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UNIQUENESS OF GENERALIZED WALDSPURGER MODEL FOR $GL(2n)$

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Let E/F be a quadratic extension of non-archimedean local field and let G be an inner form of $GL(2n, F)$ over F, which contains a subgroup H isomorphic to $GL(n, E)$. In this paper we prove that (G, H) is a Gelfand pair, i.e., the H-invariant linear functional, if there exists one, on the space of an irreducible admissible representation of G is unique up to a scalar. Globally this result will play an important role in the study of H-period integrals of cusp forms on G, and its relations to the special values of automorphic L-functions.

1. Introduction.

Let F be a nonarchimedean local field of characteristic zero, and let $E = F(\sqrt{\tau})$ be a quadratic extension field of F. We denote by $z \mapsto \bar{z}$ the Galois conjugation in E. Suppose M is a central simple algebra over F of dimension $4n^2$, which contains a subalgebra N isomorphic to $M(n, E)$ over F. We denote by G and H the multiplicative groups of M and N respectively. Then G is an inner form of the linear group $GL(2n, F)$ regarded as an algebraic group over F, and $H \subset G$ is a subgroup which is isomorphic to $GL(n, E)$ over F. Suppose that π is an irreducible admissible representation of G on a complex vector space V. Let $\mathrm{Hom}(\pi, \mathbb{C})$ be the dual space of V, and let $\mathrm{Hom}_H(\pi, \mathbb{C})$ be the set of H invariant elements in $\mathrm{Hom}(\pi, \mathbb{C})$, i.e

$$\mathrm{Hom}_H(\pi, \mathbb{C}) = \{l \in \mathrm{Hom}(\pi, \mathbb{C}) | l(\pi(h)v) = l(v), \text{ for } h \in H \text{ and } v \in V\}.$$

In this paper, we will prove the following theorem.

Theorem . *For any irreducible admissible representation π of G on a complex vector space V, we have*

$$\dim_{\mathbb{C}} \mathrm{Hom}_H(\pi, \mathbb{C}) \leq 1.$$

Furthermore if $\dim_{\mathbb{C}} \mathrm{Hom}_H(\pi, \mathbb{C}) = 1$, then π is self-contragredient.

If $\dim_{\mathbb{C}} \mathrm{Hom}_H(\pi, \mathbb{C}) = 1$, we say that π is H-distinguished. In the case that $n = 1$, then G is an inner form of $GL(2, F)$ and H is an elliptic torus of G, this theorem was proved by Waldspurger ([**W2**]), and arose in his profound studies of the Shimura correspondence and algebraicity of the special values of L-functions ([**W1**], [**W2**], [**W3**]).

In generalizing those results of Waldspurger, it is necessary to have the above theorem. To explain this, we go to a global setting. We now assume temporarily that F is a number field and F_A is the adele ring of F. Suppose π is an automorphic cuspidal representation of $G(F_A)$. Then we are interested in the period integral

$$P(\phi, \pi, H) = \int_{H(F)Z(F_A)\backslash H(F_A)} \phi(h)dh$$

where ϕ is a cusp form in the space of π and Z is the center of G. If there exists ϕ such that $P(\phi, \pi, H) \neq 0$, then we expect, as proved in the $GL(2)$ case by Waldspurger, that the square of $P(\phi, \pi, H)$ is essentially the product $L(1/2, \pi')L(1/2, \pi' \otimes \eta)$ where π' is an automorphic cuspidal representation of $GL(2n)$ related to π by the Jacquet-Langlands conditions, and η is the quadratic idele class character of F attached to E (see [**G2**] for details). As a consequence, this would imply that if the period is non-zero then

$$L(1/2, \pi')L(1/2, \pi' \otimes \eta) > 0.$$

(For the $GL(2)$ case, see [**W1**], [**J1**] and [**G1**].) On the other hand if $P(\phi, \pi, H) \neq 0$ for some ϕ, the the principle of functoriality predicts that π comes from an automorphic cuspidal representation of the orthogonal group $O(2n+1)$. In this case some properties of the Fourier coefficients of the form on $O(2n+1)$ can be read in terms of $P(\phi, \pi, H)$. Generally there are two ways to approach these problems. One is the Weil representation as used by Waldspurger and another one is the relative trace formula introduced by Jacquet ([**J1**], [**J2**]). Note that $P(\phi, \pi, H)$ defines a H-invariant linear forms on π. In both cases, it is necessary to know that $P(\phi, \pi, H)$ is Eulerian in the sense that it can be expressed as the restricted product of the local H-invariant forms. The above theorem together with its analogue in the archimedean case can be used to establish this assertion.

We now go back to the local situation. We will follow the standard approach of Gelfand-Kazhdan ([**GK**]) to prove the theorem. Let σ be the anti-involution of G defined by $\sigma(g) = g^{-1}$. Then σ preserves H. The technical part of this paper is the following proposition.

