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TWO SUBFACTORS ARISING FROM A NON-DEGENERATE COMMUTING SQUARE

An answer to a question raised by V.F.R. Jones

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When we have a non-degenerate commuting square of finite dimensional C^*-algebras, we can construct a subfactor in two ways. One is by a repetition of basic constructions in a horizontal direction and the other in a vertical direction. We prove that if one of the two is of finite depth, so is the other. Furthermore, we prove the two have the same global indices in the sense of A. Ocneanu. This gives an answer to a question V.F.R. Jones raised in his talk at Aarhus in June, 1995. We actually prove a more general result on flatness and also give an example of a new finite principal graph as its application.

1. Introduction.

We study a relation between two subfactors, "vertical" one and "horizontal"one, arising from a non-degenerate commuting square of finite dimensional C^*-algebras. We prove that if one of the two is of finite depth, so is the other. Furthermore, we prove the two have the same global indices in the sense of A. Ocneanu. This gives an answer to a question V.F.R. Jones raised in his talk at Aarhus in June, 1995. We actually prove a more general result on flatness and also give an example of a new finite principal graph as its application.

In the progress of subfactor theory, A. Ocneanu presented his striking theory "*paragroups*" in 1987. A. Ocneanu perceived a new combinatorial structure of an irreducible inclusion of AFD factors $N \subset M$ of type II_1 with finite depth and finite Jones index and axiomatized it as a paragroup. In a certain class of subfactors, namely, irreducible inclusions of AFD II_1 factors $N \subset M$ with finite Jones index and finite depth, paragroups give the complete classification. This completeness is due to a theorem of S. Popa called generating property. He also studied the necessary and sufficient condition for subfactor to be approximated by certain series of finite dimensional C^*-algebras [**P1**, **P2**].

A paragroup is constructed from two finite graphs and a complex valued function called *biunitary connection* with some unitarity conditions. In the axioms of the paragroup theory, the most important one is the flatness condition. On one hand, it makes the "parallel transport" on the graphs well-defined. On the other hand, the associativity of the group is exactly the flatness condition. It is also true that flatness condition is equivalent to the pentagonal relation in the depth two case. For details, see [S] and references cited.

V.F.R. Jones raised the following question in his talk at Aarhus in June, 1995.

Question (V.F.R. Jones).

Suppose that we have a finite dimensional non-degenerate commuting square as follows.

$$\begin{array}{ccc} R_{00} & \subset & R_{01} \\ \cap & & \cap \\ R_{10} & \subset & R_{11} \end{array} .$$

Iterating the basic constructions for the above commuting square, we get the following series of commuting squares of period 2.

$$\begin{array}{ccccccccc} R_{00} & \subset & R_{01} & \subset & R_{02} & \subset \cdots \subset & R_{0\infty} \\ \cap & & \cap & & \cap & & \cap \\ R_{10} & \subset & R_{11} & \subset & R_{12} & \subset \cdots \subset & R_{1\infty} \\ \cap & & \cap & & \cap & & \cap \\ R_{20} & \subset & R_{21} & \subset & R_{22} & \subset \cdots \subset & R_{2\infty} \\ \vdots & & \vdots & & \vdots & & \vdots \\ R_{\infty 0} & \subset & R_{\infty 1} & \subset & R_{\infty 2} & \subset \cdots & \end{array} .$$

Then is there any relation between two subfactors $R_{0\infty} \subset R_{1\infty}$ and $R_{\infty 0} \subset R_{\infty 1}$? Moreover, is the finite depth condition for $R_{0\infty} \subset R_{1\infty}$ related to that for $R_{\infty 0} \subset R_{\infty 1}$?

Note that we cannot expect simple relations between Jones indices of these two subfactors. We will answer to this question by the techniques associated with flatness in paragroup theory. More precisely, we will prove a new theorem related to flatness and will answer to the question as its application. Also, we will give a new example of a principal graph as another application. For the answer to the first question, the above two subfactors have the same global indices. For the second question, we have the affirmative answer.

The author is much indebted to Prof. Kawahigashi for valuable discussions and expresses his gratitude.

2. An answer to the question.

We consider the string algebra picture rather than that of the commuting square and freely use the notations on string algebras in [**O3**] and [**K**].

