

A QUESTION OF STARK

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One of the programs of Stark's conjectures is to find as many connections as possible between the values that Artin L -functions or their derivatives take (especially at $s = 0$) and arithmetic information associated to algebraic number fields. The most refined of Stark's conjectures involves the values of first derivatives of L -functions at $s = 0$. It was recognized early on that the conjecture should be extended to cover cases where the order of vanishing of the L -functions at $s = 0$ is greater than one. In 1980, Stark posed a question along these lines that we will consider in detail here. In particular, we will study his question for relative quadratic extensions and prove that an affirmative answer to his question exists for all cases considered.

1. Introduction.

Our aim in this section is to state Stark's question and see how it is related to his refined conjecture. Throughout this paper k will be a real quadratic number field and K will denote an abelian extension of k . We consider K and k to be subfields of the complex numbers \mathbb{C} . Three different cases arise: i) Both infinite primes of k split in K , i.e. K is totally real. ii) One infinite prime of k ramifies and the other splits. iii) Both infinite primes ramify. Stark's question involves case i). His refined conjecture deals with cases ii) and iii) (technically it deals with case i) as well but it doesn't tell us too much as we'll see below). We use the following notation for all three cases. S will denote a finite set of primes in k satisfying the following assumptions:

- (a) It contains both infinite primes,
- (b) it contains all finite primes that ramify in K ,
- (c) it contains a distinguished set of r primes that split completely in K ,
- (d) the cardinality of S , $|S|$, satisfies $|S| \geq r + 1$.

In cases ii) and iii) we will set $r = 1$ in the following way: ii) $S = \{\infty_k, \overline{\infty}_k, \mathfrak{q}_1, \dots, \mathfrak{q}_t\}$, where ∞_k is the infinite prime corresponding to the identity embedding of k in \mathbb{R} and $\overline{\infty}_k$ is the conjugate prime. The \mathfrak{q}_i are finite primes which include among them all of the ramified primes and we have $t \geq 0$. By assumption, exactly one of the infinite primes split and this will be our

distinguished prime which we denote by \mathfrak{p} . **iii)** $S = \{\infty_k, \overline{\infty}_k, \mathfrak{q}_1, \dots, \mathfrak{q}_t, \mathfrak{p}\}$, where since both infinite primes ramify by assumption, we add on as our one distinguished prime a finite prime \mathfrak{p} that splits completely in K . The \mathfrak{q}_i have the same meaning as in case **ii)**.

The abelian L -functions we use will be defined in terms of the set S . Let $G = \text{Gal}(K/k)$, which is abelian of cardinality $[K : k] = n$. Let $\widehat{G} = \text{Hom}(G, \mathbb{C}^*)$ with a typical character denoted by χ , and the trivial character denoted by χ_0 . The L -functions we work with are defined by

$$L_S(s, \chi) = \prod_{\mathfrak{q} \notin S} (1 - \chi(\sigma_{\mathfrak{q}}) N\mathfrak{q}^{-s})^{-1} \quad \text{for } \Re(s) > 1,$$

where $\sigma_{\mathfrak{q}}$ denotes the Frobenius automorphism associated to the prime \mathfrak{q} which is uniquely defined since all of the ramified primes are contained in S . It is well-known [**La**] that these abelian L -functions can all be analytically continued to the whole complex plane and all of them are actually entire functions except for when $\chi = \chi_0$, in which case there is a simple pole at $s = 1$. In particular we can expand all of these functions out as Taylor series at $s = 0$ and the general philosophy is to somehow interpret the lead term arithmetically. There is a simple formula [**Ta**] which gives the order $r(\chi)$ of the zero of $L_S(s, \chi)$ at $s = 0$ which is very useful.

$$r(\chi) = \begin{cases} |S| - 1 & \text{if } \chi = \chi_0 \\ |\{\mathfrak{q} \in S : \chi(G_{\mathfrak{q}}) = 1\}| & \text{if } \chi \neq \chi_0 \end{cases}$$

where $G_{\mathfrak{q}}$ is the decomposition group of the prime \mathfrak{q} (finite or infinite). Notice by assumption **(c)** that S contains at least r primes that split completely and for all of these primes $G_{\mathfrak{q}} = \{1\}$. Combining this fact with assumption **(d)** we see that we always have $r(\chi) \geq r$. Thus in cases **ii)** and **iii)** all of our L -functions will have at least first order zeros at $s = 0$. Before we can state Stark's refined conjecture we need two more pieces of notation. The number of roots of unity in K is denoted by w_K . If \mathfrak{P} is any prime lying above our distinguished prime \mathfrak{p} , then $|\cdot|_{\mathfrak{P}}$ denotes the usual [**O'M**, p. 65] normalized valuation on K with respect to \mathfrak{P} .

