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# CLASSIFICATION OF DIRECT LIMITS OF GENERALIZED TOEPLITZ ALGEBRAS

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**A classification is given to the class of $C^*$-algebras of real rank zero which are direct limits of so called generalized Toeplitz algebras. This class includes all so called $A\mathbb{T}$-algebras of real rank zero as well as many $C^*$-algebras which are not stably finite.**

## 1. Introduction.

A certain class of $C^*$-algebras is classified. The class consists $C^*$-algebras of real rank zero that can be expressed as direct limits of finite direct sums of matrix algebras over $\mathcal{T}$-algebras, where $\mathcal{T}$-algebras are unital essential extensions of $C(S^1)$ by compact operators $\mathcal{K}$ :

$$0 \to \mathcal{K} \to E \to C(S^1) \to 0.$$

Let $A$ be an algebra in the class. The invariant consists of the abelian semigroup $V(A)$, the Murry-von Neumann equivalence classes of projections in matrices of $A$, an abelian semigroup $k(A)_+$, some equivalence classes of homotopy classes of hyponormal partial isometries in matrices of $A$ and a homomorphism $d$ from $k(A)_+$ into $V(A)$. The main result of this paper states that the above invariant, together with the class of the identity, is complete for the class of $C^*$-algebras that we consider (cf. §5). For the subclass of the above class which consists of direct limits of finite direct sums of matrix algebras over only non-trivial extensions of the above type, the invariants are even simplier—just $V(A)$. We also show that the algebras in the class exhaust all possible invariants (cf. 3.7).

Our paper can be viewed as part of the program of classifying "amenable" $C^*$-algebras initiated by George A. Elliott. The classical model for the program is the classification of $AF$-algebras by their dimension groups [**Ell1**]. Elliott proved in [**Ell2**] that the class of real rank zero $A\mathbb{T}$-algebras, direct limits of finite direct sums of matrix algebras over $C(S^1)$ of real rank zero, could be classified by the graded group $K_0 \oplus K_1$ with its natural order. Since then a number of classification results appeared ([**BEEK**], [**D1**], [**D2**], [**DL1**], [**DL2**], [**Ell3**], [**Ell4**], [**Ell5**], [**EE**], [**EG1**], [**EG2**], [**EGL**], [**EGLP**], [**ER**], [**G1**], [**G2**], [**G3**], [**K**], [**Li**], [**Ln2**], [**Ln3**], [**Ln5**], [**LP1**], [**LP2**], [**LP3**], [**Lr**], [**Ph1**], [**Ph2**], [**Rr1**], [**Rr2**], [**Rr3**], [**Rr4**], [**Rr5**], [**Su1**], [**Su2**], [**Tm**], etc.). Many classes of $C^*$-algebras are classified. These include $C^*$-algebras of real rank zero which are direct limits of homogeneous algebras of slow dimension growth and a large class (presumably all) separable nuclear purely infinite simple $C^*$-algebras. The situation is changing even when we are typing this paper. $C^*$-algebras that we consider in this paper are usually nonsimple and their stable rank are usually greater than one. The class does contain all $A\mathbb{T}$-algebras of real rank zero. It contains many other $C^*$-algebras which are not of so called approximately homogeneous $C^*$-algebras. One of the important features which makes our class essentially different from approximately homogeneous algebras is that the torsion in $K_0$ does not arise from the torsion parts of certain metric spaces but from nontrivial extensions of $C(S^1)$ by $\mathcal{K}$.

Inductive limits of finite direct sums of matrix algebras over the classical Toeplitz algebra $\mathcal{T}_1$ (i.e. the extension with index $-1$) was first studied in [**EES**]. A stable isomorphism theorem was obtained for the special case that each summand is a single matrix algebra over $\mathcal{T}_1$ and the algebras are of real rank zero. In [**EES**], the algebras in question were shown to be absorbing extensions of real rank zero $A\mathbb{T}$-algebras by stable AF algebras, which led to the stable classification.

The class that we consider here allows building blocks to be matrix algebras over any unital essential extensions of $C(S^1)$ by $\mathcal{K}$. This gives us torsion in $K$-theory as well as non-trivial $K_1$-theory. We also obtained a classification up to isomorphism instead of stable isomorphism.

One of the technical problems that we have to deal with is to lift two "close" homomorphisms from $M_k(C(S^1))$ into $M_m(C(S^1))$ to two "close" homomorphisms from $M_k(\mathcal{T}_i)$ into $M_m(\mathcal{T}_j)$. Here, $\mathcal{T}_i$ and $\mathcal{T}_j$ are the extensions of $C(S^1)$ by $\mathcal{K}$ with indexes $-i$ and $-j$, respectively (cf. 2.1). There are several problems here. First, not every homomorphism from $M_k(C(S^1))$ into $M_m(C(S^1))$ comes from an injective homomorphism from $M_k(\mathcal{T}_i)$. Second, two quite different homomorphisms from $M_k(\mathcal{T}_i)$ into $M_m(\mathcal{T}_j)$ which map $M_k(\mathcal{T}_i)$ to $M_m(\mathcal{K})$ induce two zero homomorphisms from $M_k(C(S^1))$ to $M_m(C(S^1))$. However, there is nothing to lift. Third, assuming the life is perfect, $k = m$, $i = j$ and two homomorphisms are two automorphisms on $M_k(\mathcal{T}_i)$ which induce the identity map on $C(S^1)$. How much can we say about these two automorphisms?

It turns out that the first two problems could be avoided using some existing tricks. For the third problem, we establish a so called uniqueness theorem (Section 4). It uses the BDF-theory as well as Voiculescu's generalized Weyl-von Neumann theorem. A special case of one of the lemmas says that an automorphism $\alpha$ of the Toeplitz algebra which has the property that $\pi \circ \alpha = \pi$, where $\pi$ is the quotient map from the Toeplitz algebra onto the circle algebra $C(S^1)$, is approximately inner, a result has been just obtained by Rørdam independently with a completely different proof.

Since we allow the building blocks to include trivial extension $\mathcal{T}_0$, the invariants become more complicated. For example, $V(\mathcal{T}_0)$ is not finitely generated. This causes lots of problems when we try to employ a standard intertwining argument. For this, we have to do some surgical work.

We have some description (3.4) of the invariants that we use here. It would be nicer to have a characterization of our invariants. A version of Effros-Handelmen-Shen theorem which characterizes the dimension groups or an Elliott's theorem ([**Ell6**]) is certainly desirable. However, at present, it is beyond our reach to characterize these semigroups and we feel that it should be treated separately.

A $C^*$-algebra in the class often becomes an essential extension of an $A\mathbb{T}$-algebra by an AF-algebra. But this paper is not about classification of extensions. Classification of extensions requires to know the ideal and the quotient. Our invariant is defined for general $C^*$-algebras which does not require to know the ideal nor the quotient. In fact, a $C^*$-algebra in the class may be written into different extensions. In other words, a result of classification of extensions of a given $A\mathbb{T}$-algebra by a given AF-algebra, while is important, would not classify our class.

The paper is organized as follows. In Section 2, we define and discuss the invariants of $A\mathcal{T}$-algebras. It is rather tedious section. In Section 3, we establish an existence theorem. We further show that given a possible invariant $V_*$ there is a direct limit $A$ of finite direct sums of matrix algebras over $A\mathcal{T}$-algebras with real rank zero such that $V_*(A) = V_*$. We also show that every $A\mathbb{T}$-algebra of real rank zero can be expressed as a direct limit of finite direct sums of matrix algebras over $A\mathcal{T}$-algebras. In Section 4, we give the uniqueness theorem. In Section 5, we combine the results in the previous sections and give the main result.

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## 2. $\mathcal{T}$-algebras and their invariants.

**2.1.** Let $C(S^1)$ be the continuous functions on the unit circle and let $\mathcal{K}$ be the compact operators on an infinite dimensional separable Hilbert space. The generalized Toeplitz algebras $\mathcal{T}_k$ is an essential unital extension of $C(S^1)$ by $\mathcal{K}$ with index $-k \in \mathbf{Z}$:

$$0 \to \mathcal{K} \to \mathcal{T}_k \to C(S^1) \to 0.$$

It is well known that two extensions with the same index are isomorphic as $C^*$-algebras. (It certainly follows from [V].) We call these algebras $\mathcal{T}$-algebras. It is obvious that $\mathcal{T}_k$ is isomorphic to $\mathcal{T}_{-k}$. So we consider only those $\mathcal{T}_k$ with $k \geq 0$.

We now give another description of $\mathcal{T}_k$ (for $k \geq 0$). Let $S_1$ be the standard unilateral shift operator acting on the Hilbert space $H = l^2$. Then $\mathcal{T}_1$ is isomorphic to the universal $C^*$-subalgebra of $B(H)$ generated by $S_1$. Note that the ideal of the $C^*$-subalgebra generated by $1 - S_1S_1^*$ is the ideal of compact operators on $H$. We will identify $\mathcal{T}_1$ with this algebra whenever it is convenient without further explanation. The element $S_1$ in $\mathcal{T}_1$ will mean the shift operator. The algebra $\mathcal{T}_k$ $(k > 0)$ can be identified as follows. Consider the algebra $\mathcal{T}_1 \otimes M_k$, where $M_k$ is the $C^*$-algebra of $k \times k$ matrix algebra over the complex field. Let $S_k = \mathrm{diag}(S_1, S_1, ..., S_1)$ (there are $k$ copies of $S_1$). Then $\mathcal{T}_k$ is isomorphic to the $C^*$-subalgebras of $\mathcal{T}_1 \otimes M_k$ generated by $S_k$ and $M_k(\mathcal{K})$. We will identify these two algebras. Let $e = 1 - S_1S_1^*$, a minimum projection in $\mathcal{T}_1$, and let $\{e_{ij}\}_{i,j=1}^k$ be the matrix unit for $M_k$. One checks easily that $\mathcal{T}_k$ is generated by $S_k$ and $e \otimes e_{ij}$. Note that $1 - S_kS_k^* = \sum_{i=1}^k e \otimes e_{ii}$. We denote $g_{ij} = e \otimes e_{ij}$. Suppose that $T \in B(H)$ is an isometry and there are $k$ mutually orthogonal equivalent projections $p_1, p_2, ..., p_k$ such that $\sum_{i=1}^k p_i = 1 - TT^*$. Let $a_{ij}$ be the partial isometries such that $a_{ij}^*a_{ij} = p_j$ and $a_{ij}a_{ij}^* = p_i$. The $C^*$-subalgebra generated by $T$ is isomorphic to $\mathcal{T}_1$ and the ideal $J$ of this $C^*$-subalgebra generated by $1 - TT^*$ is isomorphic to $\mathcal{K}$. Note that $a_{11}T = a_{11}(1 - TT^*)T = 0$ and $(T^m a_{11}(T^m)^*)T^{m+l}a_{11}(T^{m+l})^* = 0$ if $l > 0$. Let $q_1 = \sum_{m=1}^\infty T^m a_{11}(T^m)^*$, a projection in $B(H)$. Then $Tq_1 = q_1T$. Set $T_1 = q_1T$. Then $T_1^*T_1 = q_1$ and $1 - T_1T_1^* = a_{11}$. Let $B$ be the $C^*$-subalgebra generated by $T$ and $\{a_{ij}\}$ and let $C$ be the $C^*$-subalgebra generated by $B$ and $q_1, q_2, ..., q_k$, where $q_i = \sum_{m=1}^\infty T^m a_{ii}(T^m)^*$. Then $C \cong \mathcal{T}_1 \otimes M_k$. It then follows that $B \cong \mathcal{T}_k$. By identifying $B$ with $\mathcal{T}_k$, we will say that $\mathcal{T}_k$ is generated by $S_k$ and $\{a_{ij}\}$. We would like to point out that the ideal of $B$ generated by $\{a_{ij}\}$ is not the set of all compact operators on $H$ even if $a_{11}$ is a rank one operator. Let $S_0$ be a unitary in $B(H)$ with essential spectrum $S^1$. Then $\mathcal{T}_0$ is isomorphic to the $C^*$-subalgebra of $B(H)$ generated by $S_0$ and $\mathcal{K}(H)$.

In this paper, we will consider direct limits of finite direct sums of matrix algebras over $\mathcal{T}$-algebras.

**2.2.** Let $k$ be a nonnegative integer. We will determine the $K$-theory of $\mathcal{T}_k$. If $k \neq 0$, the index map $\delta : K_1(C(S^1)) \to K_0(\mathcal{K})$ is the multiplication by $-k$. Hence $\delta$ is injective. From the six-term exact sequence in $K$-theory, we obtain:

$$0 \to K_1(M_n(C(S^1))) \to K_0(M_n(\mathcal{K})) \to K_0(M_n(\mathcal{T}_k)) \to K_0(M_n(C(S^1))) \to 0$$

where $K_1(M_n(C(S^1))) = \mathbf{Z}$, $K_0(M_n(\mathcal{K})) = \mathbf{Z}$ and $K_0(M_n(C(S^1))) = \mathbf{Z}$. Therefore,

$$K_0(\mathcal{T}_k) = \mathbf{Z} \oplus \mathbf{Z}/k\mathbf{Z} \quad \text{and} \quad K_1(\mathcal{T}_k) = 0.$$

If $k = 0$, then $\delta = 0$. The six-term exact sequence breaks into two:

$$0 \to K_0(\mathcal{K}) \to K_0(\mathcal{T}_0) \to K_0(C(S^1)) \to 0$$

and

$$0 \leftarrow K_1(C(S^1)) \leftarrow K_1(\mathcal{T}_0) \leftarrow 0.$$

This implies that

$$K_0(\mathcal{T}_0) = \mathbf{Z} \oplus \mathbf{Z} \quad \text{and} \quad K_1(\mathcal{T}_0) = \mathbf{Z}.$$

We would like to point out that the positive cone of $K_0(\mathcal{T}_0)$ is not the usual one on $\mathbf{Z} \oplus \mathbf{Z}$ (cf. 2.3).

