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ON THE KORTEWEG-DE VRIES EQUATION: FREQUENCIES AND INITIAL VALUE PROBLEM

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The Korteweg-de Vries equation (KdV)

$$\partial_t v(x,t) + \partial_x^3 v(x,t) - 3\partial_x v(x,t)^2 = 0 \quad (x \in S^1, t \in \mathbb{R})$$

is a completely integrable system with phase space $L^2(S^1)$. Although the Hamiltonian $\mathcal{H}(q) := \int_{S^1} \left(\frac{1}{2}\left(\partial_x q(x)\right)^2 + q(x)^3\right) dx$ is defined only on the dense subspace $H^1(S^1)$, we prove that the frequencies $\omega_j = \frac{\partial \mathcal{H}}{\partial J_j}$ can be defined on the whole space $L^2(S^1)$, where $(J_j)_{j\geq 1}$ denote the action variables which are globally defined on $L^2(S^1)$. These frequencies are real analytic functionals and can be used to analyze Bourgain's weak solutions of KdV with initial data in $L^2(S^1)$. The same method can be used for any equation in the $KdV-$hierarchy.

1. Introduction and summary of the results.

It is well known that the Korteweg-de Vries equation (KdV) on the circle

$$\partial_t v(x,t) = -\partial_x^3 v(x,t) + 6v(x,t)\partial_x v(x,t) \tag{1.1}$$

can be viewed as a completely integrable Hamiltonian system of infinite dimension. We choose as its phase space $L^2(S^1) = L^2(S^1;\mathbb{R})$ where S^1 is the circle of length 1. The Poisson structure is the one proposed by Gardner,

$$\{F,G\} = \int_{S^1} \frac{\partial F}{\partial q(x)} \frac{d}{dx} \frac{\partial G}{\partial q(x)} dx \tag{1.2}$$

where F, G are C^1- functionals on $L^2(S^1)$, and $\frac{\partial F}{\partial q(x)}$ denotes the L^2-gradient of the functional F. The Hamiltonian $\mathcal{H}$ corresponding to KdV is then given by

$$\mathcal{H}(q) := \int_{S^1} \left(\frac{1}{2}(\partial_x q)^2 + q^3\right) dx. \tag{1.3}$$

Note that the Poisson structure (1.2) is degenerate and admits the average as a Casimir function

$$[q] = \int_{S^1} q(x)dx.$$

Moreover, the Poisson structure is regular and induces a trivial foliation whose leaves are given by

$$L^2_c(S^1) = \{q \in L^2(S^1); [q] = c\}.$$

Consider the leaf $L^2_0(S^1)$ and denote by ω_G the symplectic structure on $L^2_0(S^1)$ induced by the Poisson structure (1.2).

In previous work [**Ka**], [**BBGK**] the phase space $\left(L^2_0(S^1), \omega_G\right)$ was analyzed and it was proved that KdV admits globally defined (generalized) action-angle variables. Recall that for a finite dimensional, completely integrable Hamiltonian system action-angle variables linearize the flow and, therefore, solutions can be found by quadrature. When trying to apply this procedure to find L^2-solutions of KdV one notices that the Hamiltonian $\mathcal{H}$ is only defined on the dense subspace $H^1(S^1)$ and so are the frequencies $\omega_j = \frac{\partial \mathcal{H}}{\partial J_j}$, given by the partial derivatives of $\mathcal{H}$ with respect to the action variables J_j. In Section 3, using auxilary results proved in Section 2, we show the following main result of this paper:

Theorem 1. *Let $q_0 \in L^2_0(S^1)$. Then there exists a neighborhood U_{q_0} of q_0 in $L^2_0(S^1;\mathbb{C})$ so that the frequencies are defined and analytic on U_{q_0}. Moreover, they satisfy, uniformly on U_{q_0},*

$$\omega_j = (2\pi j)^3 + o(1). \tag{1.4}$$

Our starting point of the proof of Theorem 1 is the Its-Matveev formula which, at least for finite gap potentials, provides a formula for the frequencies ω_j $(j \geq 1)$ in terms of periods of Abelian differentials of the second kind on the hyperelliptic Riemann surface Σ_q, associated to the periodic spectrum of the Schrödinger operator $-\frac{d^2}{dx^2} + q$ where q is the initial data of (1.1) (cf. e.g. [**DKN**], [**FFM**], [**MT1**, **2**]). Using Riemann bilinear relations (cf e.g. [**EM**]) one sees that the frequencies ω_j $(j \geq 1)$ can be expressed in terms of the zeroes of a conveniently normalized basis of holomorphic differentials on Σ_q. Our strategy for the proof of Theorem 1 is to show that for elements q merely in $L^2(S^1)$, such a basis of holomorphic differentials still exists with the property that their zeroes are real analytic functions of q and that these zeroes can be rather precisely located (cf. Section 2 and Appendix).

As an application of Theorem 1 we investigate the initial value problem for KdV on the circle. Denote by $\Omega : L^2_0(S^1) \to \ell^2_{1/2}(\mathbb{R}^2)$ the symplectomorphism constructed in [**BBGK**] and by $(J_j, \alpha_j)_{j\geq 1}$ the symplectic polar coordinates in $\ell^2_{1/2}(\mathbb{R}^2)$,

$$\Omega(q) = (x, y) = (x_j, y_j)_{j\geq 1} = \left(\sqrt{2J_j}\cos\alpha_j, \sqrt{2J_j}\sin\alpha_j\right)_{j\geq 1}. \tag{1.5}$$

For q in $L^2(S^1)$, introduce $p = q - [q] \in L^2_0(S^1)$ and define

$$(x(t), y(t)) = \left(\sqrt{2J_j}\cos\left(\omega_j(q)t + \alpha_j\right), \sqrt{2J_j}\sin\left(\omega_j(q)t + \alpha_j\right)\right)_{j\geq 1} \tag{1.6}$$

where

$$\omega_j(q) = \omega_j(p) + 12[q]\pi j. \tag{1.7}$$

Introduce the solution operator $\mathcal{S}^{(1)}$ of KdV, $\mathcal{S}^{(1)} : L^2(S^1) \to C(\mathbb{R}; L^2(S^1))$ where

$$\mathcal{S}^{(1)}(q)(t) = [q] + \Omega^{-1}(x(t), y(t)). \tag{1.8}$$

Recently, Bourgain [**Bo1**] has found weak solutions of KdV, which can be analyzed further by using Theorem 1:

Theorem 2. *Let $c \in \mathbb{R}$. Then*

(i) *$\mathcal{S}^{(1)}(q)$ coincides with the weak solution $\mathcal{S}_B(q)$ constructed by Bourgain for any initial data q in $L^2_c(S^1)$.*

(ii) *Given q_1, q_2 in $L^2_c(S^1)$, there exists $M > 0$ so that for any $t \in \mathbb{R}$*

$$\left\|\left(\mathcal{S}^{(1)}(q_1)\right)(\cdot, t) - \left(\mathcal{S}^{(1)}(q_2)\right)(\cdot, t)\right\|_{L^2(S^1)} \leq M(1 + |t|)\, \|q_1 - q_2\|_{L^2(S^1)}. \tag{1.9}$$

(iii) *For any $0 < T < \infty$ $\mathcal{S}^{(1)} : L^2_c(S^1) \to C\left([-T, T]; L^2_c(S^1)\right)$ is real analytic.*

Remark 1. The estimate (1.9) strengthens a result of Bourgain [**Bo1**], Theorem 5. Instead of $M(1 + |t|)$, Bourgain obtains $e^{C|t|}$ where $C > 0$ is some constant depending on q_1 and q_2.

Remark 2. Recently, we have learnt that property (iii) has also been obtained by Zhang [**Zh**] using methods very different from ours.

Remark 3. McKean-Trubowitz [**MT1**] have constructed classical solutions of (1.1), using theta functions. However, with their approach, one has to fix an isospectral set of potentials in advance and therefore, properties of well posedness cannot be proved.

Remark 4. Note that $\mathcal{S}^{(1)}$ is *not* real analytic on all of $L^2(S^1)$: for generic initial data q with $\mathrm{Im}[q] \neq 0$ one sees from (1.6) and (1.7) that KdV is ill posed.

Similar results as the ones presented for KdV hold for any of the equations in the KdV hierarchy. As an example we consider KdV_2, i.e., the fifth order equation in the hierarchy.

$$\partial_t v = \frac{1}{4}\partial_x^5 v - \frac{5}{2} v \partial_x^3 v - 5\partial_x v \partial_x^2 v + \frac{15}{2} v^2 \partial_x v. \tag{1.10}$$

In Section 4 we analyze the frequencies corresponding to the Hamiltonian of KdV_2 and use the results to construct weak solutions $\mathcal{S}^{(2)}(q)$ of (1.10) for initial data in $H^1_{c_1,c_3}(S^1) := \{q \in H^1(S^1) \mid [q] = c_3; \frac{5}{8}\int_{S^1} q(x)^2 dx = c_1\}$, with $c_1, c_3 \in \mathbb{R}$,

$$\mathcal{S}^{(2)} : H^1_{c_1,c_3}(S^1) \to C(\mathbb{R}, H^1_{c_1,c_3}(S^1))$$

and prove that $\mathcal{S}^{(2)}$ has properties similar to those of $\mathcal{S}^{(1)}$.

We point out that, according to [**Bo2**], Bourgain's method cannot be used to obtain weak solutions of (1.10) for initial data in $H^1_c(S^1)$. Results similar to the ones presented in this paper hold most likely also for the defocusing nonlinear Schrödinger equation $\frac{1}{i}\partial_t \psi = \partial_x^2 \psi - |\psi|^2 \psi$. However we have not verified the details.

This paper is related to [**BKM**] where we present further applications of the existence of action-angle variables for KdV. Among other things, we prove that in a neighborhood in $H^1(S^1)$ of the elliptic fix point $q \equiv 0$, the Hamiltonian $\mathcal{H}$ admits a convergent Birkhoff normal form.

The notation in this paper is standard and coincides with the one used in [**Ka**] and [**BBGK**].

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2. Auxilary results.

In this section we prove auxilary results needed in the following section to show that the frequencies ω_j can be defined on the whole space $L^2_0(S^1)$.

Given a potential q in $L^2_0(S^1;\mathbb{C})$, denote by $(\lambda_j = \lambda_j(q))_{j\geq 0}$ the union of the periodic and antiperiodic spectrum of $-\frac{d^2}{dx^2} + q$ (with multiplicities). The eigenvalues are ordered so that $\operatorname{Re}\lambda_0 \leq \operatorname{Re}\lambda_1 \leq \operatorname{Re}\lambda_3 \leq \dots$ and, if $\operatorname{Re}\lambda_n = \operatorname{Re}\lambda_{n+1}$, then $\operatorname{Im}\lambda_n \leq \operatorname{Im}\lambda_{n+1}$.

Introduce $\tau_j = (\lambda_{2j} + \lambda_{2j-1})/2$, $\gamma_j = (\lambda_{2j} - \lambda_{2j-1})$ and the gap interval $I_j = \{t\lambda_{2j} + (1-t)\lambda_{2j-1}; 0 \leq t \leq 1\} \subseteq \mathbb{C}$. Note that τ_j, but not γ_j are real analytic functions on $L^2_0(S^1)$. Denote by $\Delta(\lambda) = \Delta(\lambda, q)$ the discriminant

$$\Delta(\lambda) := y_1(1,\lambda) + \frac{d}{dx} y_2(1,\lambda)$$

where $y_1(x,\lambda)$ and $y_2(x,\lambda)$ denote the fundamental solutions for $-\frac{d^2}{dx^2}+q$. Introduce $\sqrt{\Delta(\mu)^2-4}$ with the sign determined so that, in the case where q is real valued, $\sqrt{\Delta(\mu)^2-4}>0$ for $\mu\in(-\infty,\lambda_0)$.

Theorem 2.1. *Let q_0 be in $L^2_0(S^1)$. Then there exists a (sufficiently small) neighborhood U_{q_0} of q_0 in $L^2_0(S^1;\mathbb{C})$ with the following properties:*

For any $j\geq 1$ and $q\in U_{q_0}$ there exists a unique sequence $(\mu_k^{(j)})_{k\in\mathbb{N}\setminus\{j\}}$ satisfying uniformly for $j\geq 1, k\in\mathbb{N}\setminus\{j\}$ and q in U_{q_0},

$$|\mu_k^{(j)}-\tau_k|\leq|\gamma_k|^2 O\left(\frac{1}{k}\right), \tag{2.1}$$

so that, for all k in $\mathbb{N}\setminus\{j\}$,

$$\int_{\lambda_{2k-1}}^{\lambda_{2k}}\frac{\varphi_j(\lambda)d\lambda}{\sqrt{\Delta(\lambda)^2-4}}=0 \tag{2.2}$$

where $\varphi_j(\lambda)$ is the entire function

$$\varphi_j(\lambda)=\frac{1}{j^2\pi^2}\prod_{k\neq j}\frac{\mu_k^{(j)}-\lambda}{k^2\pi^2}. \tag{2.3}$$

Moreover, the $\mu_k^{(j)}$'s are analytic functions on U_{q_0}. In case where q is real valued, the $\mu_k^{(j)}$ are real valued and satisfy

$$\lambda_{2k-1}\leq\mu_k^{(j)}\leq\lambda_{2k}\quad(\forall j\neq\ell).$$

Rephrasing the above results, Theorem 2.1 states that there exist 1-forms $\Omega_j(\lambda)=\frac{\varphi_j(\lambda)d\lambda}{\sqrt{\Delta(\lambda)^2-4}}$ in the Hilbert space of quadratically integrable holomorphic 1-forms on the (open) Riemann surface $y=\sqrt{\Delta(\lambda)^2-4}$ which satisfy (2.2) and that the zeroes of these 1-forms $\Omega_j(\lambda)$ depend analytically on q. Actually, the Ω_j's are a basis in the space of these 1-forms. If $q\in U_{q_0}$ is real valued, the existence of such a basis of 1-forms $\Omega_j(\lambda)$ with a product representation (2.3) has been proved by [**MT2**] (cf also [**FKT**]). A proof of Theorem 2.1 is provided in the Appendix.

The rest of this section is devoted to a refinement of the asymptotics (2.1). First note that, in the case where $\gamma_k=0$, $\mu_k^{(j)}=\tau_k$. In case $\gamma_k\neq 0$, we make a change of coordinates in (2.2), $\lambda(t)=\tau_k+\frac{\gamma_k}{2}t$, which leads to

$$\int_{-1}^{1}\frac{\varphi_j\big(\lambda(t)\big)\frac{\gamma_k}{2}dt}{\sqrt{\Delta\big(\lambda(t)\big)^2-4}}=0. \tag{2.4}$$

It is convenient to introduce

$$\begin{aligned} \Delta_\alpha(t) &= (\lambda_{2\alpha} - \lambda(t))\,(\lambda_{2\alpha-1} - \lambda(t)) \quad (\alpha \geq 1) \\ \Delta_0(t) &= (\lambda(t) - \lambda_0)\,. \end{aligned} \tag{2.5}$$

Notice that

$$\Delta_\alpha(t) = \left(\tau_\alpha + \frac{\gamma_\alpha}{2} - \lambda(t)\right)\left(\tau_\alpha - \frac{\gamma_\alpha}{2} - \lambda(t)\right) = (\tau_\alpha - \lambda(t))^2 - \left(\frac{\gamma_\alpha}{2}\right)^2.$$

Therefore, with $\tau_k - \lambda(t) = \tau_k - \tau_k - \frac{\gamma_k}{2}t$,

$$-\Delta_k(t) = (\lambda_{2k} - \lambda(t))\,(\lambda(t) - \lambda_{2k-1}) = \left(\frac{\gamma_k}{2}\right)^2 (1 - t^2)$$

and

$$\begin{aligned} \Delta\,(\lambda(t))^2 - 4 &= 4\,(\lambda_0 - \lambda(t)) \prod_{\alpha\geq 1} \frac{(\lambda_{2\alpha} - \lambda(t))\,(\lambda_{2\alpha-1} - \lambda(t))}{(\alpha^2\pi^2)^2} \\ &= 4\Delta_0(t)\left(\frac{\gamma_k}{2}\right)^2 \frac{(1-t^2)}{k^4\pi^4} \prod_{\alpha\neq k} \frac{\Delta_\alpha(t)}{(\alpha^2\pi^2)^2} \\ &= 4\Delta_0(t)\left(\frac{\gamma_k}{2}\right)^2 \frac{1-t^2}{k^4\pi^4} \frac{\Delta_j(t)}{(j^2\pi^2)^2} \prod_{\alpha\in\mathbb{N}\setminus\{j,k\}} \frac{(\tau_\alpha - \lambda(t))^2}{(\alpha^2\pi^2)^2} \\ &\quad \cdot \prod_{\alpha\in\mathbb{N}\setminus\{j,k\}} \left(1 - \frac{(\frac{\gamma_\alpha}{2})^2}{(\tau_\alpha - \lambda(t))^2}\right). \end{aligned} \tag{2.6}$$

Further we introduce

$$\xi_\alpha^{(j)} := \mu_\alpha^{(j)} - \tau_\alpha. \tag{2.7}$$

This leads to

$$\begin{aligned} \varphi_j\,(\lambda(t)) &= \frac{1}{j^2\pi^2} \prod_{\alpha\neq j} \frac{\tau_\alpha - \lambda(t) + \xi_\alpha^{(j)}}{\alpha^2\pi^2} \\ &= \frac{1}{j^2\pi^2} \frac{\xi_k^{(j)} - \frac{\gamma_k}{2}t}{k^2\pi^2} \prod_{\alpha\in\mathbb{N}\setminus\{j,k\}} \frac{\tau_\alpha - \lambda(t)}{\alpha^2\pi^2} \left(1 + \frac{\xi_\alpha^{(j)}}{\tau_\alpha - \lambda(t)}\right). \end{aligned} \tag{2.8}$$

Combining (2.6) and (2.8) we obtain

$$\frac{\varphi_j\,(\lambda(t))}{\sqrt{\Delta\,(\lambda(t))^2 - 4}} = \frac{\xi_k^{(j)} - \frac{\gamma_k}{2}t}{2\frac{\gamma_k}{2}\sqrt{\Delta_0(t)}\sqrt{\Delta_j(t)}} \frac{1}{\sqrt{1-t^2}} \prod_{\alpha\in\mathbb{N}\setminus\{j,k\}} \left(1 + \frac{\xi_\alpha^{(j)}}{\tau_\alpha - \lambda(t)}\right) \tag{2.9}$$

$$\cdot \prod_{\alpha\in\mathbb{N}\setminus\{j,k\}} \left(1 - \frac{(\frac{\gamma_\alpha}{2})^2}{\big(\tau_\alpha - \lambda(t)\big)^2}\right)^{-1/2}.$$

The Equation (2.4) then takes the form

(2.10)

$$\begin{aligned}
&\xi_k^{(j)} \int_{-1}^{1} \frac{dt}{\sqrt{\Delta_0(t)}\sqrt{\Delta_j(t)}\sqrt{1-t^2}} \\
&\quad \cdot \prod_{\alpha\in\mathbb{N}\setminus\{j,k\}} \left(1 + \frac{\xi_\alpha^{(j)}}{\tau_\alpha - \lambda(t)}\right)\left(1 - \frac{(\frac{\gamma_\alpha}{2})^2}{\big(\tau_\alpha - \lambda(t)\big)^2}\right)^{-1/2} \\
&= \frac{\gamma_k}{2}\int_{-1}^{1} \frac{t\,dt}{\sqrt{\Delta_0(t)}\sqrt{\Delta_j(t)}\sqrt{1-t^2}} \\
&\quad \cdot \prod_{\alpha\in\mathbb{N}\setminus\{j,k\}} \left(1 + \frac{\xi_\alpha^{(j)}}{\tau_\alpha - \lambda(t)}\right)\left(1 - \frac{(\frac{\gamma_\alpha}{2})^2}{\big(\tau_\alpha - \lambda(t)\big)^2}\right)^{-1/2}.
\end{aligned}$$

Now introduce

$$A_{jk}(t) \equiv A_{jk}(t,q) := \prod_{\alpha\in\mathbb{N}\setminus\{j,k\}} \left(1 + \frac{\xi_\alpha^{(j)}}{\tau_\alpha - \lambda(t)}\right)$$

and

$$B_{jk}(t) \equiv B_{jk}(t,q) := \prod_{\alpha\in\mathbb{N}\setminus\{j,k\}} \left(1 - \frac{(\frac{\gamma_\alpha}{2})^2}{\big(\tau_\alpha - \lambda(t)\big)^2}\right)^{-1/2}.$$

After multiplying left and right hand side of (2.10) with the t-independent nonvanishing quantity

$$\frac{\sqrt{\Delta_0(0)}\sqrt{\Delta_j(0)}}{A_{jk}(0)B_{jk}(0)}$$

the Equation (2.10) reads

$$\xi_k^{(j)} = \frac{\gamma_k}{2} \frac{\int_{-1}^{1} \frac{t\,dt}{\sqrt{1-t^2}} \sqrt{\frac{\Delta_0(0)}{\Delta_0(t)}} \sqrt{\frac{\Delta_j(0)}{\Delta_j(t)}} \frac{A_{jk}(t)}{A_{jk}(0)} \frac{B_{jk}(t)}{B_{jk}(0)}}{\int_{-1}^{1} \frac{dt}{\sqrt{1-t^2}} \sqrt{\frac{\Delta_0(0)}{\Delta_0(t)}} \sqrt{\frac{\Delta_j(0)}{\Delta_j(t)}} \frac{A_{jk}(t)}{A_{jk}(0)} \frac{B_{jk}(t)}{B_{jk}(0)}}. \tag{2.11}$$

It will turn out that for $k \geq N$, with N sufficiently large, the denominator in (2.11) is different from 0. Thus the quotient in (2.11) is well defined at least for k sufficiently large. To simplify (2.11), we write

$$C_{jk}(t) \equiv C_{jk}(t,q) := \sqrt{\frac{\Delta_0(0)}{\Delta_0(t)}} \sqrt{\frac{\Delta_j(0)}{\Delta_j(t)}} \frac{A_{jk}(t)}{A_{jk}(0)} \frac{B_{jk}(t)}{B_{jk}(0)}$$

and obtain

$$\xi_k^{(j)} = \frac{\gamma_k}{2} \frac{\int_0^1 \frac{tdt}{\sqrt{1-t^2}} (C_{jk}(t) - C_{jk}(-t))}{\int_0^1 \frac{dt}{\sqrt{1-t^2}} (C_{jk}(t) + C_{jk}(-t))}. \tag{2.12}$$

To obtain asymptotics for $\xi_k^{(j)}$ we estimate

$$\sqrt{\frac{\Delta_j(0)}{\Delta_j(\pm t)}} - 1, \sqrt{\frac{\Delta_0(0)}{\Delta_0(\pm t)}} - 1, \frac{A_{jk}(\pm t)}{A_{jk}(0)} - 1 \text{ and } \frac{B_{jk}(\pm t)}{B_{jk}(0)} - 1.$$

Recall that for $q \in L_0^2(S^1), \sum_k |\gamma_k|^2 < \infty$.

