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ON THE GEOMETRY OF VARIETIES OF INVERTIBLE SYMMETRIC AND SKEW-SYMMETRIC MATRICES

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Let $\mathrm{Sym}(n, \mathbf{F})$ and $\mathrm{Sk}(n, \mathbf{F})$ denote the algebraic varieties of $n \times n$ invertible symmetric and skew-symmetric matrices over a field F, respectively. We first show how the homotopy type of $\mathrm{Sym}(n, \mathbf{R})$ and the homology groups of $\mathrm{Sk}(n, \mathbf{R})$ can be determined using an alternative method to Iwasawa decomposition. Then, using recent results of Dimca and Lehrer, the weight polynomials of $\mathrm{Sym}(n, \mathbf{C})$ and $\mathrm{Sk}(n, \mathbf{C})$ are calculated.

1. Introduction and notations.

Let $\mathbf{F}$ be a field, $\mathrm{GL}(n, \mathbf{F})$ the set of $n \times n$ invertible matrices over $\mathbf{F}$, $\mathrm{Sym}(n, \mathbf{F})$ the set of $n \times n$ invertible symmetric matrices over $\mathbf{F}$ and $\mathrm{Sk}(n, \mathbf{F})$ the set of $n \times n$ invertible skew-symmetric matrices over $\mathbf{F}$. Note that in the case of the invertible skew-symmetric matrices n has to be even. In this paper we first determine the toplogy of the real varieties $\mathrm{Sym}(n, \mathbf{R})$ and $\mathrm{Sk}(n, \mathbf{R})$. More precisely, we show that $\mathrm{Sym}(n, \mathbf{R})$ has the homotopy type of a Grassmannian by constructing a homotopy equivalence and compute the Betti numbers of $\mathrm{Sk}(n, \mathbf{R})$ by fibering this variety over the sphere and using the associated Leray spectral sequence.

The Serre-Poincaré polynomial (weight polynomial) of a complex algebraic variety X is defined to be

$$W_c(X, t) = \sum_{i,j} \sum_{p+q=i} (-1)^j h^{p,q}(H_c^j(X; \mathbf{C}))t^i$$

where $h^{p,q}(H_c^j(X; \mathbf{C}))$ are the mixed Hodge numbers of $H_c^j(X; \mathbf{C})$. The main result of this paper is the computation of the weight polynomials for the complex varieties $\mathrm{Sym}(n, \mathbf{C})$ and $\mathrm{Sk}(n, \mathbf{C})$. This is done by using recent results of Dimca and Lehrer [DL] concerning fibrations and weight polynomials. As a corollary, the weight polynomials of the symmetric and skew-symmetric determinantal varieties can also be determined.

If an algebraic variety X is defined by polynomials

$$(*) \qquad p_1(\mathbf{x}) = 0, \cdots, p_k(\mathbf{x}) = 0$$

where $p_1, \cdots, p_k \in \mathbf{Z}[x_1, \cdots, x_n]$, then it can be reduced modulo q to give a variety defined over $\mathbf{F}_q$, the finite field of q elements. For $\mathbf{K} = \mathbf{F}_q, \mathbf{R}$ or $\mathbf{C}$ let $X(\mathbf{K})$ denote the set of solutions of $(*)$ rational over $\mathbf{K}$.

We show that for the varieties considered in this paper there is an interesting relationship between the cardinality of $X(\mathbf{F}_q)$, the compact Euler characteristic of $X(\mathbf{R})$ and the weight polynomial of $X(\mathbf{C})$. See Theorems (1.8) and (1.8)$'$ below. The relations displayed in these theorems hold true for other interesting classes of algebraic varieties as well. We just mention here the hyperplane arrangement complements (see for instance [**L**]) and the complements of some classical discriminants. This follows a general philosophy in algebraic geometry (due to Weil, Serre, Grothendieck and Deligne), see for instance [**BBD**] Section 6, relating cohomological properties of schemes over finite fields and the complex numbers. We point out however that relations with the topology of the real points are not part of this general setting. Moreover, it is not obvious whether the comparison theorems can be directly applied to our schemes $\mathrm{Sym}(n, \mathbf{F})$ and $\mathrm{Sk}(n, \mathbf{F})$ as they are not projective.

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2. Main results.