Proposition. *For each element g of G, we have $g^{-1} \in HgH$. In other words, the anti-involution σ preserves every double coset.*

This implies that a H-biinvariant distribution is also σ-invariant (see, for example, Theorem 6.13 in [**BZ**]). Then by [**GK**], we have

$$\dim_{\mathbb{C}} \operatorname{Hom}_H(\pi, \mathbb{C}) \dim_{\mathbb{C}} \operatorname{Hom}_H(\tilde{\pi}, \mathbb{C}) \leq 1$$

where $\tilde{\pi}$ is the representation contragredient to π. Note that we can realize G as the subgroup of $GL(2n, E)$ of the matrices of the form

$$\begin{pmatrix} \alpha & \gamma\beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in GL(2n, E)$$

where γ is a fixed element in $F^* = F - \{0\}$ and α, β are matrices in $M(n, E)$, and H as the subgroup of G of the matrices of the form

$$\begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix}, \qquad \alpha \in GL(n, E).$$

It is clear that each matrix $g \in G$ is conjugate in $GL(2n, E)$ to the matrix

$$\tau(g) = \begin{pmatrix} \tau I_n & 0 \\ 0 & I_n \end{pmatrix} g^{tr} \begin{pmatrix} \tau I_n & 0 \\ 0 & I_n \end{pmatrix}^{-1}$$

where g^{tr} is the transpose of g. But it is easily seen $\tau(g)$ is also in G. So g is conjugate to $\tau(g)$ in G. By the theory of Gelfand-Kazhdan ([**GK**] or [**BZ**]), we can realize $\tilde{\pi}$ in the same space V of π by the formula

$$\tilde{\pi}(g) = \pi(\tau(g^{-1})).$$

Note that for any $h \in H$ we have $\tau(h^{-1})$ is also in H. Thus if there exists an non-zero element l in $\operatorname{Hom}_H(\pi, \mathbb{C})$, then we have

$$l(\tilde{\pi}(h)v) = l(\pi(\tau(h^{-1})v)) = l(v)$$

for $h \in H$ and $v \in V$. So $\operatorname{Hom}_H(\pi, \mathbb{C}) \neq 0$ implies $\operatorname{Hom}_H(\tilde{\pi}, \mathbb{C}) \neq 0$. Then our theorem follows by some standard arguments (see, for example, [**JR**]).

A similar result for the pair $(GL(2n), GL(n) \times GL(n))$ is proved by Jacquet and Rallis in [**JR**]. In that case, the anti-involution does not preserve all the double cosets, so the proof becomes much more complicated. However the influence of their paper on this one is obvious.

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2. The tangent space to the symmetric space.

Let F be a field of characteristic zero, and let $E = F(\sqrt{\tau})$ $(\tau \in F)$ be a quadratic extension field of F. We denote by $z \mapsto \bar{z}$ the Galois conjugation in E. Let N be the norm map from E to F defined by $N(z) = z\bar{z}$ for $z \in E$. Then the central simple algebra M can be realized as an algebra of the form

$$M_\gamma = \left\{ \begin{pmatrix} \alpha & \gamma\beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} | \alpha, \beta \in M(n, E) \right\}$$

in which the subalgebra

$$N_E = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix} \right\}$$

is isomorphic to $M(n, E)$. Here γ is an element in $F^* = F - \{0\}$. We let G_γ and H_E, or simply G and H, be the multiplicative groups of M_γ and N_E respectively, and we regard them as algebraic groups over F. In particular if $\gamma \in NE^*$, then G_γ is isomorphic to $GL(2n, F)$. We denote by

$$\mathfrak{g} = \left\{ \begin{pmatrix} \alpha & \gamma\beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} | \alpha, \beta \in M(n, E) \right\}$$

the Lie algebra of G. Then G acts on $\mathfrak{g}$ by conjugation. We denote the action by Ad. So we have $\mathrm{Ad}(g)X = gXg^{-1}$ for $g \in G$ and $X \in \mathfrak{g}$.

We set

$$\epsilon = \begin{pmatrix} \sqrt{\tau} I_n & 0 \\ 0 & -\sqrt{\tau} I_n \end{pmatrix}.$$

Then $H = \{g \in G | \epsilon g \epsilon^{-1} = g\}$. We denote by L the -1 eigenspace of $\mathrm{Ad}(\epsilon)$ in $\mathfrak{g}$. So for $X \in \mathfrak{g}$, we have that $X \in L$ if and only if

$$\epsilon X \epsilon^{-1} = -X.$$

Then by a simple computation, we find

$$L = \left\{ \begin{pmatrix} 0 & \gamma\alpha \\ \bar{\alpha} & 0 \end{pmatrix} | \alpha \in M(n, E) \right\},$$

which is stable under H. Our purpose in this section is the following:

Proposition 2. *Let X be an element in L. Then there exists an element $h \in H$ such that $\bar{X} = hXh^{-1}$.*

To prove this proposition, we need to describe in some detail the H-orbits in L. Actually we will prove something more than we need here, which will be used in the future work.