Lemma 2.2. *Let $q_0 \in L_0^2(S^1)$. Then there exist a neighborhood U_{q_0} of q_0 in $L_0^2(S^1;\mathbb{C})$ and $C > 0$, independent of j and k, so that, for $q \in U_{q_0}$ and $j \neq k$,*

$$\sup_{0 \le t \le 1} \left| \frac{A_{jk}(\pm t)}{A_{jk}(0)} - 1 \right| \le C \frac{|\gamma_k|}{k^3} \left(\sum_{|\alpha - k| \le \frac{k}{2}} |\gamma_\alpha|^2 + \frac{1}{k} \right). \tag{2.13}$$

Proof. Notice that

$$\sup_{0 \le t \le 1} \left| \frac{A_{jk}(\pm t)}{A_{jk}(0)} - 1 \right| \le \sup_{|t| \le 1} \left| \frac{\frac{\partial}{\partial t} A_{jk}(t)}{A_{jk}(0)} \right|.$$

Recall that $A_{jk}(t) = \prod_{\alpha \in \mathbb{N} \setminus \{j,k\}} \left(1 + \frac{\xi_\alpha^{(j)}}{\tau_\alpha - \lambda(t)}\right)$ and $\lambda(t) = \tau_k + \frac{\gamma_k}{2} t$ and therefore,

$$\frac{1}{A_{jk}(0)} \frac{\partial}{\partial t} A_{jk}(t) = \frac{A_{jk}(t)}{A_{jk}(0)} \left(\sum_{\alpha \neq j,k} \frac{1}{\left(1 + \frac{\xi_\alpha^{(j)}}{\tau_\alpha - \lambda(t)}\right)} \frac{(-1)\xi_\alpha^{(j)}}{\left(\tau_\alpha - \lambda(t)\right)^2} (-\frac{\gamma_k}{2}) \right).$$

This leads to,

$$\left| \frac{1}{A_{jk}(0)} \frac{\partial}{\partial t} A_{jk}(t) \right| \le \widehat{C} |\gamma_k| \left(\sum_{|\alpha - k| \ge \frac{k}{2}} \frac{|\xi_\alpha^{(j)}|}{|\tau_\alpha - \lambda(t)|^2} \right) + \widehat{C} |\gamma_k| \left(\sum_{|\alpha - k| \le \frac{k}{2}} \frac{|\xi_\alpha^{(j)}|}{|\tau_\alpha - \lambda(t)|^2} \right). \tag{2.14}$$

Notice that, for any α with $|\alpha - k| \ge \frac{k}{2}$,

$$\frac{1}{|\tau_\alpha - \lambda(t)|^2} \le \frac{C}{(\alpha^2 - k^2)^2} = O\left(\frac{1}{k^4}\right).$$

Therefore,

$$\sum_{|\alpha-k|\geq\frac{k}{2}} \frac{|\xi_\alpha^{(j)}|}{|\tau_\alpha-\lambda(t)|^2} \leq \frac{C}{k^4}\sum_\alpha |\xi_\alpha^{(j)}| = O\left(\frac{1}{k^4}\right). \tag{2.15}$$

For α with $|\alpha-k|\leq\frac{k}{2}$ we use the estimate $\xi_\alpha^{(j)}=\gamma_\alpha^2 O(\frac{1}{k})$ to conclude that

$$\sum_{|\alpha-k|\leq\frac{k}{2}} \frac{|\xi_\alpha^{(j)}|}{|\tau_\alpha-\lambda(t)|^2} \leq \frac{C}{k^3}\sum_{|\alpha-k|\leq\frac{k}{2}} |\gamma_\alpha|^2. \tag{2.16}$$

Combining (2.14) - (2.16) yields (2.13). □

Lemma 2.3. *Let $q_0\in L_0^2(S^1)$. Then there exist a neighborhood U_{q_0} of q_0 in $L_0^2(S^1;\mathbb{C})$ and $C>0$, independent of j and k, so that for $q\in U_{q_0}$ and $j\neq k$,*

$$\sup_{0\leq t\leq 1}\left|\frac{B_{jk}(\pm t)}{B_{jk}(0)}-1\right| \leq \frac{C|\gamma_k|}{k^3}\left(\sum_{|\alpha-k|\leq\frac{k}{2}}|\gamma_\alpha|^2+\frac{1}{k^3}\right). \tag{2.17}$$

Proof. Notice that

$$\sup_{0\leq t\leq 1}\left|\frac{B_{jk}(\pm t)}{B_{jk}(0)}-1\right| \leq \sup_{|t|\leq 1}\left|\frac{\frac{\partial}{\partial t}B_{jk}(t)}{B_{jk}(0)}\right|.$$

Recall that $B_{jk}(t)=\prod_{\alpha\neq j,k}\left(1-\frac{(\frac{\gamma_\alpha}{2})^2}{\left(\tau_\alpha-\lambda(t)\right)^2}\right)^{-1/2}$ to conclude that

$$\frac{\partial}{\partial t}B_{jk}(t)=B_{jk}(t)\left\{\sum_{\alpha\in\mathbb{N}\setminus\{j,k\}}\frac{-1/2}{\left(1-\frac{(\frac{\gamma_\alpha}{2})^2}{\left(\tau_\alpha-\lambda(t)\right)^2}\right)}\cdot\frac{2(\frac{\gamma_\alpha}{2})^2}{\left(\tau_\alpha-\lambda(t)\right)^3}\left(\frac{-\gamma_k}{2}\right)\right\}.$$

This leads to

$$\begin{aligned}\frac{1}{|B_{jk}(0)|}\left|\frac{\partial}{\partial t}B_{jk}(t)\right| &\leq \frac{|B_{jk}(t)|}{|B_{jk}(0)|}\left\{\sum_{\alpha\in\mathbb{N}\setminus\{j,k\}}C|\gamma_k|\frac{1}{|\tau_\alpha-\lambda(t)|^3}|\gamma_\alpha|^2\right\}\\ &\leq C|\gamma_k|\left(\sum_{|\alpha-k|\geq\frac{k}{2}}\frac{|\gamma_\alpha|^2}{|\tau_\alpha-\lambda(t)|^3}+\sum_{|\alpha-k|\leq\frac{k}{2}}\frac{|\gamma_\alpha|^2}{|\tau_\alpha-\lambda(t)|^3}\right)\end{aligned}$$

$$\leq C|\gamma_k| \left(\frac{C}{k^6} + \frac{1}{k^3} \sum_{|\alpha - k| \leq \frac{k}{2}} |\gamma_\alpha|^2 \right).$$

□

Lemma 2.4. *Let $q_0 \in L^2_0(S^1)$. Then there exist a neighborhood U_{q_0} in $L^2_0(S^1;\mathbb{C})$ and $C > 0$, independent of k, such that for any q in U_{q_0}*

$$\sup_{|t|\leq 1} \left| \left(\frac{\tau_k - \lambda_0}{\lambda(t) - \lambda_0} \right)^{1/2} - \left(1 - \frac{\gamma_k}{4} \frac{1}{\tau_k - \lambda_0} t \right) \right| \leq C \frac{|\gamma_k|^2}{k^4} \tag{2.18}$$

and

$$\sup_{0 \leq t \leq 1} \left| \left(\frac{\tau_k - \lambda_0}{\lambda(\pm t) - \lambda_0} \right)^{1/2} - 1 \right| \leq C \frac{|\gamma_k|}{k^2}. \tag{2.19}$$

Proof. Notice that

$$\begin{aligned} \frac{d}{dt} \left(\frac{\tau_k - \lambda_0}{\lambda(t) - \lambda_0} \right)^{1/2} &= \left(\frac{\tau_k - \lambda_0}{\lambda(t) - \lambda_0} \right)^{1/2} \frac{(-1/2)}{\lambda(t) - \lambda_0} \frac{\gamma_k}{2}, \\ \frac{d^2}{dt^2} \left(\frac{\tau_k - \lambda_0}{\lambda(t) - \lambda_0} \right)^{1/2} &= \left(\frac{\tau_k - \lambda_0}{\lambda(t) - \lambda_0} \right)^{1/2} \frac{3/4}{(\lambda(t) - \lambda_0)^2} \left(\frac{\gamma_k}{2} \right)^2, \end{aligned} \tag{2.20}$$

$$\sup_{0 \leq t \leq 1} \left| \left(\frac{\tau_k - \lambda_0}{\lambda(\pm t) - \lambda_0} \right)^{1/2} - 1 \right| \leq \sup_{|t| \leq 1} \left| \frac{d}{dt} \left(\frac{\tau_k - \lambda_0}{\lambda(t) - \lambda_0} \right)^{1/2} \right| \leq C \frac{1}{k^2} |\gamma_k|.$$

This proves (2.19). Taylor's formula and (2.20) lead to (2.18). □

The difference $\sqrt{\frac{\Delta_j(0)}{\Delta_j(\pm t)}} - 1$ needs to be treated with more care as

$$\sup_{|t| \leq 1} \left| \sqrt{\frac{\Delta_j(0)}{\Delta_j(\pm t)}} - 1 \right| \leq \frac{C|\gamma_k|}{|\tau_k - \tau_j|} = |\gamma_k| O\left(\frac{1}{k} \right).$$

Lemma 2.5. *Let $q_0 \in L^2_0(S^1)$. Then there exist a neighborhood U_{q_0} of q_0 in $L^2_0(S^1;\mathbb{C})$ and $C > 0$, independent of j and k, such that for any q in U_{q_0} and $j \neq k$,*

(2.21)

$$\left| \sqrt{\frac{\Delta_j(0)}{\Delta_j(t)}} - \left(1 + \frac{1}{2} \frac{\gamma_k(\tau_j - \tau_k)}{\Delta_j(0)} t + \left(\frac{-1}{8} \frac{\gamma_k^2}{\Delta_j(0)} + \frac{3}{8} \frac{\gamma_k^2(\tau_j - \tau_k)^2}{\Delta_j(0)^2} \right) t^2 \right) \right|$$

$$\leq C\frac{|\gamma_k|^3}{(\tau_k-\tau_j)^3}.$$

Proof. Recall that

$$\Delta_j(t)=(\tau_j-\lambda(t))^2-\left(\frac{\gamma_j}{2}\right)^2=(\tau_j-\tau_k)^2-\left(\frac{\gamma_k}{2}\right)^2-2\frac{\gamma_k}{2}(\tau_j-\tau_k)t+\left(\frac{\gamma_k}{2}\right)^2t^2$$

and therefore

$$\begin{aligned}
&\left(\frac{\Delta_j(0)}{\Delta_j(t)}\right)=\left(\frac{1}{1+\frac{\gamma_k(\tau_k-\tau_j)t+(\frac{\gamma_k}{2})^2t^2}{\Delta_j(0)}}\right)^{1/2}\\
&=1+\left\{\left(\frac{1}{1+\frac{\gamma_k(\tau_k-\tau_j)t+(\frac{\gamma_k}{2})^2t^2}{\Delta_j(0)}}\right)^{3/2}\left(-\frac{1}{2}\right)\frac{\gamma_k(\tau_k-\tau_j)+2(\frac{\gamma_k}{2})^2t}{\Delta_j(0)}\right\}\Bigg|_{t=0}t\\
&\quad+\left\{\left(\frac{1}{1+\frac{\gamma_k(\tau_k-\tau_j)t+(\frac{\gamma_k}{2})^2t^2}{\Delta_j(0)}}\right)^{5/2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{\gamma_k(\tau_k-\tau_j)+2(\frac{\gamma_k}{2})^2t}{\Delta_j(0)}\right)^2\right.\\
&\quad\left.+\left(\frac{1}{1+\frac{\gamma_k(\tau_k-\tau_j)t+(\frac{\gamma_k}{2})^2t^2}{\Delta_j(0)}}\right)^{3/2}\left(-\frac{1}{2}\right)\frac{2(\frac{\gamma_k}{2})^2}{\Delta_j(0)}\right\}\Bigg|_{t=0}\frac{t^2}{2}+\frac{\gamma_k^3}{(\tau_k-\tau_j)^3}O(1)
\end{aligned}$$

which leads to (2.21). □

Lemma 2.6. *Let $q_0\in L^2_0(S^1)$. Then there exist a neighborhood U_{q_0} of q_0 in $L^2_0(S^1;\mathbb{C})$ and $C>0$, independent of j and k, such that for any q in U_{q_0}, and $j\neq k$,*

$$\begin{aligned}
&\left|\frac{\gamma_k}{2}\frac{\int_0^1\frac{tdt}{\sqrt{1-t^2}}\left(\sqrt{\frac{\Delta_j(0)}{\Delta_j(t)}}-\sqrt{\frac{\Delta_j(0)}{\Delta_j(-t)}}\right)}{\int_0^1\frac{dt}{\sqrt{1-t^2}}\left(\sqrt{\frac{\Delta_j(0)}{\Delta_j(t)}}+\sqrt{\frac{\Delta_j(0)}{\Delta_j(-t)}}\right)}-\frac{1}{8}\frac{(-\gamma_k^2)}{\tau_k-\tau_j}\right|\\
&\leq\frac{C}{(\tau_k-\tau_j)^3}|\gamma_k|^2\left(|\gamma_j|^2+|\gamma_k|\right).
\end{aligned}\tag{2.22}$$

Proof. Using Lemma 2.5, we obtain

$$\begin{aligned}
&\int_0^1\frac{tdt}{\sqrt{1-t^2}}\left(\sqrt{\frac{\Delta_j(0)}{\Delta_j(t)}}-\sqrt{\frac{\Delta_j(0)}{\Delta_j(-t)}}\right)\\
&=\frac{\gamma_k(\tau_j-\tau_k)}{\Delta_j(0)}\int_0^1\frac{t^2dt}{\sqrt{1-t^2}}+\frac{\gamma_k^3}{(\tau_k-\tau_j)^3}O(1)
\end{aligned}\tag{2.23}$$

and

(2.24)
$$\begin{aligned}&\int_0^1 \frac{dt}{\sqrt{1-t^2}}\left(\sqrt{\frac{\Delta_j(0)}{\Delta_j(t)}}+\sqrt{\frac{\Delta_j(0)}{\Delta_j(-t)}}\right)\\ &= 2\int_0^1 \frac{dt}{\sqrt{1-t^2}} + 2\left(\frac{-1}{8}\frac{\gamma_k^2}{\Delta_j(0)}+\frac{3}{8}\frac{\gamma_k^2(\tau_j-\tau_k)^2}{\Delta_j(0)^2}\right)\int_0^1 \frac{t^2dt}{\sqrt{1-t^2}}\\ &\quad + \frac{\gamma_k^3}{(\tau_k-\tau_j)^3}O(1).\end{aligned}$$

Notice that
$$\int_0^1 \frac{dt}{\sqrt{1-t^2}} = \frac{\pi}{2}; \int_0^1 \frac{t^2dt}{\sqrt{1-t^2}} = \frac{\pi}{4}.$$

Therefore

(2.25)
$$\begin{aligned}&\frac{\gamma_k}{2}\frac{\int_0^1 \frac{tdt}{\sqrt{1-t^2}}\left(\sqrt{\frac{\Delta_j(0)}{\Delta_j(t)}}-\sqrt{\frac{\Delta_j(0)}{\Delta_j(-t)}}\right)}{\int_0^1 \frac{dt}{\sqrt{1-t^2}}\left(\sqrt{\frac{\Delta_j(0)}{\Delta_j(t)}}+\sqrt{\frac{\Delta_j(0)}{\Delta_j(-t)}}\right)}\\ &= \frac{\gamma_k}{2}\frac{\frac{\gamma_k(\tau_j-\tau_k)}{\Delta_j(0)}\frac{\pi}{4}\left(1+\frac{\gamma_k^2}{(\tau_k-\tau_j)^3}O(1)\right)}{\pi\left(1-\left(\frac{1}{16}\frac{\gamma_k^2}{\Delta_j(0)}-\frac{3}{16}\frac{\gamma_k^2(\tau_j-\tau_k)^2}{\Delta_j(0)^2}\right)+\frac{\gamma_k^3}{(\tau_k-\tau_j)^3}O(1)\right)}\\ &= \frac{\gamma_k^2}{8}\frac{\tau_j-\tau_k}{\Delta_j(0)} + \frac{\gamma_k^3}{(\tau_k-\tau_j)^3}O(1).\end{aligned}$$

Recall that $\Delta_j(0) = (\tau_j-\tau_k)^2 - (\frac{\gamma_j}{2})^2$. Therefore

$$\frac{\tau_j-\tau_k}{\Delta_j(0)} = \frac{1}{(\tau_j-\tau_k)}\frac{1}{1-\frac{(\gamma_j/2)^2}{(\tau_j-\tau_k)^2}} = \frac{1}{\tau_j-\tau_k} + \frac{(\gamma_j/2)^2}{(\tau_j-\tau_k)^3}O(1). \tag{2.26}$$

Combining (2.25) and (2.26) we obtain

$$\frac{\gamma_k}{2}\frac{\int_0^1 \frac{tdt}{\sqrt{1-t^2}}\left(\sqrt{\frac{\Delta_j(0)}{\Delta_j(t)}}-\sqrt{\frac{\Delta_j(0)}{\Delta_j(-t)}}\right)}{\int_0^1 \frac{dt}{\sqrt{1-t^2}}\left(\sqrt{\frac{\Delta_j(0)}{\Delta_j(t)}}+\sqrt{\frac{\Delta_j(0)}{\Delta_j(-t)}}\right)} = -\frac{\gamma_k^2}{8}\frac{1}{\tau_k-\tau_j} + \gamma_k^2\frac{(\gamma_k+\gamma_j^2)}{(\tau_k-\tau_j)^3}O(1).$$

□

Theorem 2.7. *Let $q_0 \in L_0^2(S^1)$. Then there exists a neighborhood U_{q_0} of q_0 in $L_0^2(S^1;\mathbb{C}), k_0 \geq 1$ and $C > 0$ such that $\forall q \in U_{q_0}$, $\forall j$ and $k \geq k_0$ with $k \neq j$,*

$$\left|\xi_k^{(j)} - \left(-\frac{1}{2}\frac{(\gamma_k/2)^2}{\tau_k-\tau_j} - \frac{\gamma_k^2}{16}\frac{1}{\tau_k-\lambda_0}\right)\right| \tag{2.27}$$

$$\leq \frac{C|\gamma_k|^2}{k^3}\left(|\gamma_k| + |\gamma_j|^2 + \sum_{|\alpha-k|\leq\frac{k}{2}} |\gamma_\alpha|^2 + O\left(\frac{1}{k}\right)\right).$$

Proof. Recall that

$$\begin{aligned}
&\xi_k^{(j)}\left(\int_0^1 \frac{dt}{\sqrt{1-t^2}}(C_{jk}(t)+C_{jk}(-t))\right)\\
&= \frac{\gamma_k}{2}\int_0^1 \frac{tdt}{\sqrt{1-t^2}}(C_{jk}(t)-C_{jk}(-t)).
\end{aligned}$$

Write

$$\begin{aligned}
C_{jk}(t) = &\sqrt{\frac{\Delta_j(0)}{\Delta_j(t)}}\\
&+\sqrt{\frac{\Delta_j(0)}{\Delta_j(t)}}\left(\sqrt{\frac{\Delta_0(0)}{\Delta_0(t)}}-1\right)\\
&+\sqrt{\frac{\Delta_j(0)}{\Delta_j(t)}}\sqrt{\frac{\Delta_0(0)}{\Delta_0(t)}}\left(\frac{A_{jk}(t)}{A_{jk}(0)}-1\right)\\
&+\sqrt{\frac{\Delta_j(0)}{\Delta_j(t)}}\sqrt{\frac{\Delta_0(0)}{\Delta_0(t)}}\frac{A_{jk}(t)}{A_{jk}(0)}\left(\frac{B_{jk}(t)}{B_{jk}(0)}-1\right).
\end{aligned}$$

Notice that, by Lemma 2.5

$$\begin{aligned}
&\sqrt{\frac{\Delta_j(0)}{\Delta_j(t)}}\left(\sqrt{\frac{\Delta_0(0)}{\Delta_0(t)}}-1\right)\\
&=\left(1+\frac{1}{2}\frac{\gamma_k(\tau_j-\tau_k)}{\Delta_j(0)}t+\left(-\frac{1}{8}\frac{\gamma_k^2}{\Delta_j(0)}+\frac{3}{8}\frac{\gamma_k^2(\tau_j-\tau_k)^2}{\Delta_j(0)^2}\right)t^2+O\left(\frac{|\gamma_k|^3}{k^3}\right)\right)\\
&\quad\cdot\left(-\frac{\gamma_k}{4}\frac{1}{\tau_k-\lambda_0}t+O\left(\frac{|\gamma_k|^2}{k^4}\right)\right)\\
&=-\frac{\gamma_k}{4}\frac{1}{\tau_k-\lambda_0}t+\frac{1}{2}\frac{\gamma_k(\tau_j-\tau_k)}{\Delta_j(0)}\left(-\frac{\gamma_k}{4}\right)\frac{1}{\tau_k-\lambda_0}t^2+O\left(\frac{|\gamma_k|^2}{k^4}\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
(2.28)\qquad &\frac{\gamma_k}{2}\int_0^1\frac{tdt}{\sqrt{1-t^2}}\left\{\left(\sqrt{\frac{\Delta_j(0)}{\Delta_j(t)}}\left(\sqrt{\frac{\Delta_0(0)}{\Delta_0(t)}}-1\right)\right.\right.\\
&\left.\left.-\sqrt{\frac{\Delta_j(0)}{\Delta_j(-t)}}\left(\sqrt{\frac{\Delta_0(0)}{\Delta_0(-t)}}-1\right)\right)\right\}
\end{aligned}$$

$$= \frac{\gamma_k}{2} \int_0^1 \frac{t^2 dt}{\sqrt{1-t^2}} 2 \left(-\frac{\gamma_k}{4} \frac{1}{\tau_k - \lambda_0} \right) + 0 + O\left(\frac{|\gamma_k|^3}{k^4} \right)$$
$$= -\frac{\gamma_k^2}{16} \frac{\pi}{\tau_k - \lambda_0} + O\left(\frac{\gamma_k^3}{k^4} \right)$$

and

(2.29)
$$\int_0^1 \frac{dt}{\sqrt{1-t^2}} \left\{ \sqrt{\frac{\Delta_j(0)}{\Delta_j(t)}} \left(\sqrt{\frac{\Delta_0(0)}{\Delta_0(t)}} - 1 \right) + \sqrt{\frac{\Delta_j(0)}{\Delta_j(-t)}} \left(\sqrt{\frac{\Delta_j(0)}{\Delta_j(-t)}} - 1 \right) \right\}$$
$$= + \frac{\gamma_k^2}{4} \frac{(\tau_k - \tau_j)}{\Delta_j(0)} \frac{1}{(\tau_k - \lambda_0)} \int_0^1 \frac{t^2 dt}{\sqrt{1-t^2}} + O\left(\frac{|\gamma_k|^2}{k^4} \right).$$

By Lemma 2.2, we conclude

$$\begin{aligned} &\left| \frac{\gamma_k}{2} \int_0^1 \frac{t dt}{\sqrt{1-t^2}} \left\{ \sqrt{\frac{\Delta_j(0)}{\Delta_j(t)}} \sqrt{\frac{\Delta_0(0)}{\Delta_0(t)}} \left(\frac{A_{jk}(t)}{A_{jk}(0)} - 1 \right) \right. \right. \\ &\qquad \left. \left. - \sqrt{\frac{\Delta_j(0)}{\Delta_j(-t)}} \sqrt{\frac{\Delta_0(0)}{\Delta_0(-t)}} \left(\frac{A_{jk}(-t)}{A_{jk}(0)} - 1 \right) \right\} \right| \\ &\leq C \frac{|\gamma_k|^2}{k^3} \left(\sum_{|\alpha - k| \leq \frac{k}{2}} |\gamma_\alpha|^2 + O\left(\frac{1}{k} \right) \right) \end{aligned} \tag{2.30}$$

and

$$\begin{aligned} &\left| \int_0^1 \frac{dt}{\sqrt{1-t^2}} \left\{ \sqrt{\frac{\Delta_j(0)}{\Delta_j(t)}} \sqrt{\frac{\Delta_0(0)}{\Delta_0(t)}} \left(\frac{A_{jk}(t)}{A_{jk}(0)} - 1 \right) \right. \right. \\ &\qquad \left. \left. + \sqrt{\frac{\Delta_j(0)}{\Delta_j(-t)}} \sqrt{\frac{\Delta_0(0)}{\Delta_0(-t)}} \left(\frac{A_{jk}(-t)}{A_{jk}(0)} - 1 \right) \right\} \right| \\ &\leq C \frac{|\gamma_k|}{k^3} \left(\sum_{|\alpha - k| \leq \frac{k}{2}} |\gamma_\alpha|^2 + O\left(\frac{1}{k} \right) \right). \end{aligned} \tag{2.31}$$