Define an action $\mathrm{GL}(n, \mathbf{F}) \times \mathrm{Sym}(n, \mathbf{F}) \to \mathrm{Sym}(n, \mathbf{F})$ by $g \cdot A = gA^tg$. An action $\mathrm{GL}(n, \mathbf{F}) \times \mathrm{Sk}(n, \mathbf{F}) \to \mathrm{Sk}(n, \mathbf{F})$ is defined similarly.

Let $\mathrm{Sym}_i(n)$ be the orbit of

$$\begin{pmatrix} -I_i & O \\ O & I_{n-i} \end{pmatrix}$$

considered as an element in $\mathrm{Sym}(n, \mathbf{R})$, i.e. the set of all matrices in $\mathrm{Sym}(n, \mathbf{R})$ having index i. It is known that the orbits $\mathrm{Sym}_i(n)$ are exactly the connected components of $\mathrm{Sym}(n, \mathbf{R})$, see for instance [**D1**], p. 46. In particular, $\mathrm{Sym}_i(n)$ are open smooth manifolds of dimension $n(n+1)/2$. Let $G_i(\mathbf{F}^n)$ denote the Grassmann variety of i-dimensional subspaces of $\mathbf{F}^n$. Our first result is the following:

Theorem 1.1. *$\mathrm{Sym}_i(n)$ and $G_i(\mathbf{R}^n)$ are of the same homotopy type for $i = 0, \cdots, n$.*

The set of invertible skew-symmetric matrices over $\mathbf{R}$, $\mathrm{Sk}(n, \mathbf{R})$, has two connected components $\mathrm{Sk}^+(n)$ and $\mathrm{Sk}^-(n)$ corresponding to $Pf(A) > 0$ and $Pf(A) < 0$ respectively, where $Pf(A)$ is the Pfaffian of A.

Theorem 1.1′. *The algebraic varieties* $\mathrm{Sk}^+(n)$ *and* $\mathrm{Sk}^-(n)$ *are isomorphic, have dimension* $(n-1)n/2$ *and the homology of* $S^2 \times S^4 \times \cdots \times S^{n-2}$.

In general, $\mathrm{Sk}^+(n)$ is not of the same homotopy type as $S^2 \times S^4 \times \cdots \times S^{n-2}$.

Proposition 1.2. $\mathrm{Sk}^+(6)$ *is not homotopy equivalent to* $S^2 \times S^4$.

Durfee [**Du**] has shown that the weight polynomial is additive on disjoint unions of quasi-projective varieties. More recently, Dimca and Lehrer [**DL**] have shown that if E, B and F are smooth complex algebraic varieties and $p : E \to B$ is a locally trivial fibration (in the strong toplogy) such that the local system $R^i p_* \mathbf{C}_E$ is constant for any i then

$$W_c(E) = W_c(B) W_c(F).$$

If G is a connected algebraic group and H a connected subgroup then we can apply this result to the fibration

$$p : G \to G/H$$

to obtain

$$W_c(G/H, t) = W_c(G, t)/W_c(H, t). \tag{1.3}$$

If H is not connected, let H^o denote the connected component of the identity in H. Then $N = H/H^o$ is a finite group and $G/H = (G/H^o)/N$. If H is discrete then, since G is connected, the action of N on the cohomology of G/H^o is trivial. It follows that

$$H^\bullet(G/H) = H^\bullet(G/H^o)$$

and

$$W(G/H) = W(G/H^o).$$

Let $M_n(\mathbf{C})$ denote the set of $n \times n$ matrices over $\mathbf{C}$ and $SD(i, n)$ the symmetric determinantal variety, i.e.

$$SD(i, n) = \left\{ A \in M_n(\mathbf{C}) : {}^t A = A \quad \text{and} \quad \mathrm{rk}\, A \le i \right\}.$$

Consider the stratification

$$\begin{aligned} \mathbf{C}^{(i+1)(i+2)/2} = {} & (SD(i+1, i+1) \setminus SD(i, i+1)) \cup \cdots \\ & \cup (SD(2, i+1) \setminus SD(1, i+1)) \cup SD(1, i+1). \end{aligned} \tag{1.4}$$

Define

$$p : SD(i,n) \setminus SD(i-1,n) \to G_{n-i}(\mathbf{C}^n) : A \mapsto \ker A. \tag{1.5}$$