For

$$X = \begin{pmatrix} 0 & \gamma A \\ \bar{A} & 0 \end{pmatrix} \in L,$$

let

$$X = X_s + X_n$$

be the Jordan decomposition of X in $\mathfrak{g}$ where X_s is semisimple and X_n is nilpotent which commutes with X_s. Since $\mathrm{Ad}(\epsilon)X = -X$, we have $\mathrm{Ad}(\epsilon)X_s + \mathrm{Ad}(\epsilon)X_n = (-X_s) + (-X_n)$. It follows that $\mathrm{Ad}(\epsilon)X_s = -X_s$ and $\mathrm{Ad}(\epsilon)X_n = -X_n$. Thus both X_s and X_n are in L. We first study the semisimple elements.

Lemma 2.1. *Suppose $X \in L$ is semisimple. Then X is H-conjugate to an element of the form*

$$\begin{pmatrix} 0 & 0 & \gamma B & 0 \\ 0 & 0 & 0 & 0 \\ \bar{B} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where B is invertible semisimple.

Proof. We write

$$X = \begin{pmatrix} 0 & \gamma A \\ \bar{A} & 0 \end{pmatrix}.$$

As in [**JR**], X is semisimple implies that A, $\bar{A}$, $A\bar{A}$ and $\bar{A}A$ are semisimple of the same rank. Let V be the vector space E^n of column vectors, which contains the F-vector space F^n. Then $M(n, E)$ operates on V by $B(v) = Bv$ for $v \in V$ and $B \in M(n, E)$. We call a vector $v \in V$ is defined over F if it is in F-form (i.e. $v \in F^n$), and we call a subspace V' of V is defined over F if there is a basis of V' such that each vector in this basis is in F-form. It is easily seen that V' is defined over F if and only if $\bar{V}' = V'$. Let V_0 be the kernel of A and V_1 its image. Then $\bar{V}_0$ is the kernel and $\bar{V}_1$ is the image of $\bar{A}$. Since A is semisimple, we have $V = V_0 \oplus V_1$. It is clear that $\bar{V}_0$ is contained in the kernel of $A\bar{A}$. Since $A\bar{A}$ has the same rank as $\bar{A}$, we conclude that $\bar{V}_0$ is the kernel of $A\bar{A}$. Now we claim that $V_0 \cap \bar{V}_1 = 0$. Indeed if v is in the intersection then $v = \bar{A}u$ and $Av = 0$. Thus $A\bar{A}u = 0$. Hence u is in the kernel of $A\bar{A}$ which is $\bar{V}_0$. So we get $v = \bar{A}u = 0$. We have therefore

$$V = V_0 \oplus \bar{V}_1 = \bar{V}_0 \oplus V_1.$$

Let $(v_1, ..., v_m)$ be a basis of V_0 and $(v_{m+1}, ..., v_n)$ be a basis of $\bar{V}_1$. Then both $(v_1, ..., v_n)$ and $(\bar{v}_1, ..., \bar{v}_n)$ are bases of V. So there exists an element $k \in GL(n, E)$ such that

$$k(v_1, ..., v_n) = (\bar{v}_1, ..., \bar{v}_n).$$

Thus we have

$$\bar{k}(\bar{v}_1, ..., \bar{v}_n) = (v_1, ..., v_n) = k^{-1}(\bar{v}_1, ..., \bar{v}_n).$$

Therefore $k\bar{k}(\bar{v}_1, ..., \bar{v}_n) = (\bar{v}_1, ..., \bar{v}_n)$. It follows that $k\bar{k} = I_n$. Clearly we also have that $kV_0 = \bar{V}_0$ and $k\bar{V}_1 = V_1$. Then the map

$$a : Gal(E/F) \to GL(n, E), \quad \mathrm{id} \mapsto a_{\mathrm{id}} = I_n, \quad \sigma \mapsto a_\sigma = k$$

defines a cocycle. Here σ is the nontrivial element in $Gal(E/F)$. It is well known that $H^1(Gal(E/F), GL(n, E)) = \{1\}$. So the cocycle a_s is cohomologous to the unit cocycle $b_s = 1$. This means that there is $g \in GL(n, E)$ such that $a_s = g^{-1} b_s s(g) = g^{-1} s(g)$. If $s = \sigma$, we get $k = g^{-1}\bar{g}$. Then

$$\bar{g}V_0 = g\bar{V}_0 = \overline{\bar{g}V_0}, \quad \bar{g}\bar{V}_1 = gV_1 = \overline{\bar{g}\bar{V}_1}.$$

If we set $B' = gA\bar{g}^{-1}$, then the kernel of B' is $\bar{g}V_0$ and the image of B' is gV_1. We have

$$V = V_0 \oplus \bar{V}_1 = \bar{g}V_0 \oplus \bar{g}\bar{V}_1 = \bar{g}V_0 \oplus gV_1.$$