**2.3.** When the stable rank of a $C^*$-algebra $A$ is not one, there may be too much information lost in $K_*(A)$. Hence, we should also consider $V(A)$, the Murray -von Neumann equivalence classes of projections in matrices over $A$. Note that $V(A)$ is a commutative semigroup.

It is well known and easy to check that if $p \in \mathcal{T}_1 \setminus \mathcal{K}$ is a projection, then $p$ is equivalent to 1. From this and the fact that $V(C(S^1)) = \mathbf{Z}_+$, one computes that

$$V(\mathcal{T}_1) = \mathbf{Z}_+ \sqcup \mathbf{N},$$

where the addition in $\mathbf{Z}_+$ and $\mathbf{N}$ are the usual ones. And if $x \in \mathbf{Z}_+$, $y \in \mathbf{N}$, then $x + y = y$.

For $k > 1$, let $S_k$ be the isometry in $\mathcal{T}_k$ such that $1 - S_k S_k^*$ is a projection in $\mathcal{K}$ with rank $k$. Then the $C^*$-subalgebra $B$ generated by $S_k$ is isomorphic to $\mathcal{T}_1$. Note also that the index of every isometry in $\mathcal{T}_k$ which is essentially unitary is of multiple $k$. It is then routinely checked that $V(\mathcal{T}_k)$ may be identified with

$$\mathbf{Z}_+ \sqcup \mathbf{N} \oplus \mathbf{Z}/k\mathbf{Z}$$

with the addition defined as follows. Addtions are the usual ones in $\mathbf{Z}_+$ and $\mathbf{N} \oplus \mathbf{Z}/k\mathbf{Z}$. If $z \in \mathbf{Z}_+$ and $y \in \mathbf{N} \oplus \mathbf{Z}/k\mathbf{Z}$, then $z + y = y + \bar{z}$, where $\bar{z}$ is the class containing $z$ in $\mathbf{Z}/k\mathbf{Z}$.

When $k = 0$, one computes that

$$V(\mathcal{T}_0) = \{(m, n) : m \in \mathbf{Z}, n \in \mathbf{Z}_+, \text{ if } n = 0, m \geq 0\}$$

with addition defined in $\mathbf{Z} \oplus \mathbf{Z}$.

We denote $V(\mathcal{T}_k)$ by $F_k$. Let $I(F_k) = \mathbf{Z}_+$ if $k \neq 0$ , and $I(F_0) = \{(m, 0) : m \in \mathbf{Z}_+\}$.

We note that if $p$ and $q$ are two projections in $M_n(\mathcal{T}_k)$ such that $[p] = [q]$ in $V(\mathcal{T}_k)$ then there exists an isometry, or a coisometry $W \in M_n(\mathcal{T}_k)$ such that $WpW^* = q$ and $W^*qW = p$.

If $V$ is a semigroup, we will denote $G(V)$ the Grothendieck group of $V$. Let $j_k : F_k \to G(F_k)$ be the natural map from $F_k$ into $G(F_k)$. In the case that $k = 0$, $F_0$ is a subsemigroup of $\mathbf{Z} \oplus \mathbf{Z}$. So $j_0$ is injective. For $k > 0$, $\ker j_k = k\mathbf{Z}_+$. In both cases, $G(F_k) = K_0(\mathcal{T}_k)$. Note that $\ker j_k \subset I(F_k)$. Denote $Q(F_k) = F_k/I(F_k) = j_k(F_k)/G(j_k(I(F_k))) \cong \mathbf{Z}_+$. Let $\eta : F_k \to F_m$ be a homomorphism of abelian semigroups, i.e., $\eta$ preserves the additions of the semigroups. A direct computation shows that $\eta$ maps $\ker j_k$ into $\ker j_m$. It follows that $\eta$ gives a group homomorphism from $G(F_k)$ into $G(F_m)$. If $V$ is a direct sum of $F_{k_i}$, $i = 1, 2, ..., n$, $Q(V)$ is the direct sum of $Q(F_{k_i})$, $i = 1, 2, ..., n$.

**Lemma 2.4.** *Let $V_1$ be a direct sum of $F_{k_i}$ $i = 1, 2, ..., n$, let $V_2$ be a direct sum of $F_{k'_j}$, $j = 1, 2, .., l$ and let $\eta : V_1 \to V_2$ be a homomorphism from semigroup $V_1$ into the semigroup $V_2$. Then*

(a) *$\eta$ maps $I(V_1)$ into $I(V_2)$,*

(b) *$\eta$ induces a quotient map $\bar{\eta} : Q(V_1) \to Q(V_2)$,*

(c) *if $\bar{\eta} = 0$, $\eta_{I(V_1)} = 0$,*

(d) *if $V_1 = F_k$ and either $\eta|_{I(F_k)}$ or $\bar{\eta}$ is not injective, then $\eta|_{I(F_k)} = 0$,*

(e) *if $V_1 = F_0$, $V_2 = F_0$ and $\eta$ is not injective, then $\eta|_{I(F_0)} = 0$, and*

(f) *if $V_1 = F_k$ with $k > 0$ and $V_2 = F_0$ then $\eta|_{I(F_k)} = 0$.*

*Proof.* It is obvious that (b) follows from (a). For (a), it is clear that we only need to show that a homomorphism $\eta : F_k \to F_m$ maps $I(F_k)$ into $I(F_m)$.

*Case* (1): $k = 0$ and $m = 0$.

Since $\ker j_0 = 0$, the homomorphism $\eta$ can be extended to a group homomorphism from $G(F_0)$ into $G(F_0)$. We still use $\eta$ for the extension. Note $G(F_0) = \mathbf{Z} \oplus \mathbf{Z}$. Suppose that $\eta((1, 0)) = (l_1, l_2)$ with $l_2 > 0$. Suppose also that $\eta((0, 1)) = (l'_1, l'_2)$ with $l'_2$ nonnegative. Then, for any $n > 0$

$\eta((-n,1)) = (l'_1 - nl_1, l'_2 - nl_2)$. If $n$ is large enough, $l'_2 - nl_2 < 0$. But since $(-n,1) \in F_0$, $\eta((-n,1))$ should be in $F_0$. This is impossible. Therefore, $l_2 = 0$. Thus $\eta$ maps $I(F_0)$ into $I(F_0)$.

*Case* (2): $k = 0$ and $m > 0$.

The map $j_m \circ \eta$ extends to a group homomorphism $\tilde{\eta} : G(F_0) \to G(F_m)$ $(= \mathbf{Z} \oplus \mathbf{Z}/m\mathbf{Z})$. Suppose that

$$\tilde{\eta}((1,0)) = x \oplus \bar{y} \quad \text{and} \quad \tilde{\eta}((0,1)) = x_1 \oplus \bar{y}_1.$$

Then

$$\tilde{\eta}((-n,1)) = (-nx + x_1) \oplus (-n\bar{y} + \bar{y}_1).$$

Since $\tilde{\eta}$ must map $F_0$ into $j_m(F_m)$, $x$ has to be zero. Since $j_m(I(F_m)) = \mathbf{Z}/m\mathbf{Z}$ and since $\ker j_m \subset I(F_m)$, this implies that $\eta$ maps $I(F_0)$ into $I(F_m)$.

*Case* (3): $k > 0$.

Since $\eta$ maps $\ker j_k$ into $\ker j_m$, the map $\eta$ gives a group homomorphism $\tilde{\eta} : G(F_k) \to G(F_m)$. Since $j_k(I(F_k))$ is torsion, so is $\tilde{\eta}(j_k(I(F_k)))$. Therefore $\eta$ maps $I(F_k)$ into $I(F_m)$.

Now for (c), $\bar{\eta} = 0$, implies that $\eta$ maps $F_k$ into $\mathbf{Z}_+$. If $k > 0$, $\eta$ maps $\ker j_k$ into zero, since $k + (1,0) = (1,0)$ in $F_k$ and since the map from $\mathbf{Z}_+$ into $G(\mathbf{Z}_+)$ $(= \mathbf{Z})$ is injective. So $\eta$ maps $k\mathbf{Z}_+$ into zero. The map $\eta$ induces a group homomorphism which maps $\mathbf{Z} \oplus \mathbf{Z}/k\mathbf{Z}$ into $\mathbf{Z}$. Therefore it must map the torsion part to zero. This implies that $\eta$ maps $I(V_1)$ into zero. If $k = 0$, let us assume that $\eta((1,0)) = l > 0$ and $\eta((0,1)) = l' \geq 0$. Let us use $\eta$ for the extension from $\mathbf{Z} \oplus \mathbf{Z}$ into $\mathbf{Z}$. Then $\eta((-n,1)) = -nl + l'$ which is negative if $n$ is large enough. But this is impossible. Therefore $\eta((1,0)) = 0$. Thus $\eta$ maps $I(V_1)$ into zero.

For (f), by (a), we may write $\eta(x) = k$ and $\eta(y) = (n,z)$, where $x \in I(F_k), k \in I(F_0)$, $y \in \mathbf{N}$ and $z \neq 0$. So $\eta(x) + \eta(y) = (k+n, z) \in F_0$. Let $\tilde{\eta} : G(F_k) \to G(F_0)$ be the group homomorphism induced by $\eta$. Then $\tilde{\eta}(\bar{x}) = 0$, if $\bar{x} \in \mathbf{Z}/k\mathbf{Z}$. Since $\eta$ and $\tilde{\eta}$ agree on $\mathbf{N} \oplus \mathbf{Z}/k\mathbf{Z}$, $\eta(\bar{x} + y) = \eta(y)$. So we have $\eta(x+y) = \eta(\bar{x}+y) = \eta(y) = (n,z)$. Therefore $k = 0$. This means that $\eta|_{I(F_0)} = 0$.

To prove (d), we may assume that $V_2 = F_m$. It follows from (a) that $\eta|_{I(F_k)}$ is either injective or zero, since $I(F_k) \cong I(F_m) \cong \mathbf{Z}_+$. Suppose that $\bar{\eta}$ is not injective. But $Q(F_k) \cong \mathbf{Z}_+$ and $Q(F_m) \cong \mathbf{Z}_+$ and $\bar{\eta}$ has to be zero if it is not injective. It follows from (c) that $\eta|_{I(F_k)} = 0$.

To prove (e), we assume that $\eta|_{I(F_0)} \neq 0$. Let $\tilde{\eta} : G(F_0) \to G(F_0)$ be the group homomorphism induced by $\eta$. Then $\tilde{\eta}$ has to be injective. Consequencely, $\eta$ has to be injective. $\square$

**Lemma 2.5.** *Let $V = \varinjlim(V_n, \phi_{n,n+1})$, where each $V_n$ is a finite direct sum of $F'_{k_i}s$. Then there is $n > 0$ such that $\phi_{n,\infty}$ is injective on $\phi_{1,n}(V_1)$.*

*Proof.* It is enough to prove the case that $V_1 = F_k$.

We first assume that $k > 0$. It is clear that there are only finitely many possible homomorphic images of $F_k$, namely, 0, $F_k$, $F_m$ (where $m|k$) or $\mathbf{Z}_+$. There is an integer $n$ such that the image of $\phi_{1,n}(F_k)$ equals to $\phi_{1,\infty}(F_k)$. If $\phi_{1,\infty}(F_k) = 0$, or $= \mathbf{Z}_+$, it is clear that $\phi_{n,\infty}$ on $\phi_{1,n}(F_k)$ is injective. Since there are only finitely many images, it is enough to show that any surjective homomorphism $\eta$ from $F_k$ (or $F_m$ ) onto $F_k$ (or $F_m$) is injective. By (d) of 2.4, both $\eta|_{I(F_k)}$ and $\bar{\eta}$ are injective. This immediately implies that $\eta$ maps $I(F_k)$ onto $I(F_k)$ and $\eta(\mathbf{N}) \cap I(F_k) = \{0\}$. Therefore $\eta$ maps $\mathbf{N} \oplus \mathbf{Z}/k\mathbf{Z}$ onto itself. Since the only surjective homomorphism from $\mathbf{Z} \oplus \mathbf{Z}/k\mathbf{Z}$ onto itself is injective, we conlude that $\eta$ has to be injective. This proves the case for $k > 0$.