Conjecture [**St4**, **Ta**]. *There exists $\varepsilon \in K$ such that*

(1) *For all characters $\chi \in \widehat{G}$ we have*

$$L_S^{(1)}(0, \chi) = -\frac{1}{w_K} \sum_{\sigma \in G} \chi(\sigma) \log |\varepsilon^{\sigma}|_{\mathfrak{P}},$$

(2) *$K(\varepsilon^{1/w_K})$ is an abelian Galois extension of k .*

Note. Tate has proved this when $|S| = 2$. For $|S| \geq 3$ it is also called for that

$$(3) \quad \varepsilon \in \{u \in K : |u|_{\mathfrak{p}'} = 1 \quad \forall \mathfrak{p}' \nmid \mathfrak{p}\}.$$

We have already stated what choice we have made for S in cases [ii](#)) and [iii](#)). In case [i](#)) we can set $r = 1$ by taking either of the split infinite primes to be our distinguished prime \mathfrak{p} . We see in this case that if $|S| \geq 3$, then all of the L -functions satisfy $L_S^{(1)}(0, \chi) = 0$ and by choosing $\varepsilon = 1$ the conjecture is trivially satisfied. There is only one situation of interest, namely, $S = \{\infty_k, \overline{\infty}_k\}$. This situation is already special since it implies that K is contained inside the Hilbert class field of k and we see by the formula for $r(\chi)$ that $L_S^{(1)}(0, \chi) = 0$ for all $\chi \neq \chi_0$. Tate [[Ta](#), p. 91] gives a nice argument for this situation.

This conjecture has been studied in great detail both numerically and theoretically by Stark [[St1-St4](#)] and has some fascinating consequences. This conjecture for case [iii](#)), where the distinguished prime is a finite prime is often referred to as the Brumer–Stark conjecture. Hayes [[H](#)] has given a beautiful exposition of this situation which can be reformulated [[Ta](#)] to give an intriguing generalization of Stickelberger’s classic theorem on factoring Gauss sums.

For the rest of this paper we will focus on case [i](#)), so for now on K denotes a totally real abelian extension of the real quadratic field k . As we saw above, we automatically have $L_S^{(1)}(0, \chi) = 0$ for $\chi \neq \chi_0$, and since we want to glean arithmetic information from the lead term of the Taylor series expansions at $s = 0$, it is natural instead to study the values $L_S^{(2)}(0, \chi)$. We stick to our assumptions on S and take the two infinite primes of k as our distinguished primes, so now $r = 2$. Let τ denote an embedding of K into \mathbb{R} that extends the non-trivial \mathbb{Q} -automorphism of k . Note that $w_K = 2$ since K is totally real. Let S_K consist of the set of all primes in K that extend the primes in S . Finally, let $U = \{u \in K : |u|_{\mathfrak{p}} = 1 \text{ for all } \mathfrak{p} \notin S_K\}$ be the group of S_K -units of K .

Question [[St5](#)]. Are there S_K -units $\varepsilon_1, \varepsilon_2 \in U$ such that all of the following conditions hold

- (1) For all $\chi \in \widehat{G}$ we have

$$\frac{1}{2!} L_S^{(2)}(0, \chi) = \det \left[\sum_{\sigma \in G} \chi(\sigma) \begin{pmatrix} -\frac{1}{2} \log |\varepsilon_1^\sigma| & -\frac{1}{2} \log |\varepsilon_1^{\sigma\tau}| \\ -\frac{1}{2} \log |\varepsilon_2^\sigma| & -\frac{1}{2} \log |\varepsilon_2^{\sigma\tau}| \end{pmatrix} \right],$$

- (2) both $K(\sqrt{\varepsilon_1})$ and $K(\sqrt{\varepsilon_2})$ are abelian Galois extensions of k ,
 (3) we have $K(\sqrt{\varepsilon_1}) = K(\sqrt{\varepsilon_2})$ in (2),
 (4) the numbers ε_1^σ generate the same principal ideal as σ runs through G , and the same holds among the ε_2^σ ?

Note. Notice that the usual absolute value $|\cdot|$ is a normalized valuation extending the distinguished prime ∞_k , and $|\cdot|^\tau$ is a normalized valuation extending $\overline{\infty}_k$.

Note. Now we are assuming that $|S| \geq 3$ in order to insure that all L -functions have at least 2nd order zeros at $s = 0$. If there are any ramified finite primes this is automatic, if not we have to throw in a prime \mathfrak{q}_1 .

2. Relative quadratic extensions.

The first serious testing ground for Stark's [question](#) is the case where $[K : k] = 2$. We study this situation in depth here. Let $G = \text{Gal}(K/k) = \{1, \sigma\}$. The problem splits naturally into two pieces that seem to require different methods: i) $|S| = 3$, ii) $|S| \geq 4$. We will only consider case ii) in this paper. What makes this case more accessible is that $L_S^{(2)}(0, \chi_0) = 0$. Part (1) above for χ_0 will then follow from our work on the non-trivial character.

We have $S = \{\infty_k, \overline{\infty}_k, \mathfrak{q}_1, \dots, \mathfrak{q}_t\}$ with $t \geq 2$. If any of the \mathfrak{q}_i are split then $r(\chi) \geq 3$ for both $\chi \in \widehat{G}$ and the [question](#) is answered by choosing $\varepsilon_1 = \varepsilon_2 = 1$. So we assume that the \mathfrak{q}_i are either ramified or inert. There are four infinite primes in S_K and t finite primes $\mathfrak{Q}_1, \dots, \mathfrak{Q}_t$ where the numbering is set up so that $\mathfrak{Q}_i | \mathfrak{q}_i$. Thus $|S_K| = |S| + 2$. Our main result is the following

Theorem. *If $[K : k] = 2$ and $|S| \geq 4$ then Stark's [question](#) can be answered in the affirmative.*