Here both $\bar{g}V_0$ and gV_1 are defined over F. Suppose $(v_1, ..., v_m)$ is a F-form basis of $\bar{g}V_0$, and $(v_{m+1}, ..., v_n)$ is a F-form basis of gV_1. Let g_1 be the matrix

$$(v_{m+1}, ..., v_n, v_1, ..., v_m).$$

Then $g_1 \in GL(n, F)$ and

$$g_1^{-1} B' \bar{g}_1 = g_1^{-1} B' g_1 = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$$

where B is invertible. So

$$\begin{pmatrix} g_1^{-1} g & 0 \\ 0 & \bar{g}_1^{-1}\bar{g} \end{pmatrix} X \begin{pmatrix} g_1 g^{-1} & 0 \\ 0 & \bar{g}_1 \bar{g}^{-1} \end{pmatrix}$$

is in the form required. □

Next we study the nilpotent elements of L. For

$$X = \begin{pmatrix} 0 & \gamma A \\ \bar{A} & 0 \end{pmatrix},$$

we have

$$X^{2n} = \begin{pmatrix} \gamma^n (A\bar{A})^n & 0 \\ 0 & \gamma^n (\bar{A}A)^n \end{pmatrix}.$$

So X is nilpotent if and only if $A\bar{A}$ is nilpotent. For convenience we denote the matrix

$$\begin{pmatrix} g & 0 \\ 0 & \bar{g} \end{pmatrix}, \quad g \in GL(n, F)$$

by $h(g)$.

Lemma 2.2. *Suppose X is nilpotent. Then there exists $h(g) \in H$ such that $gA\bar{g}^{-1}$ is upper triangular with null diagonal entries and with all the null rows on the bottom.*

Proof. We use the induction on n. Our assertion is trivial for $n = 1$, since $A\bar{A} = 0$ implies $A = 0$. To continue, we assume that our lemma is true for $n' < n$. Let m be the rank of A. Then $m < n$. At the cost of conjugating by an element $h(g)$ such that g is a permutation matrix, we may assume that the first m-rows of A are linearly independent. Then after a conjugation by an element $h(g)$ where g is of the form

$$g = \begin{pmatrix} I_m & 0 \\ D & I_{n-m} \end{pmatrix},$$

we can assume

$$A = \begin{pmatrix} B & C \\ 0 & 0 \end{pmatrix}$$

where B has size $m \times m$. Since $A\bar{A}$ is nilpotent, $B\bar{B}$ is also nilpotent. So by induction hypothesis, there is a $g_1 \in GL(m, E)$ such that $g_1 B \bar{g}_1^{-1}$ has the appropriate form. Conjugating X by $h(g)$, where

$$g = \begin{pmatrix} g_1 & 0 \\ 0 & I_{n-m} \end{pmatrix},$$

we get a matrix of the required form. □

Lemma 2.3. *Suppose $X \in L$ is nilpotent. Then there is $h(g) \in H$ such that the twisted conjugate $gA\bar{g}^{-1}$ of A by g is in the Jordan normal form.*

Proof. By the previous lemma, we can assume that A is in the form

$$A = \begin{pmatrix} B & C \\ 0 & 0 \end{pmatrix}$$

where B is of size $(n-1)\times(n-1)$ with $B\bar{B}$ nilpotent. By induction there is $g\in GL(n-1,E)$ such that

$$g B \bar{g}^{-1} = \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_k \end{pmatrix}$$

where A_i is the standard $n_k \times n_k$ Jordan block

$$\begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}.$$

Thus under the twisted conjugation by the matrix $\mathrm{diag}(g,1)$ we can assume that

$$A = \begin{pmatrix} A_1 & & & C_1 \\ & \ddots & & \vdots \\ & & A_k & C_k \\ 0 & \dots & 0 & 0 \end{pmatrix}$$

where we write the column vector C_i of n_i-components as

$$C_i = \begin{pmatrix} D_i \\ d_i \end{pmatrix}$$

with $d_i \in E$.

Then we can make D_i $(i=1,2,\dots,k)$ vanishing if we twisted-conjugate A by the matrix

$$\begin{pmatrix} I_{n_1} & & & E_1 \\ & \ddots & & \vdots \\ & & I_{n_k} & E_k \\ & & & 1 \end{pmatrix}, \quad E_i = \begin{pmatrix} 0 \\ \bar{D}_i \end{pmatrix}.$$

Now we assume $D_i = 0$ for $i=1,\dots,n$. If $d_i \neq 0$ for any $i\in\{1,2,\dots,k\}$, we consider the $n_i\times n_i$ diagonal matrix $F_i = \mathrm{diag}(\dots,\bar{d}_i^{-1},d_i^{-1})$ with d_i^{-1} and $\bar{d}_i^{-1}$ appearing in the diagonal alternately. Then we have $F_i A_i \bar{F}_i^{-1} = A_i$ and

$$F_i \begin{pmatrix} 0 \\ \vdots \\ 0 \\ d_i \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Thus twisted-conjugating A by the matrix $\mathrm{diag}(I_{n_1},\dots,F_i,\dots,I_{n_k},1)$, we made d_i to be 1 and all the other entries unchanged.