Now we consider the case that $k = 0$. There is an integer $n_1 > 0$ such that the image of $\phi_{1,n_1}(F_0)$ equals to $\phi_{1,\infty}(F_0)$. If $\phi_{n_1,\infty}$ is injective on $\phi_{1,n_1}(F_0)$, then we are done. So we assume that it is not injective. Without loss of generality, we may assume that $\phi_{n_1,n_2}$ on $\phi_{1,n_1}(F_0)$ is not injective. Therefore, $\phi_{1,n_2}$ is not injective. Let $V_{n_2} = B_1 \oplus B_2 \oplus \cdots \oplus B_m$ and $p_i : V_{n_2} \to B_i$ be the projection, where each $B_i \cong F_{k_i}$. We may assume that $B_1, B_2, ..., B_j \cong F_0$, and $B_{j+k} \cong F_{r_k}$ for some $r_k > 0$, $k = 1, 2, ..., m - j$. Then every $p_i \circ \phi_{1,n_2}$ is not injective. By (e) of 2.4, $p_i \circ \phi_{1,n_2}(F_0) = 0$, or $\mathbf{Z}_+$. Therefore, there is $n_3 > 0$ such that $\phi_{n_2,\infty}$ is injective on $\phi_{n_2,n_3} \circ p_i \circ \phi_{1,n_2}(F_0)$ for $i = 1, 2, ..., j$. From the case that $k > 0$, we conclude that there is also an integer $n_4 > 0$ such that $\phi_{n_4,\infty}$ is injective on each $\phi_{n_2,n_4}(B_{j+k})$. Therefore, there is a large $n > 0$ such that $\phi_{n,\infty}$ is injective on $\phi_{1,n}(F_0)$. □

**2.6.** Let $E_1$ and $E_2$ be two finite direct sums of matrix algebras over $\mathcal{T}$-algebras. Suppose that $E_1 = \oplus_{j=1}^{L_1} M_{l_j}(\mathcal{T}_{k_j})$ and $E_2 = \oplus_{j=1}^{L_2} M_{m_j}(\mathcal{T}_{r_j})$. Each $E_i$ is an essential extension

$$0 \to I(E_i) \to E_i \to Q(E_i) \to 0,$$

where $I(E_1) = \oplus_{j=1}^{L_1} M_{l_j}(\mathcal{K})$, $I(E_2) = \oplus_{j=1}^{L_2} M_{m_j}(\mathcal{K})$, $Q(E_1) = \oplus_{j=1}^{L_1} M_{l_j}(C(S^1))$, and $Q(E_2) = \oplus_{j=1}^{L_2} M_{m_j}(C(S^1))$. If $\phi : E_1 \to E_2$ is a homomorphism, then $\phi$ maps $I(E_1)$ into $I(E_2)$. This happens because $M_m(C(S^1))$ can not contain any copy of $\mathcal{K}$. Thus we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 \to & I(E_1) & \to & E_1 & \to & Q(E_1) & \to 0 \\ & \downarrow \phi & & \downarrow \phi & & \downarrow \bar{\phi} & \\ 0 \to & I(E_2) & \to & E_2 & \to & Q(E_2) & \to 0 \end{array}$$

where $\bar{\phi} : Q(E_1) \to Q(E_2)$ is the (unique) induced map.

One also has

$$0 \to V(I(E_i)) \to V(E_i) \to V(Q(E_i)) \to 0.$$

The six-term exact sequence of the K-groups of this extension has the form

$$0 \to K_1(E_i) \to K_1(Q(E_i)) \to K_0(I(E_i)) \to K_0(E_i) \to K_0(Q(E_i)) \to 0.$$

We will denote this exact sequence by $\mathbf{K}(E_i)$. The index map from $K_1(Q(E_i))$ to $K_0(I(E_i))$ will be denoted by $\delta_i$. By a map from $\mathbf{K}(E_1)$ to $\mathbf{K}(E_2)$ we mean five group homomorphisms $\alpha = \{\alpha_i\}_{i=1}^5$ such that the following diagram commutes

$$\begin{array}{ccccccccccc} 0 \to & K_1(E_1) & \to & K_1(Q(E_1)) & \to & K_0(I(E_1)) & \to & K_0(E_1) & \to & K_0(Q(E_1)) & \to 0 \\ & \downarrow \alpha_1 & & \downarrow \alpha_2 & & \downarrow \alpha_3 & & \downarrow \alpha_4 & & \downarrow \alpha_5 & \\ 0 \to & K_1(E_2) & \to & K_1(Q(E_2)) & \to & K_0(I(E_2)) & \to & K_0(E_2) & \to & K_0(Q(E_2)) & \to 0. \end{array}$$

**Definition 2.7.** Recall that an element $s$ in a $C^*$-algebra is called hyponormal if $s^*s \geq ss^*$. For a unital $C^*$-algebra $A$, let $S_n(A)$ be the set of all nonzero hyponormal partial isometries in $M_n(A)$ and $S(A) = \cup_{n=1}^{\infty} S_n(A)$ ($S_n(A) \to S_{n+1}(A)$ is defined by $s \to diag(s, 1)$). Let $S(A)_0$ be the set of (nonzero) hyponormal partial isometries which is homotopic to a projection. We denote by $k(A)_+ = S(A)/S(A)_0$. For two hyponormal partial isometries $v_1$ and $v_2$ in $M_n(A)$, $[v_1] = [v_2]$ in $k(A)_+$ if and only if $(1 - v_1^*v_1 + v_1) + 1_m$ is homotopic to $(1 - v_2^*v_2 + v_2) + 1_m$ in $S_{m+n}(A)$, where $1_m$ is the identity in $M_m(A)$. By defining the usual orthogonal addition, $k(A)_+$ becomes an abelian semigroup. There is a natural embedding $em : K_1(A) \to k(A)_+$. When $A$ has cancellation, $K_1(A) = k(A)_+$. We think that $k(A)_+$ may be useful for $C^*$-algebra with stable rank other than one.

It is easy to see that $k(\mathcal{T}_0)_+ \cong K_1(\mathcal{T}_0)$. One can also verify that $k(M_m(\mathcal{T}_k))_+ \cong \mathbf{Z}_+$, if $k > 0$. Let $\pi : M_m(\mathcal{T}_k) \to M_m(C(S^1))$ be the quotient map. Then $\pi$ induces a map from $k(M_m(\mathcal{T}_k))_+$ to $k(M_m(C(S^1)))_+ \cong k(C(S^1))_+$. Clearly, this is an injection. Since $k(C(S^1))_+ \cong K_1(C(S^1))$, there is an injective homomorphism $\Delta : k(\mathcal{T}_k)_+ \to K_1(C(S^1))$. Note also that $K_1(M_m(\mathcal{T}_k)) = \{0\}$, if $k > 0$.

**2.8.** Associate to each $C^*$-algebra $A$ a semigroup

$$\bar{V}(A) = \{([u^*u], [u]) : \quad u \in S(A) \quad \text{and} \quad [u] \in k(A)_+\},$$

where $[u^*u]$ is in $V(A)$. Let $d : k(A)_+ \to V(A)$ be defined by $d([u]) = [u^*u - uu^*]$. Let $\{u_t\}$ be a path of hyponormal partial isometries. Then

$\{u_t^* u_t - u_t u_t^*\}$ is a path of projections. Thus $d$ is a well defined semigroup homomorphism.

If $E$ is a finite direct sum of matrix algebras over $\mathcal{T}$-algebras, then $d = -\delta \circ \Delta$, where $\delta : K_1(Q(E)) \to K_0(I(E))$ is the usual index map.

**Definition 2.9.** We write the triples

$$V_*(A) = \{([u^*u], [u], d([u])) : u \in S(A) \quad \text{and} \quad [u] \in k(A)_+\}.$$

Let $A$ and $B$ be two $C^*$-algebras. Suppose that $V_*(A)$ and $V_*(B)$ are two such triples, then a homomorphism $\eta : V_*(A) \to V_*(B)$ is a homomorphisms $\alpha : \bar{V}(A) \to \bar{V}(B)$ for which

$$\begin{array}{ccc} V(A) & \overset{d}{\leftarrow} & k(A)_+ \\ \downarrow \alpha|_V & & \downarrow \alpha|_k \\ V(B) & \overset{d}{\leftarrow} & k(B)_+ \end{array}$$

commutes.

If $\phi : A \to B$ is a homomorphism, then $\phi$ induces not only a homomorphism from $\bar{V}(A)$ into $\bar{V}(B)$ but also a homomorphism $\phi_* : V_*(A) \to V_*(B)$. We will use the notation $F_{k*}$ for $V_*(\mathcal{T}_k)$.

Let $E$ be a finite direct sum of matrix algebras over $\mathcal{T}$-algebras. It is important to note that the image of $d$ is contained in $I(V(E)) \subset V(E)$.

A fact that we will use later is that any homomorphism from $V_*(\mathcal{T}_0)$ into $V_*(\mathcal{T}_k)$ with $k > 0$ maps the second variable to zero. This is clear, since the only homomorphism from the semigroup $\mathbf{Z}$ into the semigroup $\mathbf{Z}_+$ is zero.

**2.10.** Suppose that there is a homomorphism $\eta : V_*(E_1) \to V_*(E_2)$ where each $E_i$ is a finite direct sum of matrix algebras over $\mathcal{T}$-algebras. It follows from 2.6 that $\eta$ induces the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V(I(E_1)) & \longrightarrow & V(E_1) & \longrightarrow & V(Q(E_1)) & \longrightarrow & 0 \\ & & \downarrow \alpha_3 & & \downarrow \alpha_4 & & \downarrow \alpha_5 & & \\ 0 & \longrightarrow & V(I(E_2)) & \longrightarrow & V(E_2) & \longrightarrow & V(Q(E_2)) & \longrightarrow & 0. \end{array}$$

This in turn gives uniquely the following commutative diagram:

$$\begin{array}{ccccccc} K_0(I(E_1)) & \longrightarrow & K_0(E_1) & \longrightarrow & K_0(Q(E_1)) & \longrightarrow & 0 \\ \downarrow \alpha_3 & & \downarrow \alpha_4 & & \downarrow \alpha_5 & & \\ K_0(I(E_2)) & \longrightarrow & K_0(E_2) & \longrightarrow & K_0(Q(E_2)) & \longrightarrow & 0. \end{array}$$

Furthermore, we have the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \bar{V}(I(E_1)) & \longrightarrow & \bar{V}(E_1) & \longrightarrow & \bar{V}(Q(E_1)) \\ & & \downarrow \alpha_3 & & \downarrow \alpha_4 & & \downarrow \alpha_5 \\ 0 & \longrightarrow & \bar{V}(I(E_2)) & \longrightarrow & \bar{V}(E_2) & \longrightarrow & \bar{V}(Q(E_2)). \end{array}$$

The map $\eta$ also gives a homomorphism from $k(E_1)_+$ into $k(E_2)_+$ which, in turn, gives a homomorphism

$$\alpha_2 : K_1(Q(E_1)) \to K_1(Q(E_2)).$$

Define $\alpha_1 = em^{-1} \circ \eta|_{em(K_1(E_1))} \circ em : K_1(E_1) \to K_1(E_2)$ (note that $em$ is injective). Note that image of $d$ lies in $I(V(E_i)) = V(I(E_i))$. From the commutative diagram

$$\begin{array}{ccc} V(I(E_1)) & \overset{d}{\leftarrow} & k(E_1)_+ \\ \downarrow \eta|_V & & \downarrow \eta|_k \\ V(I(E_2)) & \overset{d}{\leftarrow} & k(E_2)_+ \end{array}$$

and using the map $\Delta_i$ and the natural embedding from $V(I(E_i))$ into $K_0(E_i)$, we obtain the following commutative diagram:

$$\begin{array}{ccccccccccc} 0 \to & K_1(E_1) & \to & K_1(Q(E_1)) & \to & K_0(I(E_1)) & \to & K_0(E_1) & \to & K_0(Q(E_1)) & \to 0 \\ & \downarrow \alpha_1 & & \downarrow \alpha_2 & & \downarrow \alpha_3 & & \downarrow \alpha_4 & & \downarrow \alpha_5 & \\ 0 \to & K_1(E_2) & \to & K_1(Q(E_2)) & \to & K_0(I(E_2)) & \to & K_0(E_2) & \to & K_0(Q(E_2)) & \to 0. \end{array}$$

Set $\alpha = \{\alpha_i\}_{i=1}^5$. Then $\alpha$ is a a homomorphism $\alpha : \mathbf{K}(E_1) \to \mathbf{K}(E_2)$ which is uniquely determined by $\eta$.

Let $E$ be a finite direct sum of matrix algebras over $\mathcal{T}$-algebras. We denote by $\Gamma$ a subset of $K_0(Q(E)) \oplus K_1(Q(E))$, as follows:

$$\begin{aligned} \Gamma &= \{([u^*u],[u]) : u \text{ is a normal partial isometry } \in Q(E)\} \\ &= \{(x,y) : x \in V(Q(E)), y \in K_1(Q(E)), \text{ if } x = 0, y = 0\}. \end{aligned}$$

Denote by $K_*(Q(E))$ the graded group $K_0(Q(E)) \oplus K_1(Q(E))$ with the partial order generated by $\Gamma$. It is clear that a homomorphism $\alpha : V_*(E_1) \to V_*(E_2)$ also induces a homomorphism $\alpha_5 \oplus \alpha_2 : K_*(Q(E_1)) \to K_*(Q(E_2))$ (which preserves the order).