Finally, using Lemma 2.3, we obtain

(2.32)
$$\begin{aligned} &\left| \frac{\gamma_k}{2} \int_0^1 \frac{t dt}{\sqrt{1-t^2}} \left\{ \sqrt{\frac{\Delta_j(0)}{\Delta_j(t)}} \sqrt{\frac{\Delta_0(0)}{\Delta_0(t)}} \frac{A_{jk}(t)}{A_{jk}(0)} \left(\frac{B_{jk}(t)}{B_{jk}(0)} - 1 \right) \right. \right. \\ &\qquad \left. \left. - \sqrt{\frac{\Delta_j(0)}{\Delta_j(-t)}} \sqrt{\frac{\Delta_0(0)}{\Delta_0(-t)}} \frac{A_{jk}(-t)}{A_{jk}(0)} \left(\frac{B_{jk}(-t)}{B_{jk}(0)} - 1 \right) \right\} \right| \end{aligned}$$

$$\leq C\frac{|\gamma_k|^2}{k^3}\left(\sum_{|\alpha-k|\leq\frac{k}{2}}|\gamma_\alpha|^2+O\left(\frac{1}{k^3}\right)\right)$$

and

(2.33)
$$\begin{aligned}&\left|\int_0^1\frac{dt}{\sqrt{1-t^2}}\left\{\sqrt{\frac{\Delta_j(0)}{\Delta_j(t)}}\sqrt{\frac{\Delta_0(0)}{\Delta_0(t)}}\frac{A_{jk}(t)}{A_{jk}(0)}\left(\frac{B_{jk}(t)}{B_{jk}(0)}-1\right)\right.\right.\\&\quad\left.\left.+\sqrt{\frac{\Delta_j(0)}{\Delta_j(-t)}}\sqrt{\frac{\Delta_0(0)}{\Delta_0(-t)}}\frac{A_{jk}(-t)}{A_{jk}(0)}\left(\frac{B_{jk}(-t)}{B_{jk}(0)}-1\right)\right\}\right|\\&\leq C\frac{|\gamma_k|}{k^3}\left(\sum_{|\alpha-k|\leq\frac{k}{2}}|\gamma_\alpha|^2+O\left(\frac{1}{k^3}\right)\right).\end{aligned}$$

We combine the above estimates to conclude

(2.34)
$$\begin{aligned}&\frac{\gamma_k}{2}\int_0^1\frac{tdt}{\sqrt{1-t^2}}\left(C_{jk}(t)-C_{jk}(-t)\right)\\&=\frac{\gamma_k}{2}\int_0^1\frac{tdt}{\sqrt{1-t^2}}\left(\sqrt{\frac{\Delta_j(0)}{\Delta_j(t)}}-\sqrt{\frac{\Delta_j(0)}{\Delta_j(-t)}}\right)\\&\quad-\frac{\gamma_k^2}{16}\frac{\pi}{\tau_k-\lambda_0}+|\gamma_k|^2O\left(\frac{1}{k^3}\right)\left(\sum_{|\alpha-k|\leq\frac{k}{2}}|\gamma_\alpha|^2+O\left(\frac{1}{k}\right)\right)\end{aligned}$$

and

(2.35)
$$\begin{aligned}&\int_0^1\frac{dt}{\sqrt{1-t^2}}\left(C_{jk}(t)+C_{jk}(-t)\right)\\&=\int_0^1\frac{dt}{\sqrt{1-t^2}}\left(\sqrt{\frac{\Delta_j(0)}{\Delta_j(t)}}+\sqrt{\frac{\Delta_j(0)}{\Delta_j(-t)}}\right)+O\left(\frac{|\gamma_k|^2}{k^2}\right)\\&\quad+O\left(\frac{\gamma_k}{k^3}\right)\left(\sum_{|\alpha-k|\leq k/2}|\gamma_\alpha|^2+O\left(\frac{1}{k}\right)\right)\\&=\pi\left(1-\left(\frac{1}{16}\frac{\gamma_k^2}{\Delta_j(0)}-\frac{3}{16}\frac{\gamma_k^2(\tau_j-\tau_k)^2}{\Delta_j(0)^2}\right)\right)+\frac{\gamma_k^3}{k^3}O(1)+O\left(\frac{\gamma_k^2}{k^2}\right)\\&\quad+O\left(\frac{\gamma_k}{k^3}\right)\left(\sum_{|\alpha-k|\leq k/2}|\gamma_\alpha|^2+O\left(\frac{1}{k}\right)\right).\end{aligned}$$

Therefore, there exists $k_0 \geq 1$ and a neighborhood U_{q_0} of q_0 in $L_0^2(S^1;\mathbb{C})$ such that for q in U_{q_0}, for any $k \geq k_0$ and $j \geq 1$, $\int_0^1 \frac{dt}{\sqrt{1-t^2}}(C_{jk}(t) + C_{jk}(-t)) \neq 0$ and its inverse satisfies the estimate

(2.36)

$$\begin{aligned}
&\left(\int_0^1 \frac{dt}{\sqrt{1-t^2}}\left(C_{jk}(t) + C_{jk}(-t)\right)\right)^{-1} \\
&= \left(\int_0^1 \frac{dt}{\sqrt{1-t^2}}\left(\sqrt{\frac{\Delta_j(0)}{\Delta_j(t)}} + \sqrt{\frac{\Delta_j(0)}{\Delta_j(-t)}}\right)\right)^{-1} \\
&\quad \cdot \left(1 + O\left(\frac{|\gamma_k|^2}{k^2}\right) + O\left(\frac{\gamma_k}{k^3}\right)\left(\sum_{|\alpha-k|\leq k/2} |\gamma_\alpha|^2 + O\left(\frac{1}{k}\right)\right)\right).
\end{aligned}$$

Combining (2.34) and (2.36)

(2.37)

$$\begin{aligned}
&\frac{\frac{\gamma_k}{2}\int_0^1 \frac{tdt}{\sqrt{1-t^2}}\left(C_{jk}(t) - C_{jk}(-t)\right)}{\int_0^1 \frac{dt}{\sqrt{1-t^2}}\left(C_{jk}(t) + C_{jk}(-t)\right)} \\
&= \frac{\frac{\gamma_k}{2}\int_0^1 \frac{tdt}{\sqrt{1-t^2}}\left(\sqrt{\frac{\Delta_j(0)}{\Delta_j(t)}} - \sqrt{\frac{\Delta_j(0)}{\Delta_j(-t)}}\right) - \frac{\gamma_k^2}{16}\frac{\pi}{\tau_k - \lambda_0} + O\left(\frac{|\gamma_k|^2}{k^3}\right)\left(\sum_{|\alpha-k|\leq\frac{k}{2}} |\gamma_\alpha|^2 + O\left(\frac{1}{k}\right)\right)}{\int_0^1 \frac{dt}{\sqrt{1-t^2}}\left(\sqrt{\frac{\Delta_j(0)}{\Delta_j(t)}} + \sqrt{\frac{\Delta_j(0)}{\Delta_j(-t)}}\right)\left(1 + O\left(\frac{|\gamma_k|^2}{k^2}\right)\right)} \\
&= \left\{-\frac{1}{8}\frac{\gamma_k^2}{\tau_k - \tau_j} - \frac{1}{16}\frac{\gamma_k^2}{\tau_k - \lambda_0} + \frac{|\gamma_k|^2}{k^3}\left(O(|\gamma_k|) + O(|\gamma_j|^2)\right)\right. \\
&\quad \left. + O\left(\frac{|\gamma_k|^2}{k^3}\right)\left(\sum_{|\alpha-k|\leq\frac{k}{2}} |\gamma_\alpha|^2 + O\left(\frac{1}{k}\right)\right)\right\}\left\{1 + O\left(\frac{|\gamma_k|^2}{k^2}\right)\right\} \\
&= -\frac{1}{8}\frac{\gamma_k^2}{\tau_k - \tau_j} - \frac{1}{16}\frac{\gamma_k^2}{\tau_k - \lambda_0} + O\left(\frac{|\gamma_k|^2}{k^3}\right)\left\{\gamma_k| + |\gamma_j|^2 + \sum_{|\alpha-k|\leq\frac{k}{2}} |\gamma_\alpha|^2\right\} \\
&\quad + \frac{\gamma_k^4}{\tau_k - \tau_j} O\left(\frac{1}{k^2}\right) + \gamma_k^2 O\left(\frac{1}{k^4}\right).
\end{aligned}$$

□

3. Frequencies of KdV.

The aim of this section is to prove Theorem 1, as stated in the introduction and to apply it to the initial value problem of KdV. In the case where $q_0 \in H_0^1(S^1)$, the frequencies $\omega_j = \frac{\partial \mathcal{H}}{\partial J_j}$ are well defined and can be obtained, using Riemann bilinear relations, from the expansion at infinity of the L_2- integrable holomorphic differentials $\Omega_j = \frac{\varphi_j(\lambda)d\lambda}{\sqrt{\Delta(\lambda)^2-4}}$ constructed in Theorem 2.1. More explicitly, ω_j is given by (cf. e.g. [**EM**], [**BKM**])

$$\omega_j = 8\pi \left(\int_{\Gamma_j} \frac{\varphi_j(\mu)d\mu}{\sqrt{\Delta(\mu)^2-4}} \right)^{-1} \left(\tau_j - \sum_{k\in\mathbb{N}\setminus\{j\}} (\mu_k^{(j)} - \tau_k) + \frac{\lambda_0}{2} \right) \tag{3.1}$$

where Γ_j is a counterclockwise oriented circle with center τ_j and sufficiently small radius $\rho_j > |\gamma_j|/2$ so that all eigenvalues $(\lambda_k)_{k\in\mathbb{N}\setminus\{2j,2j-1\}}$ lie outside Γ_j.

Due to Theorem 2.1, the right hand side of formula (3.1) is well defined even if $q_0 \in L_0^2(S^1)$ and can be extended to an analytic function in a sufficiently small complex neighborhood U_{q_0} of q_0 in $L_0^2(S^1;\mathbb{C})$. For q in U_{q_0} we therefore *define* the frequency ω_j by formula (3.1). In view of Theorem 2.1 we obtain the following

Theorem 3.1. *Let $q_0 \in L_0^2(S^1)$. Then there exists a neighborhood U_{q_0} of q_0 in $L_0^2(S^1;\mathbb{C})$ so that, for any $j \geq 1$, the frequency ω_j, given by (3.1), is well defined and analytic.*

It remains to prove the asymptotics for ω_j as $j \to \infty$ as stated in Theorem 1. We start by analyzing $\int_{\Gamma_j} \frac{\varphi_j(\mu)d\mu}{\sqrt{\Delta(u)^2-4}}$. Note that in the case where $\gamma_j = 0$ we have by the residue theorem, taking into account the determination of the root,

$$\int_{\Gamma_j} \frac{\varphi_j(\mu)d\mu}{\sqrt{\Delta(\mu)^2-4}} = 2\pi \frac{\varphi_j(\tau_j)}{\sqrt{\frac{\Delta(\mu)^2-4}{(\tau_j-\mu)(\mu-\tau_j)}}}\Bigg|_{\mu=\tau_j}. \tag{3.2}$$

In the case where $\gamma_j \neq 0$, introduce $\mu(t) = \tau_j + \frac{\gamma_j}{2} t \, (-1 \leq t \leq 1)$ and

$$\psi_j(t) := \frac{\varphi_j\big(\mu(t)\big)}{\sqrt{\frac{\Delta\left(\mu(t)\right)^2-4}{\left(\lambda_{2j}-\mu(t)\right)\left(\mu(t)-\lambda_{2j-1}\right)}}}. \tag{3.3}$$

Then, with $\big(\lambda_{2j} - \mu(t)\big)\big(\mu(t) - \lambda_{2j-1}\big) = \frac{\gamma_j}{2}(1-t)\frac{\gamma_j}{2}(t+1) = (\frac{\gamma_j}{2})^2(1-t^2)$

$$\begin{aligned} \int_{\Gamma_j} \frac{\varphi_j(\mu)d\mu}{\sqrt{\Delta(\mu)^2-4}} &= 2\int_{-1}^{1} \psi_j(t) \frac{\frac{\gamma_j}{2}dt}{\frac{\gamma_j}{2}\sqrt{1-t^2}} \\ &= 2\int_0^1 \big(\psi_j(t) + \psi_j(-t)\big)\frac{dt}{\sqrt{1-t^2}}. \end{aligned} \tag{3.4}$$

Summarizing, we obtain

$$\int_{\Gamma_j} \frac{\varphi_j(\mu)d\mu}{\sqrt{\Delta(\mu)^2-4}} = \begin{cases} 2\pi\psi_j(0) & \text{for } \gamma_j = 0 \\ 2\int_0^1 \left(\psi_j(t)+\psi_j(-t)\right)\frac{dt}{\sqrt{1-t^2}} & \text{for } \gamma_j \neq 0. \end{cases} \tag{3.5}$$

The function $\psi_j(t)$ has a product representation

$$\left(\prod_{k\neq j} \frac{\mu_k^{(j)}-\mu(t)}{k^2\pi^2}\right)\frac{1}{2}(\mu(t)-\lambda_0)^{-1/2}\prod_{k\neq j}\left(\frac{\left(\lambda_{2k}-\mu(t)\right)\left(\lambda_{2k-1}-\mu(t)\right)}{(k^2\pi^2)^2}\right)^{-1/2}. \tag{3.6}$$

Notice that

$$\begin{aligned}\left(\lambda_{2k}-\mu(t)\right)\left(\lambda_{2k-1}-\mu(t)\right) &= \left(\tau_k-\mu(t)\right)^2-\left(\frac{\gamma_k}{2}\right)^2 \\ &= \left(\tau_k-\mu(t)\right)^2\left(1-\frac{(\gamma_k/2)^2}{\left(\tau_k-\mu(t)\right)^2}\right)\end{aligned}$$

and, with $\xi_k^{(j)} = \mu_k^{(j)}-\tau_k$,

$$\mu_k^{(j)}-\mu(t) = \left(\tau_k-\mu(t)\right)\left(1+\frac{\xi_k^{(j)}}{\tau_k-\mu(t)}\right).$$

Therefore,

$$2\psi_j(t) = \left(\mu(t)-\lambda_0\right)^{-1/2}\prod_{k\neq j}\left(1+\frac{\xi_k^{(j)}}{\tau_k-\mu(t)}\right)\prod_{k\neq j}\left(1-\frac{(\gamma_k/2)^2}{\left(\tau_k-\mu(t)\right)^2}\right)^{-1/2}. \tag{3.7}$$

Similarly, as in Section 2, it is convenient to introduce

$$\begin{aligned}\Delta_0(t) &:= \mu(t)-\lambda_0; \\ A_j(t) &:= \prod_{k\neq j}\left(1+\frac{\xi_k^{(j)}}{\tau_k-\mu(t)}\right) \\ B_j(t) &:= \prod_{k\neq j}\left(1-\frac{(\gamma_k/2)^2}{\left(\tau_k-\mu(t)\right)^2}\right)^{-1/2}.\end{aligned}$$

This allows us to write $\psi_j(t)$ as follows

$$2\psi_j(t) = \Delta_0(t)^{-1/2}A_j(t)B_j(t). \tag{3.8}$$

Further, for some $\theta_\pm(t)$ with $-t < \theta_-(t) < \theta_+(t) < t$,

$$2(\psi(t) + \psi_j(-t)) = 4\psi_j(0)\left(1 + \frac{\psi_j''(\theta_+(t)) - \psi_j''(\theta_-(t))}{2\psi_j(0)}\frac{t^2}{2}\right). \tag{3.9}$$

Lemma 3.2. *There exists $C > 0$ such that, for q in U_{q_0} and $j \geq 1$,*

$$\Bigg|(2\psi_j(0))^{-1} - (\tau_j - \lambda_0)^{1/2}\Bigg\{1 - \sum_{k\neq j}\frac{\xi_k^{(j)}}{\tau_k - \tau_j} - \frac{1}{2}\sum_{k\neq j}\frac{(\gamma_k/2)^2}{(\tau_k-\tau_j)^2} + \frac{1}{2}\left(\sum_{k\neq j}\frac{\xi_k^{(j)}}{\tau_k-\tau_j}\right)^2 + \frac{1}{2}\sum_{k\neq j}\left(\frac{\xi_k^{(j)}}{\tau_k-\tau_j}\right)^2\Bigg\}\Bigg| \leq \frac{C}{j^3}\left(\sum_{k\geq j/2}|\gamma_k|^2\right)^2 + \frac{C}{j^5}. \tag{3.10}$$

Proof. Notice that $\mu(0) = \tau_j$ and therefore

$$(2\psi_j(0))^{-1} = (\tau_j - \lambda_0)^{1/2}\prod_{k\neq j}\left(1 + \frac{\xi_k^{(j)}}{(\tau_k - \tau_j)}\right)^{-1}\prod_{k\neq j}\left(1 - \frac{(\gamma_k/2)^2}{(\tau_k-\tau_j)^2}\right)^{1/2}. \tag{3.11}$$

Further

$$\prod_{k\neq j}\left(1 + \frac{\xi_k^{(j)}}{(\tau_k-\tau_j)}\right)^{-1} = 1 - \sum_{k\neq j}\frac{\xi_k^{(j)}}{\tau_k-\tau_j} + \sum_{\substack{k_1,k_2\neq j\\ k_1<k_2}}\frac{\xi_{k_1}^{(j)}}{\tau_{k_1}-\tau_j}\frac{\xi_{k_2}^{(j)}}{\tau_{k_2}-\tau_j} + \sum_{k\neq j}\frac{(\xi_k^{(j)})^2}{(\tau_k-\tau_j)^2} + O\left(\frac{1}{j^6}\right) \tag{3.12}$$

where for the estimate of the error term we used that, for $|k-j| \geq j/2$

$$\frac{1}{|\tau_k - \tau_j|} = O\left(\frac{1}{j^2}\right)$$

and for $|k-j| \leq \frac{j}{2}$

$$|\xi_k^{(j)}| = \frac{|\gamma_k|^2}{k}O(1) = O\left(\frac{|\gamma_k|^2}{j}\right) \quad \text{and} \quad \frac{1}{|\tau_k-\tau_j|} = O\left(\frac{1}{j}\right)$$

to conclude that

$$\left|\frac{\xi_k^{(j)}}{\tau_k-\tau_j}\right| \le |\gamma_k|^2 O\left(\frac{1}{j^2}\right). \tag{3.13}$$

Similarly

$$\begin{aligned}\prod_{k\neq j}\left(1-\frac{(\gamma_k/2)^2}{(\tau_k-\tau_j)^2}\right)^{1/2} &= 1-\frac{1}{2}\sum_{k\neq j}\frac{(\gamma_k/2)^2}{(\tau_k-\tau_j)^2}-\frac{1}{8}\sum_{k\neq j}\frac{(\gamma_k/2)^4}{(\tau_k-\tau_j)^4}\\ &\quad+\frac{1}{4}\sum_{\substack{k_1,k_2\neq j\\ k_1<k_2}}\frac{(\gamma_{k_1}/2)^2(\gamma_{k_2}/2)^2}{(\tau_{k_1}-\tau_j)^2(\tau_{k_2}-\tau_j)^2}+O\left(\frac{1}{j^6}\right)\\ &= 1-\frac{1}{2}\sum_{k\neq j}\frac{(\gamma_k/2)^2}{(\tau_k-\tau_j)^2}+O\left(\frac{1}{j^4}\right)\sum_{k\geq j/2}\gamma_k^4\\ &\quad+O\left(\frac{1}{j^4}\right)\sum_{\substack{k_1<k_2\\ k_2,k_1\geq j/2}}\gamma_{k_1}^2\gamma_{k_2}^2+O\left(\frac{1}{j^6}\right).\end{aligned} \tag{3.14}$$

Combining (3.11)-(3.14) we obtain

$$\begin{aligned}(2\psi_j(0))^{-1} &= (\tau_j-\lambda_0)^{1/2}\cdot\left\{1-\sum_{k\neq j}\frac{\xi_k^{(j)}}{\tau_k-\tau_j}-\frac{1}{2}\sum_{k\neq j}\frac{(\gamma_k/2)^2}{(\tau_k-\tau_j)^2}\right.\\ &\quad\left.+\frac{1}{2}\left(\sum_{k\neq j}\frac{\xi_k^{(j)}}{\tau_k-\tau_j}\right)^2+\frac{1}{2}\sum_{k\neq j}\left(\frac{\xi_k^{(j)}}{\tau_k-\tau_j}\right)^2+R_j\right\}\end{aligned} \tag{3.15}$$

where, with $\left(\sum_{k\neq j}\frac{\xi_k^{(j)}}{\tau_k-\tau_j}\right)\left(\sum_{k\neq j}\frac{(\gamma_k/2)^2}{(\tau_k-\tau_j)^2}\right) = O(\frac{1}{j^4})\left(\sum_{k\geq j/2}|\gamma_k^2|\right)^2+O(\frac{1}{j^6})$, we have

$$R_j := O\left(\frac{1}{j^4}\right)\sum_{k\geq j/2}\gamma_k^4+O\left(\frac{1}{j^4}\right)\sum_{\substack{k_1\geq j/2\\ k_2\geq j/2}}\gamma_{k_1}^2\gamma_{k_2}^2+O\left(\frac{1}{j^6}\right). \tag{3.16}$$

Finally use that $\frac{1}{2}(\tau_j-\lambda_0)^{1/2}=\frac{j\pi}{2}+O(\frac{1}{j^2})$ to obtain (3.10). □

Lemma 3.3.

$$\sup_{0\leq|t|\leq 1}\frac{|\frac{d^2}{dt^2}\psi_j(t)|}{\psi_j(0)}\leq C\frac{|\gamma_j|^2}{j^4}. \tag{3.17}$$

Proof. In a straightforward fashion one computes

$$\frac{d}{dt}\psi_j(t) = \frac{\gamma_j}{2}\psi_j(t)\left\{\frac{-1/2}{\tau_j - \lambda_0 + \frac{\gamma_j}{2}t} + \sum_{k\neq j}\frac{1}{1+\frac{\xi_k^{(j)}}{\tau_k-\mu(t)}}\frac{(-1)\xi_k^{(j)}}{\left(\tau_k-\mu(t)\right)^2}(-1)\right.$$

$$\left. + \sum_{k\neq j}\frac{(-1/2)}{1-\frac{(\gamma_k/2)^2}{\left(\tau_k-\mu(t)\right)^2}}\frac{(-1)(-2)(\gamma_k/2)^2}{\left(\tau_k-\mu(t)\right)^3}(-1)\right\}$$

$$= \frac{\gamma_j}{2}\psi_j(t)\left\{\frac{-1/2}{\tau_j - \lambda_0 + \frac{\gamma_j}{2}t} + \sum_{k\neq j}\frac{1}{1+\frac{\xi_k^{(j)}}{\left(\tau_k-\mu(t)\right)}}\frac{\xi_k^{(j)}}{\left(\tau_k-\mu(t)\right)^2}\right.$$

$$\left. + \sum_{k\neq j}\frac{1}{1-\frac{(\gamma_k/2)^2}{\left(\tau_k-\mu(t)\right)^2}}\frac{(\gamma_k/2)^2}{\left(\tau_k-\mu(t)\right)^3}\right\}.$$

(3.18)

$$\frac{d^2}{dt^2}\psi_j(t) = \frac{1}{\psi_j(t)}\left(\frac{d}{dt}\psi_j(t)\right)^2 + \left(\frac{\gamma_j}{2}\right)^2\psi_j(t)$$

$$\cdot\left\{\frac{1/2}{(\tau_j-\lambda_0+\frac{\gamma_j}{2}t)^2} + \sum_{k\neq j}\frac{-1}{\left(1+\frac{\xi_k^{(j)}}{\tau_k-\mu(t)}\right)^2}\left(\frac{\xi_k^{(j)}}{\left(\tau_k-\mu(t)\right)^2}\right)^2\right.$$

$$+ \sum_{k\neq j}\frac{1}{1+\frac{\xi_k^{(j)}}{\left(\tau_k-\mu(t)\right)^2}}\frac{2\xi_k^{(j)}}{\left(\tau_k-\mu(t)\right)^3}$$

$$+ \sum_{k\neq j}\frac{(-1)}{\left(1-\frac{(\gamma_k/2)^2}{\left(\tau_k-\mu(t)\right)^2}\right)^2}\left(\frac{(\gamma_k/2)^2}{\left(\tau_k-\mu(t)\right)^3}\right)^2(-1)$$

$$\left. + \sum_{k\neq j}\frac{1}{1-\frac{(\gamma_k/2)^2}{\left(\tau_k-\mu(t)\right)^2}}\frac{(-3)(\gamma_k/2)^2}{\left(\tau_k-\mu(t)\right)^4}(-1)\right\}.$$