Consider the sequence of string algebras

$$\begin{array}{ccccccc}
A_{0,0} & \subset & A_{0,1} & \subset & A_{0,2} & \subset \cdots \subset & N = A_{0,\infty} \\
\cap & & \cap & & \cap & & \cap \\
A_{1,0} & \subset & A_{1,1} & \subset & A_{1,2} & \subset \cdots \subset & M = A_{1,\infty} \\
\cap & & \cap & & \cap & & \cap \\
A_{2,0} & \subset & A_{2,1} & \subset & A_{2,2} & \subset \cdots \subset & M_1 = A_{2,\infty} \\
\vdots & & \vdots & & \vdots & & \vdots \\
P = A_{\infty,0} & \subset & Q = A_{\infty,1} & \subset & Q_1 = A_{\infty,2} & \subset \cdots &
\end{array}$$

constructed from four finite, bipartite connected graphs, a biunitary connection W on those graphs and a fixed starting vertex $*$. We get Jones towers of AFD II_1 factors $N \subset M \subset M_1 \subset \cdots$ in the vertical direction and $P \subset Q \subset Q_1 \subset \cdots$ in the horizontal direction as above. Observe that

$$\begin{array}{ccc}
N' \cap M_{k-1} & \subset & A_{k,0} \\
\cap & & \cap \\
N' \cap M_k & \subset & A_{k+1,0}
\end{array} \tag{2.1}$$

is a commuting square since the conditional expectations are implemented by the Jones projections of the sequence of string algebras above ([**E-K**], Proposition 3.1). We denote the biunitary connection corresponding to this commuting square by W'. Note that the horizontal inclusion graphs in (2.1) are not necessarily connected. However, all inclusion graphs in the following commuting square are trivially connected (since "disconnected $\times$ connected $=$ connected").

$$\begin{array}{ccc}
N' \cap M_{k-1} & \subset & A_{k,1} \\
\cap & & \cap \\
N' \cap M_k & \subset & A_{k+1,1}
\end{array}. \tag{2.2}$$

We call the biunitary connection corresponding to this commuting square the *composite connection* and denote this by $W' \cdot W$.

We have the following theorem under the above notations.

Theorem 2.1. *Suppose that the subfactor $N \subset M$ is of finite depth. The composite connection $W' \cdot W$ gives a flat connection.*

Proof. First, we establish some notations. We denote the set of even vertices in the initial upper horizontal graph by V_0 and the set of even vertices in the principal graph of $N \subset M$ by V_0'. We have the distinguished vertex $*' \in V_0'$ as the starting point of the string algebras $\{C_{k,l}\}_{k,l\geq 0}$ for the composite connection $W' \cdot W$. Also we have the distinguished vertex $* \in V_0$ as the starting point for the initial string algebras $\{A_{k,l}\}_{k,l\geq 0}$. See Figure 1.

Take a string $\sigma^0 \in N' \cap M_{k-1}$. Note that $N' \cap M_{k-1} \subset A_{k,0}$ by Ocneanu's compactness argument [**O3**, II.6] and we have $\sigma^0 \in A_{k,0}$. Apply the canonical shift Γ in the sense of [**O1**] to $\sigma^0 \in N' \cap M_{k-1}$ for n times (see [**B2**] for more detailed exposition on the canonical shift), then we have $\Gamma^n(\sigma^0) \in M_{2n-1}' \cap M_{k+2n-1} \subset A_{k+2n,0}$. Because $\Gamma^n(\sigma^0)$ commutes with any element in M_{2n-1}, we have the following expression for $\Gamma^n(\sigma^0) \in N' \cap M_{k+2n-1}$ and denote this string by σ.

$$\sigma = \Gamma^n(\sigma^0) = \sum_{x \in {V_0}'} \mathrm{id}^{(2n)} \cdot z(x),$$

where $\mathrm{id}^{(2n)}$ means an identity field of length $2n$, that is, $\mathrm{id}^{(2n)} = \sum_{|\rho|=2n}(\rho,\rho) \in A_{2n,0}$ and $\cdot$ means a concatenation of strings. Thus we get a field of strings $z = \{z(x)\}_{x\in V_0'}$. Because the canonical shift is the same as parallel transport [**O3**, II.5] on the canonical commuting square of $N \subset M$, we get $\sigma(*) = \sigma^0$ by flatness of the biunitary connection arising from the canonical commuting square of $N \subset M$.