From the above we have the following:

**Corollary 2.11.** *Let $E_1$ and $E_2$ be as in 2.10. Suppose that $\alpha$ is a homomorphism from $V_*(E_1)$ to $V_*(E_2)$. Then $\alpha$ induces*

(i) *uniquely two homomorphisms $\gamma$ and $\beta$ such that the following diagram commutes*

$$\begin{array}{ccccccccc} 0 \longrightarrow & V(I(E_1)) & \longrightarrow & V(E_1) & \longrightarrow & V(Q(E_1)) & \longrightarrow 0 \\ & \downarrow \gamma & & \downarrow \alpha & & \downarrow \beta & \\ 0 \longrightarrow & V(I(E_2)) & \longrightarrow & V(E_2) & \longrightarrow & V(Q(E_2)) & \longrightarrow 0, \end{array}$$

(ii) *uniquely a map $\{\alpha_i\}_{i=1}^5$, from $\mathbf{K}(E_1)$ to $\mathbf{K}(E_2)$, and*

(iii) *the map* $\alpha_5 \oplus \alpha_2$ *from* $K_*(Q(E_1))$ *to* $K_*(Q(E_2))$ *preserves the order.*

**Lemma 2.12.** *Let* $E_1$ *and* $E_2$ *be two finite direct sums of matrix algebras over* $\mathcal{T}$*-algebras such that there is no summand in the sums which is isomorphic to a matrix algebra over* $\mathcal{T}_0$*. Then a homomorphism* $\alpha : V(E_1) \to V(E_2)$ *extends to a unique homomorphism* $\sigma : V_*(E_1) \to V_*(E_2)$*. Furthermore, the conclusion of 2.11 holds.*

*Proof.* It follows from the definition that $d$ maps $k(\mathcal{T}_k)_+$ injectively onto $ker\, j_k$, where $k > 0$ and $j_k$ is the natural map from $F_k$ into $G(F_k)$. We also know that homomorphism from $F_k$ into $F_r$ (with $r > 0$) maps $ker\, j_k$ into $ker\, j_r$. □

**Definition 2.13.** Let $V_*^{(i)} = \lim\limits_{\to}(V_{n*}^{(i)}, \eta_{n,n+1}^{(i)})$, where $V_{n*}^{(i)}$ is a finite direct sum of $F_{r_j*}$, and where $\eta_{n,n+1}^{(i)}$ is a homomorphism from $V_{n*}^{(i)}$ into $V_{n+1*}^{(i)}$, $i = 1, 2$. Suppose that $V_n^{(i)}$ is the first variable, $k_n^{(i)}$ is the second variable and $d_n^{(i)}$ is the third variable of $V_{n*}^{(i)}$, respectively, for $i = 1, 2$. Let $V^{(i)} = \lim\limits_{\to} V_n^{(i)}$, $k = \lim\limits_{\to} k_n^{(i)}$, $d^{(i)} = \lim\limits_{\to} d_n^{(i)}$ and $\bar{V}^{(i)} = \lim\limits_{\to}(\bar{V}_n^{(i)}, \eta_{n,n+1}^{(i)})$. A homomorphism $\phi : V_*^{(1)} \to V_*^{(2)}$ is a homomorphism $\alpha : \bar{V}^{(1)} \to \bar{V}^{(2)}$ for which

$$\begin{array}{ccc} V^{(1)} & \stackrel{d^{(1)}}{\leftarrow} & k^{(1)} \\ \downarrow \alpha|_{V^{(1)}} & & \downarrow \alpha|_{k(A)_+} \\ V^{(2)} & \stackrel{d^{(2)}}{\leftarrow} & k^{(2)} \end{array}$$

is commutative.

Let $E = \lim\limits_{\to}(E_i, \phi_{i,i+1})$ be a direct limit of finite direct sums of matrix algebras over $\mathcal{T}$-algebras and let $I(E) = \lim\limits_{\to}(I(E_i), \phi_{i,i+1})$. Then $I(E)$ is an ideal of $E$. Set $Q(E) = E/I(E)$. Then it is clear that $Q(E) = \lim\limits_{\to}(Q(E_i), \bar{\phi}_{i,i+1})$, where $\bar{\phi}_{i,i+1}$ is the map induced by $\phi$ (see 2.6).

**Lemma 2.14.** *Let* $E = \lim\limits_{\to}(E_i, \phi_{i,i+1})$ *and* $E' = \lim\limits_{\to}(E_i', \psi_{i,i+1})$ *be two inductive limit* $C^*$*-algebras where each* $E_i$ *or each* $E_i'$ *is a finite direct sum of matrix algebras over* $\mathcal{T}$*-algebras. Suppose that each connecting map in the two inductive limit systems satisfies the following condition: any partial map from* $M_n(\mathcal{T}_0)$ *to another algebra vanishes on* $M_n(\mathcal{K})$*. Let* $\alpha$ *be a homomorphism from* $V_*(E)$ *to* $V_*(E')$*. Then* $\alpha$ *induces*

(i) *uniquely two homomorphisms* $\gamma$ *and* $\beta$ *such that the following diagram commutes*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V(I(E)) & \longrightarrow & V(E) & \longrightarrow & V(Q(E)) & \longrightarrow & 0 \\ & & \downarrow \gamma & & \downarrow \alpha & & \downarrow \beta & & \\ 0 & \longrightarrow & V(I(E')) & \longrightarrow & V(E') & \longrightarrow & V(Q(E')) & \longrightarrow & 0, \end{array}$$

(ii) *$\alpha$ induces uniquely a map $\{\alpha_i\}_{i=1}^5$ from $\mathbf{K}(E)$ to $\mathbf{K}(E')$.*

*Furthermore, $\alpha_5 \oplus \alpha_2$ from $K_*(Q(E))$ to $K_*(Q(E'))$ preserves the order. In particular, if $\alpha$ is an isomorphism, all the maps induced by $\alpha$ are isomorphisms.*

*Proof.* Let $V_*(E) = \varinjlim(V_*(E_n), \eta_{n,n+1})$ and $V_*(E') = \varinjlim(V_*(E'_n), \eta'_{n,n+1})$. There exists $n'_1 > 1$ such that $\alpha(\eta_{1,\infty}(\bar{V}(E_1))) \subset \eta_{n'_1,\infty}(\bar{V}(E'_{n'_1}))$. To see this, it is enough to assume that $E_1 = M_n(\mathcal{T}_k)$. If $k > 0$, $\bar{V}(E_1)$ is finitely generated. If $k = 0$, although $\bar{V}(E_1)$ is not finitely generated, the connecting map $\bar{V}(E_1) \to \bar{V}(E_2)$ vanishes on $\bar{V}(I(E_1))$. The image in $\bar{V}(E_2)$ is generated by a single element. In both cases, the inclusion follows. There is $n''_1$ such that $\eta_{n''_1,m}$ maps $\eta_{n''_1,n'_1}(V(E'_{n'_1}))$ injectively into $V(E_m)$ for any $m > n''_1$. This is possible because of 2.5. Choose $n_1 = n''_1 + 1$. Let $h$ be the inverse of $\eta'_{n_1,\infty}$ on $\eta'_{n_1,\infty}(\eta_{n'_1,n_1}(\bar{V}(E'_{n'_1})))$. Define

$$\alpha^{(1)} = \eta'_{n'_1,n_1} \circ h \circ \alpha \circ \eta_{1,\infty}.$$

Then the following diagram commutes:

$$\begin{array}{ccc} \bar{V}(E_1) & \longrightarrow & \bar{V}(E) \\ \downarrow \alpha^{(1)} & & \downarrow \alpha \\ \bar{V}(E'_{n_1}) & \longrightarrow & \bar{V}(E'). \end{array}$$

Since $k(E_1)_+$ is finitely generated, we can choose $\alpha^{(1)}$ so that the following diagram commutes:

$$\begin{array}{ccc} V_*(E_1) & \longrightarrow & V_*(E) \\ \downarrow \alpha^{(1)} & & \downarrow \alpha \\ V_*(E'_{n_1}) & \longrightarrow & V_*(E'). \end{array}$$

As above, there is $n'_2$ such that

$$\begin{array}{ccccc} \bar{V}(E_1) & \longrightarrow & \bar{V}(V_2) & \longrightarrow & \bar{V}(E) \\ \downarrow & & \downarrow & & \downarrow \\ \bar{V}(E'_{n_1}) & \longrightarrow & \eta_{n'_2,\infty}(\bar{V}(E'_{n_2})) & \longrightarrow & \bar{V}(E') \end{array}$$

commutes. The above argument actually says that one has the following commutative diagram

$$\begin{array}{ccccccc} \bar{V}(E_1) & \longrightarrow & \bar{V}(E_2) & & \longrightarrow & & \bar{V}(E) \\ \downarrow & & \downarrow & & & & \downarrow \\ \bar{V}(E'_{n_1}) & \longrightarrow & \eta_{n'_2,\infty}(\bar{V}(E'_{n'_2})) & \to & \bar{V}(E'_{n_2}) & \to & \bar{V}(E'). \end{array}$$

This will give the following commutative diagram:

$$\begin{array}{ccccc} \bar{V}(E_1) & \longrightarrow & \bar{V}(E_2) & \longrightarrow & \bar{V}(E) \\ \downarrow \alpha^{(1)} & & \downarrow \alpha^{(2)} & & \downarrow \alpha \\ \bar{V}(E'_{n_1}) & \longrightarrow & \bar{V}(E'_{n_2}) & \longrightarrow & \bar{V}(E'). \end{array}$$

The above argument further gives us the following diagram:

$$\begin{array}{ccccc} V_*(E_1) & \longrightarrow & V_*(E_2) & \longrightarrow & V_*(E) \\ \downarrow \alpha^{(1)} & & \downarrow \alpha^{(2)} & & \downarrow \alpha \\ V_*(E'_{n_1}) & \longrightarrow & V_*(E'_{n_2}) & \longrightarrow & V_*(E'). \end{array}$$

Continuing this way, we have a commutative diagram:

$$\begin{array}{ccccccc} V_*(E_1) & \longrightarrow & V_*(E_2) & \longrightarrow \cdots & \longrightarrow & V_*(E) \\ \downarrow \alpha^{(1)} & & \downarrow \alpha^{(2)} & & & \downarrow \alpha \\ V_*(E'_{n_1}) & \longrightarrow & V_*(E'_{n_2}) & \longrightarrow \cdots & \longrightarrow & V_*(E') \end{array}$$

where each map is homomorphism. We will call this diagram (1).

If $\alpha$ is an isomorphism, by the similar argument as above, by choosing $m_2$ first (after choosing $n_1$), then choosing $n_2$, one has the following commutative diagram:

$$\begin{array}{ccccccc} V_*(E_1) & \longrightarrow & V_*(E_{m_2}) & \longrightarrow & \cdots \longrightarrow & V_*(E) \\ \downarrow \alpha^{(1)} & \nearrow \alpha^{(1)'} & \downarrow \alpha^{(2)} & \nearrow \alpha^{(2)'} & & \alpha \downarrow\uparrow \beta \\ V_*(E'_{n_1}) & \longrightarrow & V_*(E'_{n_2}) & \longrightarrow & \cdots \longrightarrow & V_*(E'). \end{array}$$

We will call this diagram (2).

By passing to a subsequence and changing the notation, we may assume that $n_k = k$ and $m_k = k$ in both diagram (1) and diagram (2).

For each $k$, diagram (1) gives the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V(I(E_k)) & \longrightarrow & V(E_k) & \longrightarrow & V(Q(E_k)) & \longrightarrow & 0 \\ & & \downarrow \gamma^{(k)} & & \downarrow \alpha^{(k)} & & \downarrow \beta^{(k)} & & \\ 0 & \rightarrow & V(I(E'_k)) & \longrightarrow & V(E'_k) & \longrightarrow & V(Q(E'_k)) & \longrightarrow & 0. \end{array}$$

It also gives commutative diagrams

$$\begin{array}{ccccc} V(I(E_1)) & \longrightarrow & V(I(E_2)) & \longrightarrow & V(I(E)) \\ \downarrow & & \downarrow & & \downarrow \\ V(I(E'_1)) & \longrightarrow & V(I(E'_2)) & \longrightarrow & V(I(E')) \end{array}$$

and

$$\begin{array}{ccccc} V(Q(E_1)) & \longrightarrow & V(Q(E_2)) & \longrightarrow & V(Q(E)) \\ \downarrow & & \downarrow & & \downarrow \\ V(Q(E'_1)) & \longrightarrow & V(Q(E'_2)) & \longrightarrow & V(Q(E')). \end{array}$$

Combining these commutative diagrams, we obtain the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V(I(E)) & \longrightarrow & V(E) & \longrightarrow & V(Q(E)) & \longrightarrow & 0 \\ & & \downarrow \gamma & & \downarrow \alpha & & \downarrow \beta & & \\ 0 & \rightarrow & V(I(E')) & \longrightarrow & V(E') & \longrightarrow & V(Q(E')) & \longrightarrow & 0. \end{array}$$

Furthermore, if $\alpha$ is an isomorphism, then diagram (2) ensures that $\alpha$, $\beta$ and $\gamma$ are invertible.