We begin by estimating $|\frac{\frac{d}{dt}\psi_j(t)}{\psi_j(t)}|$,

$$\left|\frac{\frac{d}{dt}\psi_j(t)}{\psi_j(t)}\right| \leq C|\gamma_j|\left\{\frac{1}{j^2} + \sum_{k\neq j}\left|\frac{\xi_k^{(j)}}{\tau_k-\mu(t)}\right|\frac{1}{|\tau_k-\mu(t)|} + \sum_{k\neq j}\frac{|\gamma_k|^2}{|\tau_k-\mu(t)|^3}\right\}$$

$$\leq C\frac{|\gamma_j|}{j^2}.$$

Therefore,

$$\sup_{|t|\leq 1}\left|\frac{\left(\frac{d}{dt}\psi_j(t)\right)^2}{\psi_j(0)\psi_j(t)}\right| \leq C\frac{|\gamma_j|^2}{j^4}. \tag{3.19}$$

All the other terms are treated similarly. □

Theorem 3.4. *Let $q_0 \in L^2_0(S^1)$. Then there exists a neighborhood U_{q_0} of q_0 in $L^2_0(S^1;\mathbb{C})$ and $C > 0$ so that for any q in U_{q_0} and $j \geq 1$,*

$$\left|\left(\int_{\Gamma_j}\frac{\varphi_j(\mu)d\mu}{\sqrt{\Delta(\mu)^2-4}}\right)^{-1} - \left(2\pi\psi_j(0)\right)^{-1}\right| \leq C\frac{|\gamma_j|^2}{j^3} \tag{3.20}$$

and

$$\begin{aligned}&\left|\left(2\psi_j(0)\right)^{-1} - \left\{\tau_j^{1/2} - \frac{1}{2}\frac{\lambda_0}{\tau_j^{1/2}} - \tau_j^{1/2}\sum_{k\leq j/2}\frac{\xi_k^{(j)}}{\tau_k-\tau_j}\right\}\right| \\ &\leq \frac{C}{j^2}\left(\sum_{k\geq j/2}|\gamma_k|^2 + \frac{1}{j}\right).\end{aligned} \tag{3.21}$$

Proof. Notice that, according to (3.5), (3.9) and as $\int_0^1 \frac{dt}{\sqrt{1-t^2}} = \frac{\pi}{2}$

$$\begin{aligned}&\left(\int_{\Gamma_j}\frac{\varphi_j(\mu)d\mu}{\sqrt{\Delta(\mu)^2-4}}\right)^{-1} \\ &= \left(2\pi\psi_j(0)\right)^{-1}\left(1 + \int_0^1\frac{\psi_j''(\theta_+(t)) - \psi_j''(\theta_-(t))}{4\pi\psi_j(0)}\frac{t^2dt}{\sqrt{1-t^2}}\right)^{-1}.\end{aligned}$$

Thus, by Lemma 3.3

$$\begin{aligned}\left|\left(\int_{\Gamma_j}\frac{\varphi_j(\mu)d\mu}{\sqrt{\Delta(\mu)^2-4}}\right)^{-1} - \left(2\pi\psi_j(0)\right)^{-1}\right| &\leq \frac{C}{|\psi_j(0)|}\sup_{|t|\leq 1}\left|\frac{\psi_j''(t)}{\psi_j(0)}\right| \\ &\leq Cj\frac{|\gamma_j|^2}{j^4} \leq C\frac{|\gamma_j|^2}{j^3}.\end{aligned}$$

This proves (3.20). To prove the estimates (3.21) use (3.10) and notice that

$$\left|(\tau_j-\lambda_0)^{1/2}\sum_{k\leq j/2}\frac{\frac{1}{2}(\gamma_k/2)^2}{(\tau_k-\tau_j)^2}\right| \leq Cj\left(\sum_{k\leq j/2}|\gamma_k|^2\right)\frac{1}{j^4} = O\left(\frac{1}{j^3}\right) \tag{3.22}$$

and, in view of (3.13),

$$\left|(\tau_j-\lambda_0)^{1/2}\left(\sum_{k\neq j}\left|\frac{\xi_k^{(j)}}{\tau_k-\tau_j}\right|\right)^2\right| \leq Cj\left(\sum_k \frac{|\gamma_k|^2}{j^2}\right)^2 = O\left(\frac{1}{j^3}\right). \tag{3.23}$$

According to Theorem 2.7

$$\begin{aligned}\left|(\tau_j-\lambda_0)^{1/2}\sum_{k\geq j/2}\left(\frac{\xi_k^{(j)}}{\tau_k-\tau_j}+\frac{\frac{1}{2}(\gamma_k/2)^2}{(\tau_k-\tau_j)^2}\right)\right| \\ \leq Cj\sum_{k\geq j/2}\frac{|\gamma_k|^2}{|\tau_k-\tau_j|\,|\tau_k-\lambda_0|}+Cj\sum_{k\geq j/2}\frac{|\gamma_k|^2}{j^3|\tau_k-\tau_j|} \\ =\frac{C}{j^2}\sum_{k\geq j/2}|\gamma_k|^2+O\left(\frac{1}{j^3}\right).\end{aligned} \tag{3.24}$$

Finally

$$\begin{aligned}(\tau_j-\lambda_0)^{1/2} &= \tau_j^{1/2}\left(1-\frac{\lambda_0}{\tau_j}\right)^{1/2} = \tau_j^{1/2}\left(1-\frac{1}{2}\frac{\lambda_0}{\tau_j}+O\left(\frac{1}{j^4}\right)\right) \\ &= \tau_j^{1/2}-\frac{1}{2}\frac{\lambda_0}{\tau_j^{1/2}}+O\left(\frac{1}{j^3}\right).\end{aligned} \tag{3.25}$$

Combining these estimates we obtain

$$\begin{aligned}(2\psi_j(0))^{-1} &= \left(\tau_j^{1/2}-\frac{1}{2}\frac{\lambda_0}{\tau_j^{1/2}}+0\left(\frac{1}{j^3}\right)\right)\cdot\left(1-\sum_{k\leq j/2}\frac{\xi_k^{(j)}}{\tau_k-\tau_j}\right) \\ &\quad +O\left(\frac{1}{j^2}\right)\left(\sum_{k\geq j/2}|\gamma_k|^2+\frac{1}{j}\right) \\ &= \tau_j^{1/2}-\frac{1}{2}\frac{\lambda_0}{\tau_j^{1/2}}-\tau_j^{1/2}\sum_{k\leq j/2}\frac{\xi_k^{(j)}}{\tau_k-\tau_j}+O\left(\frac{1}{j^3}\right) \\ &\quad +O\left(\frac{1}{j^2}\right)\left(\sum_{k\geq j/2}|\gamma_k|^2+\frac{1}{j}\right)\end{aligned}$$

which proves (3.21). □

Combining Theorem 3.4 and Theorem 2.7 we obtain the following result which completes the proof of Theorem 1:

Theorem 3.5. *Let $q_0 \in L^2_0(S^1)$. Then there exists a neighborhood U_{q_0} of q_0 in $L^2_0(S^1;\mathbb{C})$ and $C>0$ so that for any q in U_{q_0} and $j \geq 1$,*

$$|\omega_j - (2\pi j)^3| \leq C\left(\sum_{k\geq j/2} |\gamma_k|^2 + (\sqrt{\tau_j} - j\pi)j^2\pi^2 + O\left(\frac{1}{j}\right)\right). \tag{3.26}$$

Proof. According to (3.1) and using (3.20)

$$\begin{aligned}
\omega_j &= 8\pi\left(\int_{\Gamma_j} \frac{\varphi_j(\mu)d\mu}{\sqrt{\Delta(\mu)^2-4}}\right)^{-1}\left(\tau_j - \sum_{k\in\mathbb{N}\setminus\{j\}}(\mu_k^{(j)} - \tau_k) + \frac{\lambda_0}{2}\right)\\
&= 8\pi((2\pi\psi_j(0))^{-1})\left\{\tau_j - \sum_{k\leq j/2}(\mu_k^{(j)} - \tau_k) + \frac{\lambda_0}{2}\right\} + O\left(\sum_{k\geq j}|\gamma_k|^2\right)\\
&\quad + O\left(\frac{1}{j}\right).
\end{aligned}$$

By (3.21),

$$\begin{aligned}
\omega_j &= \frac{8\pi}{\pi}\left(\tau_j^{1/2} - \frac{1}{2}\frac{\lambda_0}{\tau_j^{1/2}} - \tau_j^{1/2}\sum_{k\leq j/2}\frac{\xi_k^{(j)}}{\tau_k - \tau_j}\right)\left(\tau_j - \sum_{k\leq j/2}\xi_k^{(j)} + \frac{\lambda_0}{2}\right)\\
&\quad + O(1)\left(\sum_{k\geq j}|\gamma_k|^2 + \frac{1}{j}\right)\\
&= 8\left\{\tau_j^{3/2} - \frac{1}{2}\lambda_0\tau_j^{1/2} - \tau_j^{3/2}\sum_{k\leq j/2}\frac{\xi_k^{(j)}}{\tau_k-\tau_j} - \tau_j^{1/2}\sum_{k\leq j/2}\xi_k^{(j)} + \frac{1}{2}\lambda_0\tau_j^{1/2}\right\}\\
&\quad + O(1)\left(\sum_{k\geq j}|\gamma_k|^2 + \frac{1}{j}\right).
\end{aligned} \tag{3.27}$$

Notice that

$$\begin{aligned}
&\left|-\tau_j^{3/2}\sum_{k\leq j/2}\frac{\xi_k^{(j)}}{\tau_k-\tau_j} - \tau_j^{1/2}\sum_{k\leq j/2}\xi_k^{(j)}\right|\\
&= \left|\tau_j^{1/2}\sum_{k\leq j/2}\xi_k^{(j)}\left(\frac{\tau_j}{\tau_j-\tau_k} - 1\right)\right|\\
&\leq C|j|\sum_{k\leq j/2}|\xi_k^{(j)}|\left|\frac{\tau_k}{\tau_j-\tau_k}\right|
\end{aligned} \tag{3.28}$$

$$\leq C\frac{1}{j}\sum_{k\leq j/2} k^2|\xi_k^{(j)}| = O\left(\frac{1}{j}\right)$$

where we used that, by Theorem 2.7, $\sum_{k\leq j/2} k^2|\xi_k^{(j)}| \leq C$.

Finally, write $\sqrt{\tau_j} = j\pi + \frac{\overset{\wedge}{\tau}_j}{(j\pi)^2}$, to obtain (with $\overset{\wedge}{\tau}_j \in l^2(j)$)

$$\begin{aligned}\tau_j^{3/2} &= \left((j\pi)^2 + 2\frac{\overset{\wedge}{\tau}_j}{(j\pi)} + O\left(\frac{1}{j^4}\right)\right)\left(j\pi + \frac{\overset{\wedge}{\tau}_j}{(j\pi)^2}\right)\\ &= (j\pi)^3 + 3\overset{\wedge}{\tau}_j + O\left(\frac{1}{j^3}\right).\end{aligned} \tag{3.29}$$

Combining (3.27)-(3.29) leads to the claimed estimate. □

The above results can be applied to the initial value problem for KdV,

$$\partial_t v : = -\partial_x^3 v + 3\partial_x(v^2) \qquad (t > 0; x \in S^1) \tag{3.30}$$

$$v(x,0) = q(x). \tag{3.31}$$

Recently, using a fixed point argument, Bourgain [**Bo1**] has constructed weak solutions $v_B(x,t) = \mathcal{S}_B(q)(x,t)$ of (3.30)-(3.31) for initial data in $L^2(S^1)$ globally in time and proved that these solutions are unique within a certain space of functions $f : \mathbb{R} \times [-T,T] \to \mathbb{R}$ (and $T > 0$) and that they can be approximated within this space by smooth solutions corresponding to a smooth approximation of the initial data.

We obtain Bourgain's weak solutions $\mathcal{S}_B(q)$ of (3.30)-(3.31) by using action-angle coordinates as follows: Given q in $L^2(S^1)$, define $p := q - [q] \in L_0^2(S^1)$ where $[q] = \int_{S^1} q(x)dx$ and introduce ($j \geq 1$), with $\omega_j(p)$ given by Theorem 1,

$$\omega_j(q) = \omega_j(p) + 12[q]\pi j. \tag{3.32}$$

In view of Theorem 3.1, the ω_j's are real analytic functions on $L^2(S^1)$. Denote by $\Omega : L_0^2(S^1) \to \ell^2_{1/2}(\mathbb{R}^2)$ the symplectomorphism $\Omega(p) = (x(p), y(p)) = (x_j(p), y_j(p))_{j\geq 1}$ constructed in [**BBGK**] and denote by J_j, α_j the symplectic polar coordinates corresponding to x_j, y_j, i.e.

$$(x_j, y_j) = \left(\sqrt{2J_j}\cos\alpha_j, \sqrt{2J_j}\sin\alpha_j\right). \tag{3.33}$$

Define $(x(t), y(t)) \equiv (x(t,q), y(t,q))$ by setting

$$(x(t), y(t)) = \left(\sqrt{2J_j}\cos\left(\omega_j(q)t + \alpha_j\right), \sqrt{2J_j}\sin\left(\omega_j(q)t + \alpha_j\right)\right)_{j\geq 1} \tag{3.34}$$

and introduce the solution operator $\mathcal{S}^{(1)}$

$$\mathcal{S}^{(1)} : L^2(S^1) \to C(\mathbb{R}; L^2(S^1))$$

defined by

$$\mathcal{S}^{(1)}(q)(t) = [q] + \Omega^{-1}(x(t), y(t)). \tag{3.35}$$

To prove that (3.35) defines a weak solution of KdV, we approximate the initial data q by finite gap potentials $(q_N)_{N\geq 1}$ with

$$q_N := \Omega^{-1}\Big(\Pi_N(\Omega(p))\Big) + [q] \tag{3.36}$$

where Π_N denotes the projection

$$\Pi_N : \ell^2_{1/2}(\mathbb{R}^2) \to \ell^2_{1/2}(\mathbb{R}^2), (x_j, y_j)_{j\geq 1} \mapsto \big((x_j, y_j)_{1\leq j\leq N}, 0\big). \tag{3.37}$$

Note that $q = \lim_{N\to\infty} q_N$ in $L^2(S^1)$ and that for any $j \geq 1, \omega_j(q) = \lim_{N\to\infty} \omega_j(q_N)$ uniformly in j. Therefore

$$(x(\cdot, q), y(\cdot, q)) = \lim_{N\to\infty} (x(\cdot, q_N), y(\cdot, q_N))$$

in $C\big([-T, T], \ell^2_{1/2}(\mathbb{R}^2)\big)$ (for any $T > 0$), which in turn implies that

$$\mathcal{S}^{(1)}(q) = \lim_{N\to\infty} \mathcal{S}^{(1)}(q_N)(\text{in } \ C([-T, T]; L^2(S^1))). \tag{3.38}$$

Notice that for arbitrary $N \geq 1$, $\mathcal{S}^{(1)}(q_N)(t)$ is an N-gap potential for any t and therefore a classical solution of KdV with initial data q_N. For $c \in \mathbb{R}$, we have introduced the symplectic leaf $L^2_c(S^1) = \{q \in L^2(S^1); \int_{S^1} q(x)dx = c\}$.

Theorem 3.6. *Let $c \in \mathbb{R}$. Then*

(i) *$\mathcal{S}^{(1)}(q) = \mathcal{S}_B(q)$ for q in $L^2_c(S^1)$.*

(ii) *Given q_1, q_2 in $L^2_c(S^1)$ there exists $M > 0$ so that for any $t \in \mathbb{R}$,*

$$\left\| \left(\mathcal{S}^{(1)} q_1\right)(\cdot, t) - \left(\mathcal{S}^{(1)} q_2\right)(\cdot, t)\right\|_{L^2(S^1)} \leq M(1 + |t|)\|q_1 - q_2\|_{L^2(S^1)}. \tag{3.39}$$

(iii) *For any $T > 0$, $\mathcal{S}^{(1)} : L^2_c(S^1) \to C\big([-T, T]; L^2_c(S^1)\big)$ is real analytic.*

For remarks concerning results related to Theorem 3.6 we refer the reader to the introduction.

Proof of Theorem 3.6. Let q be in $L^2(S^1)$ and q_N $(N \geq 1)$ be defined as in (3.36). As classical solutions of KdV are unique we conclude that $\mathcal{S}^{(1)}(q_N) = \mathcal{S}_B(q_N)$ for any $N \geq 1$. Moreover Bourgain proved that $\lim_{N\to\infty} \mathcal{S}_B(q_N) =$

$\mathcal{S}_B(q)$ in $C\big([-T,T];L^2(S^1)\big)$. Together with (3.38) we conclude that $\mathcal{S}^{(1)}(q) = \mathcal{S}_B(q)$.

Statements (ii) and (iii) rely both on Theorem 3.5 which, combined with (3.32) leads to the following result: Let $c \in \mathbb{R}$ and $q_0 \in L^2_c(S^1)$. Then there exists a neighborhood U_{q_0} of q_0 in $L^2_c(S^1;\mathbb{C})$ so that

$$\omega_j(q) \ \text{ can be analytically extended to } \ U_{q_0} \tag{3.40}$$

and

$$\omega_j(q) = (2\pi j)^3 + 12c\pi j + o(1) \quad (\text{as } \ j \to \infty) \tag{3.41}$$

uniformly for q in U_{q_0}.

Fix $0 < T < \infty$ arbitrarily. By choosing U_{q_0} smaller, if necessary, we can assume that Ω can be analytically extended to $\{\mathcal{S}^{(1)}(q)(t) - c; q \in U; |t| \leq T\}$ and that its inverse is analytic as well. Further note that $\ell^2_{1/2}\big(C([-T,T];\mathbb{C}^2)\big)$ can be embedded into $C\big([-T,T];\ell^2_{1/2}(\mathbb{C}^2)\big)$, the complexification of the Banach space $C\big([-T,T];\ell^2_{1/2}(\mathbb{R}^2)\big)$. It therefore suffices to prove that $\overset{\approx}{\mathcal{S}}{}^{(1)} : U_1 \to \ell^2_{1/2}\big(C([-T,T];\mathbb{C}^2)\big)$ is analytic where $U_1 = \Omega(U_{q_0} - c)$ and where, with $\overset{\approx}{\omega}_j = \omega_j(\Omega^{-1}(x,y)) + 12c\pi j$ (cf. (3.32)),

$$\begin{aligned}
&\overset{\approx}{\mathcal{S}}{}^{(1)}\left(\left(\sqrt{2J_j}\cos\alpha_j,\ \sqrt{2J_j}\sin\alpha_j\right)_{j\geq 1}\right) \\
&\quad = \left(\sqrt{2J_j}\cos\left(\overset{\approx}{\omega}_j t + \alpha_j\right),\ \sqrt{2J_j}\sin\left(\overset{\approx}{\omega}_j t + \alpha_j\right)\right)_{j\geq 1}.
\end{aligned}$$

Due to the assumption that c is real and due to (3.41) we conclude that $\overset{\approx}{\mathcal{S}}{}^{(1)}$ is locally bounded on U_1. Moreover each component $\big(\sqrt{2J_j}\cos(\overset{\approx}{\omega}_j t + \alpha_j); \sqrt{2J_j}\sin(\overset{\approx}{\omega}_j t + \alpha_j)\big)$ is analytic on U_1. By a version of [**PT**], Appendix A, Theorem 3, we conclude that $\overset{\approx}{\mathcal{S}}{}^{(1)}$ is analytic on U_1 and (iii) is proved. To prove (ii) first note that, in particular, Ω is locally Lipschitz continuous. Taking into account that $\operatorname{Iso}(q_j) = \{\tilde{q} \in L^2(S^1); \operatorname{spec}(-\frac{d^2}{dx^2} + \tilde{q}) = \operatorname{spec}(-\frac{d^2}{dx^2} + q_j)\}$ are compact in $L^2(S^1)(j = 1,2)$ one concludes that there exists $M_1 > 0$, depending on q_1 and q_2, such that for any $\tilde{q}_j \in \operatorname{Iso}(q_j)$

$$\left\|\Omega\left(\tilde{q}_1 - c\right) - \Omega\left(\tilde{q}_2 - c\right)\right\|_{\ell^2_{1/2}(\mathbb{R}^2)} \leq M_1 \left\|\tilde{q}_1 - \tilde{q}_2\right\|_{L^2(S^1)} \tag{3.42}$$

where $c = [q_1] = [q_2]$. Using Ω^{-1} instead of Ω, one concludes that there exists $M_2 > 0$ such that

$$\left\|\tilde{q}_1 - \tilde{q}_2\right\|_{L^2(S^1)} \leq M_2 \left\|\Omega\left(\tilde{q}_1 - c\right) - \Omega\left(\tilde{q}_2 - c\right)\right\|_{\ell^2_{1/2}(\mathbb{R}^2)}. \tag{3.42$'$}$$

Therefore it suffices to show that, for any t,

$$\begin{aligned}&\left\|\Omega(\mathcal{S}^{(1)}(q_1)(\cdot,t)-c)-\Omega(\mathcal{S}^{(1)}(q_2)(\cdot,t)-c)\right\|_{\ell^2_{1/2}}\\ &\leq M\|q_1-q_2\|_{L^2}(1+|t|).\end{aligned}\tag{3.43}$$

To simplify notation, write for $k=1,2$

$$(x^{(k)}(t),y^{(k)}(t))=\Omega(\mathcal{S}^{(1)}(q_k)(\cdot,t)-c);\ \omega_j^{(k)}=\omega_j(q_k).$$

Recall that for $k=1,2$ and $j\geq 1$

$$\frac{d}{dt}\begin{pmatrix}x_j^{(k)}(t)\\ y_j^{(k)}(t)\end{pmatrix}=\omega_j^{(k)}\begin{pmatrix}y_j^{(k)}(t)\\ -x_j^{(k)}(t)\end{pmatrix}.$$

Therefore, with $\widetilde{\omega}_j^{(j)}:=\omega_j^{(k)}-(2\pi j)^3-12[c]\pi j$,

$$\begin{aligned}\frac{d}{dt}\begin{pmatrix}x_j^{(1)}(t)-x_j^{(2)}(t)\\ y_j^{(1)}(t)-y_j^{(2)}(t)\end{pmatrix}&=\widetilde{\omega}_j^{(1)}\begin{pmatrix}y_j^{(1)}(t)\\ -x_j^{(1)}(t)\end{pmatrix}-\widetilde{\omega}_j^{(2)}\begin{pmatrix}y_j^{(2)}(t)\\ -x_j^{(2)}(t)\end{pmatrix}\\ &\quad+((2\pi j)^3+12[c]\pi j)\begin{pmatrix}y_j^{(1)}(t) & -y_j^{(2)}(t)\\ -(x_j^{(1)}(t) & -x_j^{(2)}(t))\end{pmatrix}.\end{aligned}\tag{3.44}$$

Scalar multiply (3.44) by $\begin{pmatrix}x_j^{(1)}(t) & -x_j^{(2)}(t)\\ y_j^{(1)}(t) & -y_j^{(2)}(t)\end{pmatrix}$ to conclude that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\begin{vmatrix}x_j^{(1)}(t)-x_j^{(2)}(t)\\ y_j^{(1)}(t)-y_j^{(2)}(t)\end{vmatrix}^2\\ &\leq\left|\widetilde{\omega}_j^{(1)}\begin{pmatrix}y_j^{(1)}(t)\\ -x_j^{(1)}(t)\end{pmatrix}-\widetilde{\omega}_j^{(2)}\begin{pmatrix}y_j^{(2)}(t)\\ -x_j^{(2)}(t)\end{pmatrix}\right|\begin{vmatrix}x_j^{(1)}(t)-x_j^{(2)}(t)\\ y_j^{(1)}(t)-y_j^{(2)}(t)\end{vmatrix}.\end{aligned}\tag{3.45}$$

Notice that $\left(\widetilde{\omega}_j(y_j(t),-x_j(t))\right)_{j\geq 1}$ is analytic on U with values in $\ell^2_{1/2}(\mathbb{C}^2)$ due to the fact that $\widetilde{\omega}_j=0(1)$ uniformly on U (cf. Theorem 1). Therefore

$$\left\|\left(\widetilde{\omega}_j^{(1)}\begin{pmatrix}y_j^{(1)}(t)\\ -x_j^{(1)}(t)\end{pmatrix}-\widetilde{\omega}_j^{(2)}\begin{pmatrix}y_j^{(2)}(t)\\ -x_j^{(2)}(t)\end{pmatrix}\right)_{j\geq 1}\right\|_{\ell^2_{1/2}(\mathbb{C}^2)}\leq M\|q_1-q_2\|_{L^2}\tag{3.46}$$

where $M>0$ is uniform for $t\in\mathbb{R}$. (Use compactness of the isospectral sets.)

