)) \supset \mathrm{Span}\, Q_0(Z) \supset \mathrm{Span}\{Q(0), Q(0) + \Delta_{11}, Q(0) + \Delta_{12}, \dots, Q(0) + \Delta_{33}\} = \mathbb{I}_2(T(0))$, i.e. $\mathrm{Span}\, Q_0(Z) = \mathbb{I}_2(T(0))$. This proves $(*)$ for the general T.

On the other hand, the graded ideal $I(T) = \oplus I_d(T)$ of $T \subset \mathbb{P}^8$ is generated by the component $I_2(T)$, and $\mathbb{I}_2(T) = \mathbb{P}(I_2(T))$.

Therefore the quadrics of $Q_0(Z)$ (resp. - the projective tangent cones of Θ at the points of Z) cut the projective subvariety $T \subset \mathbb{P}^8$ out (see also [25, Prop. 4.14]).

□

5.7.

Remarks.

(i). See [24]: Let $Z_T = \mathrm{Sing}_1^{st}\,\Theta \cup \mathrm{Sing}_2^{st}\,\Theta$. Then $Z \subset Z_T$ can be separated among the components of Z_T by the numerical property:

$Z=$ the union of all the irreducible components of Z_T not having class $12{\cdot}\Theta^6/6!$.

(ii). Let $z \in Z$ be general, and let $C \in \mathcal{D}$ be such that $\Phi(C) = z$. Let $Q_0(C) = Q_0(z) \in Q_0(Z) \subset D_6(T)$ be the rank 6 quadric attached to z, let Λ be the ruling of $Q_0(z)$ defined by $\mathbb{P}^5(C) \in \Lambda$, and let $\bar{\Lambda}$ be the complementary ruling of $Q_0(z)$. Let $\bar{\mathbb{P}^5} \in \bar{\Lambda}$, let $\bar{C} = T \cap \bar{\mathbb{P}^5}$, and let $\bar{z} = \Phi\left(\bar{C}\right)$ be the Abel-Jacobi image of the curve $\bar{C} \in \mathcal{D}$. Obviously, $Q_0\left(\bar{C}\right) = Q_0(\bar{z}) = Q_0(z)$, i.e. the degree of the finite map $Q_0 : Z \to Q_0(Z)$ is at least two. In fact, as it follows from the definition of the map Q_0, the only preimages of the quadric $Q_0(z)$ are the two points z and $\bar{z}$, identified with the two rulings Λ and $\bar{\Lambda}$ of the quadric.

6. The nodal T.

6.1. The tetragonal triples of Donagi connected with the nodal T. Here we describe the two tetragonal triples which corresponds to the 4-gonal systems on the two nodal discriminant sextics of the nodal T - see e.g. [10]:

Let $T = W \cap Q$ have a simple node at the point $(z)_0 = (x)_0 \times (y)_0$. Let $p = p_1 : T \to \mathbb{P}^2$ and $q = p_2 : T \to \mathbb{P}^2$ be the natural projections. Then the discriminant sextic Δ_p of p (resp. - Δ_q of q) has a simple node at the point $(x)_0$ (resp. - at the point $(y)_0$). Let $\mathbb{P}^1_p = |\mathcal{O}_{\mathbb{P}^2}(1-(x)_0)|$ be the pencil of lines through $(x)_0$ (resp. $\mathbb{P}^1_q = |\mathcal{O}_{\mathbb{P}^2}(1-(y)_0)|$). Let $p^{-1}\left((x)_0\right) = L + \bar{L}$, $q^{-1}\left((y)_0\right) = M + \bar{M}$. Clearly, $L \cap \bar{L} = M \cap \bar{M} = (z)_0$.

Let $\mathrm{pr} : T \to \mathbb{P}^7$ be the rational projection from $(z)_0$, and let T_+ be the image of T. In particular, the images $\mathrm{pr}(L), \mathrm{pr}\left(\bar{L}\right), \mathrm{pr}(M)$, and $\mathrm{pr}\left(\bar{M}\right)$ are 4 isolated singular points of T_+, which lie on the exceptional quadric $Q_0 \subset T_+$ of pr.

Let $l \in \mathbb{P}^1_p$ and $m \in \mathbb{P}^1_q$ be general, and let $C(l,m) = p^{-1}(l) \cap q^{-1}(m) \cap T$. A straightforward check gives that $C(l,m)$ is a curve of bidegree $(2,2)$ and of arithmetic genus one, which has a simple node at the point $(z)_0$. Let $q(l,m) = \mathrm{pr}(C(l,m)) \subset T_+$ be the proper image of $C(l,m)$. It follows that $q(l,m)$ is a conic. Thus, T_+ is birational to a conic bundle $s^+ : T^+ \to \mathbb{P}^1_p \times \mathbb{P}^1_q$. It is not hard to see that the birational map $T_+ \to T^+$ is a composition of the blow-ups of the singular points $\mathrm{pr}(L), \mathrm{pr}\left(\bar{L}\right), \mathrm{pr}(M)\,\mathrm{pr}\left(\bar{M}\right)$, followed by contracting of the four exceptional divisors along their rulings. Because of the complexity of the notation, caused by the additional exceptional sets, we shall work on T_+, disregarding the difference between T_+ and T^+; the statement will not change substantially, if we work on T^+.