By 2.11 and diagram (1), we obtain the following commutative diagram

$$\begin{array}{ccccccc} \mathbf{K}(E_1) & \longrightarrow & \mathbf{K}(E_2) & \longrightarrow \cdots & \longrightarrow & \mathbf{K}(E) \\ \downarrow & & \downarrow & & & \\ \mathbf{K}(E_1') & \longrightarrow & \mathbf{K}(E_2') & \longrightarrow \cdots & \longrightarrow & \mathbf{K}(E') \end{array}$$

which will induces a map from $\mathbf{K}(H)$ to $\mathbf{K}(E')$. It is clear that

$$K_*(Q(E)) = \lim_{\rightarrow} K_*(Q(E_n)) \quad \text{and} \quad K_*(Q(E')) = \lim_{\rightarrow} K_*(E_n).$$

Thus we also have that $\alpha_5 \oplus \alpha_2$ preserves the order.

If $\alpha$ is a isomorphism, from diagram (2), we have

$$\begin{array}{ccccccc} \mathbf{K}(E_1) & \longrightarrow & \mathbf{K}(E_2) & \longrightarrow \cdots & \longrightarrow & \mathbf{K}(E) \\ \downarrow & \nearrow & \downarrow & \nearrow & & \\ \mathbf{K}(E_1') & \longrightarrow & \mathbf{K}(E_2') & \longrightarrow \cdots & \longrightarrow & \mathbf{K}(E'). \end{array}$$

This gives an isomorphism from $\mathbf{K}(E)$ onto $\mathbf{K}(E')$. In particular, $\alpha_i$ is isomorphism, $i = 1, 2, ..., 5$. □

**Corollary 2.15.** *Let $E$ and $E'$ be as in Lemma 2.13 with additional assumption that no $\mathcal{T}_0$ ever appears in $E_i$ and $E_i'$. Let $\alpha$ be a homomorphism from $V(E)$ into $V(E')$. Then all conclusions in Lemma 2.13 hold.*

## 3. Existence.

In this section, we first prove an existence Theorem (3.1). Then we show another type of existence Theorem (3.7), which says that given any possible invariant $V_*$, there is an inductive limit $A$ of finite direct sums of matrix algebras over $\mathcal{T}$-algebras of real rank zero such that $V_*(A) = V_*$. We also show that every real rank zero $A\mathbb{T}$ algebra is an inductive limit of finite direct sums of matrix algebras over $\mathcal{T}$-algebras, i.e., an $A\mathcal{T}$-algebra.

If $\phi : A \to B$ is a homomorphism from $C^*$-algebra $A$ into $C^*$-algebra $B$, we will use the notation $\phi_{*0}$ and $\phi_{*1}$ for the maps from $K_0(A)$ into $K_0(B)$ and from $K_1(A)$ into $K_1(B)$, respectively.

**Theorem 3.1.** *Let $E_1$ and $E_2$ be two finite direct sums of matrix algebras over $\mathcal{T}$-algebras. Suppose that there is a homomorphism $\alpha$ from $V_*(E_1)$ into $V_*(E_2)$ which maps $[1_{E_1}]$ into $[P]$ for some projection $P$ in $E_2$. Then*

(i) *$\alpha$ induces two homomorphisms $\gamma$ and $\beta$ such that the following diagram commutes:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bar{V}(I(E_1)) & \longrightarrow & \bar{V}(E_1) & \longrightarrow & \bar{V}(Q(E_1)) & \longrightarrow & 0 \\ & & \downarrow \gamma & & \downarrow \alpha & & \downarrow \beta & & \\ 0 & \longrightarrow & \bar{V}(I(E_2)) & \longrightarrow & \bar{V}(E_2) & \longrightarrow & \bar{V}(Q(E_2)) & \longrightarrow & 0 \end{array}$$

(ii) *$\alpha$ induces a map, consisting of five group homomorphisms $\{\alpha_i\}_{i=1}^5$, from $\mathbf{K}(E_1)$ to $\mathbf{K}(E_2)$.*

*Suppose further that $\psi$ is a $*$-homomorphism from $Q(E_1)$ to $Q(E_2)$ with $\psi_{*0} = \alpha_5$ and $\psi_{*1} = \alpha_2$ which satisfies the following condition: if $M_m(\mathcal{T}_0)$ is a summand of $E_1$ and $M_{m'}(\mathcal{T}_k)$ is a summand of $E_2$ and the partial map of $\gamma$ from $I(F_0)$ into $I(F_k)$ is not zero, then the partial map of $\psi$ from $Q(M_m(\mathcal{T}_0))$ to $Q(M_{m'}(\mathcal{T}_k))$ is injective. Then there exists a homomorphism $\phi$ from $E_1$ to $E_2$ such that $\phi$ induces $\alpha$, $\phi(1_{E_1}) = P$ and such that $\pi_2 \circ \phi = \psi \circ \pi_1$, where $\pi_i$ is the quotient map from $E_i$ to $Q(E_i)$.*

*Proof.* First, we note from Corollary 2.11 that the commuting diagrams in the theorem comes automatically from map $\alpha$. Also if $\phi$ is a homomorphism from $E_1$ into $E_2$ such that $\phi$ induces $\alpha$ from $V_*(E_1)$ to $V_*(E_2)$ and such that $\pi_2 \circ \phi = \psi \circ \pi_1$, then $\phi$ induces $\psi$. Clearly, we may assume that $E_2$ has only one summand, say $M_m(\mathcal{T}_r)$.

Write $E_1 = \oplus_{j=1}^k E_1^{(j)}$ where $E_1^{(j)} = M_{n_j}(\mathcal{T}_{k_j})$ and let $1_{n_j}$ be the unit of $E_1^{(j)}$. Associated with each $E_1^{(j)}$ we have an extension

$$0 \to I(E_1^{(j)}) \to E_1^{(j)} \to Q(E_1^{(j)}) \to 0.$$

Write $\psi = \oplus_{i=1}^k \psi_i$ where each $\psi_i$ is a homomorphism from $M_{n_i}(C(S^1))$ into $M_m(S^1)$. There exists a unitary $V \in M_m(C(S^1))$ such that $\operatorname{Ad} V \circ \psi = \operatorname{diag}(\psi_1, \cdots, \psi_k)$. Let $W = \operatorname{diag}(W_1, \cdots, W_k)$ be a unitary in $M_m(C(S^1))$. Then $\operatorname{Ad}(WV) \circ \psi$ still has diagonal form. We can choose $W$ such that $WV$ can be lifted to a unitary in $E_2$. So we may assume that $\psi$ already is of diagonal form:

$$\psi = \operatorname{diag}(\psi_1, \cdots, \psi_k).$$

Let $e_i = \operatorname{diag}(0, \cdots, \psi_i(1_{n_i}), 0, \cdots, 0)$. If $[e_i] = m_i$, then $m \geq m_1 + m_2 + \cdots + m_k$. Notice that $e_i M_m(C(S^1)) e_i \cong M_{m_i}(C(S^1))$. We note that some of $\psi_j$ may be zero. This means that the corresponding $m_j = 0$.

Since $\alpha(\sum_{i=1}^k [1_{n_i}]) = [P]$, there are mutually orthogonal projections $d_1$, $d_2, ..., d_k$ in $PM_m(\mathcal{T}_r)P$ such that $P = \sum_{i=1}^k d_i$, $\alpha([1_{n_j}]) = [d_j]$ and $\pi_2(d_j) =$

$\psi(1_{n_j})$. Note that

$$\pi(d_i)M_m(C(S^1))\pi(d_i) \cong M_{m_i}(C(S^1)),$$

if $d_i$ is not in $M_m(\mathcal{K})$. Let $\psi_j = \psi|_{Q(E_1^{(j)})}$. It is enough to show that, for each $j$, there exists a unital homomorphism $\phi_j : E_1^{(j)} \to d_j M_m(\mathcal{T}_r) d_j$ such that $\pi_2 \circ \phi_j = \psi_j \circ \pi_1$ and the homomorphism from $V(E_1^{(j)})$ into $V(E_2)$ induced by $\phi_j$ agrees with $\alpha|_{V(E_1^{(j)})}$.

We first consider the case that $d_j \in M_m(\mathcal{K})$. Let $[d_j] = b_j \in \mathbf{Z}_+ \setminus \{0\}$ (if $b_j = 0$, we define $\phi_j = 0$). So $d_i M_m(\mathcal{K}) d_i \cong M_{b_i}$. Since $\alpha$ preserves the addition, $n_i$ divides $b_i$. Let $\Phi$ be the quotient map from $M_{n_j}(\mathcal{T}_{k_j})$ onto $M_{n_j}$. Since $n_i$ divides $b_i$, there is a unital homomorphism $\Psi$ from $M_{n_j}$ into $d_j M(\mathcal{K}) d_j$. We define $\phi_j = \Psi \circ \Phi$. It is worthy of pointing out that, in this case, $\phi_j$ vanishes on $I(E_1^{(j)})$ and $\psi_j = 0$.

Now the homomorphism $\phi_j$ maps $E_1^{(j)}$ to $d_j E_2 d_j$ and sends the identity to $d_j$. Furthermore, $\phi_j$ induces a map from $V(E_1^{(j)})$ to $V(E_2)$ which agrees $\alpha$ on $V(E_1^{(j)})$.

Let us now assume that $d_j$ is not in $M_m(\mathcal{K})$. Let $\{e_{il}\}$ be a matrix unit for $M_{n_j}(\mathbf{C})$ which is in $M_{n_j}(\mathcal{T}_{k_j})$. Let $\psi_j(\pi(e_{il})) = \bar{q}_{il}$. Then $\bar{q}_{il}$ is a matrix unit (for $M_{n_j}$). There is a matrix unit $\{q'_{il}\} \in d_j M_m(\mathcal{T}_k) d_j$ such that $\pi(q'_{il}) = \bar{q}_{il}$ (see [**Eff**]). Note $\alpha([e_{ii}]) = \alpha([e_{11}])$, $[q'_{ii}] = [q'_{11}]$ in $V(d_j M_m(\mathcal{T}_k) d_j)$ and $\alpha_5([e_{11}]) = [\bar{q}'_{11}]$. There is a projection $q' \in d_j M_m(\mathcal{T}_k) d_j$ such that $\alpha([e_{11}]) = [q']$ in $V(d_j M_m(\mathcal{T}_k) d_j)$. Then either $[q'] - [q'_{11}]$, or $[q'_{11}] - [q']$ is in $I(V(d_j M_m(\mathcal{T}_k) d_j)) \cong \mathbf{Z}_+$. Since $\sum_{i=1}^{n_j} \alpha([e_{ii}]) = [d_j]$, the second case will not happen. Let $d'_j = d_j - \sum_{i=1}^{n_j} q'_{ii}$. Then $[d'_j] = n_j[q'] - n_j[q'_{11}] = n_j([q'] - [q'_{11}]) \geq 0$. If $[q'] - [q'_{11}] \neq 0$, there are mutually orthogonal and mutually equivalent projections $a_i$ in $d'_j M_m(\mathcal{K}) d'_j$ such that $[a_i] = [q'] - [q'_{11}]$ and $\sum_{i=1}^{n_j} a_i = d'_j$. There is a matrix unit $\{a_{il}\} \subset d'_j M_m(\mathcal{K}) d'_j$ such that $a_{ii} = a_i$, $i = 1, 2, ..., n_j$. Let $q_{il} = q'_{il} + a_{il}$. Then $\pi(q_{il}) = \bar{q}_{il}$ and $\alpha([e_{ii}]) = [q_{ii}]$.

Let $g : M_{n_j}(\mathbf{C}) \to d_j M_m(\mathcal{T}_r) d_j$ be the monomorphism which maps $e_{il}$ into $q_{il}$. Let $\bar{g} = \pi_2 \circ g$. Then we can write

$$\psi_j = \bar{g} \otimes \tilde{\psi}_j,$$

where $\tilde{\psi}_j$ is a unital homomorphism from $C(S^1)$ into $\pi_2(q_{11}) Q(M_m(\mathcal{T}_r)) \pi_2(q_{11}) \cong M_t(C(S^1))$ and $t = m_j/n_j$. It is enough to show that there exists a homomorphism $\tilde{\phi}_j : e_{11} M_{n_j}(\mathcal{T}_{k_j}) e_{11} \to g(e_{11}) M_m(\mathcal{T}_r) g(e_{11})$ such that $\pi_2 \circ \tilde{\phi}_j(S_{k_j}) = \psi_j(z)$ where $z$ is the canonical generator of $C(S^1)$ and where $S_{k_j}$ is the canonical isometry of $\mathcal{T}_{k_j}$ and such that the map from $V_*(e_{11} M_{n_j}(\mathcal{T}_{k_j}) e_{11})$ into $V_*(g(e_{11}) M_m(\mathcal{T}_r) g(e_{11}))$ agrees with the restriction of $\alpha$. Note that $\pi_1(e_{11} M_{n_j}(\mathcal{T}_{k_j}) e_{11}) \cong C(S^1)$.

*Case* (1): $r = 0$.

First, consider the case $k_j = 0$. Note that $e_{11}M_{n_j}(\mathcal{T}_0)e_{11} \cong \mathcal{T}_0$.

Since $r = 0$, every normal partial isometry in $M_m(C(S^1))$ lifts to a normal partial isometry in $M_m(\mathcal{T}_0)$. In particular, $\psi_j(z)$ lifts to a unitary $W \in g(e_{11})M_m(\mathcal{T}_r)g(e_{11})$. Let $A$ be the $C^*$-algebra generated by $W$ and $g(e_{11})M_m(\mathcal{K})g(e_{11})$. Then $A \cong \mathcal{T}_0$. Note that $W$ has essential spectrum $S^1$.

