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TRANSITION OPERATORS, GROUPS, NORMS, AND SPECTRAL RADII

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Let P be a transition operator over a countable set which is invariant under the action of a locally compact group G with compact point stabilizers. We give upper bounds for the norm and spectral radius of P acting on $\ell^s(X,\mu)$, where $1 < s < \infty$ and μ is a measure on X satisfying a compatibility condition with respect to G. When G is amenable, our inequalities become equalities involving the modular function of G. When G, besides being amenable, acts with finitely many orbits then this allows easy computation of norms and spectral radii via reduction to a finite matrix. For unimodular groups there are further simplifications. A variety of examples is given, including the (linear) buildings of type $\tilde{A}_{n-1}$ associated with $PGL(n,\mathfrak{F})$ over a local field $\mathfrak{F}$. These results extend previous work of Soardi and Woess, Salvatori, and Saloff-Coste and Woess, where only reversible Markov operators and the case $s = 2$ were studied.

1. Introduction.

The papers by Soardi and Woess [20], Salvatori [16] and Saloff-Coste and Woess [15] develop a technique to compute the convergence radius ρ of discrete reversible Markov chains which are invariant under the action of an amenable, non necessarily discrete and typically nonunimodular group. For a discrete, irreducible transition matrix P with kernel $p(x,y)$, $x,y \in X$, the convergence radius $\rho = \rho(P)$ is defined by

$$\rho(P) = \limsup_{n\to+\infty} \left[p^{(n)}(x,y)\right]^{1/n}$$

where (x,y) is any fixed pair of points in $X \times X$. If P is reversible with respect to a measure μ on X then $\rho = \|P\|_{2\to2} = \lim_{n\to+\infty} \|P^n\|_{2\to2}^{1/n}$ where the norm is taken with respect to the space $\ell^2(X,\mu)$.

In [15], under the assumption that there is an amenable group G acting on X with (typically finite) quotient I and such that $p(gx,gy) = p(x,y)$ for all $g \in G$, $x,y \in X$, we reduced the computation of ρ to that of the spectral radius of a symmetric matrix with rows and columns indexed by I. The transitive case where I is a singleton had previously been studied in [20].

The aim of the present paper is to extend these results in two directions. First, we shall consider non-reversible chains. Second, we aim to compute the operator norms and spectral radii of P acting on $\ell^s(X,\mu)$, where $1 < s < +\infty$ and μ is any "reasonable" measure on X. (We shall see examples with more than one choice for μ.) That is, we shall try to compute the quantities

$$\sigma_s(P) = \|P\|_{s\to s} = \sup_{\|f\|_s\le 1} \|Pf\|_s \quad \text{and} \quad \rho_s(P) = \lim_{n\to+\infty} \|P^n\|_{s\to s}^{1/n}.$$

In fact, we shall work with operators P associated with nonnegative kernels p, assuming only that $\sum_y p(x,y) < +\infty$.

To put in perspective the approach developed here, let us review briefly the other methods available to compute some of these quantities.

The most typical approach consists in computing the convergence radius $\rho(P)$ by studying the singularities of the Green function $g(x,y|z) = \sum_{n=0}^{\infty} p^{(n)}(x,y)z^n$: one looks for an exact closed formula for the Green function via combinatorical considerations and a functional equation. See for instance the survey by Mohar and Woess [12]. When it works, this method not only gives $\rho(P)$ but often also yields a more precise description of the behavior of $p^{(n)}$ (i.e., local limit theorems). When P is reversible with respect to μ, this gives $\sigma_2(P,\mu) = \rho_2(P,\mu)$ because these are equal to $\rho(P)$, but yields no information concerning ρ_s or σ_s, $s \ne 2$.

Another approach available that yields exact values for ρ_s in rather specific cases consists in using sophisticated information concerning the relevant harmonic analysis, e.g., the theory of group represention and more or less explicit formulas for spherical functions. When applicable, this method offers deep insight on the objects involved. In particular, a complete description of the ℓ^s spectrum may be available through this approach; see for instance Figà-Talamanca and Picardello [7] or Cartwright, Młotkowski and Steger [4, 5]. Further references can be found in these papers and in [12]. It is not clear to us what the machinery of representation theory says about whether or not $\rho_s = \sigma_s$ and about computing σ_s when $\rho_s \ne \sigma_s$. More recently, a more C^*-algebraic approach is due to de la Harpe, Robertson and Valette [9], which yields good informations on the ℓ^2-spectrum in several cases of finitely generated groups.

The present work provides a simple combinatorial technique to compute both ρ_s and σ_s. The class of examples for which our method and results apply is closely related to the class of examples where the Green function and / or the representation techniques apply. Roughly, all these techniques require a tree like (or building like) structure.

We are able to identify a large class of examples where $\rho_s = \sigma_s$ (all graphs on which an amenable group acts transitively, see Theorem 3.1). We also give many examples where $\rho_s \neq \sigma_s$ and where computations of both these numbers are feasible. Let us emphazise here that exact formulas for σ_s are seldom available even when ρ_s is known. This is not surprising since, for a two by two matrix M, computing $\rho_s(M)$ is essentially trivial (it amounts to solving a quadratic equation) but computing the exact value of $\sigma_s(M)$ is hard (if not impossible) in general.

Let us now recall some well known facts about σ_s and ρ_s. Clearly, $\rho_s \leq \sigma_s$. It may well happen that $\rho_s < \sigma_s$ and, in general, these are non-constant functions of s. We always have

$$\sigma_1(P) = \sup_y \sum_x \frac{\mu(x)}{\mu(y)} p(x,y) \quad \text{and} \quad \sigma_\infty(P) = \sup_x \sum_y p(x,y).$$

Interpolation yields

$$\sigma_s \leq \sigma_r^\theta \sigma_t^{1-\theta}$$

where $1/s = \theta/r + (1-\theta)/t$, $r \leq s \leq t$. Duality shows that for s' defined by $1/s + 1/s' = 1$ $(1 \leq s, s' \leq +\infty)$,

$$\sigma_s(P,\mu) = \sigma_{s'}(P^*,\mu).$$

If P is self-adjoint with respect to μ (that is, (P,μ) is reversible), this yields $\sigma_{s'} = \sigma_s$. By interpolation, it follows that σ_s is minimal at $s = 2$ in this case. In general, σ_s can be minimal for $s \neq 2$. All these facts are also satisfied by the ρ_t's. Note also that when P is not reversible with respect to μ, the convergence radius $\rho(P)$ may well be strictly smaller than ρ_2 and even smaller than the smallest ρ_s. Vere-Jones' papers [22, 23] contain useful information concerning operators given by nonnegative kernels and their spectral theory. For instance, Vere-Jones notes that $\rho_s(P)$ belongs to the ℓ^s-spectrum of P (see also Schaefer [17]). We will use this fact in §4.

One of our basic tools will be the relation between norms and spectral radii of convolution operators and amenability of locally compact groups.

Let G be a locally compact group with left Haar measure dg. Let q be a bounded Borel measure on G and write $\mathcal{L}_q$ for the associated left convolution operator:

$$\mathcal{L}_q f(g) = \int_G f(h^{-1}g)\, dq(h) = q \star f(g)$$

for any compactly supported continous function f. Let $q(G)$ denote the total mass of q and consider the norm $\|\mathcal{L}_q\|_{s\to s} = \sup\{\|\mathcal{L}_q f\|_s : \|f\|_s \leq 1\}$, where

$\|f\|_s^s = \int_G |f(g)|^s \, dg$. Also, let $\mathrm{supp}(q)$ denote the support of q.

Theorem 1.1 (**Berg-Christensen** [2, 3]). *Fix* $1 < s < +\infty$. *Then*

(i) $\|Q\|_{s\to s} = q(G)$ *if and only if the closed subgroup generated by* $\mathrm{supp}(q)\,\mathrm{supp}(q)^{-1}$ *is amenable.*

(ii) $\lim_{n\to\infty} \|Q^n\|_{s\to s}^{1/n} = q(G)$ *if and only if the closed subgroup generated by* $\mathrm{supp}(q)$ *is amenable.*

The case where $s = 2$ is explicitly stated in [3]. The case where $1 < s < \infty$ follows easily by interpolation using the fact that one always has $\|Q\|_{1\to 1} = \|Q\|_{\infty\to\infty} = q(G)$. Earlier references can be found in [2, 3].

This paper is organized as follows. In §2, we study norms of transition operators between *two* discrete homogeneous spaces of the same group G and extend the method of [20, 15] of lifting such operators to a convolution operator over G (Proposition 2.1). We use this to relate the norms with amenability and with the modular function of the group.

In §3, we specialize the results of §2 to group invariant transition operators on *one* homogeneous space, thereby extending the results of [20]. We give a first collection of examples of random walks on graphs where our method allows explicit computation of norms and / or spectral radii.

In §4, we consider transition operators on a discrete set which are invariant under a group acting with finitely many orbits. Letting each pair of orbits play the role of the two homogeneous spaces in §2, in the amenable case we can reduce computations to a finite matrix. This extends the results of [15]. Our examples comprise certain random walks on discrete, nonamenable groups where it is possible to find an amenable, nondiscrete group acting with finitely many orbits.

It is worthwhile noting that a good part of the methods used here is not restricted to groups acting on *discrete* sets (graphs). We shall consider the continuous setting in future work.

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2. Norms of operators between homogeneous spaces.

Let X, Y be two discrete, countable sets on which a locally compact group G acts continuously and transitively with compact point stabilizers. Throughout this paper, dg denotes a fixed left Haar measure on G. Given any $x \in X$, its stabilizer in G is open and compact, so that its measure $|G_x|$ is positive and finite. In particular, the restriction of dg to G_x is a (left and right) Haar measure on G_x (this is a special feature of discrete homogeneous spaces).

X can be identified with G/G_x in a canonical way, and for any integrable function f on G, we have

$$\int_G f(g)\,dg = \sum_{z\in X}\int_{G_x} f(g_z h)\,dh,$$

where, for each z, g_z is any element of G which is sends x to z. We shall use analogous notation for Y. Let μ and ν be positive measures on X and Y, respectively, such that, for all $x\in X$ and all $y\in Y$,

$$\mu(gx)/\mu(x) = \nu(gy)/\nu(y). \tag{2.1}$$

Let P be a transition operator from X to Y, given by

$$Pf(x) = \sum_{y\in Y} p(x,y)f(y)$$

with kernel $p(x,y)$, $x\in X$, $y\in Y$. Thus, P transforms any function $f: Y\to\mathbb{R}$ with finite support into a function $Pf: X\to\mathbb{R}$. Assume that

$$p(x,y)\geq 0,\quad \sum_{y\in Y} p(x,y) < +\infty \quad\text{and}\quad p(gx,gy)=p(x,y)$$

for all x,y and for all $g\in G$. We want to compute the norm $\|P\|_{s\to s}$ of P from $\ell^s(Y,\nu)$ to $\ell^s(X,\mu)$. Define

$$\begin{cases} p_s(x,y) = \mu(x)^{1/s}p(x,y)\nu(y)^{-1/s},\\[2ex] \Phi[Q](g) = \Phi(g) = \frac{1}{|G_{y_0}|}p(x_0,g^{-1}y_0),\\[2ex] \Phi_s[Q](g) = \Phi_s(g) = \frac{1}{|G_{y_0}|}p_s(x_0,g^{-1}y_0), \end{cases} \tag{2.2}$$

where $g\in G$ and $x_0\in X$, $y_0\in Y$ are chosen once for all as "origins" in X and Y, respectively. Observe that (2.1) implies that also p_s is a G-invariant kernel. Let $\mathcal{R}_v: u\mapsto u\star v$ be the operator of convolution by v on the right, where

$$u\star v(g) = \int_G u(h)v(h^{-1}g)\,dh = \int_G u(gh)v(h^{-1})\,dh,$$

and denote by $\|\mathcal{R}_v\|_{s\to t}$ the norm of this operator from $L^s(G)$ to $L^t(G)$.