Combining (3.45) and (3.46) we conclude that

$$\frac{d}{dt}\|(x^{(1)}(t),y^{(1)}(t))-(x^{(2)}(t),y^{(2)}(t))\|_{\ell^2_{1/2}} \leq M\|q_1-q_2\|_{L^2}.$$

This proves estimate (3.43). □

Using again properties of the map Ω one obtains in a straight forward way a result due to McKean-Trubowitz [**MT1**], and in its more general version, due to Bourgain [**Bo1**], concerning the almost periodicity of weak solutions of KdV. Recall (cf. [**MT1**], [**Bo1**]) that a function $u \in C(\mathbb{R};L^2(S^1))$ is called *almost periodic* if for any $\epsilon>0$ there exists $0<T=T(\epsilon)<\infty$ so that, given any interval I of length at least T, there exists $\tau_0 \in I$ with the property

$$\|u(\cdot,t)-u(\cdot,t+\tau_0)\|_{L^2}<\epsilon \qquad \forall t\in\mathbb{R}.$$

Proposition 3.7. *Let $q \in L^2(S^1)$. The weak solution $\mathcal{S}^{(1)}(q)(\cdot,t)$ of KdV is almost periodic.*

Proof. For ease of exposition we assume that $[q]=0$ and write $v(\cdot,t)=\mathcal{S}^{(1)}(q)(\cdot,t)$. Note that $v(\cdot,t)\in \operatorname{Iso}(q)$ and $\operatorname{Iso}(q)$ is compact. Therefore $\Omega\big(\operatorname{Iso}(q)\big)$ is compact as well and Ω^{-1} is uniformly continuous on $\Omega\big(\operatorname{Iso}(q)\big)$. In particular, given $\epsilon>0$, there exists $\delta=\delta(\epsilon)>0$ such that for $z^{(1)}=(x^{(1)},y^{(1)})$, $z^{(2)}=(x^{(2)},y^{(2)})$ in $\Omega\big(\operatorname{Iso}(q)\big)$ with $\|z^{(1)}-z^{(2)}\|_{\ell^2_{1/2}}<\delta$ one has $\|\Omega^{-1}(z^{(1)})-\Omega^{-1}(z^{(2)})\|_{L^2_0}<\epsilon$. It therefore suffices to prove that, given any $\delta>0$, there exists $0<T_\delta<\infty$ so that given any interval I of length at least T_δ, there exists $\tau_0\in I$ with the property that, for any $t\in R$,

$$\|(x(t),y(t))-(x(t+\tau_0),y(t+\tau_0),)\|_{\ell^2_{1/2}}<\delta. \tag{3.47}$$

To prove (3.47) we first note that there exists $N\geq 1$ such that, for any $t\in\mathbb{R}$,

$$\sum_{j=N+1}^{\infty} j\big(x_j(t)^2+y_j(t)^2\big)<(\delta/4)^2$$

where $\big(x_j(t),y_j(t)\big)$ is given by (3.34). Note that $\big(x_j(t),y_j(t)\big)_{1\leq j\leq N}$ is a *quasi-periodic* function of t. Therefore, given $\delta>0$, there exists $0<T_\delta<\infty$ so that in any interval I of length at least T_δ one can find $\tau_0\in I$ with the property that, for any $t\in\mathbb{R}$,

$$\sum_{j=1}^{N} j\left\{\big(x_j(t)-x_j(t+\tau_0)\big)^2+\big(y_j(t)-y_j(t+\tau_0)\big)^2\right\}<\left(\frac{\delta}{2}\right)^2$$

and therefore (3.47) follows. □

4. Frequencies of KdV_2.

In this section we investigate the frequencies of the second Hamiltonian in the KdV hierarchy and apply the results to the initial value problem of KdV_2, (cf. e.g. [MM])

$$\partial_t v = \frac{1}{4}\partial_x^5 v - \frac{5}{2} v\partial_x^3 v - 5\partial_x v \partial_x^2 v + \frac{15}{2} v^2 \partial_x v \tag{4.1}$$

$$v(x;0) = q(x). \tag{4.2}$$

We want to construct weak solutions of (4.1)-(4.2) in $C\big(\mathbb{R}; H^1(S^1)\big)$ for initial data q in $H^1(S^1)$. The construction of the solution map $\mathcal{S}^{(2)}$ is very similar as in the case of KdV. Recall that the Hamiltonian $\mathcal{H}^{(2)}$ corresponding to KdV_2 is given by (cf. e.g. [MM])

$$\mathcal{H}^{(2)}(q) = \int_{S^1} \left(\frac{1}{8}(\partial_x^2 q)^2 + \frac{5}{4} q(\partial_x q)^2 + \frac{5}{8} q^4 \right) dx. \tag{4.3}$$

We point out that $\mathcal{H}^{(2)}$ is only defined for q in $H^2(S^1)$. Nevertheless we will show that one can define the frequencies $\frac{\partial \mathcal{H}^{(2)}}{\partial J_j}$ corresponding to $\mathcal{H}^{(2)}$.

First notice that, with $p := q - [q]$,

$$\begin{aligned} \mathcal{H}^{(2)}(q) &= \mathcal{H}^{(2)}(p + [q]) \\ &= \mathcal{H}^{(2)}(p) + \frac{5}{2}[q] \cdot \mathcal{H}^{(1)}(p) + \frac{15}{2}[q]^2 \frac{1}{2} \int_{S^1} p^2 dx + \frac{5}{8}[q]^4. \end{aligned} \tag{4.4}$$

Therefore it suffices to define the frequencies for $q \in H_0^1(S^1)$. Recall that for the construction of the L_2-integrable, holomorphic 1-forms Ω_j on $y = \sqrt{\Delta^2 - 4}$ we have introduced the functions $f_j(\lambda) := C_j \frac{\varphi_j(\lambda)}{\sqrt{\Delta(\lambda)^2 - 4}}$ with

$$\phi_j(\lambda) := \frac{1}{j^2\pi^2} \prod_{k \neq j} \frac{\mu_k^{(j)} - \lambda}{j^2 \pi^2}; C_j := \left(\int_{\Gamma_j} \frac{\phi_j(u) d\mu}{\sqrt{\Delta(\mu)^2 - 4}} \right)^{-1} \tag{4.5}$$

where Γ_j is the counterclock oriented circle with center τ_j and sufficiently small radius $\rho_j > \gamma_j/2$ and C_j is a normalizing constant. The frequencies $\omega_j^{(2)}$ are then defined by

$$\omega_j^{(2)} := \frac{\pi i}{3} f_j^{(\text{iv})}(\infty) \tag{4.6}$$

where $f_j^{(\text{iv})}(\infty)$ denotes the fourth deriviative of $f_j(\lambda)$ evaluated at infinity with respect to local coordinates $\tau^2 = \frac{1}{\lambda}$ near $\lambda = \infty$. A straightforward calculation shows that

(4.7)
$$i \cdot C_j^{-1} \frac{1}{4!} f_j^{(\mathrm{iv})}(\infty) = \tau_j^2 + \tau_j \left(\frac{\lambda_0}{2} - \sum_{k \neq j} \xi_k^{(j)} \right) + \frac{3}{2} \left(\frac{\lambda_0}{2} \right)^2$$
$$- \frac{\lambda_0}{2} \sum_{k \neq j} \xi_k^{(j)} + \frac{1}{8} \sum_k \gamma_k^2 + \frac{1}{2} \left(\sum_{k,l \neq j, k \neq l} \xi_k^{(j)} \xi_l^{(j)} \right) - \sum_{k \neq j} \tau_k \xi_k^{(j)}$$

where $\xi_k^{(j)} := \mu_k^{(j)} - \tau_k$ (as in Section 2). In view of Theorem 2.1, given $q_0 \in H_0^1(S^1)$, there exists a neighborhood U_{q_0} of q_0 in $H_0^1(S^1;\mathbb{C})$ so that the right hand side of (4.7) is well defined and analytic for any $j \geq 1$. In order to derive an asymptotic expansion of the $\omega_j^{(2)}$'s we need a number of auxilary results.

From Theorem 2.7 we deduce the following

Lemma 4.1. *Let $q_0 \in L_0^2(S^1)$. Then there exist a neighborhood U_{q_0} of q_0 in $L_0^2(S^1;\mathbb{C})$, $j_0 \geq 1$ and $C > 0$ such that for $q \in U_{q_0}, j \geq j_0, k \geq j/2$*

(4.8i)
$$\left| \xi_k^{(j)} - \left(-\frac{1}{2} \frac{(\gamma_k/2)^2}{\tau_k - \tau_j} - \frac{(\gamma_k/2)^2}{4} \frac{1}{\tau_k - \lambda_0} \right) \right| \leq \frac{C}{j^5} |k\gamma_k|^2$$

and, for $j \geq j_0$, $k \leq j/2$

(4.8ii)
$$\left| \xi_k^{(j)} - \left(-\frac{(\gamma_k/2)^2}{4} \frac{1}{\tau_k - \lambda_0} \right) \right| \leq C \frac{|k\gamma_k|^2}{k^5}.$$

Lemma 4.2. *Let $q_0 \in H_0^1(S^1)$. Then there exists a neighborhood U_{q_0} of q_0 in $H_0^1(S^1;\mathbb{C})$ so that*

(4.9i)
$$\left(\int_{\Gamma_j} \frac{\varphi_j(\mu) d\mu}{\sqrt{\Delta(\mu)^2 - 4}} \right)^{-1} = \left(2\pi\psi_j(0)\right)^{-1} + O\left(\frac{|j\gamma_j|^2}{j^5} \right)$$

(4.9ii)
$$\left(2\psi_j(0)\right)^{-1} = (\tau_j - \lambda_0)^{1/2} \left\{ 1 - \sum_{k \leq j/2} \frac{1}{\tau_k - \tau_j} \left(\xi_k^{(j)} + \frac{1}{2} \frac{(\gamma_k/2)^2}{\tau_k - \tau_j} \right) \right.$$
$$\left. + \frac{1}{2} \left(\sum_{k \leq j/2} \frac{\xi_k^{(j)}}{\tau_k - \tau_j} \right)^2 + \frac{1}{2} \sum_{k \leq j/2} \left(\frac{\xi_k^{(j)}}{\tau_k - \tau_j} \right)^2 \right\}$$
$$+ O\left(\frac{1}{j^4} \right) \sum_{k \geq j/2} |k\gamma_k|^2 + O\left(\frac{1}{j^5} \right).$$

Proof. (i) Follows from (3.20).

(ii) By Lemma 3.2,

$$(2\psi_j(0))^{-1} = (\tau_j - \lambda_0)^{1/2} \left\{ 1 - \sum_{k\neq j} \frac{1}{\tau_k - \tau_j} \left(\xi_k^{(j)} + \frac{1}{2} \frac{(\gamma_k/2)^2}{\tau_k - \tau_j} \right) \right.$$
$$\left. + \frac{1}{2} \left(\sum_{k\neq j} \frac{\xi_k^{(j)}}{(\tau_k - \tau_j)} \right)^2 + \frac{1}{2} \sum_{k\neq j} \left(\frac{\xi_k^{(j)}}{(\tau_k - \tau_j)} \right)^2 \right\} + O\left(\frac{1}{j^5}\right).$$

By (4.8i) one has

$$\left| \sum_{k\geq j/2} \frac{1}{\tau_k - \tau_j} \left(\xi_k^{(j)} + \frac{1}{2} \frac{(\gamma_k/2)^2}{\tau_k - \tau_j} \right) \right| \leq O\left(\frac{1}{j^5}\right) \sum_{k\geq j/2} |\gamma_k k|^2.$$

Similarly, by (4.8i) and (4.8ii),

$$\sum_{k\geq j/2} \frac{\xi_k^{(j)}}{\tau_k - \tau_j} = O\left(\frac{1}{j^4}\right); \quad \sum_{k\leq j/2} \frac{\xi_k^{(j)}}{\tau_k - \tau_j} = O\left(\frac{1}{j^2}\right).$$

This leads to (4.9ii). □

Lemma 4.3. *Let $q_0 \in H_0^1(S^1)$. Then there exists a neighborhood U_{q_0} of q_0 in $H_0^1(S^1;\mathbb{C})$ and $C > 0$ so that for any q in U_{q_0} and $j \geq 1$*

$$\left| \omega_j^{(2)} - 8\left((j\pi)^5 + j\pi 5\tilde{\tau}_j \right) \right| \leq C \left(\sum_{k\geq j/2} |k\gamma_k|^2 + \frac{1}{j} \right) \tag{4.10}$$

where $\sqrt{\tau_j} = j\pi + \frac{\tilde{\tau}_j}{(j\pi)^3}$.

Proof. Using (4.7), (4.6) and (4.5) one obtains

$$\omega_j^{(2)} = 8\pi \left(\int_{\Gamma_j} \frac{\varphi_j(\mu)d\mu}{\sqrt{\Delta(\mu)^2 - 4}} \right)^{-1} \left\{ \tau_j^2 + \tau_j \left(\frac{\lambda_0}{2} - \sum_{k\neq j} \xi_k^{(j)} \right) \right.$$
$$\left. + \frac{3}{8}\lambda_0^2 - \frac{\lambda_0}{2} \sum_{k\neq j} \xi_k^{(j)} + \frac{1}{8} \sum_{k\neq j} \gamma_k^2 + \frac{1}{2} \sum_{k,l\neq j, k\neq l} \xi_k^{(j)} \xi_l^{(j)} - \sum_{k\neq j} \tau_k \xi_k^{(j)} \right\}.$$

Therefore with $\left(\int_{\Gamma_j} \frac{\varphi_j(\mu)d\mu}{\sqrt{\Delta(\mu)^2-4}} \right)^{-1} = (2\pi\psi_j(0))^{-1} + O(\frac{|j\gamma_j|^2}{j^5})$ and the asymptotics of $(2\pi\psi_j(0))^{-1}$ we obtain by a straight forward computation

$$\frac{1}{8}\omega_j^{(2)} = \tau_j^{5/2} + O(1) \left(\sum_{k\geq j/2} |k\gamma_k|^2 \right) + O\left(\frac{1}{j}\right).$$

With $\sqrt{\tau_j} = j\pi + \frac{\widetilde{\tau}_j}{(j\pi)^3}$ we obtain

$$\tau_j^{5/2} = (j\pi)^5 + \left(5\widetilde{\tau}_j\right) j\pi + O\left(\frac{1}{j}\right)$$

which leads to (4.10). □

Lemma 4.4. *Let $q_0 \in H_0^1(S^1)$. Then there exists a neighborhood U_{q_0} of q_0 in $H_0^1(S^1;\mathbb{C})$ and $C > 0$ so that, uniformly for q in U_{q_0},*

$$\tau_j(q) = j^2\pi^2 + \frac{1}{4}\frac{\int_0^1 q(x)^2 dx}{j^2\pi^2} + O\left(\frac{1}{j^3}\right) \tag{4.11i}$$

$$\sqrt{\tau_j(q)} = j\pi + \frac{1}{8}\frac{\int_0^1 q(x)^2 dx}{(j\pi)^3} + O\left(\frac{1}{j^4}\right). \tag{4.11ii}$$

Proof. Notice that (4.11ii) follows immediately from (4.11i), so let us concentrate on (4.11i). Recall that, for $j \geq j_0$ with j_0 sufficiently large,

$$2\tau_j = tr\left(\left(-\frac{d^2}{dx^2} + q\right) P_j\right)$$

where P_j is the Riesz projector on the subspace generated by the generalized eigenfunctions corresponding to the eigenvalues λ_{2j} and λ_{2j-1}. It is convenient to introduce, for $j \geq j_0$,

$$\begin{aligned} L(t) &:= -\frac{d^2}{dx^2} + tq, \\ P_j(t) &:= \frac{1}{2\pi i}\int_{\Gamma_j} \left(\lambda - L(t)\right)^{-1} d\lambda, \\ \tau_j(t) &:= \operatorname{tr}\left(L(t)P_j(t)\right) \end{aligned}$$

where Γ_j is a circle with center $j^2\pi^2$ and radius ρ_j so that the eigenvalues $\lambda_{2j}(tq)$ and $\lambda_{2j-1}(tq)$ are inside of Γ_j and all other eigenvalues are outside of Γ_j for any $0 \leq t \leq 1$, $q \in U_{q_0}$. (The integer j_0 has been chosen sufficiently large and the neighborhood U_{q_0} sufficiently small.) The functions $\tau_j(t)$ admit a Taylor expansion,

$$\tau_j(1) = \tau_j(0) + \tau_j'(0) + \tau_j''(0)\frac{1}{2!} + \tau_j'''(0)\frac{1}{3!} + \mathcal{E}_j \tag{4.12}$$

where $\mathcal{E}_j$ denotes the Taylor remainder term. Clearly

$$2\tau_j(0) = n^2\pi^2 tr P_j(0) = 2j^2\pi^2. \tag{4.13}$$

Further

$$2\tau_j'(0) = \operatorname{tr}\left(P_j(0)L'(0)P_j(0)\right) = \operatorname{tr}\left(qP_j(0)\right) = 2[q] = 0. \tag{4.14}$$

To compute the second derivative it is convenient to introduce

$$\hat{q}_k = \frac{1}{2}\int_{-1}^{1} e^{-i\pi kx} q(x)dx.$$

Then

$$2\tau_j''(0) = \operatorname{tr}\left(qP_j'(0)\right) = \sum_{m,k} \hat{q}_{m-k}\hat{q}_{k-m} \frac{1}{2\pi i}\int_{\Gamma_j} \frac{d\lambda}{(\lambda - m^2\pi^2)(\lambda - k^2\pi^2)}$$
$$= 4\sum_{m\neq\pm j} \frac{\hat{q}_{m-j}\hat{q}_{j-m}}{\pi^2(j^2-m^2)} = \frac{4}{\pi^2}\sum_{k\neq 0,2j} \frac{\hat{q}_n\hat{q}_{-n}}{n(2j-n)}.$$

Writing

$$\frac{1}{n(2j-n)} = \frac{1}{n}\frac{1}{2j} + \frac{1}{2j-n}\frac{1}{2j}$$
$$= \frac{1}{n}\frac{1}{2j} + \frac{1}{(2j)^2} + \frac{n}{(2j-n)(2j)^2}$$

and using that

$$\sum_{n\neq 0,2j} \frac{\hat{q}_n\hat{q}_{-n}}{n} = \sum_{n\neq 0} \frac{\hat{q}_n\hat{q}_{-n}}{n} - \frac{\hat{q}_{2j}\hat{q}_{-2j}}{2j} = 0 - \frac{\hat{q}_{2j}\hat{q}_{-2j}}{2j}, \tag{4.15}$$

we conclude from (4.14)

(4.16)
$$2\tau_j''(0) = \frac{4}{\pi^2}\sum_{n\neq 0} \frac{\hat{q}_n\hat{q}_{-n}}{(2j)^2} + \frac{4}{\pi^2}\sum_{\substack{n\neq 0\\ n\neq 2j}} \frac{n\hat{q}_n(-n)\hat{q}_{-n}}{(-n)(2j-n)(2j)^2}$$
$$= \frac{1}{\pi^2 j^2}\frac{1}{2}\int_{-1}^{1} q(x)^2dx + 0\left(\frac{1}{j^3}\right) = \frac{1}{j^2\pi^2}\int_0^1 q(x)^2dx + O\left(\frac{1}{j^3}\right).$$

Similarly, we compute $\tau_j'''(0)$,

(4.17)
$$2\tau_j'''(0) = \operatorname{tr}\left(qP_j''(0)\right)$$

$$= \operatorname{tr} \frac{1}{2\pi i} \int_{\Gamma_j} q 2 \big(\lambda - L(0)\big)^{-1} q \big(\lambda - L(0)\big)^{-1} q \big(\lambda - L(0)\big)^{-1} d\lambda$$

$$= 2 \sum_{m,k,n} \hat{q}_{m-n} \hat{q}_{n-k} \hat{q}_{k-m} \frac{1}{2\pi i} \int_{\Gamma_j} \frac{d\lambda}{(\lambda - n^2\pi^2)(\lambda - k^2\pi^2)(\lambda - m^2\pi^2)}$$

$$= 6 \cdot 2 \sum_{\substack{n \neq \pm j \\ k \neq \pm j}} \frac{\hat{q}_{j-n} \hat{q}_{n-k} \hat{q}_{k-j}}{(j^2\pi^2 - n^2\pi^2)(j^2\pi^2 - k^2\pi^2)}$$

$$= \frac{6 \cdot 2}{\pi^4} \sum_{\substack{m,\ell \\ m,\ell \neq 0, 2j}} \frac{\hat{q}_m \hat{q}_{\ell-m} \hat{q}_{-\ell}}{m(2j-m)\ell(2j-\ell)}.$$

Again, writing

$$\frac{1}{m(2j-m)} \frac{1}{(2j-\ell)\ell} = \frac{1}{m\ell(2j)^2} + \frac{m}{m(2j-m)\ell(2j)^2} + \frac{\ell}{m\ell(2j-\ell)} \frac{1}{(2j)^2} + \frac{m\ell}{\ell(2j-\ell)m(2j-m)} \frac{1}{(2j)^2}$$

and taking into account that $q \in U_{q_0} \subset H_0^1(S^1;\mathbb{C})$ we conclude that

$$2\tau_j'''(0) = \frac{6 \cdot 2}{\pi^4} \sum_{m,\ell \neq 0} \frac{\hat{q}_m \hat{q}_{\ell-m} \hat{q}_{-\ell}}{m\ell(2j)^2} + O\left(\frac{1}{j^3}\right) \tag{4.18}$$

$$= 0 + O\left(\frac{1}{j^3}\right).$$

Using the standard resolvent estimate one concludes that the Taylor remainder term $\mathcal{E}_j$ satisfies

$$\mathcal{E}_j = O\left(\frac{1}{j^3}\right). \tag{4.19}$$

Substitute (4.13), (4.14), (4.16), (4.18) and (4.19) into the Taylor expansion (4.12) we conclude

$$\tau_j(1) = j^2\pi^2 + \frac{\frac{1}{4}\int_0^1 q(x)^2 dx}{j^2\pi^2} + O\left(\frac{1}{j^3}\right).$$

This proves (4.11i). □

Theorem 4.5. *Let $q_0 \in H_0^1(S^1)$. Then there exists a neighborhood U_{q_0} of q_0 in $H_0^1(S^1;\mathbb{C})$ and $C > 0$ so that uniformly for q in U_{q_0}*

$$\left|\omega_j^{(2)} - 8\big[(j\pi)^5 - a_1 j\pi\big]\right| \leq C \tag{4.20}$$

where $a_1 = a_1(q)$ is given by

$$a_1 = \frac{5}{8} \int_0^1 q(x)^2 dx. \tag{4.21}$$

Proof. Combine the estimate (4.10) for $\omega_j^{(2)}$ with the estimate (4.11ii) for $\widetilde{\tau}_j$

$$\widetilde{\tau}_j = \frac{1}{8} \int_0^1 q(x)^2 dx + 0\left(\frac{1}{j}\right)$$

to obtain (4.20)-(4.21). □

Given real numbers c_1, c_3 we now construct weak solutions of (4.1)-(4.2) in $C\big(\mathbb{R}; H^1_{c_1,c_3}(S^1)\big)$ for initial data q in $H^1_{c_1,c_3}(S^1) := \{q \in H^1(S^1); [q] = c_3; a_1(q) = c_1\}$.