So we can assume that all the entries of A are $0,1$ under the twisted conjugation in $GL(n,E)$. Then there is a matrix $g \in GL(n,F)$ such that $g_1 A\bar{g}_1^{-1} = g_1 A g_1^{-1}$ is in the Jordan normal form. We are done. □

Proof of Proposition 2. Let $X \in L$ be an element with Jordan decomposition $X = X_s + X_n$ where X_s is semisimple and X_u is nilpotent such that $X_sX_u = X_uX_s$. By Lemma 2.1, we can assume

$$X_s = \begin{pmatrix} 0 & 0 & \gamma B & 0 \\ 0 & 0 & 0 & 0 \\ \bar{B} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where B is an invertible semisimple element in $GL(n_1,E)$ for certain $n_1 \le n$. Since X_u commutes with X_s, we have

$$X_u = \begin{pmatrix} 0 & 0 & \gamma X_1 & 0 \\ 0 & 0 & 0 & \gamma X_2 \\ \bar{X}_1 & 0 & 0 & 0 \\ 0 & \bar{X}_2 & 0 & 0 \end{pmatrix}$$

where $X_i\bar{X}_i$ are nilpotent and $\bar{B}X_1B^{-1} = \bar{X}_1$. By Lemma 2.3, there is $g_1 \in GL(n-n_1,E)$ such that $g_1X_2\bar{g}_1^{-1}$ is in the Jordan normal form. So we have $\overline{g_1X_2\bar{g}_1^{-1}} = g_1X_2\bar{g}_1^{-1}$. This implies $\bar{g}_1^{-1}g_1X_2\overline{\bar{g}_1^{-1}g_1}^{-1} = \bar{X}_2$. Then the matrix $h(g)$ where

$$g = \begin{pmatrix} \bar{B} & 0 \\ 0 & \bar{g}_1^{-1}g_1 \end{pmatrix}$$

satisfies $h(g)Xh(g)^{-1} = \bar{X}$. □

3. The symmetric space.

Let F be a nonarchimedean local field of characteristic zero, and let $E = F(\sqrt{\tau})$ $(\tau \in F)$ be a quadratic extension field of F. Let γ, G, H, ϵ be as in the last section. We consider the variety

$$S' = \{s \in G | s^2 = \tau I_{2n}\}.$$

We first explain that all elements of S' are actually conjugate to ϵ in $GL(2n,E)$. For $g \in GL(2n,E)$, we have that $g \in G$ if and only if $w^{-1}gw = \bar{g}$ where

$$w = \begin{pmatrix} 0 & \gamma I_n \\ I_n & 0 \end{pmatrix}.$$

Let $V = E^{2n}$. So for $g \in G$ and $v \in V$ we have $gv = \sqrt{\tau}v$ if and only if $g(w\bar{v}) = -\sqrt{\tau}w\bar{v}$. This implies that if $\{v_1, .., v_m\}$ is a basis of the eigenspace of g belonging to the eigenvalue $\sqrt{\tau}$, then $\{w\bar{v}_1, ..., w\bar{v}_m\}$ is a basis of the eigenspace of g belonging to the eigenvalue $-\sqrt{\tau}$. Let s be an element in S. So we have $s^2 = \tau I_{2n}$. Thus s is conjugate in $GL(2n, E)$ to an diagonal matrix with eigenvalues from the set $\{\sqrt{\tau}, -\sqrt{\tau}\}$. But s is also in G. It follows immediately from the above consideration that s is conjugate to ϵ in $GL(2n, E)$. Since both ϵ and s are in G, they are in fact conjugate in G. So we obtain

$$S' = \{g\epsilon g^{-1} | g \in G\}.$$

We denote by S the set $S'\epsilon^{-1}$. So we have

$$S = \{s \in G | s\epsilon s\epsilon = \tau I_{2n}\} = \{g\epsilon g^{-1}\epsilon^{-1} | g \in G\}.$$

Then G operates on S by the twisted action

$$(g, x) = gx(\epsilon g^{-1}\epsilon^{-1}), \quad g \in G, \quad x \in S.$$

In particular for $h \in H$, we have $\epsilon h^{-1}\epsilon^{-1} = h^{-1}$. So H acts on S by conjugation $(h, x) = hxh^{-1}$. The surjective map $\rho : G \to S$ defined by

$$\rho(g) = g\epsilon g^{-1}\epsilon^{-1}, \quad g \in G$$

satisfies

$$\rho(xgh) = x\rho(g)(\epsilon x^{-1}\epsilon^{-1})$$

for $x, g \in G$ and $h \in H$. So it induces an isomorphism between G/H and S as G-spaces. The map ρ also induces a one to one correspondence between the H-double cosets in G and H-orbits in S. Our goal is the following proposition.

Proposition 3. *For each element $g \in G$, we have $g^{-1} \in HgH$.*

Let

$$g = \begin{pmatrix} \alpha & \gamma\beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$$

be an element of G. Then we have

$$g\epsilon g\epsilon = \tau \begin{pmatrix} \alpha^2 - \gamma\beta\bar{\beta} & \gamma\beta\bar{\alpha} - \gamma\alpha\beta \\ \bar{\beta}\alpha - \bar{\alpha}\bar{\beta} & \bar{\alpha}^2 - \gamma\bar{\beta}\beta \end{pmatrix}.$$

So $g \in S$ if and only if

$$\alpha^2 = I_n + \gamma\beta\bar{\beta}, \quad \alpha\beta = \beta\bar{\alpha}.$$

We call these two algebraic equations the defining equations for S.