Proposition 2.1. *If P is G-invariant from $\ell^s(Y,\nu)$ to $\ell^s(X,\mu)$ and (2.1) holds then*

$$\|P\|_{s\to s} = \left(\frac{|G_{y_0}|}{|G_{x_0}|}\right)^{1/s}\|\mathcal{R}_{\Phi_s}\|_{s\to s}.$$

Proof. To start, assume that μ and ν are the counting measures on X and Y, respectively. In this case, (2.1) holds and $\Phi = \Phi_s$ does not depend on s. Define $\mathcal{S}_X : L^s(G) \to \ell^s(X)$ by

$$\mathcal{S}_X u(x) = \frac{1}{|G_{x_0}|} \int_{\{g\in G\,:\,gx_0=x\}} u(g)\,dg \tag{2.3}$$

and $\mathcal{T}_X : \ell^s(X) \to L^s(G)$ by

$$\mathcal{T}_X f(g) = f(gx_0). \tag{2.4}$$

Then we have

$$\|\mathcal{S}_X\|_{s\to s} \le |G_{x_0}|^{-1/s}, \quad \|\mathcal{T}_X\|_{s\to s} = |G_{x_0}|^{1/s},$$

and

$$P = \mathcal{S}_X \mathcal{R}_\Phi \mathcal{T}_Y \quad \text{and} \quad \mathcal{R}_\Phi = \mathcal{T}_X P \mathcal{S}_Y.$$

Indeed,

$$\begin{aligned}\mathcal{S}_X \mathcal{R}_\Phi \mathcal{T}_Y f(x) &= \frac{1}{|G_{x_0}|} \int_{\{g\in G\,:\,gx_0=x\}} \int_G f(hy_0)\Phi(h^{-1}g)\,dh\,dg \\ &= \frac{1}{|G_{x_0}||G_{y_0}|} \int_{\{g\in G\,:\,gx_0=x\}} \int_G f(hy_0)p(x_0, g^{-1}hy_0)\,dh\,dg \\ &= \sum_{y\in Y} p(x,y)f(y) = Pf(x).\end{aligned}$$

A similar argument yields $\mathcal{R}_\Phi = \mathcal{T}_X P \mathcal{S}_Y$, compare with [20]. This clearly proves the proposition when μ and ν are the counting measures on X and Y, respectively.

In general, we consider the operator P_s associated with the kernel p_s. Since p_s is invariant under the action of G, we can apply the result we just proved, which shows that $\left(|G_{y_0}|/|G_{x_0}|\right)^{1/s} \|\mathcal{R}_{\Phi_s}\|_{s\to s}$ is the norm of P_s from $\ell^s(Y)$ to $\ell^s(X)$ with respect to the counting measures. By construction, this is equal to $\|P\|_{s\to s}$. □

Proposition 2.1 is a special case of a more general result worth noting. Let us drop the assumption that p is invariant, and let μ, ν be arbitrary fixed measures on X, Y. Then we can compute the norm $\|P\|_{s\to t}$ of the operator P from $\ell^s(Y,\nu)$ to $\ell^t(X,\mu)$ as

$$\|P\|_{s\to t} = \left(\frac{|G_{y_0}|^{1/s}}{|G_{x_0}|^{1/t}}\right) \|K_{s,t}\|_{s\to t},$$

where $K_{s,t}$ is the operator associated with the kernel

$$k_{s,t}(h,g) = \frac{1}{|G_{y_0}|}\mu(hx_0)^{1/t}p(hx_0, gy_0)\nu(gy_0)^{-1/s}$$

on $G \times G$. Here, the norms on G are taken with respect to the fixed left Haar measure. In general, $K_{s,t}$ is not a convolution operator, even when p is invariant. However, it is a convolution operator when p is invariant, μ, ν are related by (2.1), and $s = t$, or also when p is invariant and μ and ν are constant:

Lemma 2.2. *Assume that μ and ν are proportional to the counting measures on X and Y, respectively. Then, for any invariant kernel p,*

$$\|P\|_{s\to t} = \left(\frac{|G_{y_0}|}{\nu(y_0)}\right)^{1/s}\left(\frac{\mu(x_0)}{|G_{x_0}|}\right)^{1/t}\|\mathcal{R}_\Phi\|_{s\to t}.$$

We now use Proposition 2.1 to compute norms when G is amenable. To this end, we need the modular function $\Delta : G \to]0, +\infty[$. This is the function (a multiplicative homomorphism) which carries left Haar measure to right Haar measure on G. It satisfies $|Ug| = \Delta(g)|U|$ for any measurable set $U \subset G$ and

$$\int_G f(g)\,dg = \int_G f(g^{-1})\Delta(g^{-1})\,dg. \tag{2.5}$$

For any function $u \in L^1(G)$, we denote by $\mathcal{L}_u$ the operator of convolution on the left by u (i.e., $\mathcal{L}_u f = u \star f$). A well known argument (see e.g. Hewitt and Ross [10]) shows that

$$\|\mathcal{R}_u\|_{s\to s} = \|\mathcal{L}_{\tilde{u}_s}\|_{s\to s} \le \int_G \Delta(g)^{-1/s}|u(g^{-1})|\,dg, \tag{2.6}$$

where $\tilde{u}_s = \Delta(g)^{-1/s}u(g^{-1})$. Moreover, when G is amenable,

$$\|\mathcal{L}_u\|_{s\to s} = \int_G |u(g)|\,dg \quad \text{for any} \quad u \in L^1(G).$$

Proposition 2.3. *Let X, Y be two countable sets on which a locally compact group G acts transitively and continuously with compact stabilizers. Let μ and ν be positive measures on X and Y, respectively, which satisfy (2.1). Let $p(x,y), (x,y) \in X \times Y$ be an invariant nonnegative kernel. Fix $1 \le s \le +\infty$. Then*

$$\|P\|_{s\to s} \le \left(\frac{\mu(x_0)}{|G_{x_0}|}\right)^{1/s}\sum_{y\in Y}\left(\frac{|G_y|}{\nu(y)}\right)^{1/s}p(x_0,y).$$

Furthermore, define $S = S(P) \subset G$ by $S = \{g \in G : p(x_0, gy_0) > 0\}$. Then the above inequality is an equality if and only if the group generated by SS^{-1} is amenable.

Proof. By Proposition 2.1 and (2.6) we have

$$\|P\|_{s\to s} \le \left(\frac{|G_{y_0}|}{|G_{x_0}|}\right)^{1/s} \frac{1}{|G_{y_0}|}\int_G \Delta(g)^{-1/s}\left(\frac{\mu(x_0)}{\nu(gy_0)}\right)^{1/s} p(x_0, gy_0)\, dg.$$

By definition of the modular function, we have

$$\Delta(g)|G_{gy_0}| = \Delta(g)|gG_{y_0}g^{-1}| = |G_{y_0}|$$

and this gives the proposed inequality. The last statement follows from Theorem 1.1, Proposition 2.1 and the above computation. □

While Proposition 2.3 links the norms of P with amenability, we now want to discuss unimodularity (that is, $\Delta \equiv 1$). Let us introduce some notation. Set

$$\sigma_s(P) = \sigma_s(P, \mu, \nu) = \|P\|_{s\to s}\,. \tag{2.7}$$

Write

$$\begin{aligned} a_s(P) = a_s(P,\mu,\nu) &= \left(\frac{\mu(x_0)}{|G_{x_0}|}\right)^{1/s} \sum_{y\in Y}\left(\frac{|G_y|}{\nu(y)}\right)^{1/s} p(x_0, y) \\ &= \left(\frac{|G_{y_0}|}{|G_{x_0}|}\right)^{1/s} \frac{1}{|G_{y_0}|}\int_G \Delta(g)^{-1/s}\left(\frac{\mu(x_0)}{\nu(gy_0)}\right)^{1/s} p(x_0, gy_0)\, dg \end{aligned} \tag{2.8}$$

for the upper bound in Proposition 2.3. Define P^*, the (formal) adjoint of P, by setting

$$\forall\ u \in \ell_0(Y),\ v \in \ell_0(X) \quad \langle Pu, v\rangle_\mu = \langle u, P^*v\rangle_\nu,$$

where $\ell_0(\cdot)$ denotes finitely supported functions and each inner product is weighted with the respective measure. The kernel p^* of P^* is

$$p^*(y,x) = \frac{\mu(x)}{\nu(y)} p(x,y)$$

which is invariant by (2.1). Observe that p^* depends also on μ and ν. Now consider

(2.9)

$$b(P) = \sum_{y\in Y} p(x_0, y) \quad \text{and} \quad b(P^*) = b(P^*, \nu, \mu) = \sum_{x\in X}\frac{\mu(x)}{\nu(y_0)} p(x, y_0).$$

Note that

$$b(P) = \sigma_\infty(P)\,, \qquad b(P^*) = \sigma_1(P)$$

and

$$a_s(P) = a_{s'}(P^*), \tag{2.10}$$

where s' is defined as usual by $1/s + 1/s' = 1$. To see (2.10), write

$$\begin{aligned}
a_s(P) &= \left(\frac{|G_{y_0}|}{|G_{x_0}|}\right)^{1/s} \frac{1}{|G_{y_0}|} \int_G \Delta(g)^{-1/s} \left(\frac{\mu(x_0)}{\nu(gy_0)}\right)^{1/s} p(x_0, gy_0)\, dg \\
&= \left(\frac{|G_{x_0}|}{|G_{y_0}|}\right)^{1-1/s} \frac{1}{|G_{x_0}|} \\
&\quad \cdot \int_G \Delta(g^{-1})^{-(1-1/s)} \left(\frac{\mu(y_0)}{\nu(g^{-1}x_0)}\right)^{1-1/s} p^*(y_0, g^{-1}x_0)\Delta(g^{-1})\, dg \\
&= a_{s'}(P^*),
\end{aligned}$$

where (2.5) has been used in the last identity.

Proposition 2.4. *Let X, Y, G, P, μ, ν be as in Proposition 2.3. Then*

$$a_s(P) \leq b(P)^{1/s'}\, b(P^*)^{1/s}$$

and the following properties are equivalent:

(1) $a_s(P) = b(P)^{1/s'}\, b(P^*)^{1/s}$.

(2) *There exists a constant $c > 0$ such that $\Delta(g) = c\mu(x_0)/\nu(gy_0)$ for all $g \in S = S(P)$.*

(3) *$\Delta(g) = \mu(y_0)/\mu(gy_0)$ for all g in the closed subgroup generated by SS^{-1}.*

Proof. We have

$$\begin{aligned}
a_s(P) &= \frac{1}{|G_{x_0}|^{1/s}} \frac{1}{|G_{y_0}|^{1/s'}} \int_G \Delta(g)^{-1/s} \left(\frac{\mu(x_0)}{\nu(gy_0)}\right)^{1/s} p(x_0, gy_0)\, dg \\
&= \frac{1}{|G_{x_0}|^{1/s}} \frac{1}{|G_{y_0}|^{1/s'}} \int_G \left[\Delta(g^{-1}) p^*(gy_0, x_0)\right]^{1/s} p(x_0, gy_0)^{1/s'}\, dg.
\end{aligned}$$

Now, Hölder's inequality and (2.5) yield

$$\begin{aligned}
a_s(P) &\leq \left(\frac{1}{|G_{x_0}|} \int_G p^*(y_0, gx_0)\, dg\right)^{1/s} \left(\frac{1}{|G_{y_0}|} \int_G p(x_0, gy_0)\, dg\right)^{1/s'} \\
&= b(P)^{1/s'} b(P^*)^{1/s},
\end{aligned}$$

with equality if and only if $\Delta(g)^{-1} p^*(gy_0, x_0)$ and $p(x_0, gy_0)$ are constant multiples of each other, that is, $\Delta(g)\nu(gy_0) = c\,\mu(x_0)$ for some positive constant c and all $g \in S$. The other equivalences easily follow. □

There is an analogous result of this type worth noting here. Set

$$b_s(P) = b_s(P,\mu,\nu) = b(P_s) = \sum_{y\in Y} p(x_0,y)\left(\frac{\mu(x_0)}{\nu(y)}\right)^{1/s}. \tag{2.11}$$

Note that $a_s(P,\mu,\nu) = a_s(P_s)$, the latter taken with respect to the counting measures. Also, $p^*_{s'}(gy_0,x_0) = p_s(x_0,gy_0)$ and hence

$$b_{s'}(P^*) = b_{s'}(P^*,\nu,\mu) = b(P^*_s) = \sum_{x\in X} p(x,y_0)\left(\frac{\mu(x)}{\nu(y_0)}\right)^{1/s}.$$

Applying Proposition 2.4 to P_s and the counting measures on X and Y, we obtain the following.