Theorem 4.6. *Let $c_1, c_3 \in \mathbb{R}$. Then there exists a solution map $\mathcal{S}^{(2)}$*

$$\mathcal{S}^{(2)} : H^1_{c_1,c_3}(S^1) \to C\big(\mathbb{R}, H^1_{c_1,c_3}(S^1)\big)$$

with the following properties:

(i) *$\mathcal{S}^{(2)}(q)$ is a weak solution of* (4.1)-(4.2);

(ii) *given q_1, q_2 in $H^1(S^1)$ with*

$$[q_1] = [q_2] = c_3;\ a_1(q_1) = a_1(q_2) = c_1$$

there exists $M > 0$ so that for any t

$$\left\| \mathcal{S}^{(2)}(q_1)(\cdot,t) - \mathcal{S}^{(2)}(q_2)(\cdot,t) \right\|_{H^1(S^1)} \leq M(1+|t|)\|q_1 - q_2\|_{H^1(S^1)}.$$

(iii) *For any $0 < T < \infty$, $\mathcal{S}^{(2)} : H^1_{c_1,c_3}(S^1) \to C\big([-T,T]; H^1_{c_1,c_3}(S^1)\big)$ is real analytic.*

Remark 1. According to [**Bo2**], Bourgain's method cannot be applied to solve the initial value problem (4.1)-(4.2) in $H^1(S^1)$.

Remark 2. Note that the KdV Hamiltonian $\mathcal{H}(q) = \int_{S^1} \big(\frac{1}{2}(\partial_x q)^2 + q^3\big)dx$ is a conserved quantity for (4.1), as well as the average $[q]$ and $\int_{S^1} q(x)^2 dx$. Thus, given a real valued smooth solution $v(x,t)$ of (4.1) one obtains an a priori bound for $\int_{S^1} |\partial_x v(x,t)|^2 dx$ which leads to the existence of a weak solution of (4.1)-(4.2). This solution can be approximated by finite gap solutions of (4.1). Of course, the main point of the statement in Theorem 4.6 is that the solution map $\mathcal{S}^{(2)}$ is real analytic.

Proof of Theorem 4.6. The construction of the solution map $\mathcal{S}^{(2)}$ and the proofs of its properties are similar to the ones of $\mathcal{S}^{(1)}$ and we therefore omit it. □

Appendix: Proof of Theorem 2.1.

In the case where the potential $q \in L_0^2(S^1;\mathbb{C})$ is real valued, the entire functions $\phi_j(\lambda,q)(j \geq 1)$ are constructed in [**MT2**]. Actually, instead of ϕ_j, [**MT2**] construct entire functions $1_j(\lambda,q)$ which coincide with $\phi_j(\lambda,q)$ up to a normalization factor. (In the sequel, we use the normalization introduced by McKean-Trubowitz.) By a straightforward perturbation argument one shows that for given $q_0 \in L_0(S^1;\mathbb{R})$ there exist a (sufficiently small) neighborhood U_{q_0} of q_0 in $L_0^2(S^1;\mathbb{C})$ so that for $q \in U_{q_0}$ the functions $1_j(\lambda,q)(j \geq 1)$ are uniquely defined and analytic with respect to q. The main part of the proof of Theorem 2.1 consists in analyzing the zeroes of $1_j(\lambda,q)$ for q in a neighborhood $U \subseteq U_{q_0}$ of q_0 in $L_0^2(S^1;\mathbb{C})$ which does not depend on j and proving the estimates (2.1). We proceed in two steps:

In Section A.2 we show, using Rouché's theorem that, given $q_0 \in L_0^2(S^1;\mathbb{R})$, $0 < K \leq K(q_0)$ (cf. definition after (A.2)) and $N \in \mathbb{N}$ arbitrary, there exist a neighborhood $U_{q_0,K,N}$ of q_0 in $L_0^2(S^1;\mathbb{C})$ (depending on q_0, K and N) so that, for any $j \geq 1, q \in U_{q_0,K,N}$ and $1 \leq k \leq N, k \neq j$, the entire function $1_j(\lambda,q)$ has precisely one zero, denoted by $\mu_k^{(j)}(q)$, inside the circle $\Gamma_k(K)$ (defined in Section A.1). We point out that $\bigcap_{N\geq 1} U_{q_0,K,N}$ might consist of $\{q_0\}$ only and therefore we need an additional argument for the proof of Theorem 2.1.

In Section A.3, we show that there exist a neighborhood U of q_0 in $L_0^2(S^1;\mathbb{C})$ and $N \geq 1$, depending on U only, so that the system of nonlinear equations for the zeroes $\mu_k^{(j)}(q)$ of $1_j(\lambda,q)$ with $k \geq N+1, k \neq j$, obtained from (2.2), can be solved by a contraction argument. At the same time we obtain the estimates for $\mu_k^{(j)}(q) - \tau_k(q)$ claimed in Theorem 2.1. Finally the analyticity of the zeroes $\mu_k^{(j)}(q)$ of $1_j(\lambda)$ follows from Cauchy's integral formula. We remark that by using the contraction argument of section A.3 we can obtain a new proof for the existence of $1_j(\lambda,q)$ and their product representation for real valued potentials q in L_0^2.

In Section A.4 we present the proof of Theorem 2.1, combining the results of Sections A.1, A.2 and A.3.

In order to avoid introducing cumbersome notation the same letter C will denote various constants and the same letter U will denote various neighborhoods.

A.1. Normalized Riesz basis of holomorphic differentials.

Let us first recall some notions, notations and results from [**MT1**] (cf. also [**MT2**]). Denote by $I_{3/2}$ the complex Hilbert space of entire functions

ϕ of order $\leq 1/2$ and type ≤ 1 with

$$\|\phi\|^2_{I_{3/2}} := \int_0^\infty |\phi(\lambda)|^2 \lambda^{3/2} d\lambda < \infty. \tag{A.1}$$

The inner product in $I_{3/2}$ is given by

$$\langle \phi, \psi \rangle := \int_0^\infty \phi(\lambda)\overline{\psi(\lambda)}\lambda^{3/2} d\lambda.$$

By the Paley-Wiener theorem

$$|\lambda\phi(\lambda)| \leq \|\phi\|_{I_{3/2}} e^{|Im\sqrt{\lambda}|}. \tag{A.2}$$

Introduce, for $q_0 \in L^2_0(S^1;\mathbb{R}), K(q_0) := \frac{1}{5}\min_{n\geq 0}\big(\lambda_{2n+1}(q_0) - \lambda_{2n}(q_0)\big)$ where $\big(\lambda_n(q_0)\big)_{n\geq 0}$ is the anti/periodic spectrum of $-\frac{d^2}{dx^2} + q_0$. Then $K(q_0) > 0$ and for $0 < K \leq K(q_0)$ there exist a bounded neighborhood $V \equiv V_{q_0,K}$ of q_0 in $L^2_0(S^1;\mathbb{C})$ so that for $q \in V$, the anti/periodic eigenvalues $\big(\lambda_n(q)\big)_{n\geq 0}$ satisfy

$$\sup_{j\geq 0} |\lambda_j(q) - \lambda_j(q_0)| \leq K. \tag{A .3}$$

Denote by $\Gamma_n = \Lambda_n(K, q_0)$ the counterclockwise oriented circle in $\mathbb{C}$ with center $\tau_n = \tau_n(q_0) = \frac{\lambda_{2n}(q_0)+\lambda_{2n-1}(q_0)}{2}$ and radius $r_n := \frac{\gamma_n(q_0)}{2} + 2K$ where $\gamma_n(q_0) = \lambda_{2n}(q_0) - \lambda_{2n-1}(q_0)$. Notice that, because of (A.3), for q in V, $\lambda_{2n}(q)$ and $\lambda_{2n-1}(q)$ are inside Γ_n and all other eigenvalues $\lambda_k(q)$ are outside Γ_n.

For q in V and $n \geq 1$, denote by $A_n(q) : I_{3/2} \to \mathbb{C}$ the bounded linear functional

$$A_n(q)(\phi) := \int_{\Gamma_n} \frac{\phi(\lambda)d\lambda}{\sqrt{\Delta(\lambda,q)^2 - 4}}. \tag{A .4}$$

Then $A_n(q)$ is an analytic function of $q \in V$ with values in the dual $I^*_{3/2}$ of $I_{3/2}$. Using (A.2) and (2.6) one verifies (cf.[**MT 1**, **2**]) that there exists $C \geq 1$ such that $\frac{1}{C} \leq \|nA_n(q)\|_{I^*_{3/2}} \leq C$ for q in V.

Recall (cf. [**GK**, p. 310]) that $(u_n)_{n\geq 1}$ is said to be *a Riesz basis* of a Hilbert space H if there exist an orthonormal basis $(e_j)_{j\geq 1}$ of H and an invertible bounded linear operator $B : H \to H$ such that $Be_j = u_j (j \geq 1)$. In the case where the potential q is real valued McKean-Trubowitz (cf. [**MT2**]) showed that $\big(nA_n(q)\big)_{n\geq 1}$ is a Riesz basis of $I^*_{3/2}$. To be able to consider complex valued potentials as well, we need the following auxilary result. We recall (use Marchenko's asymptotics of the eigenvalues $\lambda_j(q)$) (cf

[**Ma**, Theorem 1.5.2]) that the map $q \mapsto \left(\tau_k(q) - k^2\pi^2\right)_{k\geq 1}$ is an analytic function on V with values in $\ell_1^2(\mathbb{N};\mathbb{C})$.

Lemma A.1. *There exist $C \geq 1$ and $N \geq 1$ such that for $n \geq N$ and $q \in V$ (V defined before (A.3))*

$$n\|A_n(q) - A_n(q_0)\|_{I^*_{3/2}} \leq C\left(|\gamma_n(q)^2 - \gamma_n(q_0)^2|\right) + \frac{C}{n^2}\left(|\lambda_0(q) - \lambda_0(q_0)| + \left(\sum_{k\geq 1} k^2|\tau_k(q) - \tau_k(q_0)|^2\right)^{1/2} + \left(\sum_{k\geq 1}|\gamma_k(q)^2 - \gamma_k(q_0)^2|\right)^{1/2}\right).$$

Proof. For $\phi \in I_{3/2}, n \geq 1$ and $q \in V$

(A.5)
$$A_n(q)(\phi) - A_n(q_0)(\phi) = \int_{\Gamma_n} \frac{\Big((\Delta(\lambda,q_0)^2 - 4) - (\Delta(\lambda,q)^2 - 4)\Big)\phi(\lambda)d\lambda}{\sqrt{\Delta(\lambda,q)^2 - 4}\sqrt{\Delta(\lambda,q_0)^2 - 4}\left(\sqrt{\Delta(\lambda,q)^2 - 4} + \sqrt{\Delta(\lambda,q_0)^2 - 4}\right)}.$$

Recall that for $\lambda \in \Gamma_n$ (cf. e.g. [**Ka**, p. 547])

$$\Delta(\lambda,q)^2 - 4 = \frac{1}{n^2\pi^2}\left(\lambda_{2n}(q) - \lambda\right)\left(\lambda - \lambda_{2n-1}(q)\right)\left(1 + \beta_n(\lambda,q)\frac{\log(n+1)}{n}\right)$$

where $\beta_n(\lambda,q)$ satisfies $|\beta_n(\lambda,q)| < C$ for $n \geq 1$, $q \in V$ for some $C > 0$.

In view of the choice of K, $\Gamma_n = \Gamma_n(K)$ and V, these asymptotics can be used to obtain an estimate of the denominator of the integrand in (A.5), for q in V,

$$\sup_{\lambda\in\Gamma_n}\left|\frac{1}{\sqrt{\Delta(\lambda,q)^2 - 4}} \cdot \frac{1}{\sqrt{\Delta(\lambda,q_0)^2 - 4}} \cdot \frac{1}{\sqrt{\Delta(\lambda,q)^2 - 4} + \sqrt{\Delta(\lambda,q_0)^2 - 4}}\right| \leq Cn^3$$

where $C \geq 1$ can be chosen independently of $q \in V$ and $n \geq 1$.

To estimate the nominator of the integrand in (A.5) we write

(A .6) $$(\Delta(\lambda,q_0)^2 - 4) - (\Delta(\lambda,q)^2 - 4) = (\Delta(\lambda,q_0)^2 - 4)\left(1 - \frac{\Delta(\lambda,q)^2 - 4}{\Delta(\lambda,q_0)^2 - 4}\right).$$

Use the product representation

$$\Delta(\lambda,q)^2-4=4\big(\lambda_0(q)-\lambda\big)\prod_{k=1}^{\infty}\frac{\big(\lambda_{2k}(q)-\lambda\big)\big(\lambda_{2k-1}(q)-\lambda\big)}{k^4\pi^4}$$

and write $\big(\lambda_{2k}(q)-\lambda\big)\big(\lambda_{2k-1}(q)-\lambda\big)=\big(\tau_k(q)-\lambda\big)^2-\big(\frac{\gamma_k(q)}{2}\big)^2$ to obtain

(A.6′)

$$\frac{\Delta(\lambda,q)^2-4}{\Delta(\lambda,q_0)^2-4}=\left(1+\frac{\lambda_0(q_0)-\lambda_0(q)}{\lambda-\lambda_0(q_0)}\right)$$

$$\left(1+\frac{\big(\tau_n(q)-\tau_n(q_0)\big)\big(\tau_n(q)+\tau_n(q_0)-2\lambda\big)-\big(\frac{\gamma_n(q)}{2}\big)^2+\big(\frac{\gamma_n(q_0)}{2}\big)^2}{\big(\tau_n(q_0)-\lambda\big)^2-\big(\frac{\gamma_n(q_0)}{2}\big)^2}\right)\cdot E_n$$

where

$$E_n:=\prod_{\substack{k\neq n\\ k\geq 1}}\left(1+\frac{\big(\tau_k(q)-\tau_k(q_0)\big)\big(\tau_k(q)+\tau_k(q_0)-2\lambda\big)-\big(\frac{\gamma_k(q)}{2}\big)^2+\big(\frac{\gamma_k(q_0)}{2}\big)^2}{\big(\tau_k(q_0)-\lambda\big)^2-\big(\frac{\gamma_k(q_0)}{2}\big)^2}\right).$$

One verifies that there exist $N_0\geq 1$ and $C\geq 1$ such that for $n\geq N_0, q\in V$ and $\lambda\in\Gamma_n$

$$|E_n-1|\leq C\sum_{\substack{k\neq n\\ k\geq 1}}\left(\frac{|\tau_k(q)-\tau_k(q_0)||k^2-n^2|}{|k^2-n^2|^2}+\frac{\big|\big(\frac{\gamma_k(q)}{2}\big)^2-\big(\frac{\gamma_k(q_0)}{2}\big)^2\big|}{|k^2-n^2|^2}\right).$$

Notice that

$$\sum_{\substack{k\neq n\\ k\geq 1}}\frac{\Big|\big(\frac{\gamma_k(q)}{2}\big)^2-\big(\frac{\gamma_k(q_0)}{2}\big)^2\Big|}{|k^2-n^2|^2}\leq\frac{1}{n^2}\sum_{k\geq 1}\left|\left(\frac{\gamma_k(q)}{2}\right)^2-\left(\frac{\gamma_k(q_0)}{2}\right)^2\right|$$

and

$$\sum_{\substack{|k-n|\geq\frac{n}{2}\\ k\geq 1}}\frac{|\tau_k(q)-\tau_k(q_0)||k^2-n^2|}{|k^2-n^2|^2}\leq\frac{C}{n^2}\sum_{k\geq 1}|\tau_k(q)-\tau_k(q_0)|.$$

Further

$$\sum_{\substack{|k-n|\leq\frac{n}{2}\\ k\neq n}}\frac{|\tau_k(q)-\tau_k(q_0)||k^2-n^2|}{|k^2-n^2|^2}$$

$$\le \frac{C}{n^2} \sum_{\substack{|k-n|\le \frac{n}{2} \\ k\neq n}} \frac{1}{|k-n|} k|\tau_k(q)-\tau_k(q_0)|$$

$$\le \frac{C}{n^2} \left(\sum_{|k-n|\ge 1} \frac{1}{|k-n|^2} \right)^{1/2} \left(\sum_{k\ge 1} k^2 |\tau_k(q)-\tau_k(q_0)|^2 \right)^{1/2} .$$

Combining the above estimates we conclude that there exist $N \ge 1$ and $C \ge 1$ such that for $n \ge N, q \in V, \lambda \in \Gamma_n$

(A.6″)

$$\left|1 - \frac{\Delta(\lambda,q)^2 - 4}{\Delta(\lambda,q_0)^2 - 4}\right|$$

$$\le C|\tau_n(q)-\tau_n(q_0)| + C \left| \left(\frac{\gamma_n(q)}{2}\right)^2 - \left(\frac{\gamma_n(q_0)}{2}\right)^2 \right| + \frac{C}{n^2}|\lambda_0(q)-\lambda_0(q_0)|$$

$$+ \frac{C}{n^2} \left(\sum_{k\ge 1} k^2|\tau_k(q)-\tau_k(q_0)|^2 \right)^{1/2} + \frac{C}{n^2} \left(\sum_{k\ge 1} |\gamma_k(q)^2 - \gamma_k(q_0)^2| \right)^{1/2} .$$

In view of (A.2) and $\sup_{\lambda \in \bigcup_{n\ge 1} \Gamma_n} e^{|\operatorname{Im}\sqrt{\lambda}|} < \infty$ one concludes that there exists $C \ge 1$ such that for $\phi \in I_{3/2}$ and $n \ge 1$

$$(A.6''') \qquad \sup_{\lambda\in\Gamma_n} |\phi(\lambda)| \le \frac{C}{n^2} \|\phi\|_{I_{3/2}}.$$

Combining the estimates (A.6)-(A.6‴) we deduce from (A.5) that there exist $N \ge 1$ and $C \ge 1$ such that for $n \ge N, q \in V$ the claimed estimate for $n\|A_n(q) - A_n(q_0)\|_{I^*_{3/2}}$ holds. □

Since $A_n(q)$ is continuous on $V \subseteq L^2_0(S^1;\mathbb{C})$ with values in $I^*_{3/2}$, Lemma A.1 leads to (use the asymptotics of the eigenvalues $\lambda_j(q)$ to estimate the right hand side of the formula in Lemma A.3).

Corollary A.2. *Let $q_0 \in L^2_0(S^1;\mathbb{C})$ be real valued. Then, for any $\epsilon > 0$, there exists a neighborhhood $U \equiv U_\epsilon \subseteq V$ of q_0 in $L^2_0(S^1;\mathbb{C})$ so that for q in U*

$$\sum_{n\ge 1} n^2 \|A_n(q) - A_n(q_0)\|^2_{I^*_{3/2}} \le \epsilon.$$

Theorem A.3. *Let $q_0 \in L^2_0(S^1;\mathbb{C})$ be a real valued potential. Then there exists a neighborhood U of q_0 in $L^2_0(S^1;\mathbb{C})$ with the following properties:*

(i) *For any q in U, $\big(nA_n(q)\big)_{n\ge 1}$ is a Riesz basis of $I^*_{3/2}$;*

(ii) *there exists a uniquely determined sequence* $\left(\frac{1}{n}1_n(\cdot,q)\right)_{n\geq 1}$ *in* $I_{3/2}$, *each of the elements* $\frac{1}{n}1_n(\cdot,q)$ *depending analytically on* $q\in U$, *so that*

$$(nA_n(q))\left(\frac{1}{k}1_k(\cdot,q)\right)=\delta_{nk};$$

(iii) *for any* q *in* U, $\left(\frac{1}{n}1_n(\cdot,q)\right)_{n\geq 1}$ *is a Riesz basis of* $I_{3/2}$.

Proof. Let U be a neighborhood as in Corollary A.2. Since $\left(nA_n(q_0)\right)_{n\geq 1}$ is a Riesz basis of $I^*_{3/2}$, we can define a linear bounded map $T(q):I^*_{3/2}\to I^*_{3/2}$ by setting $T(q)\left(nA_n(q_0)\right):=nA_n(q)$. By Corollary A.2, $Id-T(q)$ is a Hilbert-Schmidt operator and by choosing $0<\epsilon<1$ sufficiently small we conclude from Corollary A.2 that $\|Id-T(q)\|_{HS}<1$ for any q in U. (Given a linear operator $A:H\to H$ on a Hilbert space H with an orthonormal basis $(e_j)_{j\geq 1}$, $\|A\|_{HS}$ denotes the Hilbert-Schmidt norm $\|A\|_{HS}=\left(\sum_{j,k}|\langle e_j,Ae_k\rangle|^2\right)^{1/2}$.) In particular $T(q)$ is an invertible, bounded linear operator and thus $\Big(T(q)\left(nA_n(q_0)\right)\Big)_{n\geq 1}$ is a Riesz basis of $I^*_{3/2}$ as well.

Using the same symbols as McKean-Trubowitz [**MT2**], we denote by $\left(\frac{1}{n}1_n(\lambda,q_0)\right)_{n\geq 1}$ the Riesz basis of $I_{3/2}$ uniquely determined by $nA_n(q_0)\left(\frac{1}{k}1_k(\cdot,q_0)\right)=\delta_{nk}$.

Let $\frac{1}{n}1_n(\cdot,q):=\left(T(q)^{-1}\right)^*\left(\frac{1}{n}1_n(\cdot,q_0)\right)$ where $\left(T(q)^{-1}\right)^*$ denotes the adjoint of $T(q)^{-1}$. One verifies that $nA_n(q)\left(\frac{1}{k}1_k(\cdot,q)\right)=\delta_{nk}$. It remains to prove that the $1_n(\cdot,q)$'s depend analytically on $q\in U$. Denote by $\mathcal{L}_{HS}(I^*_{3/2},I^*_{3/2})$ the Hilbert space of Hilbert-Schmidt operators on $I^*_{3/2}$ with inner product

$$\langle R,S\rangle_{HS}:=\sum_{i,j}R_{ij}\overline{S_{ji}}$$

where $R_{ij}=\langle Re_i,e_j\rangle$ are the coefficients of R with respect to an orthonormal basis $(e_i)_{i\geq 1}$ of $I^*_{3/2}$. From the analyticity of $A_n(q)$ it follows that the coefficients of $Id-T(q)$ are analytic on U. By Corollary A.2, $Id-T(q)$ is bounded on U in $\mathcal{L}_{HS}(I^*_{3/2},I^*_{3/2})$ and therefore, $Id-T(q)$ is analytic (cf. e.g. [**PT**, Appendix A]). As $\|\left(Id-T(q)\right)^{-1}\|_{HS}<1$,

$$T(q)^{-1}=(Id-(Id-T(q)))^{-1}=\sum_{k\geq 0}(Id-T(q))^k$$

where the series converges in $\mathcal{L}_{HS}(I^*_{3/2},I^*_{3/2})$. Therefore $T(q)^{-1}$ is analytic which implies that $\frac{1}{n}1_n(\cdot,q)=\left(T(q)^{-1}\right)^*\left(\frac{1}{n}1_n(\cdot,q_0)\right)$ is analytic on U with values in $I_{3/2}$ for any $n\geq 1$. ∎

Remark. For $q = 0$, the Riesz basis $\big(nA_n(q)\big)_{n\geq 1}$ of $I^*_{3/2}$ and $\big(\frac{1}{n}1_n(\cdot, q)\big)_{n\geq 1}$ of $I_{3/2}$ are given by

$$nA_n(0) = (-1)^n 2n^2\pi^2\delta_{(n^2\pi^2)}$$

and

$$\frac{1}{n}1_n(\lambda, 0) = \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}(\lambda - n^2\pi^2)}.$$

A.2 Rouché's theorem.