Lemma 3.1. *Let*
$$s = \begin{pmatrix} \alpha & \gamma\beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in S$$
where the matrix α has no eigenvalues $1, -1$. Then the H-conjugacy class of s has a representative of the form
$$\begin{pmatrix} A & \gamma B \\ \bar{B} & A \end{pmatrix}$$
where $A \in M(n, F)$. If $\rho(g) = s$, then g^{-1} is in HgH.

Proof. Since α has no eigenvalues $1, -1$, $\alpha^2 - I_n$ is invertible. Thus β is invertible. So from $\alpha\beta = \beta\bar{\alpha}$, we obtain $\beta^{-1}\alpha\beta = \bar{\alpha}$. Thus if
$$p_1(x)|p_2(x)|...|p_r(x),$$
where $p_i(x) \in E[x]$, are the elementary divisors of α, we have that in fact $p_i(x) \in F[x]$. This implies that α is conjugate under $GL(n, E)$ to an element of $M(n, F)$. To continue, We assume that α is in $M_n(F)$. Then $\beta\alpha = \alpha\beta$. Let $A = \beta(I_n + \alpha)^{-1}$ and
$$g = \begin{pmatrix} 1_n & \gamma A \\ \bar{A} & 1_n \end{pmatrix}.$$
Since we have
$$\begin{aligned} I_n - \gamma A\bar{A} &= I_n - \gamma\beta\bar{\beta}(I_n + \alpha)^{-2} \\ &= I_n - (\alpha^2 - I_n)(I_n + \alpha)^{-2} = 2(I_n + \alpha)^{-1}, \end{aligned}$$
so the matrix $I_n - \gamma A\bar{A}$ is invertible. Hence g is invertible. In fact we have
$$\begin{aligned} g^{-1} &= \begin{pmatrix} 1_n & -\gamma A \\ -\bar{A} & 1_n \end{pmatrix} \begin{pmatrix} (1_n - \gamma A\bar{A})^{-1} & 0 \\ 0 & (1_n - \gamma\bar{A}A)^{-1} \end{pmatrix} \\ &= \epsilon g \epsilon^{-1} \begin{pmatrix} (1_n - \gamma A\bar{A})^{-1} & 0 \\ 0 & (1_n - \gamma\bar{A}A)^{-1} \end{pmatrix} \in HgH. \end{aligned}$$
We can check at once that
$$\rho(g) = \begin{pmatrix} \alpha & \gamma\beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} = s.$$
The assertion of the lemma follows. □

Next we study the element

$$s = \begin{pmatrix} \alpha & \gamma\beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$$

in S where α is unipotent. We will prove that the set of such elements in S is the set of the unipotent elements in S. Let N be the set of unipotent elements of G, and let $\mathfrak{n}$ be the set of nilpotent elements of $\mathfrak{g}$. Then the exponential map

$$\exp : \mathfrak{n} \to N, \quad X \mapsto \exp(X)$$

defines an isomorphism of $\mathfrak{n}$ onto N. Let $N_S = N \cap S$ be the set of unipotent elements in S.

Lemma 3.2. *We have*

(1) $N_S = \exp(\mathfrak{n}_L)$ *where* $\mathfrak{n}_L = \mathfrak{n} \cap L$.

(2) *The set* $\{g \in G | \rho(g) \in N_S\}$ *is* HN_SH, *and for each element* g *in this set we have* $g^{-1} \in HgH$.

(3) *Let*

$$s = \begin{pmatrix} \alpha & \gamma\beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in S.$$

Then s *is unipotent if and only if* α *is unipotent.*

Proof. Let $X \in \mathfrak{g}$ and $u = \exp(X) \in N$. If $u \in N_S$, then $(u\epsilon)(u\epsilon) = \tau I_n$. Note that $\frac{1}{\tau}\epsilon = \epsilon^{-1}$. So we have $u^{-1} = \epsilon^{-1}u\epsilon$. This is equivalent to $-X = \epsilon^{-1}X\epsilon$, which is just the condition for $X \in \mathfrak{n}_L$. Thus we get (1).

For $u = \exp(X) \in N_S$ where $X \in \mathfrak{n}_L$, we have that

$$\begin{aligned} \rho(u) = u\epsilon u^{-1}\epsilon^{-1} &= \exp(X)\exp(\epsilon(-X)\epsilon^{-1}) \\ &= \exp(X)\exp(X) = \exp(2X) \in N_S. \end{aligned}$$

Therefore

$$HN_SH \subset \{g \in G | \rho(g) \in N_S\}.$$

On the other hand, if we let $v = \exp(X/2)$, then $v \in N_S$ and

$$\rho(v) = \exp(2X/2) = \exp(X) = u.$$

The first part of (2) follows. Now we have

$$u^{-1} = \exp(-X) = \exp(\epsilon X \epsilon^{-1}) = \epsilon \exp(X)\epsilon^{-1}.$$

So $u^{-1} \in HuH$. This proves the second part of (2).