Proof. The Dedekind zeta-function $\zeta_K(s)$ of K factors as

$$\zeta_K(s) = \zeta_k(s) L(s, \psi)$$

where

$$L(s, \psi) = \prod (1 - \psi(\mathfrak{p}) N\mathfrak{p}^{-s})^{-1},$$

the product running over all finite primes in k , and

$$\psi(\mathfrak{p}) = \begin{cases} 1 & \text{if } \mathfrak{p} \text{ splits,} \\ -1 & \text{if } \mathfrak{p} \text{ is inert,} \\ 0 & \text{if } \mathfrak{p} \text{ ramifies.} \end{cases}$$

Thus

$$(1) \quad \prod_{\mathfrak{p} \notin S_K} (1 - N\mathfrak{p}^{-s})^{-1} = \prod_{\mathfrak{p} \notin S} (1 - N\mathfrak{p}^{-s})^{-1} \cdot \prod_{\mathfrak{p} \in S} (1 - \psi(\mathfrak{p}) N\mathfrak{p}^{-s})^{-1}.$$

For $\mathfrak{p} \notin S$ we have $\psi(\mathfrak{p}) = \chi(\sigma_{\mathfrak{p}})$, where χ is the non-trivial character on G [Ja, p. 100]. Now the primes not in S_K are in one-to-one correspondence with the prime ideals of the Dedekind domain of S_K -integers in K . The Dedekind zeta-function for the S_K -integers is precisely the left hand side of (1) and the first product on the right hand side is similarly the Dedekind zeta-function for the S -integers of k . Denoting these functions by $\zeta_{S_K}(s)$ and $\zeta_S(s)$ respectively we can rewrite (1) as

$$(2) \quad \zeta_{S_K}(s) = \zeta_S(s) L_S(s, \chi).$$

In regard to part (1) of Stark's question we see that $\zeta_S(s) = L_S(s, \chi_0)$. On the other hand we have [Gr]

$$\lim_{s \rightarrow 0} \frac{1}{s^2} L_S(s, \chi) = \lim_{s \rightarrow 0} \frac{\frac{\zeta_{S_K}(s)}{s^{|S_K|-1}}}{\frac{\zeta_S(s)}{s^{|S|-1}}} = \frac{-\frac{h_{S_K} R_K}{w_K}}{-\frac{h_S R_k}{w_k}}$$

where h_{S_K}, h_S are the class numbers for the S_K -integers and S -integers of K and k respectively, and R_K and R_k are the regulators for the S_K -units and S -units respectively. Thus for the non-trivial character χ we have

$$(3) \quad \frac{1}{2!} L_S^{(2)}(0, \chi) = \frac{h_{S_K} R_K}{h_S R_k}$$

and now we take a closer look at these regulators.

The Dirichlet–Chevalley–Hasse Theorem [Ja] tells us that U is a direct product of the group $\mu_K = \{1, -1\}$ with $|S_K| - 1 = |S| + 1$ infinite cyclic groups. Note that $U \cap k = U_S = \{u \in k : |u|_{\mathfrak{p}} = 1 \text{ for all } \mathfrak{p} \notin S\}$.

Let $V = \langle v_1, \dots, v_{|S|+1} \rangle$ where $v_1, \dots, v_{|S|+1}$ is a system of fundamental units for U . We will assume without loss of generality that $v_i > 0$ for $i = 1, \dots, |S| + 1$. Let $V^- = \{\eta \in V : N_{K/k} \eta = 1\}$ and let $V^+ = \{\gamma \in V : \sigma \gamma = \gamma\}$. Note that if $\sigma v_i > 0$ we have $\sigma v_i / v_i = \eta_i \in V^-$ (if $\sigma v_i < 0$, then we consider $-\sigma v_i / v_i = \eta_i \in V^-$ instead). Then $v_i(\sigma v_i) = \eta_i v_i^2$ so that $v_i^2 \in V^- \cdot V^+$. We see that $V^- \cdot V^+$ is a \mathbb{Z} -submodule of V of finite index less than $[V : \langle v_1^2, \dots, v_{|S|+1}^2 \rangle] = 2^{|S|+1}$. Note that $V^+ = \{u \in U_S : u > 0\}$ and thus is a rank $|S| - 1$ submodule. The submodule V^- is necessarily free as well and since $V^- \cap V^+ = \{1\}$ we have $V^- \cdot V^+ = V^- \oplus V^+$ which implies that V^- is a free submodule of rank 2 (see [AW, p. 173]). By the invariant factor theorem, there exists a basis $w_1, \dots, w_{|S|+1}$ for V and positive integers a_1, a_2 such that $\varepsilon_1 = w_1^{a_1}, \varepsilon_2 = w_2^{a_2}$ forms a basis for V^- . If $u_1, \dots, u_{|S|-1}$ is a system of generators for V^+ , then

$$[\langle w_1, \dots, w_{|S|+1} \rangle : \langle u_1, \dots, u_{|S|-1}, \varepsilon_1, \varepsilon_2 \rangle] = R/R_K$$

where R is the regulator for the system $\langle u_1, \dots, u_{|S|-1}, \varepsilon_1, \varepsilon_2 \rangle$ [Wa, p. 41]. This index can also be written

$$(4) \quad R/R_K = [U : U_S \cdot U^-]$$

where $U^- = \{u \in U : N_{K/k}u = 1\}$.