The map p is a locally trivial fibration with fibre $\mathrm{Sym}(i, \mathbf{C})$ and since the Grassmannians are simply-connected we can apply the multiplicativity of weight polynomials proved in [**DL**] to (1.5). This together with the additivity of the weight polynomial applied to (1.4) yields:

$$\begin{aligned} &W_c(\mathrm{Sym}(i+1, \mathbf{C}), t) \\ &\quad = W_c(\mathbf{C}^{(i+1)(i+2)/2}, t) - W_c(G_1(\mathbf{C}^{i+1}), t)W_c(\mathrm{Sym}(i, \mathbf{C}), t) - \cdots \\ &\qquad - W_c(G_i(\mathbf{C}^{i+1}), t)W_c(\mathrm{Sym}(1, \mathbf{C}), t) - 1. \end{aligned} \tag{1.6}$$

This is the key step in getting our next result:

Theorem 1.7.

$$W_c(\mathrm{Sym}(n, \mathbf{C}), t) = \begin{cases} t^{n(n+2)/2}(t^2-1)(t^6-1)\cdots(t^{2n-2}-1)\,, & n \quad \textit{even} \\ t^{(n^2-1)/2}(t^2-1)(t^6-1)\cdots(t^{2n}-1)\,, & n \quad \textit{odd.} \end{cases}$$

In Theorem 9.1.5 of [**De**] Deligne determines the mixed Hodge structure of $H^\bullet(G;\mathbf{Q})$ where G is a connected linear algebraic group. For the case of $Sp(n, \mathbf{C})$ we give a more elementary construction.

Lemma 1.8.

$$W_c(Sp(n, \mathbf{C}), t) = t^{n^2/2}(t^4-1)(t^8-1)\cdots(t^{2n}-1).$$

Proof. Let

$$F_n = \left\{(\mathbf{x}_1, \mathbf{x}_2) \in \mathbf{C}^n \times \mathbf{C}^n : \mathbf{x}_1 J^t \mathbf{x}_2 = 1\right\}$$

and define an action $\mathrm{Sp}(n, \mathbf{C}) \times F_n \to F_n$ by

$$A \cdot (\mathbf{x}_1, \mathbf{x}_2) = (\mathbf{x}_1 A, \mathbf{x}_2 A).$$

$\mathrm{Sp}(n, \mathbf{C})$ acts transitively on F_n and the isotropy group of $(\mathbf{e}_1, \mathbf{e}_{n/2+1})$ is $\mathrm{Sp}(n-2, \mathbf{C})$. Therefore,

$$W_c(\mathrm{Sp}(n, \mathbf{C}), t) = W_c(\mathrm{Sp}(n-2, \mathbf{C}), t)W_c(F_n, t).$$

It is known (see for instance [**D2**], p. 244) that

$$W_c(F_n, t) = t^{2n-2}(t^{2n}-1). \qquad \square$$

The general linear group $\mathrm{GL}(n, \mathbf{C})$ acts transitively on $\mathrm{Sk}(n, \mathbf{C})$ and the isotropy group of J is the complex sympletic group $\mathrm{Sp}(n, \mathbf{C})$, where

$$J = \begin{pmatrix} O & I \\ -I & O \end{pmatrix}.$$

Since $\mathrm{Sp}(n, \mathbf{C})$ is a connected subgroup of $\mathrm{GL}(n, \mathbf{C})$, we can use (1.3) to get

$$W_c(\mathrm{Sk}(n, \mathbf{C}), t) = W_c(\mathrm{GL}(n, \mathbf{C}), t)/W_c(\mathrm{Sp}(n, \mathbf{C}), t).$$

The polynomial $W_c(\mathrm{GL}(n, \mathbf{C}), t)$ is also known, see [**DL**] Corollary 6.6, and is given by

$$W_c(\mathrm{GL}(n, \mathbf{C}), t) = t^{n(n-1)}(t^2 - 1)(t^4 - 1) \cdots (t^{2n} - 1).$$

This gives the following:

Theorem 1.7′.

$$W_c(\mathrm{Sk}(n, \mathbf{C}), t) = t^{n(n-2)/2}(t^2 - 1)(t^6 - 1) \cdots (t^{2n-2} - 1).$$

Remark. Since,

$$SD(i, n) = (SD(i, n) \setminus SD(i-1, n)) \cup \cdots \cup (SD(2, n) \setminus SD(1, n)) \cup SD(1, n)$$

we can use (1.5) and Theorem (1.7) to compute the weight polynomial of $SD(i, n)$. In an analogous way, the weight polynomial of the skew-symmetric determinantal variety can also be determined.