Suppose that $\gamma$ has multiplicity $l_1 > 0$. Embedding $A$ into $B(l^2)$ such that $I(A) = I(\mathcal{T}_0)$ maps onto $\mathcal{K}(l^2)$. Let $\lambda_1$ be an isomorphism from $M_{l_1}(B)$ onto $B(l^2)$. Let $\lambda_2 : \mathcal{T}_0 \to M_{l_1}(B(l^2))$ be an amplification such that $\lambda_2(a) = \mathrm{diag}(a, a, ..., a)$ (there are $l_1$ copies of $a$). Set $\phi' = \lambda_1 \circ \lambda_2$. Then (in $B(l^2)$) both unitaries $\phi'(S_0)$ and $W$ have the same essential spectrum $S^1$, i.e., spectrum of $\pi(\phi'(S_0)) = \psi(z) = S^1$. Therefore there is a unitary $U \in B(l^2)$ such that $U^*\phi'(S_0)U - W \in \mathcal{K}(l^2)$. Thus $U^*\phi'(S_0)U \in A$. We now define $\tilde{\phi}_j(a) = U^*\phi'(a)U$ for $a \in \mathcal{T}_0$. It is easy to see that $\tilde{\phi}$ meets the requirement in this case.

In the case that $\gamma = 0$ (i.e. $\alpha_3 = 0$), let

$$f : M_t(C(S^1)) \to g(e_{11})M_m(\mathcal{T}_r)g(e_{11})$$

be a monomorphism which splits the short exact sequence:

$$0 \to g(e_{11})M_m(\mathcal{K})g(e_{11}) \to g(e_{11})M_m(\mathcal{T}_0)g(e_{11}) \to M_t(C(S^1)) \to 0.$$

We define

$$\tilde{\phi}_j = f \circ \psi_j \circ \pi.$$

Clearly $\tilde{\phi}_j$ induces the same map from $V_*(e_{11}M_{n_j}(\mathcal{T}_{k_j})e_{11})$ into $V_*(g(e_{11})M_m(\mathcal{T}_r)g(e_{11}))$ as the restriction of $\alpha$.

If $k_j > 0$, we also define

$$\tilde{\phi}_j = f \circ \psi_j \circ \pi.$$

To see that $\tilde{\phi}_j$ meets other requirements, we only need to show that $\alpha$ maps $I(F_{k_j})$ into zero. The homomorphism $\alpha$ maps $\ker j_{k_j}$ into zero, since the map $j_0 : F_0 \to G(F_0)$ is injective. Thus $\alpha$ is not injective. By part (d) of 2.4, $\alpha$ indeed maps $I(F_{k_j})$ into zero.

*Case* (2): $r > 0$. Note that $\tilde{\psi}_j(z)$ can be lifted to an isometry $W$ in $g(e_{11})M_m(\mathcal{T}_r)g(e_{11})$. From the second diagram, one sees that

$$\alpha([1 - S_{k_j}S_{k_j}^*]) = \alpha_3([1 - S_{k_j}S_{k_j}^*]) = [g(e_{11}) - WW^*].$$

We first consider the case that $\alpha([1 - S_{k_j}S_{k_j}^*]) = 0$. In this case, we may assume that $W$ is a unitary.

Let us first consider the case that $k_j = 0$ and $\gamma$ does not map $I(F_{k_j})$ into zero. By the assumption on $\psi$, the essential spectrum of $W$ is the whole circle. In particular $sp(W) = S^1$.

Denote by $B$ the $C^*$-subalgebra generated by $W$ and $g(e_{11})I(M_{n_j}(\mathcal{T}_r))g(e_{11}) \cong \mathcal{K}$. Clearly $B$ is a unital essential extension of $C(S^1)$ by $\mathcal{K}$. Therefore $B \cong \mathcal{T}_0$. Suppose that $\gamma(n) = l_1 n$ for $n \in \mathbf{Z}_+$. From the case that $r = 0$, we know that there exists a unital homomorphism $\tilde{\phi}_j' : \mathcal{T}_0 \to B$ such that $\tilde{\phi}_j'$ induces a map from $V_*(\mathcal{T}_0)$ into $V_*(B)$ which has multiplicity $l_1$ on $I(F_0)$. Finally, we define $\tilde{\phi}_j = j \circ \tilde{\phi}_j'$, where $j : B \to g(e_{11})M_m(\mathcal{T}_r)g(e_{11})$ is the embedding. It is easily checked that this $\tilde{\phi}_j$ meets the requirements.

Now we consider the case that $\gamma$ is trivial on $I(F_{k_j})$. This is always the case if $k_j > 0$. In fact, in this case, the map $\delta_1$ is injective, so $\alpha_3 = 0$. So $\alpha$ is not injective, by part (f) of 2.4, $\alpha$ also maps $I(F_{k_j})$ into zero.

We define $\tilde{\phi}_j = \lambda \circ \pi_1$, where $\lambda$ is a homomorphism from $C(S^1)$ into $g(e_{11})M_m(\mathcal{T}_r)g(e_{11})$ by sending $z$ into $W$. Clearly, $\tilde{\phi}_j$ meets the requirements.

Now we consider the case that $\alpha([1 - S_{k_j}S_{k_j}^*]) \neq 0$. Note that in this case it is necessary that $k_j > 0$. Since $1 - S_{k_j}S_{k_j}^*$ is the sum of $k_j$ mutually orthogonal and equivalent projections, say $q_1, \cdots, q_{k_j}$, there are $k_j$ mutually orthogonal and equivalent projections $a_1, \cdots, a_{k_j}$ such that $\alpha([q_i]) = [a_i] = [a_1]$ in $V(M_m(\mathcal{T}_r))$ and

$$\sum_{i=1}^{k_j} a_i = g(e_{11}) - WW^*.$$

We now define $\tilde{\phi}_j$ from $e_{11}M_{k_j}(\mathcal{T}_{k_j})e_{11} \cong \mathcal{T}_{k_j}$ to $g(e_{11})M_m(\mathcal{T}_r)g(e_{11})$ by sending $S_{k_j}$ to $W$ and by sending $(1 - S_{k_j}S_{k_j}^*)I(\mathcal{T}_{k_j})(1 - S_{k_j}S_{k_j}^*)$ into $(g(e_{11}) - WW^*)M_m(\mathcal{K})(g(e_{11}) - WW^*)$ By 2.1, this is in fact a homomorphism. From our construction, it is clear that such defined $\tilde{\phi}_j$ induces a map from $V_*(e_{11}M_m(\mathcal{T}_{k_j})e_{11})$ into $V_*(M_m(\mathcal{T}_r))$ which agrees with the restriction of $\alpha$. □

**Remark 3.2.** We would like to remark that the condition for $\psi$ in the Theorem 3.1 is necessary. If $\phi : M_m(\mathcal{T}_0) \to M_{m'}(\mathcal{T}_k)$ is a homomorphism and $\ker \phi \cap I(M_m(\mathcal{T}_0)) = 0$, then $\ker \phi = 0$, since $I(M_m(\mathcal{T}_0))$ is an essential ideal of $M_m(\mathcal{T}_0)$. So $\phi$ is injective. To see that the induced map from $M_m(C(S^1))$ into $M_{m'}(S^1)$ is injective, we let $u$ be the unitary in $e_{11}(\mathcal{T}_0)e_{11}$ such that $\pi(u) = z$, where $z$ is the function in $e_{11}C(S^1)e_{11} \cong C(S^1)$ which is identity on $S^1$ and $\pi : M_m(\mathcal{T}_0) \to M_m(C(S^1))$ is the quotient map. Since $\phi$ is injective, $sp(\phi(u)) = S^1$. We want to show that, in fact, $sp(\pi_1(\phi(u))) = S^1$, where $\pi_1 : M_{m'}(\mathcal{T}_k) \to M_{m'}(C(S^1))$ is the quotient map. Suppose that

$sp(\phi(u)) = F$. If $F \neq S^1$, then, since $F$ is compact, $S^1 \setminus F$ contains an arc, say $C$. Suppose that $C_0 \in C$ is a sub-arc of $C$. Let $f \in C(S^1)$ such that $f(\xi) = 1$, if $\xi \in C_0$, $f(\xi) = 0$, if $\xi \in F$ and $f$ is linear otherwise. Let $a = \phi(u)\phi(f(u))$. Then $C_0 \subset sp(a)$. However, $a \in I(M_{m'}(\mathcal{T}_k))$ ($\cong \mathcal{K}$). So $sp(a)$ can have only countably many points. A contradiction. Therefore, $S^1 \setminus F$ does not contain any arc, which implies that $F = S^1$.

**3.3.** The class of $A\mathcal{T}$-algebras of real rank zero will be denoted by $\boldsymbol{\mathcal{T}}$. The subclass $\boldsymbol{\mathcal{T}}_1$ consists of those $C^*$-algebras that there are no $M_m(\mathcal{T}_0)$ appeared in any summand in the direct limit system. We will call the algebras in these class $\boldsymbol{\mathcal{T}}$-algebras and $\boldsymbol{\mathcal{T}}_1$-algebras, respectively.

If $E$ is a unital $A\mathcal{T}$-algebra then $V(E) = \lim\limits_{\to}(V(E_n), \phi^*_{n,n+1})$ is a direct limit of finite direct sums of $V(E_n)$. The homomorphism $\phi^*_{n,n+1}$ maps the subsemigroup $I(V(E_n))$ into $I(V(E_{n+1}))$. Let $I(V(E))$ be the subsemigroup $\lim\limits_{\to} I(V(E_n))$. If we assume that $E$ has real rank zero, then $Q(V(E))$ is a positive cone of an unperforated ordered group with Riesz decomposition property. This follows from the fact that $Q(E)$ is an $A\mathbb{T}$-algebra with real rank zero and Theorem 8.1 of [**Ell2**].

Let $V = \lim\limits_{\to}(V_n, \alpha^{n,n+1})$ be a direct limit of finite direct sums of $F'_{k(n)}s$. $(G(V), V)$ is an ordered group. By 2.10, there are maps $\alpha_5^{n,n+1} : Q(V_n) \to Q(V_{n+1})$ induced by $\alpha$. Let $V_* = \lim\limits_{\to}(V_{n*}, \alpha^{n,n+1})$ be a direct limit of finite direct sums of $F'_{k(n)*}s$. By 2.10, there are also maps $\alpha_5^{n,n+1}$ from $Q(V_{n*})$ into $Q(V_{n+1*})$. We will identify $V$ with the semigroup $\lim\limits_{\to}(V_n, \alpha^{n,n+1})$ of $V_*$.

**Lemma.** *Let $E = \lim\limits_{\to}(E_n, \phi_{n,n+1})$ be a direct limit of finite direct sums of $\mathcal{T}$-algebras of real rank zero. Then, for any $k$ and any summand $E_k^i \cong M_m(\mathcal{T}_0)$ of $E_k$, there exists an integer $N(k) > 0$ such that every partial homomorphism $\phi_{k,n}^{i,j}$ is either non-injective or the multiplicity of $(\psi_{k,n}^{i,j})_{*0}$ is greater than 1 for all $n \geq N$, where $\psi_{k,n}$ is the homomorphism from $Q(E_k^i)$ into $Q(E_n^j)$ induced by $\phi_{k,n}^{i,j}$.*

*Proof.* Suppose that the lemma is false. We may assume, by taking a subsequence, that $E' \cong M'_m(\mathcal{T}_0)$ is a summand of $E_1$ and $\phi_{1,n}^{1,j_n}$ is injective and $(\psi_{1,n}^{1,j_n})_{*0}$ has multiplicity 1 for $j_n$ and for each $n$. Let $\psi_{n,n+1}$ be the homomorphism from $Q(E_n)$ into $Q(E_{n+1})$ induced by $\phi_{n,n+1}$. Let $p = \psi_{1,\infty}(z^*z)$, where $z$ is the canonical unitary in $e_{11}M_m(C(S^1))e_{11}$. Then $pQ(E)p$ is an $A\mathbb{T}$-algebra of real rank zero. Let $h = (z + z^*)/2$ and $x = \psi_{1,\infty}(h)$. Since $pQ(E)p$ has real rank zero, for any $1/2\varepsilon > 0$, there is a selfadjoint element $y \in pQ(E)p$ with finite spectrum such that

$$\|x - y\| < \varepsilon/4.$$

Suppose that $y = \sum_{l=1}^{L} \gamma_l e_l$, where $\gamma_l$ are real numbers and $e_l$ are mutually orthogonal projections in $pQ(E)p$. Without loss of generality, by taking a large $n$, we may assume that $e_l \in \psi_{n,\infty}(\psi_{1,n}(z^*z)Q(E_n)\psi_{1,n}(z^*z))$. We may also assume, by taking large $n$, without loss of generality, that

$$\|\psi_{1,n}(h) - g\| < \varepsilon/2,$$

where $g$ is selfadjoint element in $\psi_{1,n}(z^*z)Q(E_n)\psi_{1,n}(z^*z)$ and $\psi_{n,\infty}(g) = y$. Let $f_l$ be selfadjoint element in $\psi_{1,n}(z^*z)Q(E_n)\psi_{1,n}(z^*z)$ such that $\psi_{n,\infty}(f_l) = e_l$. Since

$$\|\psi_{n,k}(f_l)^2 - \psi_{n,k}(f_l)\| \to 0,$$

as $k \to \infty$, by functional calculus, we may further assume that $g$ has finite spectrum.