We can also deduce a relationship between R and R_k . By definition, the regulator R is the absolute value of the following determinant

$$\begin{vmatrix} \log |u_1|_{\Omega_1} & \cdots & \log |u_1|_{\Omega_{t-1}} & \log |u_1| & \log |u_1^\tau| & \log |u_1^\sigma| & \log |u_1^{\sigma\tau}| \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \log |u_{|S|-1}|_{\Omega_1} & \cdots & \log |u_{|S|-1}|_{\Omega_{t-1}} & \log |u_{|S|-1}| & \log |u_{|S|-1}^\tau| & \log |u_{|S|-1}^\sigma| & \log |u_{|S|-1}^{\sigma\tau}| \\ \log |\varepsilon_1|_{\Omega_1} & \cdots & \log |\varepsilon_1|_{\Omega_{t-1}} & \log |\varepsilon_1| & \log |\varepsilon_1^\tau| & \log |\varepsilon_1^\sigma| & \log |\varepsilon_1^{\sigma\tau}| \\ \log |\varepsilon_2|_{\Omega_1} & \cdots & \log |\varepsilon_2|_{\Omega_{t-1}} & \log |\varepsilon_2| & \log |\varepsilon_2^\tau| & \log |\varepsilon_2^\sigma| & \log |\varepsilon_2^{\sigma\tau}| \end{vmatrix}.$$

Subtracting the fourth to the last column from the second to the last and the third to the last from the last gives us

$$(5) \quad R = 2^{|S|-1} R_k \det \left[\sum_{\sigma \in G} \chi(\sigma) \begin{pmatrix} -\frac{1}{2} \log |\varepsilon_1^\sigma| & -\frac{1}{2} \log |\varepsilon_1^{\sigma\tau}| \\ -\frac{1}{2} \log |\varepsilon_2^\sigma| & -\frac{1}{2} \log |\varepsilon_2^{\sigma\tau}| \end{pmatrix} \right]$$

where a factor of $2^{|S|-3}$ comes from converting $|u|_\Omega = |u|_{\mathfrak{q}}^2$, for $u \in k$ and \mathfrak{q} being inert or ramified, in writing down R_k . Plugging (4) and (5) into (3) we find

$$(6) \quad \frac{1}{2!} L_S^{(2)}(0, \chi) = \frac{2^{|S|-1}}{[U : U_S \cdot U^-]} \cdot \frac{h_{S_K}}{h_S} \det \left[\sum_{\sigma \in G} \chi(\sigma) \begin{pmatrix} -\frac{1}{2} \log |\varepsilon_1^\sigma| & -\frac{1}{2} \log |\varepsilon_1^{\sigma\tau}| \\ -\frac{1}{2} \log |\varepsilon_2^\sigma| & -\frac{1}{2} \log |\varepsilon_2^{\sigma\tau}| \end{pmatrix} \right]$$

and on comparison with part (1) of Stark's question we see that we just need to analyze the indices in front of the determinant more carefully. Calling the whole product in front of the determinant A , we wish to show that $A \in \mathbb{Z}$. We define a homomorphism $f: U \rightarrow U$ given by $u \mapsto u/\sigma u$. By Herbrand's lemma [Ly, p. 12] we find

$$[U : U_S \cdot U^-] = [f(U) : f(U_S \cdot U^-)] \cdot [\ker f : \ker f \cap (U_S \cdot U^-)].$$

Notice that $\ker f = U_S$ and so the second index on the right hand side is 1. The first index is just $[U^{1-\sigma} : \langle \varepsilon_1^2, \varepsilon_2^2 \rangle]$. Furthermore,

$$[U^- : \langle \varepsilon_1^2, \varepsilon_2^2 \rangle] = [U^- : U^{1-\sigma}] \cdot [U^{1-\sigma} : \langle \varepsilon_1^2, \varepsilon_2^2 \rangle]$$

and so

$$\frac{1}{[U : U_S \cdot U^-]} = \frac{|H^1(G, U)|}{[U^- : \langle \varepsilon_1^2, \varepsilon_2^2 \rangle]}$$

where $H^1(G, U)$ is the 1st cohomology group [N, p.12]. Finally, $[U^- : \langle \varepsilon_1^2, \varepsilon_2^2 \rangle] = 2^3$ and so we can write A as

$$(7) \quad A = \frac{2^{|S|-1}}{2^3} \cdot \frac{h_{S_K}}{h_S} \cdot |H^1(G, U)|.$$

Since $|S| - 1 \geq 3$ by assumption, the first index is integral. The product $|H^1(G, U)|h_{S_K}/h_S$ is integral also since S contains all of the ramified primes [Ta, p. 105] or [CH, p. 28], and so A is a positive integer.

If we consider part (1) of Stark's question for the trivial character we find

$$\det \begin{vmatrix} -\frac{1}{2} \log |N_{K/k} \varepsilon_1| & -\frac{1}{2} \log |(N_{K/k} \varepsilon_1)^\tau| \\ -\frac{1}{2} \log |N_{K/k} \varepsilon_2| & -\frac{1}{2} \log |(N_{K/k} \varepsilon_2)^\tau| \end{vmatrix} = 0$$

since $N_{K/k} \varepsilon_1 = 1$. Things match up for the trivial character since we have $r(\chi_0) = |S| - 1 \geq 3$ and so $L_S^{(2)}(0, \chi_0) = 0$. Note that $\varepsilon_1 \in U$ and so its prime factorization looks like $(\varepsilon_1) = \mathfrak{Q}_1^{b_1} \cdots \mathfrak{Q}_t^{b_t}$. Recalling that all of these ideals are ramified or inert we see that $(\sigma \varepsilon_1)$ has the same factorization. Since $(N_{K/k} \varepsilon_1) = (1)$, we have $b_1 = \cdots = b_t = 0$ and so ε_1 is a “true unit”. The same goes for ε_2 , so part (4) is satisfied.