Corollary 2.5. *One always has $a_s(P) \le b_s(P,\mu,\nu)^{1/s'}\, b_{s'}(P^*,\nu,\mu)^{1/s}$. Equality holds if and only if there exists $c>0$ such that $\Delta(g)=c$ for all $g\in S$, i.e., if and only if the closed subgroup generated by SS^{-1} is unimodular.*

A final simple observation will be crucial in the sequel. Let X,Y,Z be three countable sets on which the locally compact group G acts continuously and transitively with compact point stabilizers. Let μ,ν,ξ be positive measures on X,Y,Z, respectively, such that μ,ν and ν,ξ satisfy (2.1). Fix origins $x_0\in X_0$, $y_0\in Y_0$, $z_0\in Z_0$. Let $P:\ell^s(Y,\nu)\to\ell^s(X,\mu)$ be a G-invariant operator with kernel $p(x,y)>0$ satisfying $\sum_y p(x,y) < +\infty$. Also, let $Q:\ell^s(Z,\xi)\to\ell^s(Y,\nu)$ be a G-invariant operator with kernel $q(y,z)>0$ satisfying $\sum_z q(y,z) < +\infty$. Consider $\Phi[P]$, $\Phi_s[P]$, $\Phi[Q]$ and $\Phi_s[Q]$, as defined by (2.2). Then the following relations follow easily from the definitions (2.2), (2.8), (2.9) and (2.11).

Lemma 2.6. *Under the above hypotheses and notation,*

$$\Phi[PQ] = \Phi[Q]\star\Phi[P] \quad \text{and} \quad \Phi_s[PQ] = \Phi_s[Q]\star\Phi_s[P].$$

Moreover,

$$a_s(PQ) = a_s(Q)a_s(P), \quad b(PQ) = b(Q)b(P) \quad \text{and} \quad b_s(PQ) = b_s(Q)b_s(P).$$

3. Invariant Markov chains: The transitive case.

This section specializes the previous considerations to the case where $X=Y$ and $Pf=\sum_y p(x,y)f(y)$ is a Markov operator, i.e., $p(x,y)\ge 0$, $\sum_y p(x,y) = 1$ (this last condition is merely a normalization here since p is invariant and

G acts transitively). The results described below are more or less implicit in [20] (for $s = 2$) but it seems worth stating them more explicitly.

Assume as above that there exists a locally compact group G which acts continuously and transitively on X with compact point stabilizers and such that p is invariant under the action of G (i.e, $p(gx, gy) = p(x, y)$). Instead of two measures we only need one ($\nu = \mu$). Assume also that (2.1) holds, that is,

$$\mu(gx)/\mu(x) = \mu(gy)/\mu(y) \quad \text{for all } \ g \in G \ \text{ and all } \ x, y \in X. \tag{3.1}$$

We say that P is irreducible if for all $x, y \in X$, there exists n such that the kernel of P^n satisfies $p^{(n)}(x, y) > 0$. Irreducibility implies that

$$S = \{g \in G : p(x_0, gx_0) > 0\}$$

satisfies $\bigcup_n S^n = G$. Here, x_0 $(= y_0)$ is our fixed origin in X. When P is irreducible and μ is an invariant mesure for P (that is, $\mu(y) = \sum_x \mu(x)p(x, y)$) then (3.1) is automatically satisfied.

In addition to the quantities $\sigma_s(P, \mu) = \|P\|_{s\to s}$, $a_s(P, \mu)$, $b(P) = 1$, $b(P^*, \mu)$ and $b_s(P, \mu)$ introduced in Section 2 (obviously, here we omit ν in our notation), define

$$\rho_s(P) = \rho_s(P, \mu) = \lim_{n\to+\infty} \|P^n\|_{s\to s}^{1/n}. \tag{3.2}$$

Now fix $1 < s < +\infty$.

Theorem 3.1.

(1) *If the Markov operator P is G-invariant, G acts transitively on X and (3.1) holds, then*

$$\sigma_s(P, \mu) \le a_s(P, \mu).$$

Equality holds if and only if the closed subgroup of G generated by SS^{-1} is amenable.

(2) *Also,*

$$\rho_s(P, \mu) \le a_s(P, \mu),$$

and equality holds if and only if the closed subgroup generated by S is amenable. In particular, if the closed subgroup generated by S is amenable, then

$$\sigma_s(P, \mu) = \rho_s(P, \mu) = a_s(P, \mu) = \left(\frac{\mu(x_0)}{|G_{x_0}|}\right)^{1/s} \sum_{y\in Y} \left(\frac{|G_y|}{\mu(y)}\right)^{1/s} p(x_0, y). \tag{3.3}$$

(3) *Furthermore,*

$$a_s(P,\mu) \le b(P^*,\mu)^{1/s}.$$

This is an equality if and only if $\Delta(g) = \mu(x_0)/\mu(gx_0)$ *for all* g *in the closed subgroup generated by* SS^{-1}. *If* P *is irreducible, equality holds if and only if* $\Delta(g) = \mu(x_0)/\mu(gx_0)$ *for all* $g \in G$.

(4) *Finally,*

$$a_s(P,\mu) \le b_s(P,\mu)^{1/s'}\, b_{s'}(P^*,\mu)^{1/s},$$

and equality holds if and only if the closed subgroup generated by SS^{-1} *is unimodular. If* P *is irreducible, equality holds if and only if* G *is unimodular.*

Proof. Parts (1), (3) and (4) follow immediately from the results of Section 2. For (3) and (4), in the irreducible case, write id $= g_1 \dots g_k$ with $g_i \in S$ to show that the constant c of Proposition 2.4 and Corollary 2.5 must be equal to 1. Part (2) uses Lemma 2.6, which yields $a_s(P^n) = [a_s(P)]^n$, and Theorem 1.1. □

Example 1: The homogeneous tree. This is the basic and most typical example of a graph where an amenable, nonunimodular group acts transitively. Let T_r ($r \ge 2$) denote the homogeneous tree where each vertex has exactly $r+1$ neighbours. Choose one end ω_0 of the tree and draw the tree as a layer of horocycles with ω_0 on top of the picture and all the other ends at the bottom of the picture (see Figure 1).

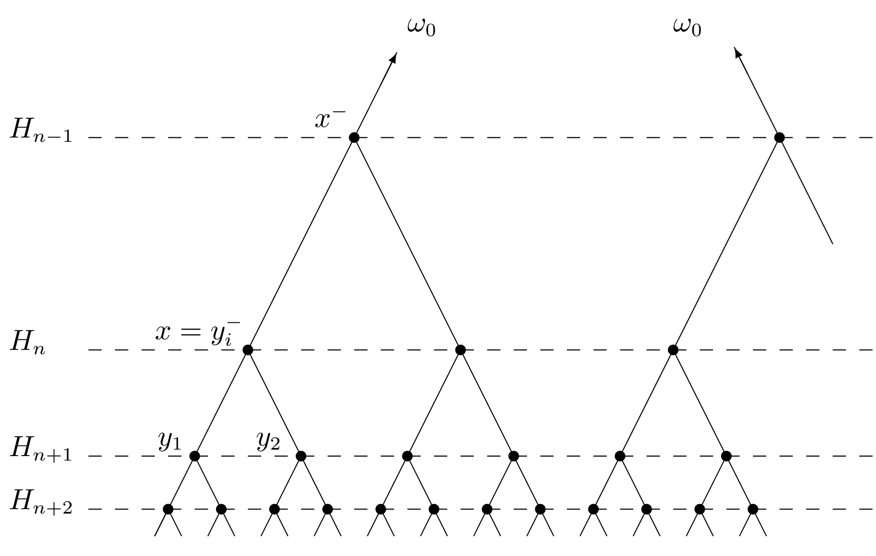


Figure 1. The tree T_2.

Let G be the subroup of the automorphism group of the T_r which fixes ω_0. This is an amenable, nonunimodular group, see Nebbia [13] and [20]. For any $x \in T_r$, let x^- be the neighbour of x which is on the geodesic from x to ω_0. Besides, x has r neighbours $y_1, \cdots, y_r$ such that $y_i^- = x$. It is not hard to see [20] that

$$\Delta(g) = \frac{|G_x|}{|G_{gx}|} = \begin{cases} 1/r & \text{if } \; gx = x^- \\ \;r & \text{if } \; gx = y_i, \quad 1 \le i \le r. \end{cases}$$

Consider the invariant kernel p given by

$$p(x^-, x) = \alpha, \quad p(x, x^-) = \beta, \quad \text{where } \; r\alpha + \beta = 1, \; \alpha, \beta \ge 0.$$

We will consider P as an operator acting on the ℓ^s spaces with respect to three different measures.

(1) To start, let us work with the counting mesure. Since G is amenable, we can compute

$$\rho_s(P) = \sigma_s(P) = a_s(P) = \sum_{y \in Y} \left(\frac{|G_y|}{|G_x|} \right)^{1/s} p(x,y) = r^{1/s}\beta + r^{1/s'}\alpha.$$

In the case $\alpha = \beta = 1/(r+1)$, P is the simple random walk, and we find the well known values

$$a_2(P) = \frac{2\sqrt{r}}{r+1} \quad \text{ and } \quad a_s(P) = \frac{r^{1/s} + r^{1/s'}}{r+1}.$$

In this case the minimum of the $a_s(P)$ is attained at $s = 2$. Indeed, P is symmetric and thus reversible with respect to the counting measure. When $\alpha \ne \beta$, this is not true, nor is the counting measure invariant for P. The minimum of $a_s(P)$ for $s \in \mathbb{R}$ is attained at $s = s_0 = 2\left(1 + \log_r(\alpha/\beta)\right)^{-1}$ where $\log_r(t) = \log(t)/\log(r)$. This minimum is $2\sqrt{r\alpha\beta}$. Observe however that s_0 does not always belong to $[1, +\infty]$. When $s_0 \notin [1, +\infty]$, that is, when $\alpha/\beta \notin \,]1/r, r]$, the minimum on $[1, +\infty]$ is taken at $s = 1$ and is equal to $b(P^*) = \alpha + r\beta$.

(2) Consider now the measure

$$\nu(x) = r^{-n} \quad \text{if } \; x \in H_n \,,$$

where H_n, $n \in \mathbb{Z}$, denotes the horocycle at level n with respect to our fixed origin $x_0 \in H_0$. This measure is invariant for P and satisfies

$$\Delta(g) = \nu(x_0)/\nu(gx_0).$$

Thus, for all s,

$$\rho_s(P,\nu) = \sigma_s(P,\nu) = a_s(P,\nu) = b(P^*,\nu)^{1/s} = 1.$$

(3) Finally, when $\alpha, \beta \neq 0$, P is reversible with respect to the measure μ defined by

$$\mu(x) = (\alpha/\beta)^n \quad \text{if } \ x \in H_n.$$

This measure satisfies (3.1). Thus, setting $\theta = r\alpha/\beta$, we can compute

$$\rho_s(P,\mu) = \sigma_s(P,\mu) = a_s(P,\mu) = \theta^{1/s}\beta + r\theta^{-1/s}\alpha = \beta(\theta^{1/s} + \theta^{1/s'}).$$

This expression is minimal for $s = 2$ as it should be since (P,μ) is a reversible chain. From this last result, we deduce that

$$\rho(P) = \limsup_{n\to+\infty} [p^n(x,y)]^{1/n} = 2\sqrt{r\alpha\beta}.$$

Example 2: The distance transitive graph $D_{m,\ell}$. The above computations can be generalized to a number of tree-like graphs. For instance, it applies to the distance transitive infinite graph $X = D_{m,\ell}$ which can be viewed as the free product of m copies of the complete graph K_ℓ on ℓ vertices. In this connected graph, each vertex x belongs to exactly m copies of K_ℓ which are disjoint except for their commom vertex x. It is easy to see that this graph has infinitely many ends and that the group G of all automorphisms that fixes one given end acts transitively on X.