Denote by $I_{-1/2}$ the complex Hilbert space of entire functions ϕ of order $\leq 1/2$ and type ≤ 1 with

$$\|\phi\|^2_{I_{-1/2}} := \int_0^\infty |\phi(\lambda)|^2 \frac{d\lambda}{\sqrt{\lambda}} < \infty.$$

The inner product in $I_{-1/2}$ is given by

$$\langle\phi, \psi\rangle := \int_0^\infty \phi(\lambda)\overline{\psi(\lambda)}\frac{d\lambda}{\sqrt{\lambda}} < \infty.$$

By the Paley-Wiener theorem

$$|\phi(\lambda)| \leq \|\phi\|_{I_{-1/2}} e^{|\operatorname{Im}\sqrt{\lambda}|}. \tag{A.7}$$

Denote by $I^*_{-1/2}$ the Hilbert space dual to $I_{-1/2}$. Notice that the identity embedding $I_{3/2} \hookrightarrow I_{-1/2}$ is continuous. As in Section A.1, denote by V a neighborhood of a real valued potential $q_0 \in L^2_0$ with the property (A.3). For q in V, the functionals $A_n(q)$ introduced in (A.4) extend to linear bounded functionals on $I_{-1/2}$, $A_n(q) : I_{-1/2} \to \mathbb{C}$. Using (A.7) and the definition of $A_n(q)$ one concludes that there exists $C > 1$ such that $\frac{1}{C} \leq \|\frac{1}{n}A_n(q)\|_{I^*_{-1/2}} \leq C$ for q in V. By the same arguments used in the proof of Lemma A.1 one obtains

Lemma A.1′. *There exist $C \geq 1$ and $N \geq 1$ such that for $n \geq N$ and $q \in V$*

$$\begin{aligned}&\frac{1}{n}\|A_n(q) - A_n(q_0)\|_{I^*_{-1/2}}\\ &\leq C\big(|\gamma_n(q) - \gamma_n(q_0)|^2 + |\tau_n(q) - \tau_n(q_0)|\big)\end{aligned}$$

$$+\frac{C}{n^2}\left(|\lambda_0(q)-\lambda_0(q_0)|+\left(\sum_{k\geq 1}k^2|\tau_k(q)-\tau_k(q_0)|^2\right)^{1/2}\right.$$
$$\left.+\left(\sum_{k\geq 1}|\gamma_k(q)^2-\gamma_k(q_0)^2|\right)^{1/2}\right).$$

Next we need to estimate the $I_{-1/2}$-norm of the functions $n1_n(\lambda,q_0)$ for a real valued potential q_0.

Lemma A.4. *Let $q_0\in L_0^2(S^1;\mathbb{R})$. Then there exists $C'>0$ such that*

$$\|n1_n(\cdot,q_0)\|_{I_{-1/2}}\leq C' \qquad (n\geq 1).$$

Proof. It follows from the Paley-Wiener theorem that the Dirac measures $\sqrt{\pi}\delta_0$, $(\sqrt{2\pi}\delta_{n^2\pi^2})_{n\geq 1}$ are an orthonormal basis of $I^*_{-1/2}$ (Kotelnikov theorem; cf. also Remark after Theorem A.3). Therefore

$$\|j1_j(\cdot,q_0)\|^2_{I_{-1/2}}=\pi|j1_j(0,q_0)|^2+2\pi\sum_{n\geq 1}|j1_j(n^2\pi^2,q_0)|^2.$$

Further recall that, as q_0 is real valued, ([**MT1**])

$$j1_j(\lambda,q_0)=C_j\prod_{k\neq j}\frac{\mu_k^{(j)}-\lambda}{k^2\pi^2} \tag{A .8}$$

where $\lambda_{2k-1}(q_0)\leq\mu_k^{(j)}\leq\lambda_{2k}(q_0)$ and $\frac{1}{C}\leq C_j\leq C$ for all $j\geq 1$ for some constant $C\geq 1$.

For $1\leq n\neq j$, write

$$\begin{aligned}j1_j(n^2\pi^2,q_0)&=C_j\prod_{k\neq j}\frac{\mu_k^{(j)}-n^2\pi^2}{k^2\pi^2}\\&=C_j\left(\prod_{k\neq n,j}\frac{\mu_k^{(j)}-n^2\pi^2}{k^2\pi^2}\right)\frac{\tau_j-n^2\pi^2}{j^2\pi^2}\frac{j^2\pi^2}{\tau_j-n^2\pi^2}\cdot\frac{\mu_n^{(j)}-n^2\pi^2}{n^2\pi^2}.\end{aligned}$$

Notice that there exists $C\geq 1$ (cf. e.g. [**PT**, Appendix E, Lemma 3]) such that for $j,n\geq 1$ with $j\neq n$

$$\left|\left(\prod_{k\neq n,j}\frac{\mu_k^{(j)}-n^2\pi^2}{k^2\pi^2}\right)\frac{\tau_j-n^2\pi^2}{j^2\pi^2}\right|\leq C.$$

This implies that there exists $C \geq 1$ such that for $j \neq n, j, n \geq 1$,

$$|j 1_j(n^2\pi^2, q_0)| \leq C \frac{j^2}{|j^2 - n^2|} \frac{\mu_n^{(j)} - n^2\pi^2}{n^2\pi^2}.$$

Similarly, there exists $C \geq 1$ such that for $j \geq 1$

$$|j 1_j(j^2\pi^2, q_0)| \leq C \quad \text{and} \quad |j 1_j(0, q_0)| \leq C.$$

Notice that for $1 \leq |n-j| \leq j/2$, $\frac{j^2}{|j^2-n^2|}\frac{1}{n^2} \leq \frac{C}{|j^2-n^2|}$ and for $|n-j| > j/2$, $\frac{j^2}{j^2-n^2}\frac{1}{n^2} \leq Cn^2$. Therefore we conclude that there exists $C \geq 1$ such that for any $j \geq 1$,

$$\sum_{n \geq 0} |j 1_j(n^2\pi^2, q_0)|^2 \leq 2C + C \sum_{n \neq j} |\mu_n^{(j)} - n^2\pi^2|^2.$$

As $\lambda_{2n-1}(q_0) \leq \mu_n^{(j)} \leq \lambda_{2n}(q_0)$ the claimed result then follows from the asymptotics of the eigenvalues. □

In order to apply Rouché's theorem we need to prove that $|\frac{1}{j}1_j(\lambda, q) - \frac{1}{j}1_j(\lambda, q_0)| < |\frac{1}{j}1_j(\lambda, q_0)|$ for $\lambda \in \Gamma_n$ and $q \in U$. Notice that the difference $\psi_j(\lambda, q) := \frac{1}{j}1_j(\lambda, q) - \frac{1}{j}1_j(\lambda, q_0)$ is in $I_{3/2}$ and therefore, by Theorem A.3, can be written as

$$\psi_j(\lambda, q) = \sum_{n \geq 1} n A_n(q_0)(\psi_j(\cdot, q)) \frac{1}{n} 1_n(\lambda, q_0).$$

For any $\lambda \in \mathbb{C}$, the Dirac measure δ_λ is in $I^*_{-1/2}$ and $|\delta_\lambda(\psi)| \leq \|\psi\|_{I_{-1/2}} e^{|\operatorname{Im}\sqrt{\lambda}|}$ (cf. A.7). Introduce $\beta := \sup_{\lambda \in \cup_1^\infty \Gamma_k} e^{|\operatorname{Im}\sqrt{\lambda}|}$. Then $\beta < \infty$ and

$$\sup_{\lambda \in \cup_1^\infty \Gamma_k} |\psi_j(\lambda, q)| \leq \beta \|\psi_j(\cdot, q)\|_{I_{-1/2}}. \tag{A .9}$$

Further, by Lemma A.4, there exists $C' \geq 1$ such that for $j \geq 1$,

$$\begin{aligned}
&\|\psi_j(\cdot, q)\|_{I_{-1/2}} \\
&\leq \sum_{n \geq 1}^{\infty} |n A_n(q_0)(\psi_j(\cdot, q))| \left\| \frac{1}{n} 1_n(\cdot, q_0) \right\|_{I_{-1/2}} \\
&\leq C' \sum_{n \geq 1}^{\infty} \frac{1}{n^2} |n A_n(q_0)(\psi_j(\cdot, q))|.
\end{aligned}$$

Rewrite $nA_n(q_0)(\psi_j(\cdot, q))$, using the definition of ψ_j,

$$n A_n(q_0)(\psi_j(\cdot, q)) = n A_n(q_0) \left(\frac{1}{j} 1_j(\cdot, q) \right) - \delta_{nj}$$

$$
\begin{aligned}
&= \big(nA_n(q_0) - nA_n(q)\big)\left(\frac{1}{j}1_j(\cdot, q)\right) \\
&= \big(nA_n(q_0) - nA_n(q)\big)\left(\frac{1}{j}1_j(\cdot, q_0)\right) + \big(nA_n(q_0) - nA_n(q)\big)\big(\psi_j(\cdot, q)\big)
\end{aligned}
$$

to conclude that

$$
\begin{aligned}
&\sum_{n\geq 1} \frac{1}{n^2} |nA_n(q_0)(\psi_j(\cdot, q))| \\
&\leq \sum_{n\geq 1} \frac{1}{n^2} \|nA_n(q) - nA_n(q_0)\|_{I^*_{-1/2}} \left\| \frac{1}{j} 1_j(\cdot, q_0) \right\|_{I_{-1/2}} \\
&\quad + \sum_{n\geq 1} \frac{1}{n^2} \|nA_n(q) - nA_n(q_0)\|_{I^*_{-1/2}} \|\psi_j(\cdot, q)\|_{I_{-1/2}}.
\end{aligned}
$$

Apply Lemma A.1′ in a way similar as Lemma A.1 to Corollary A.2 to conclude that for any $0 < \epsilon \leq \frac{1}{2}$ there exists a neighborhood $U \equiv U_\epsilon \subset V$ of q_0 in $L^2_0(S^1; \mathbb{C})$, such that, for $q \in U$,

$$
\sum_{n\geq 1} \left\| \frac{1}{n} A_n(q) - \frac{1}{n} A_n(q_0) \right\|_{I^*_{-1/2}} \leq \epsilon.
$$

Together, the two estimates above lead to

$$
\|\psi_j(\cdot, q)\|_{I_{-1/2}} \leq \epsilon \left\| \frac{1}{j} 1_j(\cdot, q_0) \right\|_{I_{-1/2}} + \frac{1}{2} \|\psi_j(\cdot, q)\|_{I_{-1/2}}.
$$

Together with Lemma A.4, this implies $\|\psi_j(\cdot, q)\|_{I_{-1/2}} \leq 2\epsilon \|\frac{1}{j} 1_j(\cdot, q_0)\|_{I_{-1/2}} \leq 2\epsilon \frac{C'}{j^2}$. Taking into account (A.9) and the definition of ψ_j this estimate leads to

$$
\sup_{\lambda \in \cup_1^\infty \Gamma_k} \left| \frac{1}{j} 1_j(\lambda, q) - \frac{1}{j} 1_j(\lambda, q_0) \right| \leq \beta 2C' \frac{\epsilon}{j^2}, \tag{A.10}
$$

for any $q \in U$ and $j \geq 1$.

To be able to apply Rouché's theorem we also need to estimate $\frac{1}{j} 1_j(\lambda, q_0)$ from below for $\lambda \in \Gamma_n$. As q_0 is real valued, $\frac{1}{j} 1_j(\lambda, q_0)$ has product representation (A.8) and in view of the definition of the circles $\Gamma_n (n \geq 1)$, we conclude that for any integer $1 \leq N < \infty$ there exists $\rho_N > 0$ such that for any $j \geq 1$

$$
\inf_{\lambda \in \cup_1^N \Gamma_n} \left| \frac{1}{j} 1_j(\lambda, q_0) \right| \geq \rho_N \frac{1}{j^2}. \tag{A.11}
$$

Combining (A.10) and (A.11) we obtain

$$\left|\frac{1}{j}1_j(\lambda,q)\right| \geq \left|\frac{1}{j}1_j(\lambda,q_0)\right| - \left|\frac{1}{j}1_j(\lambda,q) - \frac{1}{j}1_j(\lambda,q_0)\right| \geq \frac{\rho_N - \beta 2C'\epsilon}{j^2}$$

for $\lambda \in \bigcup_1^N \Gamma_n$, $q \in U$ and $j \geq 1$.

Theorem A.5. *Let $q_0 \in L^2_0(S^1;\mathbb{C})$ be real valued, $1 \leq N < \infty$ an integer and $0 < K \leq K(q_0)$. Then there exists a bounded neighborhood $U_{N,K}$ of q_0 in $L^2_0(S^1;\mathbb{C})$ with the following properties:*

(i) *The statements of Theorem A.3 hold and (A.3) is valid;*

(ii) *for any $q \in U_{N,K}$ and $j \geq 1$, the function $\frac{1}{j}1_j(\lambda,q)$ has precisely one zero, denoted by $\mu_k^{(j)}(q)$, inside the circle $\Gamma_k = \Gamma_k(K)$ for $1 \leq k \leq N, k \neq j$. In addition, if $1 \leq j \leq N$, $\frac{1}{j}1_j(\lambda,q)$ has no zero inside $\Gamma_j(K)$;*

(iii) *the zeroes $\mu_k^{(j)}(q)$ are analytic functions of q on $U_{N,K}$.*

Proof. For ease of writing, from now on, we do not always indicate the dependence on the choice of K. Choose $\epsilon > 0$ and a neighborhood $U \equiv U_{\epsilon,N}$ of q_0 in $L^2_0(S^1;\mathbb{C})$ sufficiently small such that the statements of Theorem A.3 and (A.10) are valid and such that $\beta C\epsilon \leq \frac{1}{2}\rho_N$ where the product βC is as in (A.10) and ρ_N is given as in (A.11). Then

$$\sup_{\lambda \in \cup_1^N \Gamma_n} \left|\frac{1}{j}1_j(\lambda,q) - \frac{1}{j}1_j(\lambda,q_0)\right| \leq \frac{1}{2} \inf_{\lambda \in \cup_1^N \Gamma_n} \left|\frac{1}{j}1_j(\lambda,q_0)\right|$$

for any $j \geq 1$ and q in U. Further notice that, due to the product representation (A.8), statement (ii) holds for $q = q_0$. Therefore, by Rouché's theorem, (ii) follows. Statement (iii) is a consequence of Cauchy's integral formula

$$\mu_k^{(j)}(q) = \frac{1}{2\pi i}\int_{\Gamma_k} \lambda \frac{\frac{d}{d\lambda}1_j(\lambda,q)}{1_j(\lambda,q)} d\lambda$$

together with the analyticity of $1_j(\lambda,q)$ (cf. Theorem A.3). □

A.3. Contraction mapping.

To localize the zeroes of the $1_j(\lambda,q)$ outside the circles $\Gamma_n (1 \leq n \leq N)$, we consider the following system (cf. notation of Section 2 or (A.16) below)

$$\text{(A.12)} \quad \xi_k^{(j)}(q) = \frac{\gamma_k(q)}{2} \frac{\int_0^1 \frac{t\,dt}{\sqrt{1-t^2}}(C_{jk}(t) - C_{jk}(-t))}{\int_0^1 \frac{dt}{\sqrt{1-t^2}}(C_{jk}(t) + C_{jk}(-t))} \qquad (k \geq N+1, k \neq j)$$

where $C_{jk}(t) = C_{jk}(t, q, \xi^{(j)}(q))$ and $\mu_k^{(j)}(q) = \tau_k(q) + \xi_k^{(j)}(q)$. Let's first outline how we obtain a solution of (A.12) by using a fixed point argument:

Let U be a neighborhood of q_0 in $L_0^2(S^1; \mathbb{C})$ with the properties of Theorem A.3. Let $N \geq 1$ and chose arbitrarily $q \in U$ and $j \geq 1$. (We will choose N later sufficiently large, but N will be independent of q and j.) Let $\mathbb{N}\backslash\{j\} = \mathcal{A} \cup \mathcal{B}$ where $\mathcal{A} \equiv \mathcal{A}_{Nj} := \{1 \leq k \leq N; k \neq j\}$ and $\mathcal{B} \equiv \mathcal{B}_{Nj} := (\mathbb{N}\backslash\{j\})\backslash\mathcal{A}$.

Elements $\eta = (\eta_k)_{k \neq j}$ in $\ell^2(\mathbb{N}\backslash\{j\}; \mathbb{C})$ are decomposed $\eta = (\eta_\mathcal{A}, \eta_\mathcal{B})$ with $\eta_\mathcal{A} = (\eta_k)_{k\in\mathcal{A}} \in \ell^2(\mathcal{A}; \mathbb{C})$ and $\eta_\mathcal{B} = (\eta_k)_{k\in\mathcal{B}} \in \ell^2(\mathcal{B}; \mathbb{C})$. Given $q \in U$, denote by $\mathcal{C}_\mathcal{A}$ and $\mathcal{C}_\mathcal{B}$ the Hilbert cubes (with induced strong topology) $\mathcal{C}_\mathcal{A} \equiv \mathcal{C}_{\mathcal{A}_{Nj}} := \{\eta_\mathcal{A}; |\eta_k| \leq \gamma_k(q_0) + \frac{1}{k}K \text{ for } k \in \mathcal{A}\}$ and $\mathcal{C}_\mathcal{B} \equiv \mathcal{C}_{\mathcal{B}_{Nj}} := \{\eta_\mathcal{B}; |\eta_k| \leq \frac{1}{2}|\gamma_k(q)| + \min\left(K, \frac{1}{2}|\gamma_k(q)|\right) \text{ for } k \in \mathcal{B}\}$. (Cf. (A.3) for the constant K.)

Introduce the map $T = T_q^{(j)} : \mathcal{C}_\mathcal{A} \times \mathcal{C}_\mathcal{B} \to \ell^2(\mathcal{B}; \mathbb{C})$ where the components of $T(\eta_\mathcal{A}, \eta_\mathcal{B})$ are given by the righthand side of (A.12)

$$T(\eta_\mathcal{A}, \eta_\mathcal{B}) = \left(\frac{\gamma_k(q)}{2} \frac{\int_0^1 \frac{t\,dt}{\sqrt{1-t^2}} (C_{jk}(t) - C_{jk}(-t))}{\int_0^1 \frac{dt}{\sqrt{1-t^2}} (C_{jk}(t) + C_{jk}(-t))} \right)_{k\in\mathcal{B}} \tag{A.13}$$

with $C_{jk}(t) = C_{jk}(t, q, \eta_\mathcal{A}, \eta_\mathcal{B})$. Using the same arguments as in Section 2 one shows that there exist a (sufficiently small) neighborhood U of q_0 in $L_0^2(S^1; \mathbb{C})$ and $N \geq 1$ sufficiently large (independent of q, $j \geq 1$) so that $T(\eta_\mathcal{A}, \eta_\mathcal{B}) \in \mathcal{C}_\mathcal{B}$ (cf. Proposition A.6 below). Moreover, for any $\eta_\mathcal{A} \in \mathcal{C}_\mathcal{A}$, $T(\eta_\mathcal{A}, \cdot) : \mathcal{C}_\mathcal{B} \to \mathcal{C}_\mathcal{B}$ is a contraction (cf. Proposition A.7 below). Therefore there exists a unique element in $\mathcal{C}_\mathcal{B}$, denoted by $\kappa(\eta_\mathcal{A})$, satisfying

$$\kappa(\eta_\mathcal{A}) = T(\eta_\mathcal{A}, \kappa(\eta_\mathcal{A})). \tag{A.14}$$

In the remainder of this section we prove the statements claimed above.

Proposition A.6. *There exist a (sufficiently small) neighborhood U of q_0 in $L_0^2(S^1; \mathbb{C})$ and $N \geq 1$ so that for q in U and $j \geq 1$ the map $T = T_q^{(j)} : \mathcal{C}_\mathcal{A} \times \mathcal{C}_\mathcal{B} \to \ell^2(\mathcal{B}; \mathbb{C})$ satisfies the following estimate for $k \in \mathcal{B}$ and any $(\eta_\mathcal{A}, \eta_\mathcal{B}) \in \mathcal{C}_\mathcal{A} \times \mathcal{C}_\mathcal{B}$*

$$|T(\eta_\mathcal{A}, \eta_\mathcal{B})_k| \leq C \frac{|\gamma_k(q)|^2}{k} \leq \frac{1}{2}|\gamma_k(q)| + \min\left(K, |\gamma_k(q)|\right) \tag{A.15}$$

where $C > 0$ is independent of $q \in U, j \geq 1$ and $k \in \mathcal{B}$.

Proof. Recall that $T(\eta_\mathcal{A}, \eta_\mathcal{B})$ is given by

$$T(\eta_\mathcal{A}, \eta_\mathcal{B}) = \left(\frac{\gamma_k(q)}{2} \frac{\int_0^1 \frac{t\,dt}{\sqrt{1-t^2}} (C_{jk}(t) - C_{jk}(-t))}{\int_0^1 \frac{dt}{\sqrt{1-t^2}} (C_{jk}(t) + C_{jk}(-t))} \right)_{k\in\mathcal{B}}.$$

As in Section 2, we write $C_{jk}(t) \equiv C_{jk}(t, q, \eta_{\mathcal{A}}, \eta_{\mathcal{B}})$ as a product of the form

$$C_{jk}(t) = \sqrt{\frac{\Delta_0(0)}{\Delta_0(t)}} \sqrt{\frac{\Delta_j(0)}{\Delta_j(t)}} \frac{A_{jk}(t)}{A_{jk}(0)} \frac{B_{jk}(t)}{B_{jk}(0)}. \tag{A .16}$$

By the same proof as for Lemma 2.3-Lemma 2.5 one shows that there exists a neighborhood U of q_0 in $L_0^2(S^1;\mathbb{C})$ and $C > 0$ independent of j, k and $q \in U$ so that for $q \in U, k \neq j$

$$\begin{aligned}
\sup_{|t|\le 1} \left| \frac{A_{jk}(t)}{A_{jk}(0)} - 1 \right| &\le C|\gamma_k(q)| \sum_{|\alpha-k|\ge \frac{k}{2}} \frac{|\gamma_k(q)|}{\alpha^4} + C|\gamma_k(q)| \sum_{|\alpha-k|\le \frac{k}{2}} \frac{|\gamma_\alpha(q)|}{\alpha^2} \\
&\le C\frac{|\gamma_k(q)|}{k}, \\
\sup_{|t|\le 1} \left| \frac{B_{jk}(t)}{B_{jk}(0)} - 1 \right| &\le \frac{C|\gamma_k(q)|}{k^3} \left(\sum_{|\alpha-k|\le \frac{k}{2}} |\gamma_\alpha|^2 + \frac{1}{k^3} \right) \\
&\le C\frac{|\gamma_k(q)|}{k^3}
\end{aligned}$$

and

$$\begin{aligned}
\sup_{|t|\le 1} \left| \left(\frac{\Delta_0(0)}{\Delta_0(t)} \right)^{1/2} - 1 \right| &\le C\frac{|\gamma_k(q)|}{k^2} \\
\sup_{|t|<1} \left| \left(\frac{\Delta_j(0)}{\Delta_j(t)} \right)^{1/2} - 1 \right| &\le C\frac{|\gamma_k(q)|}{k}.
\end{aligned}$$

We proceed as in the proof of Theorem 2.7 and write

$$\begin{aligned}
C_{jk}(t) = \left(\frac{\Delta_j(0)}{\Delta_j(t)} \right)^{1/2} &+ \left(\frac{\Delta_j(0)}{\Delta_j(t)} \right)^{1/2} \left(\left(\frac{\Delta_0(0)}{\Delta_0(t)} \right)^{1/2} - 1 \right) \\
&+ \left(\frac{\Delta_j(0)}{\Delta_j(t)} \right)^{1/2} \left(\frac{\Delta_0(0)}{\Delta_0(t)} \right)^{1/2} \left(\frac{B_{jk}(t)}{B_{jk}(0)} - 1 \right) \\
&+ \left(\frac{\Delta_j(0)}{\Delta_j(t)} \right)^{1/2} \left(\frac{\Delta_0(0)}{\Delta_0(t)} \right)^{1/2} \frac{B_{jk}(t)}{B_{jk}(0)} \left(\frac{A_{jk}(t)}{A_{jk}(0)} - 1 \right).
\end{aligned}$$

Then for $|t| \le 1$, $(\eta_{\mathcal{A}}, \eta_{\mathcal{B}}) \in \mathcal{C}_{\mathcal{A}} \times \mathcal{C}_{\mathcal{B}}$

$$\begin{aligned}
&\left| \frac{\Delta_j(0)}{\Delta_j(t)} \right|^{1/2} \le 1 + C\frac{|\gamma_k(q)|}{k}, \\
&\left| \left(\frac{\Delta_j(0)}{\Delta_j(t)} \right)^{1/2} \left(\left(\frac{\Delta_0(0)}{\Delta_0(t)} \right)^{1/2} - 1 \right) \right| \le \left(1 + C\frac{|\gamma_k(q)|}{k} \right) C\frac{|\gamma_k(q)|}{k^2}
\end{aligned}$$

$$\left|\left(\frac{\Delta_j(0)}{\Delta_j(t)}\right)^{1/2}\left(\frac{\Delta_0(0)}{\Delta_0(t)}\right)^{1/2}\left(\frac{B_{jk}(t)}{B_{jk}(0)}-1\right)\right|$$
$$\le\left(1+C\frac{|\gamma_k(q)|}{k}\right)\left(1+C\frac{|\gamma_k(q)|^2}{k^2}\right)C\frac{|\gamma_k(q)|}{k^3}$$

and

$$\left|\left(\frac{\Delta_j(0)}{\Delta_j(t)}\right)^{1/2}\left(\frac{\Delta_0(0)}{\Delta_0(t)}\right)^{1/2}\frac{B_{jk}(t)}{B_{jk}(0)}\left(\frac{A_{jk}(t)}{A_{jk}(0)}-1\right)\right|$$
$$\le\left(1+C\frac{|\gamma_k(q)|}{k}\right)\left(1+C\frac{|\gamma_k(q)|}{k^2}\right)\left(1+C\frac{|\gamma_k(q)|}{k^3}\right)C\frac{|\gamma_k(q)|}{k}.$$

We conclude, similar as in the proof of Theorem 2.7 (and using that $|\eta_k|\le|\gamma_k(q)|$ for $k\in\mathcal{B}$), that for $q\in U, j\ge 1, (\eta_{\mathcal{A}},\eta_{\mathcal{B}})\in\mathcal{C}_{\mathcal{A}}\times\mathcal{C}_{\mathcal{B}}$

$$|T_q^{(j)}(\eta_{\mathcal{A}},\eta_{\mathcal{B}})_k|\le C\frac{|\gamma_k(q)|^2}{k}.$$

Notice that $|\gamma_k(q)|^2\le\sum_{\alpha\ge 1}|\gamma_\alpha(q)|^2$ is bounded on U. Therefore there exists $N\ge 1$ independent of $q\in U, j\ge 1$ such that for $k\ge N, q\in U$

$$\frac{C|\gamma_k(q)|}{k}\le 1; C\frac{|\gamma_k(q)|^2}{k}\le\frac{|\gamma_k(q)|}{2}+K.$$

□

Notice that it follows from Proposition A.6 that the range of $T_q^{(j)}:\mathcal{C}_{\mathcal{A}}\times\mathcal{C}_{\mathcal{B}}\to\ell^2(\mathcal{B};\mathbb{C})$ is contained in $\mathcal{C}_{\mathcal{B}}$.