With regard to part (2), there is a nice criterion that says that $K(\sqrt{\varepsilon})$ is an abelian extension of k if and only if $N_{K/k} \varepsilon$ is a square in K (see [T, p. 25]). We conclude that $K(\sqrt{\varepsilon_1})$ and $K(\sqrt{\varepsilon_2})$ are both abelian extensions of k . But part (3) asks that these be *common* abelian extensions of k . This can be achieved in the case where $|S| \geq 5$ since A is divisible by $2^{|S|-1}/2^3$ and is thus even. Multiplying the top row of the determinant in (6) by A amounts to raising ε_1 to the A th power. Then $\sqrt{\varepsilon_1^A} \in K$ and by adding the second row to the first row and setting $\varepsilon'_1 = \varepsilon_1^A \varepsilon_2$, we see that ε'_1 and ε_2 satisfy all the requirements of the question. This concludes the proof of our theorem when $|S| \geq 5$. If $|S| = 4$ we have S_K -units $\varepsilon_1, \varepsilon_2 \in U$ satisfying parts (1) and (2) of the question, but the abelian extensions $K(\sqrt{\varepsilon_1})$ and $K(\sqrt{\varepsilon_2})$ of k might not be same. \square

3. $|S| = 4$.

We now wish to carry out a more careful analysis of the relative quadratic case when $|S| = 4$ which we assume for the rest of this section. We first

show that we can answer Stark's [question](#) affirmatively here as well except possibly for the family of situations where *all* of the following conditions hold:

- (a) k is a real quadratic field with discriminant $d_k \equiv 1 \pmod{8}$, i.e. 2 splits, say $(2) = \mathfrak{p}_2 \bar{\mathfrak{p}}_2$,
- (b) $N(\varepsilon_k) = 1$ where $\varepsilon_k > 1$ is the fundamental unit of k , so d_k has at least two prime divisors [[Co](#), p. 185],
- (c) $K = k(\sqrt{\varepsilon_k})$,
- (d) the two prime ideals in k above 2 ramify in K .

Assuming that these conditions do hold, we see that (b) and (c) insure that K is totally real and that the only primes in k that can ramify are \mathfrak{p}_2 and $\bar{\mathfrak{p}}_2$. So our conditions imply that we must choose $S = \{\infty_k, \overline{\infty}_k, \mathfrak{p}_2, \bar{\mathfrak{p}}_2\}$. Once S is defined, then A is uniquely defined by Equation (7). If A is even, we can argue as above and finish off the [question](#). So we can also assume

- (e) A is odd.

We now proceed to reduce ourselves to this recalcitrant family of situations by showing that we can answer the [question](#) affirmatively otherwise.

We already have a choice of ε_1 and ε_2 that almost works. We just have to find a way to modify them so that $K(\sqrt{\varepsilon_1}) = K(\sqrt{\varepsilon_2})$, which is equivalent to the ratio $\varepsilon_1/\varepsilon_2$ being a square in K . Let U_K be the unit group of K and $M = \{u \in U_K : u > 0\}$. By Dirichlet's unit theorem this is a free \mathbb{Z} -module of rank 3. Recall that ε_1 and ε_2 are in U_K with $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$. We first want to show that either the basis $\langle \varepsilon_1, \varepsilon_2 \rangle$ of V^- can be extended to a basis for M or that Stark's [question](#) has an affirmative answer. If $x \in V^-$, and $x = y^a$ for some $y \in M$ and $a > 0$, we need to show that $y \in V^-$ to be able to extend the basis [[AW](#), p. 171]. Consider $N_{K/k}(y)^a = N_{K/k}(x) = 1$ which implies $N_{K/k}(y) = \pm 1$. If the only exponents for which this happens are odd we are done since an odd a implies $N_{K/k}(y) = 1$. Assume a is even for some y . Then $\varepsilon_1^{n_1} \varepsilon_2^{n_2} = y^{2b}$ with $n_1, n_2 \in \mathbb{Z}$. If n_1 and n_2 are both odd this implies $\varepsilon_1/\varepsilon_2$ is a square in K and we have our common abelian extensions. If, say, n_1 is odd and n_2 is even, then ε_1 is a square in K . Replacing ε_1 by $\varepsilon_1 \varepsilon_2$ we have the ratio $\varepsilon_1 \varepsilon_2 / \varepsilon_2$ being a square in K as desired. Say n_1, n_2 are both even. Then $\varepsilon_1^{n_1/2} \varepsilon_2^{n_2/2} = y^b$ since both sides must be positive. Continuing in this way proves our claim.