Let P be the simple random walk on $D_{m,\ell}$. Then P is G-invariant and reversible with respect to the counting measure. Each vertex x has $m(\ell-1)$ neighbours in $D_{m,\ell}$: One "father" x^-, $\ell-2$ "brothers" x_k ($k = 1,\dots,\ell-2$) such that $x_k^- = x^-$ and $(m-1)(\ell-1)$ "children" $y_{i,j}$ ($1 \le i \le m-1$, $1 \le j \le \ell-1$) such that $y_{i,j}^- = x$. We now compute the ratios $|G_y|/|G_x|$ where y and x are neighbours. In order to do so, we use the following simple formula (see, for example, the proof of Lemma 1 in [20, 15]):

$$|G_y|/|G_x| = |G_y x|/|G_x y|.$$

(Note that $|\cdot|$ stands for Haar measure on the left and for cardinality on the right hand side.) This yields

$$\frac{|G_y|}{|G_x|} = \begin{cases} (m-1)(\ell-1) & \text{if } y = x^- \\ 1 & \text{if } y \text{ is a brother of } x \\ [(m-1)(\ell-1)]^{-1} & \text{if } y \text{ is a child of } x. \end{cases}$$

Theorem 3.1 now gives

$$\sigma_s(P) = \rho_s(P) = a_s(P) = \frac{\ell - 2 + [(m-1)(\ell-1)]^{1/s} + [(m-1)(\ell-1)]^{1/s'}}{m(\ell-1)}.$$

More generally, consider the regular tree T_r with one fixed end ω_0 and $r = r_1 r_2$. Fix a connected vertex transitive graph Ξ with degree d on r_2 vertices. In T_r, partition the children of any vertex x into r_1 sets $\Xi_1, \dots \Xi_{r_1}$, of r_2 elements each. We give to each Ξ_i the graph structure of Ξ. We consider the graph with vertex set $X = T_r$ and an edge from x to y if and only if either there is an edge from x to y in T_r or x, y have the same father in T_r, belong to the same Ξ_i and are neighbours in Ξ_i. The graph X is regular of degree $r + d + 1$. For the simple random walk on X, in the same way as above, we compute

$$\sigma_s(P) = \rho_s(P) = a_s(P) = \frac{d + r^{1/s} + r^{1/s'}}{r + d + 1}.$$

For $d = 0$, (i.e., $r_2 = 1$), we recover the tree. For $d = r_2 - 1 = \ell - 2$, $\Xi = K_{\ell-1}$ and $r_1 = m - 1$, we recover $D_{m,\ell}$. Non symmetric random walks that preserve the "family structure" of these graphs can also be analysed as in the case of the tree.

We refer the reader to [12] for references concerning Examples 1 and 2.

Example 3: The buildings of $PGL(n, \mathfrak{F})$, $\mathfrak{F}$ a local field. Let $\mathfrak{F}$ be a (commutative) nonarchimedean local field with valuation $v : \mathfrak{F}^* \to \mathbb{Z}$ and $v(0) = \infty$ ($\mathfrak{F}^*$ denotes the multiplicative group). Denote by $\mathfrak{O} = \{\mathfrak{a} \in \mathfrak{F} : v(\mathfrak{a}) \geq 0\}$ the ring of *integers* and $\mathfrak{P} = \{\mathfrak{a} \in \mathfrak{F} : v(\mathfrak{a}) \geq 1\}$ the maximal ideal in $\mathfrak{O}$. Let q be the (finite) order of the *residual field* $\mathfrak{K} = \mathfrak{O}/\mathfrak{P}$. The *absolute value* of $\mathfrak{a} \in \mathfrak{F}$ is $|\mathfrak{a}| = q^{-v(\mathfrak{a})}$. It induces an ultrametric. Choose a *uniformizer* $\mathfrak{p} \in \mathfrak{P}$, that is, $|\mathfrak{p}| = q^{-1}$.

$GL(n, \mathfrak{F})$ is the group of invertible $n \times n$ matrices over $\mathfrak{F}$, and $PGL(n, \mathfrak{F}) = GL(n, \mathfrak{F})/\mathfrak{F}^*$, where, more precisely, $\mathfrak{F}^*$ stands for all nonzero multiples of the identity matrix. Associated with $PGL(n, \mathfrak{F})$, there is a symmetric space X which is called a (*linear*) *building of type* $\tilde{A}_{n-1}$. This is an $(n-1)$-dimensional simplicial complex with several particular features, see e.g., Ronan [14]. In order to understand its structure, it is enough to describe its one-skeleton, which is a countable graph. X will stand for this graph, and $PGL(n, \mathfrak{F}) \subset AUT(X)$ is closed and acts transitively. We now give a brief description of X.

A *lattice* is an $\mathfrak{O}$-submodule of $\mathfrak{F}^n$ of the form $L = \mathfrak{O}v_1 + \cdots + \mathfrak{O}v_n$, where $\{v_1, \dots, v_n\}$ is a basis of $\mathfrak{F}^n$. Two lattices L, L' are equivalent if

$L' = aL$ for some $a \in \mathfrak{F}^*$. The equivalence class of L is denoted $[L]$. Then $X = \{[L] : L \text{ a lattice}\}$. Two points $x, y \in X$ with $x = [L]$ are neighbours in the graph X if we can write $y = [L']$ such that $\mathfrak{p}L \subset L' \subset L$ strictly. For such L', the mapping $[L'] \mapsto L'/\mathfrak{p}L$ is an isomorphism from the neighbourhood graph $N(x) = \{y \in X : y \sim x\}$ of $x \in X$ onto the family of nontrivial subspaces of $\mathfrak{K}^n$, where two elements are neighbours if one strictly contains the other. Thus,

$$\deg(x) = \sum_{k=1}^{n-1} \binom{n}{k}_q , \qquad \text{where } \binom{n}{k}_q = \frac{(q^n - 1)\cdots(q^{n-k+1} - 1)}{(q^k - 1)\cdots(q - 1)} ,$$

see, for instance, Goldman and Rota [8]. It seems that there is no useful closed formula for $W(n,q) = \sum_{k=0}^{n} \binom{n}{k}_q$. However, $W(n,q)$ satisfies $W(n+1,q) = 2W(n,q-1) + (q^n - 1)W(n-1,q)$, with $W(0,q) = 1$ and $W(1,q) = 2$, see Andrews [1]. This gives $W(2,q) = q + 3$, $W(3,q) = 2(q^2 + q + 2)$ and $W(4,q) = q^4 + 3q^3 + 4q^2 + 3q + 5$. For $n = 2, 3$ one recovers the well known values of $\deg(x) = W(n,q) - 2$, that is, $q + 1$ and $2(q^2 + q + 1)$.

The building associated with $PGL(2, \mathfrak{F})$ is the tree T_q. For any $n \geq 2$, the *apartment* of X associated with a given basis $\{v_1, \dots, v_n\}$ of $\mathfrak{F}^n$ is

$$\mathfrak{A} = \mathfrak{A}(v_1, \dots, v_n) = \{[\mathfrak{p}^{\ell_1}\mathfrak{O}v_1 + \cdots + \mathfrak{p}^{\ell_n}\mathfrak{O}v_n] : \ell_i \in \mathbb{Z}\}.$$

For $n = 2$, this is a two-sided infinite geodesic in the tree.

For $n = 3$, the subgraph of X induced by $\mathfrak{A}$ is a tiling of the plane by equilateral triangles. Furthermore, every edge in X is common to $q + 1$ triangles.

For any n, the group $PGL(n, \mathfrak{F})$ acts on X by matrix multiplication from the left, i.e., $g[L] = [gL] = [\mathfrak{O}gv_1 + \cdots + \mathfrak{O}gv_n]$. Here, we typically write g as a matrix in $GL(n, \mathfrak{F})$, while thinking of it as an element of $PGL(n, \mathfrak{F})$ consisting of all its nonzero multiples.

We now want to compute the norm and spectral radius of the simple random walk operator P on X, acting on $\ell^s(X)$ with the counting measure. Let G be the image in $PGL(n, \mathfrak{F})$ of the upper-triangular subgroup of $GL(n, \mathfrak{F})$. It acts transitively on X.

To see this, we choose $x_0 = [\mathfrak{O}^n] = [\mathfrak{O}e_1 + \cdots + \mathfrak{O}e_n]$ as our basepoint in X, where $\{e_1, \dots, e_n\}$ is the standard basis of $\mathfrak{F}^n$. The stabilizer of x_0 in $PGL(n, \mathfrak{F})$ is (isomorphic to) the group $PGL(n, \mathfrak{O}) = GL(n, \mathfrak{O})/\mathfrak{O}^*$, where $\mathfrak{O}^* = \mathfrak{O} \setminus \mathfrak{P}$ is the set of invertibles in $\mathfrak{O}$, and $GL(n, \mathfrak{O})$ consists of all matrices $g \in GL(n, \mathfrak{F})$ such that g and g^{-1} have all entries in $\mathfrak{O}$. Now consider $h \in GL(n, \mathfrak{F})$. Locate an entry h_{nj} in row n whose absolute value is largest in that row. By right multiplication by a permutation matrix (an element of $GL(n, \mathfrak{O})$), we can move this entry to position (n, n). Let

$E_{i,j}$ be the matrix with entry 1 in position (i,j) and 0 elsewhere. If $\mathfrak{a} \in \mathfrak{O}$, then $I + \mathfrak{a}E_{i,j} \in GL(n,\mathfrak{O})$. By right multiplication by $n-1$ matrices $I + \mathfrak{a}_j E_{n,j}$, $j = 1, \dots, n-1$, we can replace all entries in row n, except that in position (n,n), by 0.

Now repeat with the $(n-1)\times(n-1)$ matrix at upper left of the new h obtained in this way, and continue. At the end, we get $k \in GL(n,\mathcal{O})$ such that $hk = g$ is upper triangular, i.e., in G. So $k\,x_0 = x_0$ and $h\,x_0 = g\,x_0$. As every element of X is of the form $h\,x_0$ for some h as above, we get transitivity of G.

Also, it is well known that G is amenable, being solvable. From this we can already conclude, thanks to Theorem 3.1, that the norm and spectral radius of P on $\ell^s(X)$ are equal. We now proceed to compute these numbers for $1 < s < \infty$.

For our computation, we have to understand the action of G_{x_0} on $N(x_0)$. Write $\mathcal{E} = \{0,1\}^n \setminus \{\underline{0},\underline{1}\}$, where $\underline{0} = (0,\dots,0)$ and $\underline{1} = (1,\cdots,1)$. Let $\mathfrak{A} = \mathfrak{A}(e_1,\dots,e_n)$. The neighbours of x_0 in $\mathfrak{A}$ are of the form

$$y_{\underline{\varepsilon}} = [\mathfrak{p}^{\varepsilon_1}\mathfrak{O}e_1 + \cdots + \mathfrak{p}^{\varepsilon_n}\mathfrak{O}e_n], \quad \text{where } \ \underline{\varepsilon} \in \mathcal{E}.$$

We can represent $G_{x_0} = G \cap PGL(n,\mathfrak{O})$ by the group

$$G_{x_0} = \{g = (g_{ij})_{i,j=1,\dots,n} : g_{ij} \in \mathfrak{F},\ |g_{ii}| = 1,\ |g_{ij}| \le 1\ (i< j),\ g_{ij} = 0\ (i> j)\}.$$

Also, an element of G mapping x_0 to $y_{\underline{\varepsilon}}$ is $g_{\underline{\varepsilon}} = \operatorname{diag}(\mathfrak{p}^{\varepsilon_i})_{i=1,\dots,n}$. Hence, we can represent

$$\begin{aligned} G_{y_{\underline{\varepsilon}}} = g_{\underline{\varepsilon}} G_{x_0} g_{\underline{\varepsilon}}^{-1} = \{h = (h_{ij}) : h_{ij} \in \mathfrak{F},& \\ |h_{ii}| = 1,\ |h_{ij}| \le q^{\varepsilon_j - \varepsilon_i}\ (i < j),\ h_{ij} = 0\ (i > j)\}& \end{aligned}$$

and

$$\begin{aligned} G_{x_0} \cap G_{y_{\underline{\varepsilon}}} = \{g = (g_{ij}) : g_{ij} \in \mathfrak{F},& \\ |g_{ii}| = 1,\ |g_{ij}| \le q^{-\max\{0,\varepsilon_i - \varepsilon_j\}}\ (i < j),\ g_{ij} = 0\ (i > j)\}.& \end{aligned}$$