We claim the field of strings $z = \{z(x)\}_{x\in V_0'}$ is flat with respect to the composite connection $W' \cdot W$. (See [**O3**, II.5] for the definition of flatness of a field of strings.) That is, we claim the field of strings $z = \{z(x)\}_{x\in V_0'}$ satisfies the following identity via the connection $W' \cdot W$.

$$\begin{tikzcd} \bullet \arrow[d, "z"'] & \\ \bullet \arrow[r, "\mathrm{id}^{(4)}"'] & \bullet \end{tikzcd} \quad = \quad \begin{tikzcd} \bullet \arrow[r, "\mathrm{id}^{(4)}"] & \bullet \arrow[d, "z"] \\ & \bullet \end{tikzcd},$$

where $\mathrm{id}^{(4)}$ means an identity field of length four. This means the composite connection $W' \cdot W$ is a flat connection in the sense of [**O3**, II.6]. We show this identity by embedding z successively via connections W' and W. See Figure 1.

Our claim for the first embedding is obvious because the embedding via connection W' is just a natural embedding of $N' \cap M_{k-1}$ into $A_{k,0}$. We denote this embedded field by $z' = \{z'(y)\}_{y\in V_0}$. The claim for the second and third embeddings follow from the fact that the field $\{z'(y)\}_{y\in V_0}$ is a flat field for connection W. For the fourth embedding, we have the following argument.

Recall that we have the Jones projection $e \in C_{2n,2}$ for string algebras $\{C_{k,l}\}_{k,l\geq 0}$ and that the Jones projection satisfies $ze = ez$ as shown by direct computations. Also we have $z\xi = \xi z$ for any $\xi \in C_{2n,1}$.

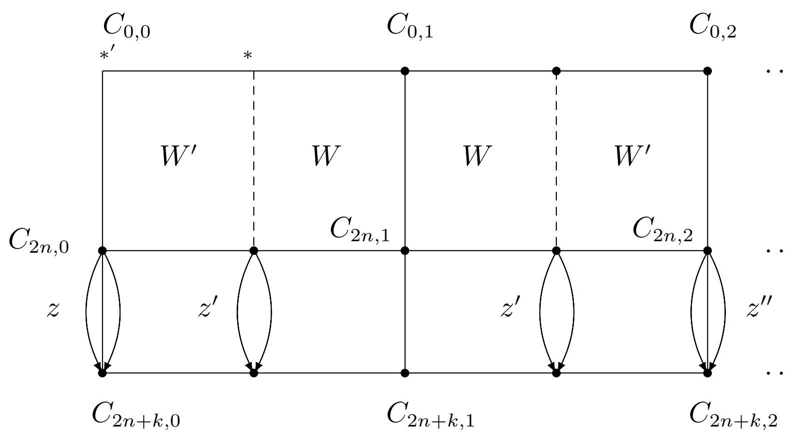
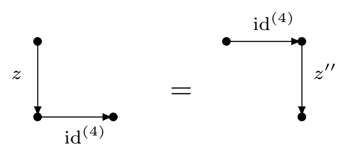


Figure 1. Composite connection $W' \cdot W$ and its string algebras.

These commutativity relations mean that we have $z\xi = \xi z$ for any $\xi \in C_{2n,2}$ because $C_{2n,2}$ is generated by $C_{2n,1}$ and Jones projection $e \in C_{2n,2}$. Thus we get the following identity via the biunitary connection $W' \cdot W$



where z'' is a field $z'' = \{z''(x)\}_{x \in V_0'}$. By the same argument as in [**O3**] (on the eleventh line in page 33 of Section II.6), we get $z'' = z$. Thus we have proved the field $z = \{z(x)\}_{x \in V_0'}$ is flat. So we are done. □

Corollary 2.2 (**An answer to the second question**). *$N \subset M$ is of finite depth iff the same holds for $P \subset Q$.*

Proof. Follows from the previous theorem and [**B1**, Theorem 2.6], since P is an intermediate subfactor of $B_\infty \subset Q$. □

In the rest of the paper, we show that the natural numerical relation between the two subfactors is given by the *global indices*. This notion was firstly introduced by A. Ocneanu [**O1**] as Jones index of the asymptotic inclusion subfactor $M \vee (M' \cap M_\infty) \subset M_\infty$.

Definition 2.3. The global index for the inclusion of II_1 factors $N \subset M$ is defined by

$$\sum_{{}_M X_M} (\dim_M X_M)^2,$$

where summation runs through all the irreducible M-M bimodules ${}_MX_M$ arising from the subfactor $N \subset M$. We denote the global index of $N \subset M$ by $[[M : N]]$.