Assuming now that $\varepsilon_1, \varepsilon_2, \varepsilon_3$ is a system of fundamental units for the unit group U_K of K , we can write the fundamental unit ε_k of k as

$$(8) \quad \varepsilon_k = \varepsilon_1^a \varepsilon_2^b \varepsilon_3^c$$

where we can assume $c \geq 0$. We have $\varepsilon_k^2 = N_{K/k} \varepsilon_k = N_{K/k}(\varepsilon_1^a \varepsilon_2^b \varepsilon_3^c) = (N_{K/k} \varepsilon_3)^c$ which implies that $c = 1$ or 2 . Now say $c = 1$, then $\varepsilon_1, \varepsilon_2, \varepsilon_k$ is a

system of fundamental units in K . We claim we are done in this case. The regulator for field K is

$$R(K) = \left\| \begin{array}{ccc} \log |\varepsilon_k| & \log |\varepsilon_k^\sigma| & \log |\varepsilon_k^{\sigma\tau}| \\ \log |\varepsilon_1| & \log |\varepsilon_1^\sigma| & \log |\varepsilon_1^{\sigma\tau}| \\ \log |\varepsilon_2| & \log |\varepsilon_2^\sigma| & \log |\varepsilon_2^{\sigma\tau}| \end{array} \right\|$$

where the double vertical lines signify that we first want to take the determinant and then take the absolute value. Subtracting the first column from the second and adding the first column to the third gives

$$R(K) = \left\| \begin{array}{ccc} \log |\varepsilon_k| & 0 & 0 \\ \log |\varepsilon_1| & -\log |\varepsilon_1| + \log |\varepsilon_1^\sigma| & \log |\varepsilon_1| + \log |\varepsilon_1^{\sigma\tau}| \\ \log |\varepsilon_2| & -\log |\varepsilon_2| + \log |\varepsilon_2^\sigma| & \log |\varepsilon_2| + \log |\varepsilon_2^{\sigma\tau}| \end{array} \right\|$$

and so

$$\frac{R(K)}{R(k)} = \left\| \begin{array}{cc} -\log |\varepsilon_1| + \log |\varepsilon_1^\sigma| & \log |\varepsilon_1| + \log |\varepsilon_1^{\sigma\tau}| \\ -\log |\varepsilon_2| + \log |\varepsilon_2^\sigma| & \log |\varepsilon_2| + \log |\varepsilon_2^{\sigma\tau}| \end{array} \right\|.$$

Adding $1/2$ the first column to the second column and using the fact that $-\log |\varepsilon_i^\tau| = \log |\varepsilon_i| + \log |\varepsilon_i^\sigma| + \log |\varepsilon_i^{\sigma\tau}|$ we obtain

$$\frac{R(K)}{R(k)} = 2 \left\| \sum_{\sigma \in G} \chi(\sigma) \begin{pmatrix} -\frac{1}{2} \log |\varepsilon_1^\sigma| & -\frac{1}{2} \log |\varepsilon_1^{\sigma\tau}| \\ -\frac{1}{2} \log |\varepsilon_2^\sigma| & -\frac{1}{2} \log |\varepsilon_2^{\sigma\tau}| \end{pmatrix} \right\|.$$

By the same reasoning employed earlier we find

$$\frac{1}{2!} L_S^{(2)}(0, \chi) = \frac{h(K)R(K)}{h(k)R(k)} \cdot 2^l$$

where l is the number of inert finite primes in S , and $h(K), h(k)$ are the usual class numbers of the fields K and k . For the situation at hand, the possible values of l are 0, 1, and 2. If $l = 0$, then there is ramification in the extension K/k and so $h(k)|h(K)$ [CH, p. 20]. We have

$$(9) \quad \frac{1}{2!} L_S^{(2)}(0, \chi) = 2 \frac{h(K)}{h(k)} \left\| \sum_{\sigma \in G} \chi(\sigma) \begin{pmatrix} -\frac{1}{2} \log |\varepsilon_1^\sigma| & -\frac{1}{2} \log |\varepsilon_1^{\sigma\tau}| \\ -\frac{1}{2} \log |\varepsilon_2^\sigma| & -\frac{1}{2} \log |\varepsilon_2^{\sigma\tau}| \end{pmatrix} \right\|$$

and with the even coefficient out front we can choose ε_1 and ε_2 in such a way that $\varepsilon_1/\varepsilon_2$ is a square in K . The same argument goes through for $l = 1$. If

$l = 2$, the extension is unramified, but the quotient $h(K)/h(k)$ has at worst a denominator of 2 [CH, p. 21] and so $2^{3\frac{h(K)}{h(k)}}$ is still even and we are done.

Now assume that $c = 2$ in Equation (8). Then we have

$$\frac{R(K)}{R(k)} = \left\| \sum_{\sigma \in G} \chi(\sigma) \begin{pmatrix} -\frac{1}{2} \log |\varepsilon_1^\sigma| & -\frac{1}{2} \log |\varepsilon_1^{\sigma\tau}| \\ -\frac{1}{2} \log |\varepsilon_2^\sigma| & -\frac{1}{2} \log |\varepsilon_2^{\sigma\tau}| \end{pmatrix} \right\|.$$