The left Haar measure on the group of upper triangular invertible $n \times n$ matrices over $\mathfrak{F}$ is given by

$$dg = |g_{11}^n g_{22}^{n-1} \cdots g_{nn}|^{-1} \prod_{i \le j} dg_{ij},$$

where dg_{ij} stands for the Lebesgue (Haar) measure λ on (the additive group) $\mathfrak{F}$, compare with [10], p. 209 (where this is stated for matrices over $\mathbb{R}$). The above two stabilizers and their intersections are compact open subgroups, so that their left Haar measure coincides with their right Haar measure, which is the restriction of the Haar measure on the whole group. If we normalize λ so that $\lambda(\mathfrak{O}) = 1$ and hence $\lambda(\mathfrak{P}) = 1/q$, $\lambda(\mathfrak{O}^*) = 1 - 1/q$, the measures of our stabilizers are $|G_{x_0}| = (1-1/q)^n$ and $|G_{x_0} \cap G_{y_{\underline{\varepsilon}}}| = (1-1/q)^n q^{-M(\underline{\varepsilon})}$, where

$$M(\underline{\varepsilon}) = \sum_{1 \le i < j \le n} \max\{0, \varepsilon_i - \varepsilon_j\}$$

is the number of times that a 1 comes before a 0 in the vector $\underline{\varepsilon}$. For instance, if $n = 5$ then $M((0,1,0,1,0)) = 3$ and $M((1,0,1,1,0)) = 4$. With this notation, we have

$$|G_{x_0} y_{\underline{\varepsilon}}| = |G_{x_0}| \Big/ |G_{x_0} \cap G_{y_{\underline{\varepsilon}}}| = q^{M(\underline{\varepsilon})}.$$

Write $|\underline{\varepsilon}| = \sum_i \varepsilon_i$ and let $W(n,k;q) = \sum_{\underline{\varepsilon} \in \{0,1\}^n : |\underline{\varepsilon}| = k} q^{M(\underline{\varepsilon})}$. Then $W(n,0;q) = W(n,n;q) = 1$, and expanding with respect to the value of ε_n, one obtains $W(n,k;q) = W(n-1,k-1;q) + q^k W(n-1,k;q)$. The q-binomial coefficients satisfy the same recursion, whence they coincide with the $W(n,k;q)$. Therefore, $\sum_{\underline{\varepsilon} \in \mathcal{E}} q^{M(\underline{\varepsilon})} = \deg(x_0)$. Thus, every element of $N(x)$ is of the form $g y_{\underline{\varepsilon}}$ with $g \in G_{x_0}$ and $\underline{\varepsilon} \in \mathcal{E}$, that is, $N(x) = \bigcup_{\underline{\varepsilon} \in \mathcal{E}} G_{x_0} y_{\underline{\varepsilon}}$ (disjoint union).

Next, $|G_{y_{\underline{\varepsilon}}} x_0| = |G_{x_0} g_{\underline{\varepsilon}}^{-1} y_{\underline{\varepsilon}}| = |G_{x_0} y_{\underline{1} - \underline{\varepsilon}}| = q^{M(\underline{1} - \underline{\varepsilon})}$. From this and Theorem 3.1, we get

$$\sigma_s(P) = \rho_s(P) = \frac{\sum_{\underline{\varepsilon} \in \mathcal{E}} q^{M(\underline{\varepsilon})/s'} q^{M(\underline{1} - \underline{\varepsilon})/s}}{\sum_{\underline{\varepsilon} \in \mathcal{E}} q^{M(\underline{\varepsilon})}}.$$

When $n = 2$, we obtain the norm of the simple random walk operator on the tree T_q of Example 1 above. When $n = 3$, we find

$$\sigma_s(P) = \rho_s(P) = \frac{q + q^{2/s} + q^{2/s'}}{1 + q + q^2}.$$

When $n = 4$, the above gives

$$\begin{aligned} \sigma_s(P) &= \rho_s(P) \\ &= \frac{q^{4/s} + q^{4/s'} + 2(q^{3/s} + q^{3/s'}) + q(2q + q^{2/s} + q^{2/s'}) + 2q(q^{1/s} + q^{1/s'})}{q^4 + 3q^3 + 4q^2 + 3q + 3} \end{aligned}$$

which, for $s = 2$, simplifies to

$$\sigma_2(P) = \rho_2(P) = \frac{6q^2 + 8q^{3/2}}{q^4 + 3q^3 + 4q^2 + 3q + 3}.$$

For general n and $s = 2$,

$$\sigma_2(P) = \rho_2(P) = \frac{\sum_{k=1}^{n-1} \binom{n}{k} q^{k(n-k)/2}}{\sum_{k=1}^{n-1} \binom{n}{k}_q}$$

(with ordinary binomial coefficients in the numerator).

Besides the cases $n = 2$ (well known) and $n = 3$, $s = 2$ (see [4]; for arbitrary s also Mantero and Zappa [11]), these results are new.

To conclude this section, observe that several of the above results have been obtained previously by other methods. The advantage of our method, when it is applicable, is that it reduces computation to simple and easy combinatorial considerations.

4. Invariant transition operators.

This section generalizes Theorem 3.1 to quasi-transitive transition operators. Namely, let X be a countable set and let $p(x, y)$ be a nonnegative kernel on X with $\sum_y p(x, y) < +\infty$. Assume that there is a locally compact group G that acts continuously on X with compact point stabilizers and such that p is G-invariant. We do not assume that G acts transitively, and we denote by $G\backslash X = I$ the quotient space. We say that G acts quasi-transitively when I is a finite set. Let us start this section with an easy corollary to the main result of [16].

Theorem 4.1. *Let P be a G-invariant Markov operator on X with invariant measure μ satisfying (3.1). Assume further that PP^* is irreducible, that μ is bounded and that $I = G\backslash X$ is finite. Then, for each fixed $1 < s < +\infty$, $\|P\|_{s\to s} = 1$ if and only if G is amenable and unimodular.*

Proof. If there exists $1 < s < +\infty$ such that $\|P\|_{s\to s} = 1$, then interpolation shows that $\|P\|_{2\to 2} = 1$. Hence, $\|PP^*\|_{2\to 2} = 1$. We can then use Theorem 2 of [16] to conclude that G is amenable and unimodular. Conversely, if G is unimodular and amenable, Theorem 2 of [16] implies $\|PP^*\|_{2\to 2} = \|P\|^2_{2\to 2} = 1$. Since P is Markovian and μ invariant, $\|P\|_{1\to 1} = \|P\|_{\infty\to\infty} = 1$. Classical interpolation now implies that we must have $\|P\|_{s\to s} = 1$ for all $1 \le s \le \infty$. □

To go further, denote by $X_i, i \in I$, the different orbits of G on X and, for each orbit, fix an origin $x_i \in X_i$. Let μ_i be the restriction of μ to X_i. Then P induces a family $\{P_{i,j} : i, j \in I\}$ of operators, where

$$P_{i,j} : \ell^s(X_j, \mu_j) \to \ell^s(X_i, \mu_i)$$

has kernel $p_{i,j}$ defined by

$$p_{i,j}(x,y) = p(x,y) \quad \text{if} \;\; x \in X_i,\; y \in X_j.$$

These kernels are G-invariant and the pairs of measures μ_i, μ_j satisfy property (2.1). Thus, the results of Section 2 apply. Referring to (2.7) and (2.8), set

$$\text{(4.1)} \qquad \sigma_s(i,j) = \sigma_s(P_{i,j}, \mu_i, \mu_j) \quad \text{and} \quad a_s(i,j) = a_s(P_{i,j}, \mu_i, \mu_j)\,.$$

Form the following matrices over I :

$$\text{(4.2)} \qquad \Sigma_s(P,\mu) = \big(\sigma_s(i,j)\big)_{i,j\in I} \quad \text{and} \quad A_s(P,\mu) = \big(a_s(i,j)\big)_{i,j\in I}\,.$$

It is useful to observe that

$$A_s(P,\mu) = A_{s'}(P^*)^t$$

where A^t is the transpose of A. This is not quite obvious but it follows from (2.10), which shows that $a_s(P_{i,j}) = a_{s'}([P_{i,j}]^*) = a_{s'}(P^*_{j,i})$.

Given a matrix M indexed by I, we denote by $\sigma_s(M)$ the norm of M acting on $\ell^s(I)$ and by $\rho_s(M)$ the spectral radius of M acting on $\ell^s(I)$. Here and in what follows, I is endowed with its counting measure.

Set

$$S = S(P) = \{g \in G :\; \exists\, i,j \in I,\; p(x_i, gx_j) > 0\}.$$

When the action of G is not transitive, it is still true that $G = \bigcup_{n\ge 1} S^n$ if P is irreducible. However, it may well be that $G = \bigcup_{n\ge 1} S^n$ and P is not irreducible, as easy examples show. Our aim is to prove the following generalization of Proposition 2.3.

Theorem 4.2. *Let P be G-invariant on X, with $I = G\backslash X$, and suppose that the measure μ satisfies (3.1). Then, with the notation of (4.1) and (4.2),*

$$\text{(4.3)} \qquad \|P\|_{s\to s} = \sigma_s(P,\mu) \le \sigma_s\Big(\Sigma_s(P,\mu)\Big) \le \sigma_s\Big(A_s(P,\mu)\Big).$$

Moreover, if the subgroup generated by SS^{-1} is amenable,

$$\text{(4.4)} \qquad \sigma_s(P,\mu) = \sigma_s\Big(A_s(P,\mu)\Big).$$

Finally, if I is finite and if PP^* is irreducible then $\sigma_2(P,\mu) = \sigma_2\Big(A_2(P,\mu)\Big)$ if and only if G is amenable.

Proof of (4.3). The inequality $\sigma_s(\Sigma_s) \le \sigma_s(A_s)$ follows from Proposition 2.3 which yields $\sigma_s(i,j) \le a_s(i,j)$. We now prove that $\|P\|_{s\to s} = \sigma_s(P) \le \sigma_s(\Sigma_s)$. To this end, write $P = \sum_{i,j} P_{i,j}$. Denote by R_i the operator of restriction to X_i of a function f initially defined on X. Set $f_i = R_i f$. Then, $f = \sum_i f_i$, and

$$Pf = \sum_{i,j} P_{i,j} f_j.$$

Pick $u \in \ell^s(X,\mu)$, $v \in \ell^{s'}(X,\mu)$ with $1/s + 1/s' = 1$. Then

$$\begin{aligned}\langle Pu, v\rangle_\mu &= \sum_{i,j} \langle P_{i,j} u_j, v_i\rangle_{\mu_i} \le \sum_{i,j} \sigma_s(i,j) \|u_j\|_s \|v_i\|_{s'} \\ &\le \sigma_s(\Sigma_s) \left(\sum_j \|u_j\|_s^s\right)^{1/s} \left(\sum_i \|v_i\|_{s'}^{s'}\right)^{1/s'} = \sigma_s(\Sigma_s) \|u\|_s \|v\|_{s'}.\end{aligned}$$

This clearly shows that $\|P\|_{s\to s} = \sigma_s(P) \le \sigma_s(\Sigma_s)$. □

Let us pause here to comment on what we just proved. Inequality (4.3) shows that we can estimate the norm of P acting on $\ell^s(X,\mu)$ by the ℓ^s-norm of the matrix $A_s(P,\mu)$ whose rows and columns are indexed by the quotient space $I = G\backslash X$. In particular, if I is finite, (4.3) leads to computing the ℓ^s-norm of a finite matrix.