Let

$$\chi_c(\Omega) = \sum_i (-1)^i \dim H_c^i(\Omega; \mathbf{R})$$

denote the compact Euler characteristic of a space Ω.

Theorem 1.9. *For all $n \in \mathbf{N}$,*

$$(**) \qquad |\,\mathrm{Sym}(n)(\mathbf{F}_q)| = W_c(\mathrm{Sym}(n, \mathbf{C}), q^{1/2})$$

and

$$\chi_c(\mathrm{Sym}(n, \mathbf{R})) = W_c(\mathrm{Sym}(n, \mathbf{C}), i).$$

Here, $|\quad|$ denotes the cardinality of the finite set $\mathrm{Sym}(n)(\mathbf{F}_q)$. An analogous result holds for the skew-symmetric matrices:

Theorem 1.8′. *For all* $n \in \mathbf{N}$

$$|\operatorname{Sk}(n)(\mathbf{F}_q)| = W_c(\operatorname{Sk}(n,\mathbf{C}), q^{1/2})$$

and

$$\chi_c(\operatorname{Sk}(n,\mathbf{R})) = W_c(\operatorname{Sk}(n,\mathbf{C}), i).$$

Remark. Let Γ be a finite group of algebraic automorphisms of the complex variety X and $R(\Gamma)$ the Grothendieck ring of Γ. Then one can define the equivariant weight polynomial $W_c^\Gamma(X)(t)$ as an element of the ring $R(\Gamma)[t]$, see [**DL**] for this construction and its basic properties. Some of the results above can be restated in terms of these equivariant weighted polynomials. More precisely, let $X = \operatorname{GL}(n,\mathbf{C})/\operatorname{SO}(n,\mathbf{C})$ and $\Gamma = O(n,\mathbf{C})/\operatorname{SO}(n,\mathbf{C}) \simeq \{\pm 1\}$ with the natural action on X.

The group Γ has just two irreducible representations, namely the trivial one τ and the nontrivial one ϵ (given by multiplication by -1 on $\mathbf{C}$). It follows that one can write

$$W_c^\Gamma(X,t) = P(t)\cdot\tau + Q(t)\cdot\epsilon.$$

Now $P(t)$ is the weight polynomial of the invariant part $H^\bullet(X)^\Gamma$, which is the same as $H^\bullet(X/\Gamma) = H^\bullet(\operatorname{Sym}(n,\mathbf{C}))$. Hence,

$$P(t) = W_c(\operatorname{Sym}(n,\mathbf{C}), t).$$

To determine $Q(t)$, note that there is an obvious morphism $\rho : R(\Gamma)[t] \to \mathbf{Z}[t]$ obtained by taking the dimension of a Γ-representation. Then we have

$$\rho(W_c^\Gamma(X)) = W_c(X) = W_c(\operatorname{GL}(n,\mathbf{C}))/W_c(\operatorname{SO}(n,\mathbf{C}))$$

by (1.3). On the other hand, we obviously have

$$\rho(W_c^\Gamma(X)) = P(t) + Q(t)$$

hence

$$Q(t) = W_c(\operatorname{GL}(n,\mathbf{C}))/W_c(\operatorname{SO}(n,\mathbf{C})) - W_c(\operatorname{Sym}(n,\mathbf{C})).$$

As a consequence of our formulas, it is interesting to note that $Q(t) = 0$ for n odd. We now show in a different way that for $n = 2$ $Q(t) \neq 0$. In this case Γ is generated by the equivalence class with representative

$$\gamma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let $Y = \{(x_0, x_1, x_2) \in \mathbf{C}^3 : x_0x_2 - x_1^2 - 1 = 0\}$, the Milnor fibre of the A_1-singularity and define $h : X \to Y \times \mathbf{C}^*$ by

$$h[g] = \left(g_{11}^2 + g_{21}^2, g_{11}g_{12} + g_{21}g_{22}, g_{12}^2 + g_{22}^2, \det g^2\right) / \det g$$

where

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}.$$

h is a homeomorphism and Γ acts on $Y \times \mathbf{C}^*$ by

$$\gamma(x_0, x_1, x_2, x_3) = (-x_0, -x_1, -x_2, -x_3).$$

Now,

$$\begin{aligned} H^\bullet(X) &= \operatorname{Tot} H^\bullet(Y) \otimes H^\bullet(\mathbf{C}^*) \\ &= \operatorname{Tot} \Lambda(\Omega) \otimes \Lambda(dx/x) \\ &= \Lambda(dx/x, \Omega) \end{aligned}$$