Recall that we have

$$\alpha^2 - 1 = \gamma\beta\bar{\beta}, \quad \alpha\beta = \beta\bar{\alpha}.$$

Then it is easily seen that

$$(s - I_{2n})^2 = 2\begin{pmatrix} \alpha - I_n & 0 \\ 0 & \bar{\alpha} - I_n \end{pmatrix} s = 2s\begin{pmatrix} \alpha - I_n & 0 \\ 0 & \bar{\alpha} - I_n \end{pmatrix}.$$

Thus

$$(s - I_{2n})^{2n} = 2^n \begin{pmatrix} (\alpha - I_n)^n & 0 \\ 0 & (\bar{\alpha} - I_n)^n \end{pmatrix} s^n.$$

Because s^n is nonsingular, we have that $(s - I_{2n})^{2n} = 0$ if and only if $(\alpha - I_n)^n = 0$. Hence s is unipotent if and only if α is unipotent. □

It remains to study the set

$$\{g \in G | -\rho(g) \in N_S\}.$$

By the above lemma this is the set of the element

$$s = \begin{pmatrix} \alpha & \gamma\beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in S$$

such that $-\alpha$ is unipotent. Recall that

$$w = \begin{pmatrix} 0 & \gamma 1_n \\ 1_n & 0 \end{pmatrix}.$$

So we have $w\epsilon w^{-1} = -\epsilon$ and $wH = Hw$.

Lemma 3.3. *The set*

$$\{g \in G | -\rho(g) \in N_S\}$$

is HN_SwH. If $g \in HN_SwH$, then $g^{-1} \in HgH$.

Proof. For $g \in G$, we have

$$\rho(gw) = gw\epsilon w^{-1} g^{-1}\epsilon^{-1} = -g\epsilon g^{-1}\epsilon^{-1} = -\rho(g).$$

Hence $-\rho(g) \in N_S$ if and only if $\rho(gw) \in N_S$. By Lemma 3.2, $\rho(gw) \in N_S$ if and only if $gw \in HN_SH$, which in turn is equivalent to say that

$$g \in HN_SHw^{-1} = HN_Sw^{-1}H = HN_SwH$$

since $w^{-1} = w\mathrm{diag}(\gamma^{-1}, \gamma^{-1}) \in wH$.

Now let $g = uw$ be an element in $N_S w$ where u is in N_S. Then we can write $u = \exp(X)$ where $X \in \mathfrak{n}_L$. So

$$g^{-1} = w^{-1}u^{-1} = w^{-1}u^{-1}ww^{-1} = \exp(-w^{-1}Xw)w^{-1} = \exp(-\bar{X})w^{-1}.$$

Recall that we have proved in the last section that there exists an element h in H such that $\bar{X} = hXh^{-1}$. So we have $h_1Xh_1^{-1} = -\bar{X}$ where $h_1 = \epsilon h \in H$. Thus

$$\begin{aligned} g^{-1} = \exp(h_1Xh_1^{-1})w^{-1} &= h_1\exp(X)h_1^{-1}w^{-1} = h_1\exp(X)w^{-1}\bar{h}_1^{-1} \\ &= h_1 g \begin{pmatrix} \gamma^{-1} & 0 \\ 0 & \gamma^{-1} \end{pmatrix} \bar{h}_1^{-1} \end{aligned}$$

which is in HgH. This ends the proof of the lemma. □

Finally we prove that a general element of S is compound of these three types of elements we just studied. We first fix a notation. Let n_1, n_2, n_3 be three nonnegative integers such that $n_1 + n_2 + n_3 = n$. If

$$g_i = \begin{pmatrix} \alpha_i & \gamma\beta_i \\ \bar{\beta}_i & \bar{\alpha}_i \end{pmatrix} \in G_{n_i},$$

then we use $\sigma(g_1, g_2, g_3)$ to denote the element

$$\begin{pmatrix} \alpha_1 & 0 & 0 & \gamma\beta_1 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 & \gamma\beta_2 & 0 \\ 0 & 0 & \alpha_3 & 0 & 0 & \gamma\beta_3 \\ \bar{\beta}_1 & 0 & 0 & \bar{\alpha}_1 & 0 & 0 \\ 0 & \bar{\beta}_2 & 0 & 0 & \bar{\alpha}_2 & 0 \\ 0 & 0 & \bar{\beta}_3 & 0 & 0 & \bar{\alpha}_3 \end{pmatrix}$$

of G. For $g_i, g_i' \in G_{n_i}$, $i = 1, 2, 3$, the following relation is easily verified

$$\sigma(g_1, g_2, g_3)\sigma(g_1', g_2', g_3') = \sigma(g_1g_1', g_2g_2', g_3g_3').$$

It is also obvious that

$$\sigma(g_1, g_2, g_3) = \sigma(g_1', g_2', g_3')$$

if and only if $g_i = g_i'$, $i = 1, 2, 3$. We use S_{n_i} , H_{n_i} and ϵ_{n_i} to denote the corresponding parts in G_{n_i} to S, H and ϵ in G.