Now $e = \psi_{1,n}^{1,j_n}(z^*z)$ commutes with both $\psi_{1,n}(h)$ and $g$. Since $\psi_{1,n}^{1,n}$ is injective and $(\psi_{1,j_n}^{1,n})_{*0}$ has multiplicity 1, $eQ(E_n)e \cong C(S^1)$. We have $\psi_{1,n}^{1,j_n}(h)$ ($= e\psi_{1,n}(h)$), $ege \in eQ(E_n)e \cong C(S^1)$, $ege$ has finite spectrum and

$$\|\psi_{1,n}^{1,j_n}(h) - ege\| < \varepsilon/2.$$

Since $sp(\psi_{1,n}^{1,j_n}(h)) = [-1, 1]$, and $ege$ has at most 1 point in its spectrum, the above inequality could not hold. □

**Proposition 3.4.** *Let $E = \lim_{\rightarrow}(E_n, \phi_{n,n+1})$ be an $A\mathcal{T}$-algebra of real rank zero. Denote by $\psi_{n,n+1}$ the $*$-homomorphism from $Q(E_n)$ to $Q(E_{n+1})$ induced by $\phi_{n,n+1}$. Then there exists a subsequence $\{n(k)\}$ of integers such that each partial homomorphism of $(\phi_{n(k),n(k+1)})_{*0}$ is either zero when restricted on any summand $I(F_0)$, or the corresponding partial homomorphism of $(\psi_{n(k),n(k+1)})_{*0}$ has multiplicity $> 2^{k+1}$.*

*Proof.* This is an immediate consequence of the proof of Lemma 3.3. □

**Definition 3.5.** Let $V = \lim_{\rightarrow}(V_n, \alpha^{n,n+1})$ be a direct limit of finite direct sums of $F_{n(k)*}$'s. We say $V$ satisfies *generalized Riesz decomposition property* if $(G(Q(V)), Q(V))$ is an unperforated ordered group with Riesz decomposition property and there exists a subsequence $\{n(k)\}$ of integers such that every partial homomorphism $\alpha'_{n(k),n(k+1)}$ restricted on any summand $I(F_0)$ is either zero or the corresponding partial homomorphism $\alpha_5^{n(k),n(k+1)'}$ has multiplicity $> 2^{k+1}$, where $\alpha_5^{n(k),n(k+1)}$ is the homomorphism from $Q(V_{n(k)})$ into $Q(V_{n(k+1)})$ induced by $\alpha_{n(k),n(k+1)}$.

So we have the following.

**Proposition 3.6.** *If $E$ is an $A\mathcal{T}$-algebra of real rank zero, then $V(E)_*$ has the generalized Riesz decomposition property.*

*Proof.* This follows immediately from 3.4 and 3.3 . □

**Proposition 3.7.**

(A) *Let* $V = \lim\limits_{\to}(V_n, \alpha^{n,n+1})$, *where* $V_n = \oplus_{i=1}^{m(n)} F_{(n,i)}$ *and* $(G(Q(V)), Q(V))$ *has the Riesz decomposition property. Then, for a fixed order unit* $q(e) \in Q(V)$, *where* $e \in V$, *there exists a unital* $\mathcal{T}_1$*-algebra* $E = \lim\limits_{\to}(E_n, \phi_{n,n+1})$ *with*

$$V(E) = V, [1] = e.$$

(B) *Let* $V_* = \lim\limits_{\to}(V_{n*}, \alpha^{n,n+1})$, *where* $V_{n*} = \oplus_{i=1}^{m(n)} F_{(n,i)*}$, *has the generalized Riesz decomposition property. Then, for a fixed order unit* $q(e) \in Q(V)$, *where* $e \in V$, *there exists a unital* $\mathcal{T}$*-algebra,* $E$ *such that*

$$V_*(E) = V_*, [1] = e.$$

*Proof.* We first find an AT-algebra $Q(E)$ of real rank zero which will gives us the required $\mathcal{T}$-algebra (or $\mathcal{T}_1$-algebra).

For case (A), let $j_n : V_n \to G(V_n)$ and $j : V \to G(V)$ be the natural homomorphism from $V_n$ and $V$ into their Grothendieck groups. It is easy to see that

$$\ker j = \lim_{\to}(\ker j_n, \alpha^{n,n+1}).$$

We will denote $k(V_n)_+ = \ker j_n$ and $k(V)_+ = \ker j$. We write $(Q(V_n) \oplus k(V_n)_+, \Gamma_n)$ for the direct sum of summands $\mathbf{Z}_+ \oplus k_i\mathbf{Z}_+$ with order in each such summand determinded by the strict order in the first component. Let $\alpha_5^{n,n+1} : Q(V_n) \to Q(V_{n+1})$ be the quotient map induced by $\alpha^{n,n+1}$ and let $\alpha_2^{n,n+1} : k(V_n)_+ \to k(V_{n+1})_+$ be the restriction of $\alpha^{n,n+1}$. One can extend $\alpha_5^{n,n+1}$ to group homomorphisms from $G(Q(V_n))$ into $G(Q(V_{n+1}))$ and from $k(V_n)_+ = G(k(V_n))_+$ into $k(V_{n+1})_+ = G(k(V_{n+1}))_+$, respectively. We will use $\alpha_5^{n,n+1}$ for the first extension, $\alpha_2^{n,n+1}$ for the second extension and let $\eta^{n,n+1} = \alpha_5^{n,n+1} \oplus \alpha_2^{n,n+1}$. Let $k(V_n) = G(k(V_n)_+)$ and $k(V) = G(k(V)_+)$ We then denote

$$(G(Q(V)) \oplus k(V), \Gamma) = \lim_{\to}(G(Q(V_n)) \oplus k(V_n), \Gamma_n, \eta^{n,n+1}).$$

For case (B), let

$$Q_*(V_n) = \{(\pi(x), y) : (x, y) \in V_{n*}\},$$

where $\pi : V_n \to Q(V_n)$ is the quotient map. Define $\eta^{n,n+1}(\pi(x), y) = (\pi(\alpha^{n,n+1}(x)), \alpha_{n,n+1}(y))$ $(x, y) \in V_{n*}$. Let

$$k(V)_+ = \lim_{\to}(k(V_n)_+, \alpha^{n,n+1}).$$

We also note that

$$Q(V) = \lim_{\rightarrow}(Q(V_n), \alpha^{n,n+1}).$$

Then $k(V) = G(k(V)_+) = \lim\limits_{\rightarrow} G(k(V_n)_+)$ and $G(Q(V)) = \lim\limits_{\rightarrow} G(Q(V_n))$. Define

$$\Gamma = \lim_{\rightarrow}(Q_*(V_n), \alpha_5^{n,n+1} \oplus \alpha_2^{n,n+1}).$$

For both case (A) and (B), since $Q(G(V))$ has Riesz decomposition property, the asymptotic condition (ii) in Theorem 7.2 of [Ell2], implies that $(Q(G(V)) \oplus k(V), \Gamma)$ (for both case (A) and (B)) has the asymptotic property (ii) in Theorem 8.1 of [Ell2]. Therefore, by Theorem 8.1 of [Ell2], $(G(Q(V)) \oplus k(V), \Gamma)$ (in both cases) has Riesz decomposition property. We also assume that $e_n \in V_n$ such that $\alpha^{n,n+1}(e_n) = e_{n+1}$ and the image of $e_n$ is $e \in V$.

Now by Lemma 7.2 of [Ell2], by passing to a subsequence if necessary, we may assume the following.

(1): For every $n \in \mathbf{N}$, every summand $\mathbf{Z}$ of $Q(V_n)$, and every summand $\mathbf{Z}$ of $Q(V_{n+1})$, the partial homomorphism $\alpha_5^{n,n+1'} : \mathbf{Z} \to \mathbf{Z}$ has multiplicity $m_0 \geq 2^{n+1}$ or the corresponding partial homomorphism $\alpha_2^{n,n+1'}$ is zero.

(2): if $\alpha_2^{n,n+1'} = 0$ but $\alpha_3^{n,n+1'} \neq 0$ the partial homomorphism $\alpha_5^{n,n+1}$ has multiplicity $m_0 > 2^{n+1}$.

To see (2), we note that if the corresponding summand is $F_{r*}$ with $r > 0$, $\alpha_3^{n,n+1'} \neq 0$, implies that $\alpha_2^{n,n+1} \neq 0$, so $m_0 > 2^{n+1}$. If $r = 0$, then we apply the second condition of the generalized Riesz decomposition property.

Let

$$A_n = \oplus_{i=1}^{m(n)} M_{k(n,i)}(C(S^1)),$$

where each summand $M_{k(n,i)}(C(S^1))$ corresponds a summand of $Q(G(V_n))$. Here $m(n)$ is determines by $Q(G(V_n))$. For $k(n,i)$, we let $k(1,i) = 1$ and $k(n,i)$ defined inductively by homomorphisms $\alpha^{n,n+1}$ as follows.

Suppose that there are $m(n+1)$ copies of $\mathbf{Z}$ in $Q(G(V_n))$ and

$$q(e_{n+1}) = (k(n+1,1), k(n+1,2), ..., k(n+1, m(n+1))),$$

where $q : G(V_n) \to Q(G(V_n))$ is the quotient map. Since $q(e)$ is an order unit, without loss of generality, by passing a subsequence if necessary, we may assume that $k(n+1,i)$ is positive for each $i$. Set

$$A_{n+1} = \oplus_{i=1}^{m(n+1)} M_{k(n+1,i)}(C(S^1)).$$

Note that

$$K_0(A_n) = Q(G(V_n)) \quad \text{and} \quad K_1(A_n) = k(V).$$

We define the partial homomorphism

$$\psi'_{n,n+1} : M_{k(n,i)}(C(S^1)) \to M_{k(n+1,j)}(C(S^1))$$

as follows. If $\alpha_2^{n,n+1'} = 0$ and $\alpha_3^{n,n+1'} = 0$, define

$$\psi'_{n,n+1}(f) = \mathrm{diag}(f(1), \cdots f(1), 0, \cdots, 0) \in M_{k(n+1,j)}(C(S^1))$$

(there are $m_0$ many $f(1)$). If $\alpha_2^{n,n+1'} = 0$ and $\alpha_3^{n,n+1} \neq 0$, (in this case, $m_0 > 2^{n+1}$), define

$$\psi'_{n,n+1}(f) = \mathrm{diag}(f(z), f(1), \cdots, f(1), 0, \cdots, 0) \in M_{k(n+1,j)}(C(S^1))$$

(there are $m_0 - 1$ many $f(1)$).

If $\alpha_2^{n,n+1'}$ maps a generator of $i$ th summand of $k(V_n)$ to $m_1$ times a generator of $j$th summand of $k(V_{n+1})$ and $m_1 \neq 0$, then set

$$\begin{aligned}&\psi'_{(n,n+1)}(f)\\ &= \mathrm{diag}(f(z^{m_1-1}), f(\omega), f(\omega^2), f(\omega^3), \cdots, f(\omega^{m_0}), 0, \cdots, 0) \in M_{m'}(C(S^1)),\end{aligned}$$

where $\omega = \exp(2\pi i/(m_0 - 2))$. The construction of $\psi_{n,n+1}$ ensures that it induces the same map $K_0(A_n) \oplus K_1(A_n) \to K_0(A_{n+1}) \oplus K_1(A_{n+1})$ as $(\alpha_5^{n,n+1} \oplus \alpha_2^{n,n+1})$. Let

$$A = \lim_{\to}(A_n, \psi_{n,n+1}).$$

Then $A$ is of real rank zero. This is certainly known. We sketch the proof as follows. Let $h$ be a selfadjoint function in a summand of $A_n$. We want to show that, for any $\varepsilon > 0$, there are $m > n$ and selfadjoint element $h'$ in $A_m$ with finite spectrum such that $\|\psi'_{n,m}(h) - h'\| < \varepsilon$ for each partial map $\psi'_{n,m}$ from $A_n$ into $A$. Using an $\varepsilon/16$-cover, dividing the spectrum into finitely many disjoint pieces, write $h$ as a direct sum of selfadjoint elements and then shift the spectrum, without loss of generality, we may assume that $sp(h)$ is in $[0, \|h\|]$ and it is $\varepsilon/8$-dense in $[0, \|h\|]$. Take a large $m > n$,

$$\psi'_{n,m}(h) = \mathrm{diag}(h(z^{m'_1-1}), h(\omega), h(\omega^2), ..., g(\omega^{m_0}), 0, ..., 0).$$

Note that $\{h(\omega), h(\omega^2), ..., h(\omega^{m_0})\}$ is $\varepsilon/4$ dense in $[0, \|h\|]$, if $m$ is large enough. One then applies (i) of Lemma 8 in [**LR**] to obtain the assertion.