where $\Omega = j^*\Delta(dx_0 \wedge dx_1 \wedge dx_2)$, and $j : X \to \mathbf{C}^3$ is the inclusion, see [**D2**] p. 192. Hence,

$$H^\bullet(X)^\Gamma = \Lambda(dx/x).$$

3. The proofs.

Proof of Theorem 1.1. First we note that $\mathrm{Sym}_0(n)$ and $\mathrm{Sym}_n(n)$ are convex subsets in $\mathbf{R}^{n^2}$, and as such they are contractible. For $0 < i < n$, define $p : \mathrm{Sym}_i(n) \to G_i(\mathbf{R}^n)$ by

$$p(A) = \langle \xi_1, \cdots, \xi_i \rangle$$

where $\xi_1, \cdots, \xi_i$ are the eigenvectors corresponding to the negative eigenvalues of A. In case of multiple eigenvalues, these vectors are not uniquely defined, but the subspace $p(A)$ is. The map p is a locally trivial fibration with contractible fibre and so is a homotopy equivalence. $\square$

Proof of Theorem 1.1′. Let G be any matrix with $\det G < 0$ and define $\phi : \mathrm{Sk}^+(n) \to \mathrm{Sk}^-(n)$ by

$$\phi(A) = GA^tG.$$

Clearly, ϕ is an algebraic isomorphism. Let

$$B = \{(\mathbf{v}, V) \in \mathbf{R}^n \setminus \{\mathbf{0}\} \times G_{n-2}(\mathbf{R}^n) : \\ v_1 = 0, \mathbf{v} \perp V \text{ and } \langle \mathbf{e}_1 \rangle \cap V = \{\mathbf{0}\}\}$$

where $\mathbf{e}_1 = (1, 0, \cdots, 0)$ and $\mathbf{v} = (v_1, \cdots, v_n)$. Define $p_1 : \mathrm{Sk}^+(n) \to B$ by

$$p_1(A) = (\mathbf{v}, \langle \mathbf{e}_1, \mathbf{v} \rangle^{\perp})$$

where if A has entries a_{ij} then $\mathbf{v} = (0, a_{12}, \cdots, a_{1\,n})/(a_{12}^2 + \cdots + a_{1\,n}^2)$. Here the orthogonality is with respect to A. The map p_1 is a locally trivial fibration with fibre $\mathrm{Sk}^+(n-2)$. Define $p_2 : B \to S^{n-2}$ by

$$p_2(\mathbf{v}, V) = (v_2, \cdots, v_n) / \left(v_2^2 + \cdots + v_n^2\right)^{1/2}.$$

Clearly, p_2 is a locally trivial fibration with contractible fibre. Let

$$p = p_2 \circ p_1 : \mathrm{Sk}^+(n) \to S^{n-2}.$$

Now $p : \mathrm{Sk}^+(4) \to S^2$ is a homotopy equivalence and so $H_\bullet(\mathrm{Sk}^+(4)) \simeq H_\bullet(S^2)$. Assume $H_\bullet(\mathrm{Sk}^+(i)) \simeq H_\bullet(S^2 \times S^4 \times \cdots \times S^{i-2})$ for some integer i. The Wang exact sequence of the fibration $p : \mathrm{Sk}^+(i+2) \to S^i$ is given by

$$\begin{aligned} \cdots \to H_{n-i+1}(\mathrm{Sk}^+(i)) \to H_n(\mathrm{Sk}^+(i)) \to H_n(\mathrm{Sk}^+(i+2)) \to \\ H_{n-i}(\mathrm{Sk}^+(i)) \to H_{n-1}(\mathrm{Sk}^+(i)) \to \cdots . \end{aligned}$$

If n is odd, $H_n(\mathrm{Sk}^+(i)) = H_{n-i}(\mathrm{Sk}^+(i)) = 0$, and so $H_n(\mathrm{Sk}^+(i+2)) = 0$. If n is even, the Wang exact sequence splits into the short exact sequence

$$0 \to H_n(\mathrm{Sk}^+(i)) \to H_n(\mathrm{Sk}^+(i+2)) \to H_{n-i}(\mathrm{Sk}^+(i)) \to 0.$$