Proof of (4.4). In order to prove (4.4), we may assume that the measure μ is the counting measure (if not, consider the kernel $p_s(x,y) = \mu(x)^{1/s} p_s(x,y) \mu(y)^{-1/s}$ and the associated operator acting on $\ell^s(X)$). We need to introduce some further notation. Fix an auxilliary homogeneous space $Z = G/G_{z_0}$, endowed with the counting measure. Using the notation (2.3), (2.4) define the operators $K_{s,i} : \ell^s(Z) \to \ell^s(X_i)$ by

$$K_{s,i} f(x) = \left(\frac{|G_{z_0}|}{|G_{x_i}|}\right)^{-1/s} \mathcal{S}_{X_i} \mathcal{T}_Z f(x) = \left(\frac{|G_{z_0}|}{|G_{x_i}|}\right)^{-1/s} \frac{1}{|G_{x_i}|} \int_{g\in G : gx_i = x} f(gz_0)\, dg,$$

and define the operators $K'_{s,i} : \ell^s(X_i) \to \ell^s(Z)$ by

$$K'_{s,i} f(z) = \left(\frac{|G_{z_0}|}{|G_{x_i}|}\right)^{1/s} \mathcal{S}_Z \mathcal{T}_{X_i} f(z) = \left(\frac{|G_{z_0}|}{|G_{x_i}|}\right)^{1/s} \frac{1}{|G_{z_0}|} \int_{g\in G : gz_0 = z} f(gx_i)\, dg.$$

These operators satisfy

$$\|K_{s,i}\|_{s\to s} = \|K'_{s,i}\|_{s\to s} = 1.$$

Observe that the formal adjoint $(K_{s,i})^* : \ell^{s'}(X_i) \to \ell^{s'}(Z)$ of $K_{s,i}$, defined by

$$\langle K_{s,i} u, v\rangle_X = \langle u, (K_{s,i})^* v\rangle_Z \quad \text{for all } u \in \ell^s(Z),\ v \in \ell^{s'}(X_i),$$

is given by $(K_{s,i})^* = K'_{s',i}$.

Let $\xi(i), i \in I$ and $\zeta(i), i \in I$ be non-negative functions on I, supported on a finite set $\Omega \subset I$ and such that

$$\sum_{i \in I} \xi(i)^s = \sum_{i \in I} \zeta(i)^{s'} = 1,$$

where $1/s + 1/s' = 1$. We now pick a homogeneous space Z adapted to Ω. Since Ω is finite, the group

$$G_\Omega = \bigcap_{i \in \Omega} G_{x_i}$$

is a compact open subgroup of G. Thus, we can choose $Z = G/G_\Omega$ which is a countable set. We also choose z_0 to be the identity element modulo G_Ω, so that $G_{z_0} = G_\Omega$. For this choice, we have $G_{z_0} \subset G_{x_i}$ for all $i \in \Omega$. Consider now the operators

$$K = \sum \xi(i) K_{s,i} : \ell^s(Z) \to \ell^s(X) \text{ and } K' = \sum_i \zeta(i) K'_{s,i} R_i : \ell(X)^s \to \ell^s(Z)\,.$$

It is easy to check that

$$\|K\|_{s \to s} \le 1, \quad \|K'\|_{s \to s} \le 1. \tag{4.5}$$

Consider also

$$K'PK = \sum_{i,j} K'_{s,i} P_{i,j} K_{s,j} : \ell^s(Z) \to \ell^s(Z).$$

This is an operator with invariant kernel

$$\begin{aligned}\kappa(z_0, gz) &= \frac{1}{|G_{z_0}|} \int_{G_{z_0}} \int_{G_{z_0}} \sum_{i,j} \frac{1}{|G_{x_j}|} \left(\frac{|G_{x_i}|}{|G_{x_j}|} \right)^{1/s} \xi(j)\, \zeta(i)\, p_{i,j}(\alpha x_i, g\beta x_j)\, d\alpha\, d\beta \\ &= |G_{z_0}| \sum_{i,j} \frac{1}{|G_{x_j}|} \left(\frac{|G_{x_i}|}{|G_{x_j}|} \right)^{1/s} \xi(j)\, \zeta(i)\, p_{i,j}(x_i, gx_j).\end{aligned}$$

This nice simplification arises because $G_{z_0} \subset G_{x_i}$ for all $i \in \Omega$. Set

$$S_\Omega = \{g \in G : \exists\, i, j \in \Omega, \;\; p(x_i, gx_j) > 0\} \subset S$$

and observe that

$$s \in S_\Omega \implies \exists\, i, j \in I \text{ such that } G_{x_i} s G_{x_j} \subset S_\Omega.$$

This yields (using once more that $G_{z_0} \subset G_{x_i}$ for all $i \in \Omega$)

$$S(K'PK) = \{g \in G : \kappa(z_0, gz_0) > 0\} = G_{z_0} S_\Omega G_{z_0} = S_\Omega \subset S.$$

By Proposition 2.4, if the closed subgroup of G generated by SS^{-1} is amenable, then

$$\begin{aligned}
\|K'PK\|_{s\to s} &= \sum_{z\in Z} \left(\frac{|G_{z_0}|}{|G_z|}\right)^{1/s} \kappa(z_0, z) \\
&= \sum_{z\in Z} \left(\frac{|G_{z_0}|}{|G_z|}\right)^{1/s} |G_{z_0}| \sum_{i,j} \frac{\xi(j)\zeta(i)}{|G_{x_j}|} \left(\frac{|G_{x_i}|}{|G_{x_j}|}\right)^{1/s} p_{i,j}(x_i, g_z x_j) \\
&= \int_G \Delta(h)^{1/s} \sum_{i,j} \frac{\xi(j)\zeta(i)}{|G_{x_j}|} \left(\frac{|G_{x_i}|}{|G_{x_j}|}\right)^{1/s} p_{i,j}(x_i, hx_j)\, dh \\
&= \int_G \sum_{i,j} \frac{\xi(j)\zeta(i)}{|G_{x_j}|} \left(\frac{|G_{x_i}|}{|G_{x_j}|}\right)^{1/s} \left(\frac{|G_{x_j}|}{|G_{hx_j}|}\right)^{1/s} p_{i,j}(x_i, hx_j)\, dh \\
&= \sum_{i,j} \xi(j)\zeta(i) \sum_{y\in X_j} \left(\frac{|G_{x_i}|}{|G_y|}\right)^{1/s} p_{i,j}(x_i, y) \\
&= \sum_{i,j} a_s(i,j)\xi(i)\zeta(j).
\end{aligned}$$

Now, since $1/s + 1/s' = 1$, we have

$$\begin{aligned}
\|P\|_{s\to s} &= \sup\{\langle P\tilde{u}, \tilde{v}\rangle_X : \|\tilde{u}\|_s \le 1,\ \ \|\tilde{v}\|_{s'} \le 1\} \\
&\ge \sup\{\langle PKu, (K')^*v\rangle_X : \|Ku\|_s \le 1,\ \ \|(K')^*v\|_{s'} \le 1\} \\
&\ge \sup\{\langle K'PKu, v\rangle_Z : \|u\|_s \le 1,\ \ \|v\|_{s'} \le 1\} \\
&= \sum_{i,j} a_s(i,j)\xi(j)\zeta(i),
\end{aligned}$$

where (4.5) has been used to obtain the second inequality. Taking suprema over ξ and ζ yields

$$\|P\|_{s\to s} = \sigma_s(P) \ge \sigma_s(A_s)$$

and thus $\sigma_s(P) = \sigma_s(A_s)$ by (4.3). This ends the proof of (4.4). □

We postpone the the proof of the last statement of Theorem 4.2, giving first its analogue for spectral radii in the place of norms.

Theorem 4.3. *Under the same assumptions as in Theorem 4.2, we have*

$$\rho_s(P,\mu) \le \rho_s(\Sigma_s(P,\mu)) \le \rho_s\Big(A_s(P,\mu)\Big). \tag{4.6}$$

Moreover, if the subgroup generated by S is amenable,

$$\rho_s(P,\mu) = \rho_s\Big(A_s(P,\mu)\Big). \tag{4.7}$$

Finally, if I is finite, $1 < s < +\infty$ and P irreducible then $\rho_s(P,\mu) = \rho_s\Big(A_s(P,\mu)\Big)$ if and only if G is amenable.

Proof. The following relations follow from Lemma 2.6.

$$\Sigma_s(P^n) \le [\Sigma_s(P)]^n \quad \text{and} \quad A_s(P^n) = [A_s(P)]^n\,. \tag{4.8}$$

Inequality (4.6) is an now easy consequence of (4.3) because

$$[\Sigma_s(P)]^n \le [A_s(P)]^n.$$

To prove (4.7), we observe that the subgroup generated by S contains the subgroup generated by S^nS^{-n} for each n. Thus, we can apply (4.4) and (4.8) to conclude that

$$\sigma_s\left(P^n,\mu\right) = \sigma_s\left(A_s^n(P)\right) = \sigma_s\left([A_s(P)]^n\right)$$

and this yields $\rho_s(P,\mu) = \rho_s\left(A_s(P,\mu)\right).$

We now prove the last statement of the theorem. Assume that I is finite and P irreducible. Form $Q = (I+P)^m$ and choose m such that

$$Q_{1,1} = R_1 Q R_1^* :\ \ \ell^s(X_1,\mu_1) \to \ell^s(X_1,\mu_1)$$

is irreducible (recall that $R_1 : \ell^s(X,\mu) \to \ell^s(X_1,\mu_1)$ is simply the operator of restriction to X_1). See [15], Sect. 5, where it is proved that such an m does exist (irreduciblility of P and finiteness of I are used here).

We now use the fact that $\rho_s(P^m)$ belongs to the spectrum of P^m acting on $\ell^s(X,\mu)$; see [23], p. 602, and the references given there. On one hand, this shows that $\rho_s(Q) = \Big(1+\rho_s(P)\Big)^m$. On the other hand, it is easy to check that $A_s(Q) = \Big(I + A_s(P)\Big)^m$ so that $\rho(A_s(Q)) = \Big(1+\rho\Big(A_s(P)\Big)\Big)^m$. Here, we write $\rho(A_s)$ instead of $\rho_s(A_s)$ since this quantity is simply the largest eigenvalue (in absolute value) of the finite dimensional matrix A_s.

Now, by hypothesis, $\rho_s(P) = \rho\Big(A_s(P)\Big)$. This implies that $\rho_s(Q) = \rho\Big(\Sigma_s(Q)\Big) = \rho\Big(A_s(Q)\Big)$. Since we also have

$$\Sigma_s(Q) \le A_s(Q)$$

elementwise and that $\Sigma_s(Q^k), A_s(Q^k)$ have all their entries positive for k large enough, we can conclude that $\Sigma_s(Q) = A_s(Q)\,$; see for example Seneta [18], p. 3. In particular, $\sigma_s(Q_{1,1}) = a_s(Q_{1,1})$. Since $Q_{1,1}$ is irreducible, Theorem 3.1 applies and shows that G is amenable. □

End of the proof of Theorem 4.2. Assume that $I = G\backslash X$ is finite, that PP^* is irreducible, and that $\sigma_2\Big(A_2(P)\Big) = \sigma_2(P)$. We want to conclude that G is amenable. Considering the self-adjoint operator $PP^* : \ell^2(X,\mu) \to \ell^2(X,\mu)$, we have

$$\rho_2(PP^*) = \sigma_2(PP^*) = \sigma_2(P)^2,$$

and

$$A_2(PP^*) = A_2(P)A_2(P^*) = A_2(P)A_2(P)^t,$$

so that

$$\rho\Big(A_2(P)A_2(P)^t\Big) = \sigma_2\Big(A_2(PP^*)\Big) = \sigma_2\Big(A_2(P)\Big)^2.$$

Thus, the hypothesis $\sigma_2(P) = \sigma_2\Big(A_2(P)\Big)$ implies $\rho_2(PP^*) = \rho\Big(A_2(PP^*)\Big)$. Since PP^* is irreducible we can apply Theorem 4.3 which shows that G is amenable. This concludes the proof of Theorem 4.2. $\square$

Remark. Concerning the last statement in Theorem 4.2 the reader may wonder why only the ℓ^2 version is stated there. Indeed, we have not been able to prove the ℓ^s version for $s \neq 2$. The obvious technique to obtain an ℓ^s result in this context is to use interpolation to reduce the problem to ℓ^2; see, e.g., [2, 3]. Here, the idea would be to apply Stein interpolation (Stein and Weiss [21], p. 205) to the family of operators $P_s = [\sigma_s(A_s(P,\mu))]^{-1}P$, but this requires to extend the map $t = 1/s \mapsto P_s$ to complex values of t and to show that this extention is analytic. It is not clear to us whether this is possible or not.