The birational conic bundle structure $\left\{q(l,m) : l, m \in \mathbb{P}^1_p \times \mathbb{P}^1_q\right\}$ on T_+ determines the non-trivial component of the discriminant curve of $s_+ :\ \Delta \subset \mathbb{P}^1_p \times \mathbb{P}^1_q$. Clearly, Δ is a smooth curve of bidegree $(4,4)$ on the quadric $\mathbb{P}^1_p \times \mathbb{P}^1_q$.

Let $l \in \mathbb{P}^1_p$ be general. Then the surface $S_4(l) = \mathrm{pr}\,(p^{-1}(l)) \subset T_+$ is an anticanonically embedded del Pezzo surface of degree 4. The map $s_+ : S_4(l) \to [l] \times \mathbb{P}^1_q$ defines a conic bundle structure on $S_4(l)$, which has degenerations at the 4 intersection points $\Delta \cap \left([l] \times \mathbb{P}^1_q\right)$. Let $\tilde{\Delta}$ be the double covering of Δ induced by s_+; $\tilde{\Delta}$ is isomorphic to the curve of components of the degenerate fibers over Δ.

Let $g_p \subset \mathrm{Symm}^4 \Delta$ be the 4-gonal system on Δ defined by the set of effective divisors $\left\{[l] \times \mathbb{P}^1_q : l \in \mathbb{P}^1_p\right\}$ (similarly - for g_q), and let $s^*_+\,(g_p) = \left\{L \in \mathrm{Symm}^4 \tilde{\Delta} : (s_+)_*(L) \in g_p\right\}$. Let $l \in \mathbb{P}^1_p$,and $S_4(l)$ be as above. The set of sixteen (-1)-curves on the anticanonically embedded $S_4(l)$ coincide with the set of lines on $S_4(l)$. The map $p = p_1$ defines a splitting of this set into two "equal" parts: Eight of these lines come from the components of the degenerate fibers $p^{-1}(x)$, $(x) \in (l \cap \Delta_p) - (x)_0$, and the second 8-tuple is the set of these lines on $S_4(l)$ which are components of the four degenerate conics $s_+^{-1}(u)$, $u \in \Delta \cap \left([l] \times \mathbb{P}^1_q\right)$. The lines from the second 8-tuple are components of the fibers of the conic bundle structure s_+. However, the lines from the first 8-tuple are "sections" of s_+ - the map s_+ sends each of these lines isomorphically onto a line on the base quadric $\mathbb{P}^1_p \times \mathbb{P}^1_q$. Each of the lines of the first 8-tuple intersects exactly 4 lines of the second 8-tuple. Moreover, if $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ is such a 4-tuple of lines (of the 2-nd system), then $(s_+)_*\,(\lambda_1 + \cdots + \lambda_4) = \left([l] \times \mathbb{P}^1_q\right) \cap \Delta$.

The last causes a splitting of the natural preimage $(s_+)^*\left(\left([l] \times \mathbb{P}^1_q\right) \cap \Delta\right)$ of $\left([l] \times \mathbb{P}^1_q\right) \cap \Delta$, in $\mathrm{Symm}^4 \tilde{\Delta}$, into the following two subsets - each of cardinality 8:

(1) The set $\tilde{\Delta}^+_p(l) =$ the set of 4-tuples defined by the intersections with the projections of the 8 components of the degenerate fibers $p^{-1}(x)$, $(x) \in (l \cap \Delta_p)\,(x)_0$;

(2) the complementary set $\tilde{\Delta}^-_p(l) := (s_+)^*\left(\left([l] \times \mathbb{P}^1_q\right) \cap \Delta\right) - \tilde{\Delta}^+_p(l)$.

Clearly, this splitting does not depend on the perticular choice of the general line $l \in \mathbb{P}^1_p$. Therefore it defines a global splitting $(s_+)^*\,(g_p) = \tilde{\Delta}^+_p \cup \tilde{\Delta}^-_p$.

Obviously, the component $\tilde{\Delta}^+_p$ is isomorphic to the non-singular model of the double covering $\tilde{\Delta}_p$ of Δ induced by the projection $p : T \to \mathbb{P}^2$.

There are naturally defined involutions $i_p^+ : \tilde{\Delta}_p^+ \to \tilde{\Delta}_p^+$ and $i_p^- : \tilde{\Delta}_p^- \to \tilde{\Delta}_p^-$, defined by interchanging the 4-tuple $\lambda_1, \ldots, \lambda_4$ with its complementary $\bar{\lambda}_1, \ldots, \bar{\lambda}_4$. (By definition, $\lambda_i + \bar{\lambda}_i$, $i = 1, \ldots, 4$, are the four degenerate conics of $s_+ : S_4(l) \to \left([l] \times \mathbb{P}^1_q\right)$).