Therefore,

$$\begin{aligned} b_n(\mathrm{Sk}^+(i+2)) &= b_n(\mathrm{Sk}^+(i)) + b_{n-i}(\mathrm{Sk}^+(i)) \\ &= b_n\left(S^2 \times S^4 \times \cdots \times S^i\right). \end{aligned}$$

□

Remark. The above results can be proved using Iwasawa decomposition as follows: if G is a Lie group and K a maximal compact subgroup then $K \subset G$ is a deformation retract, see [B] p. 70. Let $\mathrm{Iso}(A)$ denote the isotropy group of the element

$$A = \begin{pmatrix} -I_i & O \\ O & I_{n-i} \end{pmatrix}.$$

$\mathrm{GL}(n, \mathbf{R})$ has $O(n)$ as a maximal compact subgroup and so $O(n) \cap \mathrm{Iso}(A) = O(i) \times O(n-i)$ is a maximal compact subgroup in $\mathrm{Iso}(A)$. Hence $\mathrm{Sym}_i(n) = \mathrm{GL}(n, \mathbf{R}) / \mathrm{Iso}(A)$ contains the Grassmann manifold $O(n)/O(i) \times O(n-i)$ as a deformation retract. Similarly, $\mathrm{Sp}(n, \mathbf{R})$ has $\mathrm{Sp}(n, \mathbf{R}) \cap O(n) = U(n)$

as a maximal compact subgroup. Therefore, $\mathrm{Sk}(n) = \mathrm{GL}(n,\mathbf{R})/\mathrm{Sp}(n,\mathbf{R})$ contains $O(n)/U(n)$ as a deformation retract. The latter has two connected components both isomorphic to $\mathrm{SO}(n)/U(n)$, the cohomology of which is known, see for instance [**B**] p. 210.

Proof of Proposition 1.2. $\mathrm{GL}(n,\mathbf{R})$ acts transitively on $\mathrm{Sk}(n,\mathbf{R})$ and the isotropy group of J is the real sympletic group $\mathrm{Sp}(n,\mathbf{R})$. Consider the following exact sequence

$$\cdots \to \pi_i(\mathrm{Sp}(6,\mathbf{R})) \to \pi_i(\mathrm{GL}(6,\mathbf{R})) \to \pi_i(\mathrm{Sk}(6,\mathbf{R})) \to \pi_{i-1}(\mathrm{Sp}(6,\mathbf{R})) \to \cdots .$$

Now

$$\pi_i(\mathrm{GL}(6,\mathbf{R})) \simeq \pi_i(\mathbf{O}) \qquad \text{for} \quad i<5$$

and

$$\pi_i(\mathrm{Sp}(6,\mathbf{R})) \simeq \pi_i(\mathbf{U}) \qquad \text{for} \quad i<6$$

see [**MT**] p. 216. Also $\pi_4(\mathbf{O}) = 0$ and $\pi_3(\mathbf{U}) = \mathbf{Z}$, see [**MT**] p. 212, and therefore $\pi_4(\mathrm{Sk}(6,\mathbf{R}))$ cannot be $\mathbf{Z}_2 \oplus \mathbf{Z}$. $\square$

Proof of Theorem 1.7. This is by induction on n. Now $\mathrm{Sym}(1,\mathbf{C}) = \mathbf{C}^*$ and so the statement holds for $n=1$. Let

$$\mathbf{p}_n = \begin{cases} q^{n(n+2)/4}(q-1)(q^3-1)\cdots(q^{n-1}-1)\,, & n \quad \text{even} \\ q^{(n^2-1)/4}(q-1)(q^3-1)\cdots(q^{n}-1)\,, & n \quad \text{odd} \end{cases}$$

and assume the statement is true for $i \le k$. By the inductive hypothesis

$$\text{(i)} \qquad \sum_{j=0}^{k-1} \begin{pmatrix} \mathbf{k}-1 \\ \mathbf{j} \end{pmatrix} \mathbf{p}_{k-1-j} = q^{(n-1)(n-2)/2}$$

and

$$\text{(ii)} \qquad \sum_{j=0}^{k} \begin{pmatrix} \mathbf{k} \\ \mathbf{j} \end{pmatrix} \mathbf{p}_{k-j} = q^{n(n-1)/2}$$

where $\left(\begin{smallmatrix} \mathbf{n} \\ \mathbf{k} \end{smallmatrix}\right)$ is the Gaussian polynomial, see [**St**, p. 26].