By definition, we easily know the value of the global index $[[M : N]]$ is infinity if the inclusion is of infinite depth.

The following lemma is essentially contained in [**Y**, Proposition 2.4].

Lemma 2.4. *Suppose that we have an inclusion $N \subset M$ with finite index and that there exists an intermediate subfactor P. Then we have $[[M : P]] \leq [[M : N]]$.*

Proof. Recall that all the irreducible M-M bimodules for the inclusion $N \subset M$ arise from the irreducible decomposition of the relative tensor products

$$_M M \otimes_N M \otimes_M M \otimes_N \cdots \otimes_N M_M.$$

Note that ${}_NM_M \cong_N P \otimes_P M_M$ and apply this to the above decomposition, then we have

$$_M M \otimes_P P \otimes_N P \otimes_P M \otimes_M M \otimes_P P \otimes_N P \otimes_P M \otimes_M \otimes \cdots \otimes_N P \otimes_P M_M.$$

Because the P-P bimodule ${}_PP \otimes_N P_P$ contains the irreducible P-P bimodule ${}_PP_P$, we can conclude the above relative tensor product contains the relative tensor product

$$_M M \otimes_P M \otimes_M M \otimes_P \cdots \otimes_P M_M.$$

All the irreducible M-M bimodules for the inclusion $P \subset M$ arise from the irreducible decomposition of these relative tensor products. This means the equivalence classes of the irreducible M-M bimodules for the inclusion $N \subset M$ contain those for the inclusion $P \subset M$. Thus we get the above inequality. □

Corollary 2.5 (**An answer to the first question**). *The global indices for $N \subset M$ and $P \subset Q$ are equal.*

Proof. First we assume that the inclusion $N \subset M$ is of finite depth. We note that the principal graph of $B_\infty \subset Q$ and the principal graph of $N \subset M$ have the same even vertices. With this fact and Lemma 2.4, we get the following inequality for global indices.

$$[[Q : P]] \leq [[Q : B_\infty]] = [[M : N]].$$

By symmetry, we get the inequality $[[Q : P]] \geq [[M : N]]$. Thus we get $[[Q : P]] = [[M : N]]$.

Next we assume the inclusion $N \subset M$ is of infinite depth. Then the inclusion $P \subset Q$ is also of infinite depth by the previous corollary. In this case, global indices of these inclusions are both infinite. This completes the proof. □

Corollary 2.6. *If the horizontal graphs associated with the commuting square (2.1) are connected, then the connection arising from this commuting square is also flat.*

Proof. The proof is similar to that of Theorem 2.1. □

Finally, we give the following instructive example.

Example 2.7. Consider the commuting square with all the graphs being Dynkin diagram E_7 as in [**E-K**]. We know that we have two and only two such commuting squares, i.e. two and only two biunitary connections, and that these two are not flat. Flat part of this biunitary connection $B_k = N' \cap M_{k-1}$ is already studied by D. Evans and Y. Kawahigashi in [**E-K**] and the principal graph for the inclusion $N \subset M$ is given by Dynkin diagram D_{10}. Looking at the entries of Perron-Frobenius eigenvectors, the relation between the two upper horizontal graphs E_7 and D_{10} is given by Figure 2. By Theorem 2.1, we know the graph with the middle stage skipped in Figure 2 is the principal graph for the inclusion $B_\infty \subset Q$. The principal graph is given by Figure 3.

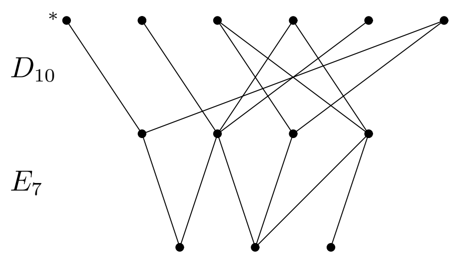


Figure 2. The connecting relation between the two upper horizontal graphs.

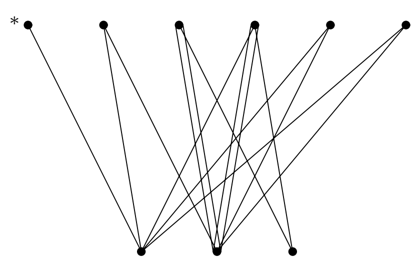


Figure 3. The principal graph for $B_\infty \subset Q$.

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