In fact, the 4-tuples $(\lambda_1, \ldots, \lambda_4) \in \tilde{\Delta}_p^+$ are (-1)-curves on $S_4(l)$; the same - for the complementary 4-tuples. Denote by $S\,(\lambda_1, \ldots, \lambda_4)$ the ruled surface, which is defined by contraction of the complementary 4-tuple $(\bar{\lambda}_1, \ldots, \bar{\lambda}_4)$. It follows from the definition of the elements of $\tilde{\Delta}_p^+$ (= the existence of a secant (-1)-curve - see above) that $S\,(\lambda_1, \ldots, \lambda_4) = \mathbb{F}_1$.

Similarly, the 4-tuples which belong to the component $\tilde{\Delta}_p^-$ correspond to the relatively minimal models $S\,(\bar{\lambda}_1, \ldots, \bar{\lambda}_4)$, of the surfaces $S_4(l)$, which are of type $\mathbb{F}_0$ (i.e. - quadrics).

Let $\Delta_p^+ = \tilde{\Delta}_p^+ / i_p^+$ and $\Delta_p^- = \tilde{\Delta}_p^- / i_p^-$ be the quotient curves. Obviously, the natural 8-sheeted coverings $\tilde{\Delta}_p^+ \to \mathbb{P}^1_p$ and $\tilde{\Delta}_p^- \to \mathbb{P}^1_p$ define the 4-sheeted coverings (the 4-gonal systems): $g_p^+ : \Delta_p^+ \to \mathbb{P}^1_p$ and $g_p^- : \Delta_p^- \to \mathbb{P}^1_p$.

In fact, we have restored the tetragonal construction of Donagi - see e.g. [10]. Therefore we have proved the following (see the notations above):

Proposition 6.2. *$\left\{\left(\tilde{\Delta}, \Delta\right), \left(\tilde{\Delta}_p^+, \Delta_p^+\right), \left(\tilde{\Delta}_p^-, \Delta_p^-\right)\right\}$ is a 4-gonal triple of Donagi - see [10]. Moreover, $\tilde{\Delta}_p^+$ is isomorphic to the smooth model of the nodal determinant plane sextic $\Delta_p \subset \mathbb{P}^2$, and the involution $i_p^+ : \tilde{\Delta}_p^+ \to \tilde{\Delta}_p^-$ is a desingularization of the involution i_p on $\tilde{\Delta}_p$, defined by the covering $\tilde{\Delta}_p \to \Delta_p$.*

The same is true also for the 4-gonal triple $\left\{\left(\tilde{\Delta}, \Delta\right), \left(\tilde{\Delta}_q^+, \Delta_q^+\right), \left(\tilde{\Delta}_q^-, \Delta_q^-\right)\right\}$ of Donagi, which corresponds to the 4-gonal system g_q on the $(4,4)$-curve Δ; just like above, the curve $\tilde{\Delta}_q^-$ is isomorphic to the smooth model of the curve $\tilde{\Delta}_q$ of components of degenerate fibers for $q : T \to \mathbb{P}^2$.

Corollary 6.3. *Let T be a general nodal bidegree $(2,2)$ divisor, and let $\left(\tilde{\Delta}_p, \Delta_p\right)$ and $\left(\tilde{\Delta}_q, \Delta_q\right)$ be the discriminant pairs for the natural projections $p : T \to \mathbb{P}^2$ and $q : T \to \mathbb{P}^2$. Let $\left(\tilde{\Delta}, \Delta\right)$ be the discriminant pair of the conic bundle structure $s_+ : T^+ \to \mathbb{P}^1 \times \mathbb{P}^1$ defined by the node of T (see above), and let g_p and g_q be the 4-gonal systems on the $(4,4)$-curve Δ defined by the rulings of the quadric $\mathbb{P}^1 \times \mathbb{P}^1$. Then g_p and g_q define the two tetragonal triples of Donagi:*

$$\left\{\left(\tilde{\Delta}, \Delta\right), \left(\tilde{\Delta}_p^+, \Delta_p^+\right), \left(\tilde{\Delta}_p^-, \Delta_p^-\right)\right\} \text{ and } \left\{\left(\tilde{\Delta}, \Delta\right), \left(\tilde{\Delta}_q^+, \Delta_q^+\right), \left(\tilde{\Delta}_q^-, \Delta_q^-\right)\right\}$$

such that $\left(\tilde{\Delta}_p^+, \Delta_p^+\right)$ *is a desingularization of the nodal pair* $\left(\tilde{\Delta}_p, \Delta_p\right)$ (*see above*), *and the pair* $\left(\tilde{\Delta}_q^+, \Delta_q^+\right)$ *is a desingularization of* $\left(\tilde{\Delta}_q, \Delta_q\right)$. *In other words, the Dixon correspondence* (*see* [9], [24]) *between the discriminant pairs of the nodal bidegree* $(2,2)$ *divisor* T *is a composition of two* 4*-gonal correspondences of Donagi.*

Acknoledgement. I would like to express my gratitude to R. Donagi and A. Verra for their substantial help, and also - to the referee for his remarks.

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Received July 8, 1994 and revised November 29, 1995. Supported in part by the Bulgarian foundation "Scientific Research" and by NSF under the US-Bulgarian project "Algebra and Algebraic Geometry".

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