Multiplying (i) by $q^k(q^k-1)$ and (ii) by $q^k(q-1)$ and adding gives

$$\sum_{j=0}^{k-1} q^j \begin{pmatrix} \mathbf{k} \\ \mathbf{j} \end{pmatrix} \mathbf{p}_{k+1-j} + q^{k+1} - q^k = q^{(k+1)(k+2)/2} - q^{k(k+1)/2}.$$

Now

$$q^j \binom{\mathbf{k}}{\mathbf{j}} = \binom{\mathbf{k}+1}{\mathbf{j}} - \binom{\mathbf{k}}{\mathbf{j}-1}$$

and so

$$\sum_{j=0}^{k-1} \binom{\mathbf{k}+1}{\mathbf{j}} \mathbf{p}_{k+1-j} - \sum_{j=0}^{k-2} \binom{\mathbf{k}}{\mathbf{j}} \mathbf{p}_{k-j} + q^{k+1} - q^k$$
$$= q^{(k+1)(k+2)/2} - q^{k(k+1)/2}.$$

Adding (ii) to this gives

$$\sum_{j=0}^{k+1} \binom{\mathbf{k}+1}{\mathbf{j}} \mathbf{p}_{k+1-j} = q^{(k+1)(k+2)/2}.$$

□

Proof of Theorem 1.9. Verification of the first relation in the theorem is by induction on n. We have $\mathrm{Sym}(1, \mathbf{C}) = \mathbf{C}^*$ and so

$$\begin{aligned} W_c\left(\mathrm{Sym}(1, \mathbf{C}), q^{1/2}\right) &= q - 1 \\ &= |\,\mathrm{Sym}(1)(\mathbf{F}_q)|. \end{aligned}$$

Suppose the statement holds for $k \le i$. From (1.4) we have

$$\begin{aligned} |\,\mathrm{Sym}(i+1)(\mathbf{F}_q)| = \left|\mathbf{F}_q^{(i+1)(i+2)/2}\right| - \left|(G_1(\mathbf{F}_q^{i+1}))\right| |\,\mathrm{Sym}(i)(\mathbf{F}_q)| - \\ \cdots - \left|(G_i(\mathbf{F}_q^{i+1}))\right| |\,\mathrm{Sym}(1)(\mathbf{F}_q)|. \end{aligned}$$

The result follows from comparing this with (1.6) and the fact that $\mathbf{C}^n$ and $G_i(\mathbf{C}^n)$ satisfy $(**)$. For the second relation we have that from Theorem (1.1)

$$\begin{aligned} \chi_c(\mathrm{Sym}(n, \mathbf{R})) &= (-1)^{n(n+1)/2} \sum_{i=0}^{n} \chi(\mathrm{Sym}_i(n)) \\ &= (-1)^{n(n+1)/2} \sum_{i=0}^{n} \chi(G_i(\mathbf{R}^n)). \end{aligned}$$

Now,

$$\chi(G_i(\mathbf{R}^{2m+1})) = \binom{m}{[i/2]}$$

and

$$\chi(G_i(\mathbf{R}^{2m})) = \begin{cases} 0 & i \quad \text{odd} \\ \binom{m}{i/2} & i \quad \text{even} \end{cases}$$

see for instance [**E**]. Hence,

$$\chi_c(\mathrm{Sym}(n, \mathbf{R})) = \begin{cases} (-2)^{m+1} & n = 2m+1 \\ (-2)^m & n = 2m. \end{cases}$$

□

Proof of Theorem 1.8′. Let $\mathrm{Sk}\, D(i,n)$ denote the skew-symmetric determinantal variety

$$\mathrm{Sk}\, D(i,n) = \left\{ A \in M_n(\mathbf{C}) : {}^t A = A \quad \text{and} \quad \mathrm{rk}\, A \leq i \right\}.$$

The first relation is verified in a way completely analogous to that of the case for $\mathrm{Sym}(n, \mathbf{C})$. For the second, Theorem (1.1)′ implies

$$\begin{aligned} \chi_c(\mathrm{Sk}(n, \mathbf{R})) &= (-1)^{n/2} 2\chi(\mathrm{Sk}^+(n, \mathbf{R})) \\ &= (-1)^{n/2} 2\chi(S^2)\chi(S^4)\cdots\chi(S^{n-2}) \\ &= (-2)^{n/2}. \end{aligned}$$

□

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