The cases where $l = 1$ and $l = 2$ can still be handled as above, but the case where $l = 0$ looks like Equation (9) without the benefit of the 2 on the right hand side. Going back to Equation (8) with $c = 2$ we make one further reduction. Notice that if we replace ε_1 by $\varepsilon_1 \varepsilon_k$, then part (1) of Stark's question is still satisfied and $K(\sqrt{\varepsilon_1 \varepsilon_k})$ is still abelian over k . If a is odd and b is even we have $\varepsilon_k = \varepsilon_1 \eta^2$ for some unit η , and by replacing ε_1 by $\varepsilon_1 \varepsilon_k \varepsilon_2$ we have the ratio $\varepsilon_1^2 \eta^2 \varepsilon_2 / \varepsilon_2$ being a square in K as desired. The same type of argument goes through for a even, b odd, and a odd, b odd. Finally, for a even, b even, and $c = 2$ we notice that $\sqrt{\varepsilon_k} \in K$ which implies that $K = k(\sqrt{\varepsilon_k})$. If $N(\varepsilon_k) = -1$, then K wouldn't be totally real and so this situation can only arise when $N(\varepsilon_k) = 1$. Looking at the field $K = k(\sqrt{\varepsilon_k}) = \mathbb{Q}(\sqrt{\varepsilon_k})$, the conjugates over \mathbb{Q} of $\sqrt{\varepsilon_k}$ are $\pm\sqrt{\varepsilon_k}$, $\pm 1/\sqrt{\varepsilon_k}$ and all possible non-trivial automorphisms have order 2 so that $\text{Gal}(K/\mathbb{Q}) = \mathbb{Z}_2 \times \mathbb{Z}_2$. The only possible ramification that can occur in $k(\sqrt{\varepsilon_k})/k$ is at the primes in k lying above 2. Since the only case left over from above is where $l = 0$, we need two prime ideals above 2 in k which both ramify. We have now reduced ourselves to the family of situations satisfying (a), (b), (c), (d) and (e).

In order to find an affirmative answer for this family as well we need a few lemmas.

Lemma 1. *If $d_k \equiv 1 \pmod{8}$ is the discriminant of a real quadratic number field k with $N(\varepsilon_k) = 1$, then $k(\sqrt{\varepsilon_k}) = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ where $d_k = d_1 d_2$ and d_1, d_2 are both integers greater than 1.*

Proof. The fundamental unit is $\varepsilon_k = a + b\sqrt{d_k}$ where $a > 1$ and b are positive integers. It is easy to verify that $\sqrt{\varepsilon_k} = (\sqrt{2(a+1)} + \sqrt{2(a-1)})/2$. Let $2(a+1) = u^2 d_1$ and $2(a-1) = v^2 d_2$ with d_1 and d_2 square-free positive integers. We have $4(a^2 - 1) = u^2 v^2 d_1 d_2$ or $x^2 d_k = t^2 d_1 d_2$. Since d_k is square-free we find $w^2 d_k = d_1 d_2$ and $w|d_1$ and $w|d_2$ since d_1 and d_2 are square-free. This implies $w|u^2 d_1 - v^2 d_2 = 4$ so $w = 1$ or 2 . We want to show that $w \neq 2$. If $d_1 d_2 = 4d_k$, then $d_1 = 2r$ and $d_2 = 2s$ with $rs = d_k$ and either $r \equiv s \equiv 1 \pmod{4}$ or $r \equiv s \equiv 3 \pmod{4}$. This implies that $u^2 r - v^2 s = 2$ or $u^2 - v^2 \equiv 2 \pmod{4}$ which has no solutions. Thus $d_k = d_1 d_2$ and $d_1, d_2 > 1$ since otherwise we would have $\sqrt{\varepsilon_k} \in k$ which can't happen. \square

Lemma 2. *If p is a prime congruent to 3 modulo 4 and $\varepsilon = a + b\sqrt{p} > 1$ is the fundamental unit of $k_1 = \mathbb{Q}(\sqrt{p})$, then 2ε is a square in k_1 .*

Proof. We automatically have $N(\varepsilon) = 1$ and so $\varepsilon = (1 + \varepsilon)/(1 + \bar{\varepsilon}) = (1 + \varepsilon)^2/2(1 + a)$. Let $2(1 + a) = u^2 d_1$ where d_1 is assumed square-free. By the same argument as above in Lemma 1 we conclude that $d_1 = 2$ or $d_1 = 2p$. \square

Comment. We can refine Lemma 2. Notice that $N(1 + \varepsilon) = 2(1 + a)$. Working modulo 8 it is easy to see that $2u^2$ is not the norm of an integer from $\mathbb{Q}(\sqrt{p})$ when $p \equiv 3 \pmod{8}$. Similarly, one finds that $2pu^2$ is not the norm of an integer from $\mathbb{Q}(\sqrt{p})$ when $p \equiv 7 \pmod{8}$. So $d_1 = 2p$ when $p \equiv 3 \pmod{8}$ and $d_1 = 2$ when $p \equiv 7 \pmod{8}$.

Our approach now boils down to a consideration of several cases and an appeal to genus theory [Co, pp. 225-226]. By Lemma 1, the three quadratic subfields of K will be $k_1 = \mathbb{Q}(\sqrt{d_1})$, $k_2 = \mathbb{Q}(\sqrt{d_2})$, and $k = \mathbb{Q}(\sqrt{d_k})$. By the conductor-discriminant formula [Wa, p. 27], the discriminant of K is given by $d_K = d_{k_1} d_{k_2} d_k$. The norm of the relative discriminant of the extension K/k can be obtained from the formula $d_K = (ND_{K/k})(d_k)^2$ [Lo, p. 82].