We now turn our attention to unimodularity. Referring to (2.9), let us consider the matrices

$$(4.9)\quad B = B(P) = \Big(b(i,j)\Big)_{i,j\in I} \quad \text{and} \quad B_* = B_*(P,\mu) = \Big(b_*(i,j)\Big)_{i,j\in I},$$

where

$$b(i,j) = b(P_{i,j}) \quad \text{and} \quad b_*(j,i) = b(P^*_{j,i},\mu_i,\mu_j)\,.$$

(Recall that $P^*_{j,i} = R_jP^*R_i$, where the adjoint P^* is taken with respect to μ.) Proposition 2.4 yields

$$a_s(i,j) \le b(i,j)^{1/s'}b_*(j,i)^{1/s}\,.$$

Hence, for any $\xi(\cdot)\,,\ \zeta(\cdot) \geq 0$ such that $\sum_j \xi(j)^s = \sum_i \zeta(i)^{s'} = 1\,$,

$$\begin{aligned}\sum_{i,j} a_s(i,j)\xi(j)\zeta(i) &\leq \sum_{i,j} [b(i,j)\xi(j)\zeta(i)]^{1/s'}\,[b_*(j,i)\xi(j)\zeta(i)]^{1/s} \\ &= \left(\sum_{i,j} b(i,j)\xi(j)\zeta(i)\right)^{1/s'} \left(\sum_{i,j} b_*(j,i)\xi(j)\zeta(i)\right)^{1/s} \\ &\leq \sigma_s(B)^{1/s'}\sigma_{s'}(B_*)^{1/s}.\end{aligned}$$

Thus, $\sigma_s(A_s) \leq \sigma_s(B)^{1/s'}\sigma_{s'}(B_*)^{1/s}$. Furthermore, if we apply this to P^n, (4.8) yields

$$\sigma_s(A_s^n) \leq \sigma_s(B^n)^{1/s'}\sigma_{s'}(B_*^n)^{1/s},$$

whence

$$\rho_s(A_s) \leq \rho_s(B)^{1/s'}\rho_{s'}(B_*)^{1/s}.$$

In order to study the case when this last inequality is an equality, consider the matrix $C = \Big(c(i,j)\Big)_{i,j\in I}$, where

$$c(i,j) = b(i,j)^{1/s'}b_*(j,i)^{1/s}.$$

Observe that

$$\Big[C^n\Big]_{i,j} \leq \Big[B^n\Big]_{i,j}^{1/s'}\Big[B_*^n\Big]_{j,i}^{1/s}.$$

Indeed, by induction,

$$\begin{aligned}\Big[C^n\Big]_{i,j} &= \sum_\ell \Big[C^{n-1}\Big]_{i,\ell}\Big[C\Big]_{\ell,j} \\ &\leq \sum_\ell \Big([B^{n-1}]_{i,\ell}\, b(\ell,j)\Big)^{1/s'}\Big(b_*(j,\ell)\,[B_*^{n-1}]_{\ell,i}\Big)^{1/s} \\ &\leq \Big[B^n\Big]_{i,j}^{1/s'}\Big[B_*^n\Big]_{j,i}^{1/s}.\end{aligned}$$

This shows that

$$\rho_s(A_s) \leq \rho_s(C) \leq \rho_s(B)^{1/s'}\rho_{s'}(B_*)^{1/s}.$$

Now, assume that I is finite. If P is irreducible, so are the nonnegative matrices A_s and C. If the equality

$$\rho_s(A_s) = \rho_s(B)^{1/s'}\rho_s(B_*)^{1/s}$$

holds true, it follows that $\rho_s(A_s) = \rho_s(C)$, and this implies

$$a_s(i,j) = c(i,j) = b(i,j)^{1/s'}b_*(j,i)^{1/s}$$

since $A_s \le C$ elementwise (see [18], p. 22). Therefore, we can now apply Proposition 2.4 to obtain that there exist positive constants $\delta(i,j)$ $(i,j \in I)$ such that

$$\Delta(g)\frac{\mu(gx_j)}{\mu(x_i)} = \delta(i,j) \quad \text{for all } i,j \in I \;\text{ and all }\; g \in S(i,j)\,, \tag{4.10}$$

where $S(i,j) = S(P_{i,j}) = \{g \in G : p_{i,j}(x_i, gx_j) > 0\}$. Using all this we obtain the following result.

Proposition 4.4. *Let P be G-invariant, $I = G\backslash X$, and suppose that μ satisfies (3.1).*

(1) *Referring to (4.2) and (4.9), we have*

$$\sigma_s(A_s) \le \sigma_s(B)^{1/s'}\sigma_{s'}(B_*)^{1/s}$$

and

$$\rho_s(A_s) \le \rho_s(B)^{1/s'}\rho_{s'}(B_*)^{1/s}.$$

(2) *Assume that I is finite and P irreducible. Then*

$$\rho(A_s) = \rho(B)^{1/s'}\rho(B_*)^{1/s}$$

if and only if $\Delta(g) = \mu(x)/\mu(gx)$ for all $g \in G$.

Proof. Only (2) still needs to be proved. Observe that we have used the notation ρ instead of ρ_s because, for finite non-negative matrices, ρ_s is simply the largest eigenvalue (in modulus).

Assume first that P is irreducible and aperiodic. This means that, for any finite set $F \subset X$, there exists an integer $n = n(F)$ such that $P^n(x,y) > 0$ for all $x, y \in F$. Also, for any k, P^k is aperiodic and irreducible and $A_s(P^k) = [A_s(P)]^k$, $B(P^k) = [B(P)]^k$, $B_*(P^k) = [B_*(P)]^k$.

Assume that $\rho_s(A_s) = \rho_s(B)^{1/s'}\rho_{s'}(B_*)^{1/s}$. It follows that, for any k,

$$\rho_s(A_s(P^k)) = \rho_s(B(P^k))^{1/s'}\rho_{s'}(B_*(P^k))^{1/s}.$$

Set $F = \{x_i, i \in I\}$, $n = n(F)$ as above (I finite is used here). Define $S_n(i,j) = \{g \in G : p^n(x_i, gx_j)\}$. Applying (4.10) to P^n, and observing that $\mathrm{id} \in S_n(i,j)$ for all i,j, we get $\Delta(g) = \mu(x)/\mu(gx)$ for all $g \in S_n = \bigcup_{i,j} S_n(i,j)$ and all x. As $G = \bigcup_k S_n^k$ because P^n is irreducible, it follows that $\Delta(g) = \mu(x)/\mu(gx)$ for all $g \in G$.

If P is periodic of period $d > 1$, let $Z_0, \dots, Z_{d-1}$ be the cyclic (i.e., periodicity) classes where the numbering is such that $P^k(x,y) > 0$ if and only if

$x \in Z_i,\ y \in Z_j$ with $k = j - i \bmod (k)$. Let $\widetilde{p}$ be the kernel $p^{(d)}$ restricted to Z_0. Let $H = \{g \in G : gZ_0 \subset Z_0\}$. The following facts are either well known or can be checked by simple elementary arguments:

(1) The operator $\widetilde{P}$ associated to $\widetilde{p}$ on Z_0 is irreducible and aperiodic.

(2) H is a normal subgroup of G, G/H is isomorphic to the finite cyclic group of order d and $H = \{g \in G : gZ_i \subset Z_i\}$ for each $i = 0, \dots, d-1$.

(3) For any $z_i \in Z_i$, $H = \{g \in G : gz_i \in Z_i\}$. In particular, for any z, $G_z \subset H$.

(4) H acts continuously on Z_0 with compact stabilizers $H_z = G_z$. It has finitely many orbits equal to those of the $X_i \cap Z_o$, $i \in I$, that are non-empty. The kernel $\widetilde{p}$ is invariant and the mesure μ restricted to Z_0 satisfies (3.1) under the action of H.

(5) Define J to be the subset of I given by

$$J = \{j \in I : X_j \cap Z_0 \neq \emptyset\}.$$

Then the finite matrices $A_s(H)$, $B(H)$, $B_*(H)$ indexed on J are equal to the corresponding submatrices of $A_s(\widetilde{P}), B(\widetilde{P}), B_*(\widetilde{P})$, respectively. Further, if $k \in J$ and $\ell \notin J$, or if $k \notin J$ and $\ell \in J$, the (k, ℓ) entries of $A_s(\widetilde{P}), B(\widetilde{P}), B_*(\widetilde{P})$, are zero.

(6) G is unimodular if and only if H is unimodular. More generally, two multiplicative positive functions are equal on G if and only if they are equal on H.

We now use these facts to finish the proof of Proposition 4.4. If $\rho(A_s) = \rho(B)^{1/s'}\rho(B_*)^{1/s}$, then the Perron-Frobenius theorem and point five above show that

$$\rho(A_s(\widetilde{P})) = \rho(B(\widetilde{P}))^{1/s'}\rho(B_*(\widetilde{P}))^{1/s}.$$

As $\widetilde{P}$ is irreducible and aperiodic, we obtain

$$\forall\, h \in H,\ \ \Delta(h) = \mu(x)/\mu(hx).$$

Point six shows that this identity extends to G. □

If we specialize to the case where P is a Markov kernel and μ an invariant measure for P, we have $\rho_s(B), \rho_s(B_*) \leq 1$ with equality if I is finite. We obtain

Theorem 4.5. *Let P be a G-invariant Markov operator on X with invariant measure μ satisfying (3.1). Suppose that $I = G\backslash X$ is finite. Fix $1 < s < \infty$.*

(1) *If P is irreducible then $\rho(A_s) = 1$ if and only if $\Delta(g) = \mu(x)/\mu(gx)$ for all $g \in G$.*

Furthermore, $\rho_s(P) = 1$ *if and only if* G *is amenable and* $\Delta(g) = \mu(x)/\mu(gx)$ *for all* $g \in G$.

(2) *If* PP^* *is irreducible then* $\sigma_s(A_s) = 1$ *if and only if* $\Delta(g) = \mu(x)/\mu(gx)$ *for all* $g \in G$.
Furthermore, $\sigma_s(P) = 1$ *if and only if* G *is amenable and* $\Delta(g) = \mu(x)/\mu(gx)$ *for all* $g \in G$.

The second statement improves upon Theorem 4.1.

In the same spirit, consider the matrix $B_s(P) = \Big(b_s(i,j)\Big)_{i,j\in I}$ where, referring to (2.11),

$$b_s(i,j) = b_s(P_{i,j}, \mu_i, \mu_j).$$

Using the above line of reasoning and Corollary 2.5, we obtain the following.

Proposition 4.6. *Under the assumptions of Theorem 4.3,*

$$\sigma_s\Big(A_s(P,\mu)\Big) \le \sigma_s\Big(B_s(P,\mu)\Big)^{1/s'} \sigma_{s'}\Big(B_{s'}(P^*,\mu)\Big)^{1/s}$$

and

$$\rho_s\Big(A_s(P,\mu)\Big) \le \rho_s\Big(B_s(P,\mu)\Big)^{1/s'} \rho_{s'}\Big(B_{s'}(P^*,\mu)\Big)^{1/s}.$$

Assume further that that I *is finite and* P *irreducible. Then*

$$\rho_s(A_s) = \rho\Big(B_s(P)\Big)^{1/s'} \rho\Big(B_{s'}(P^*)\Big)^{1/s}$$

if and only if G *is unimodular.*

Remarks.

(1) When G is not unimodular the bounds in Propositions 4.4 and 4.6 appear to be poor in general, compare with [16], Ex. 2.

(2) In our previous paper [15] the "only if" stated in Lemma 5(2) is wrong as one can see from Example 3, Case 2 of the same paper (in the formula for $a(i, i+1)$ in this example, $1/d_i$ under the square root should be replaced by d_i). The correct statement for Lemma 5(2) of [15] is that, assuming that I is finite, $\rho(A) = \rho(B)$ if and only if G is unimodular. This is a special case of Proposition 4.6, with $s = s'$ and $P = P^*$.