Lemma 3.4. *Each element s of S is H-conjugate to an element of the form*

$$\sigma(s_1, s_2, s_3)$$

where $s_i \in S_{n_i}$ $(i = 1, 2, 3)$ for certain nonnegative integers n_1, n_2, n_3 such that $n_1 + n_2 + n_3 = n$, and if we write

$$s_i = \begin{pmatrix} \alpha_i & \gamma\beta_i \\ \bar{\beta}_i & \bar{\alpha}_i \end{pmatrix}, \quad i = 1, 2, 3,$$

then α_1 has no eigenvalues $1, -1$, α_2 is unipotent, and $-\alpha_3$ is unipotent.

Proof. Let

$$s = \begin{pmatrix} \alpha & \gamma\beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$$

be an element in S. Then up to a conjugation by an element in $GL(n, E)$, we can assume that

$$\alpha = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}$$

where α_1 has no eigenvalues $1, -1$, α_2 is unipotent, and $-\alpha_3$ is unipotent. Let $\beta = (\beta_{ij})$ where β_{ij} $(1 \le i, j \le 3)$ is a matrix of size $n_i \times n_j$. Since s is in S, we have $\alpha\beta = \beta\bar{\alpha}$. This implies

$$\alpha_i\beta_{ij} = \beta_{ij}\bar{\alpha}_j, \quad 1 \le i, j \le 3.$$

If $i \ne j$, then α_i and $\bar{\alpha}_j$ have no common eigenvalues. So we must have $\beta_{ij} = 0$ if $i \ne j$. Thus

$$s = \begin{pmatrix} \alpha_1 & 0 & 0 & \gamma\beta_1 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 & \gamma\beta_2 & 0 \\ 0 & 0 & \alpha_3 & 0 & 0 & \gamma\beta_3 \\ \bar{\beta}_1 & 0 & 0 & \bar{\alpha}_1 & 0 & 0 \\ 0 & \bar{\beta}_2 & 0 & 0 & \bar{\alpha}_2 & 0 \\ 0 & 0 & \bar{\beta}_3 & 0 & 0 & \bar{\alpha}_3 \end{pmatrix} = \sigma(s_1, s_2, s_3)$$

where we set $\beta_i = \beta_{ii}$ and

$$s_i = \begin{pmatrix} \alpha_i & \gamma\beta_i \\ \bar{\beta}_i & \bar{\alpha}_i \end{pmatrix}.$$

We still need to show that s_i is in S_{n_i}. Note that

$$\epsilon = \sigma(\epsilon_{n_1}, \epsilon_{n_2}, \epsilon_{n_3}).$$

So we have

$$s\epsilon s\epsilon = \sigma(s_1, s_2, s_3)\sigma(\epsilon_{n_1}, \epsilon_{n_2}, \epsilon_{n_3})\sigma(s_1, s_2, s_3)\sigma(\epsilon_{n_1}, \epsilon_{n_2}, \epsilon_{n_3})$$
$$= \sigma(s_1\epsilon_{n_1}s_1\epsilon_{n_1}, s_2\epsilon_{n_2}s_2\epsilon_{n_2}, s_3\epsilon_{n_3}s_3\epsilon_{n_3}).$$

On the other hand we have

$$s\epsilon s\epsilon = \tau I_{2n} = \sigma(\tau I_{2n_1}, \tau I_{2n_2}, \tau I_{2n_3}).$$

Thus we get

$$s_i\epsilon_{n_i}s_i\epsilon_{n_i} = \tau I_{2n_i}, \quad i = 1, 2, 3,$$

which means $s_i \in S_{n_i}$. This ends the proof of the lemma. □

We are now ready to prove Proposition 3.

Proof of Proposition 3. Let g be an element of G and let $s = \rho(g)$ be in S. Then by Lemma 3.4, we can assume that

$$s = \sigma(s_1, s_2, s_3)$$

where s_i, $i = 1, 2, 3$, satisfy the conditions of that lemma. By Lemmas 3.1, 3.2 and 3.3, there exist $g_i \in G_{n_i}$ for $i = 1, 2, 3$ such that

$$g_i\epsilon_{n_i}g_i^{-1}\epsilon_{n_i}^{-1} = s_i$$

and such that $g_i^{-1} = h_i g_i h_i'$ for some $h_i, h_i' \in H_{n_i}$. So we have

$$\rho(\sigma(g_1, g_2, g_3)) = \sigma(s_1, s_2, s_3) = s.$$

Thus we can assume that $g = \sigma(g_1, g_2, g_3)$. In this case we have

$$g^{-1} = \sigma(g_1^{-1}, g_2^{-1}, g_3^{-1}) = \sigma(h_1g_1h_1', h_2g_2h_2', h_3g_3h_3') = hgh'$$

where $h = \sigma(h_1, h_2, h_3), h' = \sigma(h_1', h_2', h_3') \in H$. The assertion of the proposition follows. □

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