Case 1. d_k is the product of two primes p_1, p_2 where either $p_1 \equiv p_2 \equiv 1 \pmod{4}$ or $p_1 \equiv p_2 \equiv 3 \pmod{4}$. In either case we have $K = \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2})$. If $p_1 \equiv p_2 \equiv 1 \pmod{4}$ we find $ND_{K/k} = 1$ which implies there is no ramification and so these fields do not belong to our family. On the other hand, if $p_1 \equiv p_2 \equiv 3 \pmod{4}$ we find $ND_{K/k} = 16$ and so both primes above 2 in k ramify. As Kubota showed (see in particular [K, p. 74]), a fundamental system of units for K in this special case is $\varepsilon_1 = \gamma_1, \varepsilon_2 = \sqrt{\gamma_1 \gamma_2}$, and $\varepsilon_3 = \sqrt{\varepsilon_k}$ where $\gamma_1 > 1, \gamma_2 > 1$ are the fundamental units for k_1 and k_2 . The class number of K in this particular case is given by $h(K) = h(k_1)h(k_2)h(k)$ [K, p. 80] and so

$$\frac{1}{2!} L_S^{(2)}(0, \chi) = h(k_1)h(k_2) \left\| \sum_{\sigma \in G} \chi(\sigma) \begin{pmatrix} -\frac{1}{2} \log |\varepsilon_1^\sigma| & -\frac{1}{2} \log |\varepsilon_1^{\sigma\tau}| \\ -\frac{1}{2} \log |\varepsilon_2^\sigma| & -\frac{1}{2} \log |\varepsilon_2^{\sigma\tau}| \end{pmatrix} \right\|.$$

We now verify that the two S_K -units $\eta_1 = 2\varepsilon_1 \varepsilon_2$ and $\eta_2 = \varepsilon_2^{h(k_1)h(k_2)}$ satisfy Stark's question. Clearly $N_{K/k} \varepsilon_1 = 1$. To see that $N_{K/k} \varepsilon_2 = 1$, let $\sqrt{\gamma_1} =$

$(u_1\sqrt{2} + v_1\sqrt{2p_1})/2$ and $\sqrt{\gamma_2} = (u_2\sqrt{2} + v_2\sqrt{2p_2})/2$. Then

$$\begin{aligned} N_{K/k}\varepsilon_2 &= \left(\frac{u_1\sqrt{2} + v_1\sqrt{2p_1}}{2}\right) \left(\frac{u_1\sqrt{2} - v_1\sqrt{2p_1}}{2}\right) \\ &\quad \cdot \left(\frac{u_2\sqrt{2} + v_2\sqrt{2p_2}}{2}\right) \left(\frac{u_2\sqrt{2} - v_2\sqrt{2p_2}}{2}\right) \\ &= 1 \end{aligned}$$

by the comment above. The L -function values match up and η_1/η_2 is a square in K by Lemma 2 and the fact that $h(k_1)h(k_2)$ is odd by genus theory.

Case 2. d_k is the product of three primes p_1, p_2 and q where $p_1 \equiv p_2 \equiv 3 \pmod{4}$ and $q \equiv 1 \pmod{4}$ (the situation where all three are congruent to 1 modulo 4 is eliminated since $ND_{K/k} = 1$). The three quadratic subfields look like $k_1 = \mathbb{Q}(\sqrt{p_1})$, $k_2 = \mathbb{Q}(\sqrt{p_2q})$, and k by Lemma 1 (we can eliminate the situation where $k_1 = \mathbb{Q}(\sqrt{q})$ and $k_2 = \mathbb{Q}(\sqrt{p_1p_2})$ since $ND_{K/k} = 1$). If $q \equiv 5 \pmod{8}$, then we find using Hilfssatz 10 in [K] that $\varepsilon_1 = \gamma_1, \varepsilon_2 = \gamma_2$, and $\varepsilon_3 = \sqrt{\varepsilon_k}$ are a system of fundamental units for K . Now we have $h(K)/h(k) = h(k_1)h(k_2)/2$ and we know by genus theory that $2|h(k_2)$. The most interesting situation would be if $2||h(k_2)$. Then we would use $\eta_1 = 2\varepsilon_1\varepsilon_2$ and $\eta_2 = \varepsilon_2^{h(k_1)h(k_2)/2}$. If $q \equiv 1 \pmod{8}$, we find a fundamental system of units that looks like the $q \equiv 5 \pmod{8}$ situation above (type ii) in Satz 1 of [K]) or like $\varepsilon_1 = \gamma_1, \varepsilon_2 = \sqrt{\gamma_1\gamma_2}$, and $\varepsilon_3 = \sqrt{\varepsilon_k}$ (type v) in Satz 1). The latter set up is even easier since $h(K)/h(k) = h(k_1)h(k_2)$ is now even and we can choose $\eta_1 = \varepsilon_1, \eta_2 = \varepsilon_1\varepsilon_2^{h(k_1)h(k_2)}$.

Case 3. We now assume that d_k is the product of 4 or more primes. We use Lemma 1 to describe the various possibilities we can have for the three quadratic subfields. We can eliminate a lot of possibilities by calculating $ND_{K/k}$ and finding it equal to 1. For the remaining possibilities, an appeal to genus theory supplies us with enough powers of 2 to always enable us to find an affirmative answer.

Notice that all solutions $\varepsilon_1, \varepsilon_2 \in U$ we found above satisfy part (4) of the question. The most interesting parts of Stark's question are (1) and (2) because of the obvious analogy with his conjecture. Parts (3) and (4) are more provisional and might need to be modified or dropped based on further study.

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