Proposition A.7. *There exist a (sufficiently small) neighborhood U of q_0 in $L_0^2(S^1;\mathbb{C})$ and $N\ge 1$ so that for $q\in U$ and $j\ge 1$, $T=T_q^{(j)}:\mathcal{C}_{\mathcal{A}}\times\mathcal{C}_{\mathcal{B}}\to\mathcal{C}_{\mathcal{B}}$ satisfies, for $(\eta_{\mathcal{A}},\eta_{\mathcal{B}})\in\mathcal{C}_{\mathcal{A}}\times\mathcal{C}_{\mathcal{B}}$*

$$\|T(\eta'_{\mathcal{A}},\eta'_{\mathcal{B}})-T(\eta_{\mathcal{A}},\eta_{\mathcal{B}})\|^2_{\ell^2(\mathcal{B};\mathbb{C})}\le\frac{1}{4}(\|\eta'_{\mathcal{A}}-\eta_{\mathcal{A}}\|^2+\|\eta'_{\mathcal{B}}-\eta_{\mathcal{B}}\|^2).$$

In particular, T is continuous and, for any $\eta_{\mathcal{A}}\in\mathcal{C}_{\mathcal{A}}$ fixed, the map $T(\eta_{\mathcal{A}},\cdot):\mathcal{C}_{\mathcal{B}}\to\mathcal{C}_{\mathcal{B}}$ is a contraction.

Proof. Notice that in the decomposition (A.16) of $C_{jk}(t)$ only the term $\frac{A_{jk}(t)}{A_{jk}(0)}$ depends on $\eta=(\eta_{\mathcal{A}},\eta_{\mathcal{B}})$. Recall that $A_{jk}(t)=\prod_{\alpha\in\mathbb{N}\setminus\{j,k\}}\left(1+\frac{\eta_\alpha}{\tau_\alpha-\lambda(t)}\right)$. Thus, with $D_{jk}(t)\equiv D_{jk}(t,q,\eta'_{\mathcal{A}},\eta'_{\mathcal{B}},\eta_{\mathcal{A}},\eta_{\mathcal{B}})$,

$$D_{jk}(t):=\frac{A_{jk}(t,q,\eta'_{\mathcal{A}},\eta'_{\mathcal{B}})}{A_{jk}(0,q,\eta'_{\mathcal{A}},\eta'_{\mathcal{B}})}-\frac{A_{jk}(t,q,\eta_{\mathcal{A}},\eta_{\mathcal{B}})}{A_{jk}(0,q,\eta_{\mathcal{A}},\eta_{\mathcal{B}})}$$

$$= \sum_{\substack{\beta=1 \\ \beta\neq k,j}}^{\infty} \left(\prod_{\substack{\alpha<\beta \\ \alpha\neq k,j}} \frac{(1+\frac{\eta'_\alpha}{\tau_\alpha-\lambda(t)})}{1+\frac{\eta'_\alpha}{\tau_\alpha-\tau_k}} \prod_{\substack{\alpha>\beta \\ \alpha\neq k,j}} \frac{(1+\frac{\eta_\alpha}{\tau_\alpha-\lambda(t)})}{1+\frac{\eta_\alpha}{\tau_\alpha-\tau_k}} \right) \cdot \left(\frac{1+\frac{\eta'_\beta}{\tau_\beta-\lambda(t)}}{1+\frac{\eta'_\beta}{\tau_\beta-\tau_k}} - \frac{1+\frac{\eta_\beta}{\tau_\beta-\lambda(t)}}{1+\frac{\eta_\beta}{\tau_\beta-\tau_k}} \right).$$

Using the estimates in the proof of Proposition A.6 one shows that there exists $C > 1$ independent of $q \in U, j \geq 1$ and $(\eta'_{\mathcal{A}}, \eta'_{\mathcal{B}}), (\eta_{\mathcal{A}}, \eta_{\mathcal{B}}) \in \mathcal{C}_{\mathcal{A}} \times \mathcal{C}_{\mathcal{B}}$ such that

$$|D_{jk}(t)| \leq C \sum_{\beta\neq k,j} \frac{|\eta_\beta - \eta'_\beta|}{|\beta^2 - k^2|}.$$

This leads to the estimate (for some $C > 0$ independent of q, j, η, η')

$$\begin{aligned} &\|T(\eta'_{\mathcal{A}}, \eta'_{\mathcal{B}}) - T(\eta_{\mathcal{A}}, \eta_{\mathcal{B}})\|^2_{\ell^2(\mathcal{B};\mathbb{C})} \\ &\leq C \sum_{k\in\mathcal{B}} \left|\frac{\gamma_k(q)}{2}\right|^2 \left|\sum_{\beta\neq k,j} \frac{|\eta_\beta - \eta'_\beta|}{|\beta^2 - k^2|}\right|^2 \leq C \sum_{k\in\mathcal{B}} \left|\frac{\gamma_k(q)}{2}\right|^2 \frac{1}{k^2} \|\eta - \eta'\|^2. \end{aligned}$$

By choosing $N \geq 1$ sufficiently large, but independent of j, q it follows that for any q in U

$$C \sum_{k\in\mathcal{B}} \left|\frac{\gamma_k(q)}{2}\right|^2 \frac{1}{k^2} \leq \frac{1}{4}.$$

□

Corollary A.8. *There exist a (sufficiently small) neighborhood U of q_0 and $N \geq 1$ so that for $q \in U$ and for all $j \geq 1$, $T = T_q^{(j)} : \mathcal{C}_{\mathcal{A}} \times \mathcal{C}_{\mathcal{B}} \to \mathcal{C}_{\mathcal{B}}$ has for any $\eta_{\mathcal{A}} \in \mathcal{C}_{\mathcal{A}}$ a unique fixed point $\kappa(\eta_{\mathcal{A}}) = \kappa^{(j)}(\eta_{\mathcal{A}}, q) \in \mathcal{C}_{\mathcal{B}}$,*

$$T(\eta_{\mathcal{A}}, \kappa(\eta_{\mathcal{A}})) = \kappa(\eta_{\mathcal{A}}).$$

Moreover, for any $\eta_{\mathcal{A}}, \eta'_{\mathcal{A}}$ in $\mathcal{C}_{\mathcal{A}}$,

$$\|\kappa(\eta'_{\mathcal{A}}) - \kappa(\eta_{\mathcal{A}})\|^2_{\ell^2(\mathcal{B};\mathbb{C})} \leq \frac{1}{3} \|\eta'_{\mathcal{A}} - \eta_{\mathcal{A}}\|^2_{\ell^2(\mathcal{A},\mathbb{C})}.$$

In particular, the map $\kappa : \mathcal{C}_{\mathcal{A}} \to \mathcal{C}_{\mathcal{B}}$ is continuous.

Proof. Fix an arbitrary element $\eta_{\mathcal{A}} \in \mathcal{C}_{\mathcal{A}}$. Then, by Proposition A.7, for any $\eta'_{\mathcal{B}}, \eta_{\mathcal{B}} \in \mathcal{C}_{\mathcal{B}}$

$$\|T(\eta_{\mathcal{A}}, \eta'_{\mathcal{B}}) - T(\eta_{\mathcal{A}}, \eta_{\mathcal{B}})\|_{\ell^2(\mathcal{B};\mathbb{C})} \leq \frac{1}{4} \|\eta'_{\mathcal{B}} - \eta_{\mathcal{B}}\|_{\ell^2(\mathcal{B};\mathbb{C})}.$$

Therefore $T(\eta_{\mathcal{A}}, \cdot)$ is a contraction map on $\mathcal{C}_{\mathcal{B}}$ and there exists a unique fixed point $\kappa(\eta_{\mathcal{A}}) \in \mathcal{C}_{\mathcal{B}}$, i.e., $\kappa(\eta_{\mathcal{A}}) = T(\eta_{\mathcal{A}}, \kappa_{\mathcal{B}}(\eta_{\mathcal{A}}))$.

For $\eta_{\mathcal{A}}, \eta'_{\mathcal{A}} \in \mathcal{C}_{\mathcal{A}}$ one obtains, again by Proposition A.7,

$$\begin{aligned}\|\kappa(\eta'_{\mathcal{A}}) - \kappa(\eta_{\mathcal{A}})\|^2 &= \|T(\eta'_{\mathcal{A}}, \kappa(\eta'_{\mathcal{A}})) - T(\eta_{\mathcal{A}}, \kappa(\eta_{\mathcal{A}}))\|^2 \\ &\le \frac{1}{4}(\|\eta'_{\mathcal{A}} - \eta_{\mathcal{A}}\|^2 + \|\kappa(\eta'_{\mathcal{A}}) - \kappa(\eta_{\mathcal{A}})\|^2).\end{aligned}$$

As a consequence, as claimed,

$$\|\kappa(\eta'_{\mathcal{A}}) - \kappa(\eta_{\mathcal{A}})\|^2 \le \frac{1}{3}\|\eta'_{\mathcal{A}} - \eta_{\mathcal{A}}\|^2.$$

□

Proof of Theorem 2.1.

To prove Theorem 2.1 we need one more auxilary result. To formulate it, introduce for a real valued potential $q_0 \in L^2_0(S^1)$ circles $\tilde{\Gamma}_k(q_0)$ in $\mathbb{C}$ of radius $\frac{1}{2}\gamma_k(q_0) + \frac{1}{k}2K(q_0)$ and center $\tau_k(q_0)$ where $K(q_0) := \frac{1}{5}\min_{n\ge 0}(\lambda_{2n+1}(q_0) - \lambda_{2n}(q_0))$. Denote by $\tilde{D}_k(q_0)$ the closed disk with boundary $\tilde{\Gamma}_k(q_0)$.

Lemma A.9. *Let $q_0 \in L^2_0(S^1;\mathbb{C})$ be real valued. Then there exists $N(q_0) \ge 1$ so that for any $N \ge N(q_0)$ there exists a neighborhood U of q_0 in $L^2_0(S^1;\mathbb{C})$ with the following properties: For any $q \in U$, $|\lambda_j(q) - \lambda_j(q_0)| \le \frac{1}{k}K(q_0)$ $(1 \le k \le N, j = 2k, 2k-1)$, and, given arbitrary complex numbers μ_k $(1 \le k \le N)$ inside $\tilde{\Gamma}_k(q_0)$, the Dirac measures $\left((-1)^k 2\pi^2 k^2 \delta_{\mu_k}\right)_{1\le k\le N}$ together with $\left(nA_n(q)\right)_{n\ge N+1}$ form a Riesz basis of $I^*_{3/2}$.*

Proof. The proof relies on the following three observations:

(1) For any sequence $(\mu_k)_{k\ge 1}$ with μ_k in $\tilde{D}_k(q_0)$, the Dirac measures $(-1)^k 2\pi^2 k^2 \delta_{\mu_k}$ form a Riesz basis of $I^*_{3/2}$. To see this we first observe that there exists $N_0 \ge 1$ such that

$$\sum_{n\ge N_0+1} \|(-1)^k 2\pi^2 k^2 \delta_{\mu_k} - (-1)^k 2\pi^2 k^2 \delta_{\pi^2 k^2}\|^2_{I^*_{3/2}} < \frac{1}{4}$$

for any choice of μ_k in $\tilde{D}_k(q_0)$.

Recall (cf. Remark after Theorem A.3) that $\left((-1)^k 2\pi^2 k^2 \delta_{\pi^2 k^2}\right)_{k\ge 1}$ is an orthonormal basis of $I^*_{3/2}$. Thus $\left((-1)^k 2\pi^2 k^2 \delta\mu_k\right)_{k\ge N_0+1}$ spans a subspace of $I^*_{3/2}$ of codimension N_0. By Bari's theorem, adding precisely N_0 Dirac measures $\left((-1)^k 2\pi^2 k^2 \delta\mu_k\right)_{1\le k\le N_0}$ to this set will give rise to a Riesz basis as soon as $\left((-1)^k 2\pi^2 k^2 \delta\mu_k\right)_{k\ge 1}$ is ω-linearly independent. To prove the linear independence, introduce $\psi_j(\lambda) := \frac{1}{j^2}\prod_{k\neq j}\frac{\mu_k-\lambda}{k^2\pi^2}$. These functions are elements of $I_{3/2}$ and satisfy $\delta_{\mu_k}(\psi_j) = 0$ for $k \neq j$ and $\delta_{\mu_j}(\psi_j) \neq 0$. This proves that $\left((-1)^k 2\pi^2 k^2 \delta\mu_k\right)_{k\geq 1}$ is ω-linearly independent.

(2) Introduce for any sequence $\mu = (\mu_k)_{k\geq 1}$ with μ_k in $\tilde{D}_k(q_0)$ the map $T_\mu : I^*_{3/2} \to I^*_{3/2}$ given by $T_\mu\left((-1)^k 2\pi^2 k^2 \delta_{\pi^2 k^2}\right) = (-1)^k 2\pi^2 k^2 \delta_{\mu_k}$, corresponding to the change of basis from the orthonormal basis $\left((-1)^k 2\pi^2 k^2 \delta_{\pi^2 k^2}\right)_{k\geq 1}$ to the Riesz basis (cf. (1)) $\left((-1)^k 2\pi^2 k^2 \delta_{\mu_k}\right)_{k\geq 1}$. Then T_μ is a bounded invertible operator, $\|T_\mu\|_{\mathcal{L}(I^*_{3/2})} < \infty$ and $\|T_\mu^{-1}\|_{\mathcal{L}(I^*_{3/2})} < \infty$. One verifies that T_μ depends continuously on $\mu = (\mu_k)_{k\geq 1} \in \prod_{k\geq 1}\tilde{D}_k(q_0) \subseteq \ell^2(\mathbb{N};\mathbb{C})$. Due to the fact that the Hilbert cube $\prod_{k\geq 1}\tilde{D}_k(q_0)$ is compact in $\ell^2(\mathbb{N};\mathbb{C})$ we conclude that there exists $C(q_0) > 0$ such that $\|T_\mu\|^2_{\mathcal{L}(I^*_{3/2})}, \|T_\mu^{-1}\|^2_{\mathcal{L}(I^*_{3/2})} \leq C(q_0)$ for all $\mu \in \prod_{k\geq 1}\tilde{D}_k(q_0)$.

(3) By standard arguments one shows that there exists $N(q_0) \geq 1$ such that (with $C(q_0) > 0$ given as in (2))

$$\sum_{k\geq N(q_0)+1} \|(-1)^k 2\pi^2 k^2 \delta_{\tau_k(q_0)} - kA_k(q_0)\|^2_{I^*_{3/2}} \leq \frac{1}{4}\frac{1}{C(q_0)}. \tag{A.17}$$

We now combine (1), (2) and (3) to prove the Lemma A.9:

For $N \geq N(q_0)$ given, choose a neighborhood U of q_0 in $L^2_0(S^1;\mathbb{C})$ so that for q in U (cf. Corollary A.2)

$$\sum_{n\geq 1} \|nA_n(q) - nA_n(q_0)\|^2_{I^*_{3/2}} \leq \frac{1}{4}\frac{1}{C(q_0)} \tag{A.18}$$

and, in addition, for $1 \leq k \leq N$ and $j \in \{2k, 2k-1\}$, $|\lambda_j(q) - \lambda_j(q_0)| \leq \frac{1}{k}K$ as well as $|\lambda_0(q) - \lambda_0(q_0)| \leq K$. Combining these estimates with (A.18) and (A.17) one concludes that, for $q \in U$,

$$\sum_{n\geq N+1} \|nA_n(q) - (-1)^n 2n^2\pi^2 \delta_{\tau_n(q_0)}\|^2_{I^*_{3/2}} \leq \frac{1}{2}\frac{1}{C(q_0)}.$$

In view of observation (2) and Bari's theorem (cf. [**GK**, p. 310]) one sees that $(\delta_{\mu_k})_{1\leq k\leq N(\epsilon)'}$ $\left(nA_n(q)\right)_{n\geq N(\epsilon)+1}$ is a Riesz basis for any $q \in U$ and arbitrary $\mu_k \in \tilde{D}_k(q_0)$, $1 \leq k \leq N(\epsilon)$. □

Proof of Theorem 2.1. For a given real valued potential $q_0 \in L^2_0(S^1;\mathbb{C})$ let $K_1 := K(q_0)$ and choose a neighborhood U_1 of q_0 in $L^2_0(S^1:\mathbb{C})$ so that the statements of Theorem A.3 hold. Next choose $N \geq 1$ and a neighborhood $U_2 \subset U_1$ of q_0 in $L^2_0(S^1;\mathbb{C})$ so that the statements of Propositions A.6, A.7, Corollary A.8 and Lemma A.9 are valid. For this N, let $K_2 := \frac{1}{N}K(q_0)$ and choose a neighborhood $U_3 \subset U_2$ of q_0 in $L^2_0(S^1;\mathbb{C})$ so that the statements of Theorem A.5 hold. In particular, for any q in U_3 and $j \geq 1$, the function $\frac{1}{j}1_j(\lambda, q)$ has precisely one zero $\mu_k^{(j)}(q)$ inside the circle $\Gamma_k(K_2)$ for k in $\mathcal{A}_{Nj} := \{1 \leq k \leq N, k \neq j\}$. Let $U_{\mathcal{A}_{Nj}} := \big(\mu_k^{(j)}(q)\big)_{k\in\mathcal{A}_{Nj}}$ $(q \in U_3, j \geq 1)$ and denote by $\eta_{\mathcal{B}} \equiv \eta_{\mathcal{B}_{Nj}}$ the unique fixed point $\kappa(\eta_{\mathcal{A}_{Nj}})$, provided by Corollary A.8. Define the entire function ψ_j in $I_{3/2}$ given by

$$\psi_j(\lambda, q) := \frac{1}{j^2}\prod_{k\in\mathcal{A}_{Nj}}\frac{\mu_k^{(j)}(q)-\lambda}{k^2\pi^2}\prod_{k\in\mathcal{B}_{Nj}}\frac{(\eta_{\mathcal{B}})_k-\lambda}{k^2\pi^2}.$$

By construction,

$$\delta_{\mu_k^{(j)}(q)}(\psi_j(\cdot, q)) = 0 = \delta_{\mu_k^{(j)}(q)}\left(\frac{1}{j}1_j(\cdot, q)\right) \qquad (k \in \mathcal{A}_{Nj})$$

and

$$A_k(q)(\psi_j(\cdot, q)) = 0 = A_k(q)\left(\frac{1}{j}1_j(\cdot, q)\right) \qquad (k \in \mathcal{B}_{Nj}).$$

For $1 \leq j \leq N$ it follows from Theorem A.5 that $\delta_{\tau_j(q_0)}\big(\frac{1}{j}1_j(\cdot, q)\big) \neq 0$ as $\frac{1}{j}1_j(\cdot, q)$ has no zeroes inside Γ_j. By construction $\delta_{\tau_j(q_0)}(\psi_j) \neq 0$. Therefore, there exists $C_j \equiv C_j(q) \neq 0$ such that

$$\delta_{\tau_j(q_0)}(C_j\psi_j) = \delta_{\tau_j(q_0)}\left(\frac{1}{j}1_j(\cdot, q)\right).$$

By Lemma A.9, $\big((-1)^k 2\pi^2k^2\delta_{\mu_k^{(j)}(q)}\big)_{k\in\mathcal{A}_{Nj}}, (-1)^j2\pi^2j^2\delta_{\tau_j(q_0)}, \big(kA_k(q)\big)_{k\geq N+1}$ is a Riesz basis of $I^*_{3/2}$. This implies that $C_j\psi_j(\lambda, q) = \frac{1}{j}1_j(\lambda, q)$ for $1 \leq j \leq N$ and $q \in U_3$. For $j \geq N+1$, one shows, using standard asymptotic results, that $A_j(q)\big(\psi_j(\cdot, q)\big) \neq 0$. Recall that $jA_j(q)\big(\frac{1}{j}1_j(\cdot, q)\big) = 1$ and thus there exists $C_j \equiv C_j(q) \neq 0$ with $jA_j(q)\big(C_j\psi_j(\cdot, q)\big) = 1$.

By Lemma A.9, $\big((-1)^k2\pi^2k^2\delta_{\mu_k^{(j)}(q)}\big)_{1\leq k\leq N}, \big(\frac{1}{k}A_k(q)\big)_{k\geq N+1}$ is a Riesz basis of $I^*_{3/2}$. This implies that $C_j\psi_j(\lambda, q) = \frac{1}{j}1_j(\lambda, q)$ for $j \geq N+1$ and $q \in U_3$. By definition of the Hilbert cube $\mathcal{C}_{\mathcal{B}}$ (cf. Section A.3) $\big(\mu_k^{(j)}(q) - \tau_k(q)\big) =: \xi_k^{(j)}$ $(k \in \mathcal{B})$ satisfy the estimates stated in Theorem 2.1 and the analyticity of the zeroes $\mu_k^{(j)}(q)$ $(k \in \mathcal{B})$ follows again from Cauchy's integral formula

$$\mu_k^{(j)}(q) = \frac{1}{2\pi i}\int_{\Gamma_k}\lambda\frac{\frac{d}{d\lambda}1_j(\lambda, q)}{1_j(\lambda, q)}d\lambda.$$

□

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