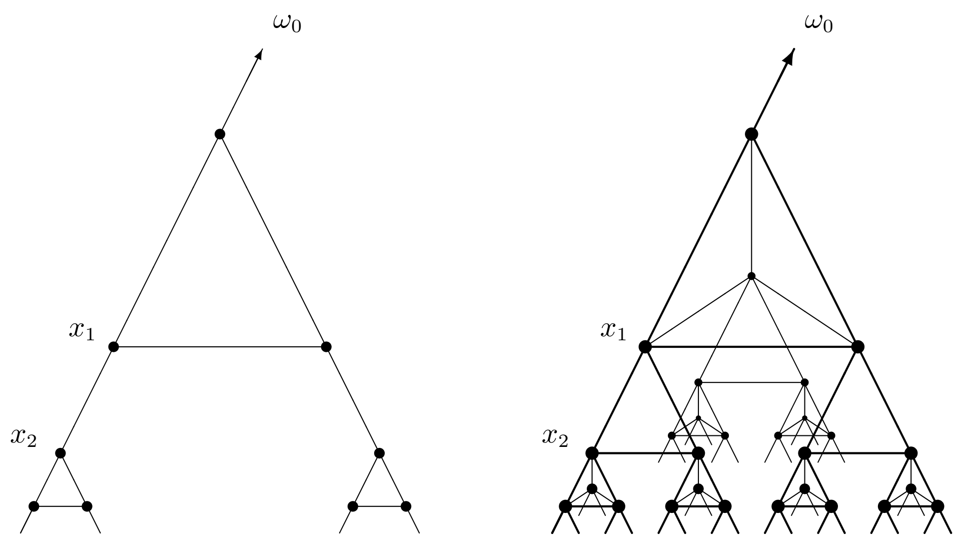
We now give some examples. It must be emphasized that being able to deal with quasi-transitive actions is of interest even in the context of Cayley graphs, as in the following case.

Example 4: The free products $\mathbb{Z}_\alpha * \mathbb{Z}_\beta$. Let a and b denote the generators of the finite cyclic groups $\mathbb{Z}_\alpha$ and $\mathbb{Z}_\beta$, respectively (written multiplicatively). The Cayley graph of the group $\mathbb{Z}_\alpha * \mathbb{Z}_\beta$ $(\alpha, \beta \geq 2)$ with respect to the generating set

$$S = \{a^i : i = 1, \dots, \alpha - 1\} \cup \{b^j : j = 1, \dots, \beta - 1\}$$

looks as in Figure 2 (we give two examples).

The group G of automorphisms of the Cayley graph which fix the end ω_0 acts on the graph with two orbits, which are represented in the figure by x_1 and x_2. This group is amenable and nonunimodular.



Cayley graph of $\mathbb{Z}_2 * \mathbb{Z}_3$ Cayley graph of $\mathbb{Z}_3 * \mathbb{Z}_4$

Figure 2.

Let P correspond to the simple random walk on this graph ($\equiv$ the random walk on the free product whose law q is the equidistribution on S) and let μ be the counting measure. We can easily compute the 2-by-2 matrices $A_s(P, \mu)$ whose entries are indexed by $I \times I$ with $I = \{1, 2\}$. We have

$$a_s(i,j) = \frac{1}{\alpha + \beta - 2} \sum_{y \in X_j} \left(\frac{|G_y|}{|G_{x_i}|} \right)^{1/s}.$$

To compute, we use once more the identity $|G_y|/|G_{x_i}| = |G_y x_i|/|G_{x_i} y|$ and obtain

$$A_s = \frac{1}{\alpha+\beta-2}\begin{pmatrix} \alpha-2 & (\alpha-1)^{1/s}+(\beta-1)^{1-1/s} \\ (\beta-1)^{1/s}+(\alpha-1)^{1-1/s} & \beta-2 \end{pmatrix}$$

for $1 \le s \le \infty$. This gives

$$\rho_s(P) = \rho(A_s) = \frac{\alpha+\beta-4}{2(\alpha+\beta-2)} + \frac{\sqrt{(\alpha+\beta-4)^2+4\left((\alpha-1)^{1/s}+(\beta-1)^{1/s'}\right)\left((\alpha-1)^{1/s'}+(\beta-1)^{1/s}\right)}}{2(\alpha+\beta-2)}.$$

For $s = 2$, this value is known: See Woess [24] and Cartwright and Soardi [6], where a precise description of the ℓ^2-spectrum is given. The above computation is however much simpler than the arguments used in [24, 6].

It is well known that the **modular group** $PSL_2(\mathbb{Z})$, is isomorphic to the free product of $\mathbb{Z}_2$ with $\mathbb{Z}_3$. More precisely, if $S : z \to -1/z$ and $T : z \to z+1$ are viewed as elements of $PSL_2(\mathbb{Z})$, then S and ST can be interpreted as the canonical generators of $\mathbb{Z}_2 * \mathbb{Z}_3 \simeq PSL_2(\mathbb{Z})$; See, e.g., [19], pp. 127-131. Thus, for the simple random walk on the Cayley graph of $PSL_2(\mathbb{Z})$ with generators $\{S; ST; (ST)^{-1}\}$, we obtain

$$\rho_s(P) = \rho(A_s) = \frac{1+\sqrt{1+4\left(1+2^{1/s}\right)\left(1+2^{1/s'}\right)}}{6}.$$

For $s = 2$, we get

$$\rho_2(P) = \rho(A_2) = \frac{1+\sqrt{13+8\sqrt{2}}}{6},$$

compare again with [24, 6].

The slightly more general random walks on the group $\mathbb{Z}_\alpha * \mathbb{Z}_\beta$ with law q_θ, given for $s \in S$ by

$$q_\theta(s) = \begin{cases} \frac{\theta}{\alpha-1} & \text{if } s = a^i, \\ \frac{1-\theta}{\beta-1} & \text{if } s = b^j, \end{cases}$$

where $0 < \theta < 1$ is a fixed parameter, can also be studied by the present method. The ℓ^2-radius of these walks was first computed in [24].

Example 5: Biregular trees. Let T_{r_1,r_2} be the biregular tree where the origin o has r_1+1 neighbours, the vertices at odd distance from o have r_2+1 neighbours and the vertices at even distance from o have r_1+1 neighbours. Fix an end ω_0 of this tree. The group of the automorphisms of T_{r_1,r_2} that fixes ω_0 acts on T_{r_1,r_2} with two orbits. Let X_i be the orbit consisting of the vertices with degree r_i+1, $i=1,2$. Consider the simple random walk P on this tree. It is reversible with respect to the measure given by $\mu(x)=r_i+1$ for $x\in X_i$. Here

$$a_s(i,j)=\frac{1}{r_i+1}\sum_{y\in X_j}\left(\frac{\mu(x_i)|G_y|}{\mu(y)|G_{x_i}|}\right)^{1/s}.$$

We compute

$$\begin{aligned}
a_s(1,1)&=a_s(2,2)=0,\\
a_s(1,2)&=\left(r_2^{1/s}+r_1^{1-s}\right)(r_1+1)^{-1/s'}(r_2+1)^{-1/s},\\
a_s(2,1)&=\left(r_1^{1/s}+r_2^{1-s}\right)(r_2+1)^{-1/s'}(r_1+1)^{-1/s}.
\end{aligned}$$

This gives

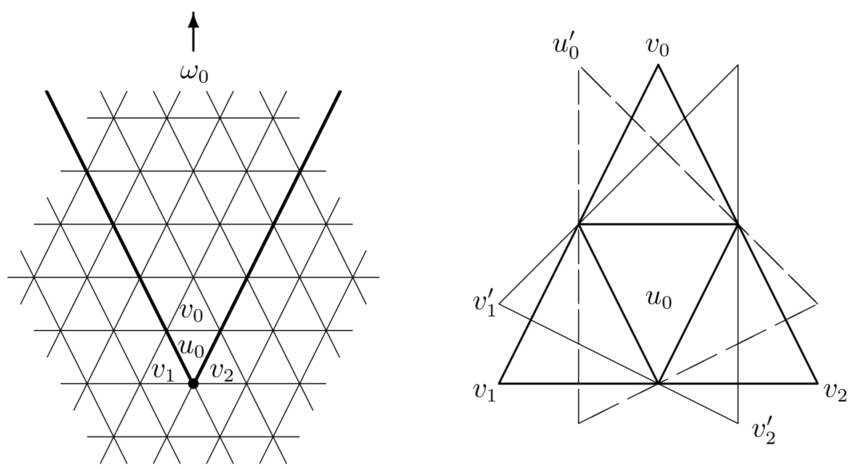
$$\rho_s(P,\mu)=\left[\left(r_2^{1/s}+r_1^{1-s}\right)\left(r_1^{1/s}+r_2^{1-s}\right)\right]^{1/2}\left[(r_1+1)(r_2+1)\right]^{-1/2}.$$

Here, we can also explicitly compute $\|P\|_{s\to s}=\sigma_s(P,\mu)$ which is equal to

$$\begin{aligned}
&\max\{a_s(1,2),a_s(2,1)\}\\
&\qquad=\max\left\{\frac{r_2^{1/s}+r_1^{1-s}}{(r_1+1)^{1/s'}(r_2+1)^{1/s}},\frac{r_1^{1/s}+r_2^{1-s}}{(r_2+1)^{1/s'}(r_1+1)^{1/s}}\right\}.
\end{aligned}$$

Example 6: Another random walk on the building of $PGL(3,\mathfrak{F})$. Let X be the building associated with $PGL(3,\mathfrak{F})$, as considered in Example 3. The vertices of the *dual graph* Y is the collection of all triangles (two-dimensionial simplices) in the building, and two triangles are adjacent if they have a common side. In this graph, $\deg(t)=3q$ for any triangle t. We consider P, the transition operator of the simple random walk on Y, and μ equal to the counting measure.

If $\mathfrak{A}$ is an apartment of X and x,x',x'' are the vertices of a triangle t in $\mathfrak{A}$, then we write w' and w'' for the half lines in $\mathfrak{A}$ starting at x and going through x' and x'', respectively. The *sector* in X with base vertex x and base chamber t is the "sixth" of $\mathfrak{A}$ bounded by the two "walls" w' and w''. Two sectors in X are called *equivalent* if their intersection is again a sector. We write Ω for the set of equivalence classes of sectors. The action of $PGL(3,\mathfrak{F})$ preserves equivalence and hence extends to Ω, see, for example, [5]. Now, the action of G, the projective group of upper triangular invertible matrices, is such that it fixes a (unique) $\omega_0 \in \Omega$: This is the equivalence class of the sector $\{[\mathfrak{p}^{\ell_1}\mathfrak{O}e_1 + \mathfrak{p}^{\ell_2}\mathfrak{O}e_2 + \mathfrak{p}^{\ell_3}\mathfrak{O}e_3] : \ell_1 \geq \ell_2 \geq \ell_3\}$. Our group acts on the dual graph Y with two orbits: one consists of the triangles pointing away from ω_0, like u_0 in the figure below, which represents a typical apartment $\mathfrak{A}$ of X; the other consists of the triangles pointing towards ω_0 (like v_0), see Figure 3.



The regular triangulation of the apartment $\mathfrak{A}$

The neighbours of u_0 when $q = 3$

Figure 3.

Besides v_0, the triangle u_0 has two more neighbours v_1, v_2 in $\mathfrak{A}$. Let u_0' be one of the $q-1$ triangles adjacent both with u_0 and v_0. For $i = 1, 2$, let v_i' be one of the $q-1$ triangles adjacent with u_0 and v_i. (Notation is such that the u-triangles are in one G-orbit and the v-triangles in the other.) We compute

$$|G_{u_0}u_0'| = q-1, \quad |G_{u_0}v_0| = 1, \qquad |G_{u_0}v_1| = |G_{u_0}v_2| = q$$

and

$$|G_{v_0}u_0| = q, \qquad |G_{v_1}u_0| = |G_{v_2}u_0| = 1, \quad |G_{v_1}v_1'| = |G_{v_2}v_2'| = q-1.$$

Hence

$$A_s = \frac{1}{3q}\begin{pmatrix} q-1 & q^{1/s}+2q^{1/s'} \\ 2q^{1/s}+q^{1/s'} & 2(q-1) \end{pmatrix},$$

and

$$\rho_s(P) = \frac{3(q-1)+\sqrt{(q-1)^2+20q+8q^{2/s}+8q^{2/s'}}}{6q}.$$

In particular, $\rho_2(P) = \sigma_2(P)$ is the norm of P.

It is clear that this example can be generalized to the dual graph (on $(n-1)$-simplices) of the building associated with $PGL(n,\mathfrak{F})$.

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