Let $M_{k(n,i)}(C(S^1))$ correspond to a summand $F_{l(n,i)}$. We will lift $M_{k(n,i)}$ to $M_{k(n,i)}(\mathcal{T}_{l(n,i)})$. Define $E'_n = \oplus_{i=1}^{m(n)} M_{k(n,i)}(\mathcal{T}_{l(n,i)})$. Then $V(E'_n) \oplus V''_n = V_n$ for case (A) and $V_*(E'_n) \oplus V''_{n*} = V_{n*}$ for case (B), where $\alpha^{n,n+1}(V''_n) \in I(V_{n+1})$ in both cases. There are, for each $n$, a finite direct sum of $\mathcal{T}$-algebras $E''_n$ such that $V''_n = V(E''_n)$ (for case (A)) or $V''_{n*} = V_*(E''_n)$. Let $E_n = E'_n \oplus E''_n$. We have

$V(E_n) = V_n$, $V_*(E_n) = V_{n*}$ for case (B), $V(I(E_n)) = I(V_n)$, $K_0(Q(E_n)) = G(Q((V_n)))$ and $K_1(Q(E_n)) = G(k(V_n))$. Remember that $\alpha^{n,n+1}$ induces $\alpha_5^{n,n+1} \oplus \alpha_2^{n,n+1}$.

We now use the second condition in the generalized Riesz decomposition property to apply 3.1. By 3.1, we obtain the following commutative diagram:

$$\begin{array}{ccc} E_n & \longrightarrow & E_{n+1} \\ \downarrow & & \downarrow \\ Q(E_n) & \overset{\psi_{n,n+1}}{\longrightarrow} & Q(E_{n+1}) \end{array}.$$

Then the direct limit $E = \varinjlim E_n$ has $V(E) = V$, $[1] = e$, or $V_*(E) = V_*$ for case (B). Note $A_n = Q(E_n)$. We have

$$Q(E) = \varinjlim(A_n, \psi_{n,n+1}).$$

Since $A$ has real rank zero, it follows from 2.3 of [**Zh1**] and 2.11 of [**Zh2**] that $E$ has real rank zero. $\square$

**Lemma 3.8.** *Every AF-algebra is an $\mathcal{T}_1$-algebra. Furthermore, we can choose $E_n$ in the direct limit such that each summand is a matrix algebra over $\mathcal{T}_1$.*

*Proof.* Let $A$ be a unital $AF$-algebra and $G$ be its dimension group. Let $G = \varinjlim(G_n, \eta_n)$, where each $G_n$ is a direct sum of finitely many copies of $\mathbf{Z}$. Let $V_n$ be the corresponding sum of $\mathbf{Z}_+$. Let $V_n^{(i)}$ be a summand ($V_n^{(i)} = \mathbf{Z}_+$). Let $R_n^{(i)} = \mathbf{Z}_+ \sqcup V_n^{(i)}$. Define $x + y = y$, if $x \in \mathbf{Z}_+$, $y \neq 0$ and $y \in V_n^{(i)}$. Let $V_n' = \oplus R_n^{(i)}$.

Now define a map $\eta_n' : V_n' \to V_{n+1}'$ by defining $\eta_n'|_{R_n^{(i)}}(x) = 0$, if $x \in \mathbf{Z}_+$ and $\eta_n'|_{R_n^{(i)}}(y) = \eta_n|_{V_n^{(i)}}(y)$. Then $V = \varinjlim(V_n', \eta_n') = \varinjlim(V_n, \eta_n)$. It follows from 3.7 that there is a $\mathcal{T}_1$-algebra $E$ such that $V(E) = V$. Since $I(V) = \{0\}$, $I(E) = 0$. Therefore, $E \cong Q(E)$. This implies that $K_0(Q(E)) = G$ as ordered group and $K_1(Q(E)) = \{0\}$. Since $Q(E)$ is an $A\mathbb{T}$-algebra, by Theorem 7.1 of [**Ell2**], $E \cong Q(E) \cong A$. $\square$

**Lemma 3.9.** *Every $A\mathbb{T}$-algebra of real zero is in $\mathcal{T}$.*

*Proof.* Let $A = \varinjlim(A_n, \rho_{n,n+1})$ be an $A\mathbb{T}$-algebra, where

$$A_n = \oplus_{j=1}^{L_n} M_{k_j}(C(S^1)).$$

Set

$$E_n = \oplus_{j=1}^{L_n} M_{k_j}(\mathcal{T}_0).$$

Let $\pi_n : E_n \to A_n$ be the quotient map. There is a unital monomorphism $j_n : A_n \to E_n$ such that $j_n \circ \pi_n$ is the identity map on $A_n$. We define $\phi_{n,n+1} = j_{n+1} \circ \rho_{n,n+1} \circ \pi_n$. So $E = \lim_{\to}(E_n, \phi_{n,n+1})$ is an $A\mathcal{T}$-algebra. Clearly $I(E) = 0$ and $E \cong A$. So $E$ has real rank zero. Therefore $E \in \mathbb{T}$. $\square$

**3.10.** Our invariants for $\mathbb{T}$ are $V_*(-)$. However, we would like briefly describe the $K$-theory of $\mathbb{T}$. If $E \in \mathbb{T}$, then $K_0(E) = G(V(E))$ and $K_1(E) = G(\ker d)$. The map from $\ker d$ into $G(\ker d)$ is injective. In particular, if $E \in \mathbb{T}_1$, $K_1(E) = 0$. In general, one may write $K_0(E) = G_0 \oplus G_1$, where $G_1$ can be any countable torsion group that every element in it has finite order. The order of $K_0(E)$ is determined by $G_0$, where $G_0/K_0(I(E))$ is any countable abelian unperforated ordered group with the Riesz decomposition property. Note that $I(E)$ is a stable $AF$-algebra.

## 4. Uniqueness.

In this section, we will prove a so called uniqueness theorem for homomorphisms from one matrix algebra over $\mathcal{T}$-algebra to another.

**Lemma 4.1** ([Ln5, 2.1])**.** *Every derivable automorphism on a unital separable $C^*$-algebra $A$ is approximately inner.*

**4.2.** Denote the function $f_1((e^{i\theta}, e^{i\gamma})) = e^{i\theta}$ and $f_2((e^{i\theta}, e^{i\gamma})) = e^{i\gamma}$ in $C(S^1 \times S^1)$, respectively. Then $C(S^1 \times S^1)$ is generated by $f_1$ and $f_2$. Let $X$ be a compact subset of $S^1 \times S^1$. Then there is a homomorphism $\phi : C(S^1 \times S^1) \to C(X)$. We will also use the notation $f_1$ and $f_2$ for $\phi(f_1)$ and $\phi(f_2)$, if there is no confusion. Note that $f_1$ and $f_2$ also generate $C(X)$. In the rest of this section, $f_1$ and $f_2$ will always mean these functions.

**Lemma.** *Let $X$ be a compact subset of $S^1 \times S^1$ and let $\tau : C(X) \to B(H)/\mathcal{K}$ be a unital trivial essential extension of $C(X)$ by $\mathcal{K}$. Suppose that $sp(\tau(f_2)) = F(\subset S^1)$. Then there exists a norm continuous path of unitaries $\{w_t\}$ in $B(H)/\mathcal{K}$ such that*

$$w_0 = \tau(f_1), \quad [w_t, \tau(f_2)] = 0,$$

*and the $C^*$-subalgebra $C^*(w_1, \tau(f_2))$ is isomorphic to $C(S^1 \times F)$.*

*Proof.* Suppose that $\lambda \in X$. Then there exists a nonzero projection $q \in B(H)/\mathcal{K}$ such that

$$\tau(f)q = q\tau(f) = f(\lambda)q$$

for all $f \in C(X)$(see [BDF2]). By taking a subprojection of $q$ if necessary, we may assume that the map $\tau'(f) = \tau(f)(1 - q)$ is still injective. Note that $f_2(\lambda) = \exp(ia)$ for some nonnegative number $a \leq 2\pi$. Let $Q$ be a projection in $B(H)$ such that $\pi(Q) = q$, where $\pi : B(H) \to B(H)/\mathcal{K}$ is the quotient map. Then $\tau'$ gives a unital trivial essential extension of $C(X)$ by $(1-Q)\mathcal{K}(1-Q)$.

By the BDF theory (see [**BDF1**]), there is an abelian AF-algebra $B \subset (1-q)(B(H)/\mathcal{K})(1-q)$ such that im $\tau' \subset B$. Therefore there is $0 \leq h_1 \leq (2\pi)(1-q)$ in $B$ such that $\tau'(f_1) = \exp(ih_1)(1-q)$. Thus $[h_1, \tau'(f_2)] = 0$. Note also that

$$h_1\tau(f_2) = h_1(1-q)\tau(f_2) = h_1\tau'(f_2) = \tau'(f_2)h_1 = \tau(f_2)(1-q)h_1 = \tau(f_2)h_1.$$

Let $\tau_0 : C(S^1 \times F) \to q(B(H)/\mathcal{K})q$ be a unital trivial essential extension of $C(S^1 \times F)$ by $Q\mathcal{K}Q$. Again, there is $0 \leq h_2 \leq (2\pi)q$ in $B(H)/\mathcal{K}$ such that

$$\exp(ih_2)q = \tau_0(f_1) \quad \text{and} \quad [h_2, \tau_0(f_2)] = 0.$$

Set $h_3 = h_1 + h_2$. Then $0 \leq h_3 \leq 2\pi$ and

$$[h_3, \tau'(f_2) + \tau_0(f_2)] = 0.$$

Note that $sp(\tau'(f_2)) = F$ and $[\tau'(f) + \tau_0(f_2)] = [\tau(f_2)] = 0$ in $K_1(B(H)/\mathcal{K})$. There is a unitary $W \in B(H)/\mathcal{K}$ such that

$$W^*(\tau'(f_2) + \tau_0(f_2))W = \tau(f_2).$$

Set $h_4 = W^*h_3W$. Then $[h_4, \tau(f_2)] = 0$ and the $C^*$-subalgebra generated by $\exp(ih_4)(= W^*(\tau'(f_1) + \tau_0(f_1))W)$ and $\tau(f_2)$ is isomorphic to $C(S^1 \times F)$. Now define $h_1' = h_1 + aq$ and

$$w_t = \exp(i(1-t)h_1' + th_4) \quad t \in [0,1].$$

Note that

$$w_0 = \exp(ih_1') = \tau'(f_1) + \exp(ia)q = \tau(f_1) \quad \text{and} \quad w_1 = \exp(ih_4).$$

Since $[\tau(f_2), h_1'] = [\tau(f_2), h_4] = 0$,

$$[w_t, \tau(f_2)] = 0 \quad t \in [0,1].$$

□

**Lemma 4.3.** *Let $u$ and $v$ be two commuting unitaries in $B(H)/\mathcal{K}$ with $[u] = 0$ in $K_1(B(H)/\mathcal{K})$ and $sp(v) = S^1$. Then there is a norm continuous path of unitaries $\{u_t\} \subset B(H)/\mathcal{K}$ such that*

$$u_0 = u,\ u_1 = 1 \quad \textit{and} \quad [u_t, v] = 0$$

*for* $0 \le t \le 1$.

*Proof.* Let $\phi : C(X) \to B(H)/\mathcal{K}$ be the monomorphism generated by $u$ and $v$ such that $\phi(f_1) = u$ and $\phi(f_2) = v$. We will assume that $sp(u) = S^1$. Otherwise, there is a selfadjoint element $h \in B(H)/\mathcal{K}$ such that $\exp(ih) = u$ and $[h, v] = 0$. We can take $u_t = \exp(i(1-t)h)$.

We consider two cases.

(1) We assume that $C^*(u, v) = C(S^1 \times S^1)$. Take a unital trivial essential extension $\tau_0 : C(S^1 \times S^1) \to B(H)/\mathcal{K}$. Since $\tau_0$ is trivial, as in the proof of Lemma 4.2, there is $0 \le h_1 \le 2\pi$ in $B(H)/\mathcal{K}$ such that $\tau_0(f_1) = \exp(ih_1)$ and $[h_1, \tau_0(f_2)] = 0$.

Define a unital extension $\tau_1$ of $C(S^1 \times S^1)$ by defining $\tau_1(f_1) = 1 \oplus \tau_0(f_1)$ and $\tau_1(f_2) = v \oplus \tau_0(f_2)$. Let $(\tau_1)_*$ and $\phi_*$ be the map from $K_1(C(S^1 \times S^1))$ into $K_1(B(H)/\mathcal{K})$ induced by $\tau_1$ and $\phi$. It is easy to check that $(\tau_1)_* = \phi_*$. By the BDF theory (see Theorem 6.6 of [**BDF2**]), there exists a unitary $W \in B(H)/\mathcal{K}$ such that

$$W^*(\tau_1)W = \phi.$$

In particular

$$W^*\tau_1(f_1)W = u \quad \text{and} \quad W^*\tau_1(f_2)W = v.$$

Therefore, by setting $h_1' = W^*(1 \oplus h_1)W$,

$$u = \exp(i(h_1')) \quad \text{and} \quad [h_1', v] = 0.$$

Now set

$$w_t = \exp(i(1-t)h_1').$$

This proves the case (1).

(2) General case, $C^*(u, v) = C(X)$, for some compact subset of $S^1 \times S^1$. By the BDF theory, $\phi = \tau_2 \oplus \tau_0$, where $\tau_2$ is unitarily equivalent to $\phi$ and $\tau_0 : C(X) \to B(H)/\mathcal{K}$ is a unital essential trivial extension. Write

$$\tau_2(f_1) = u',\ \tau_2(f_2) = v',\ \tau_0(f_1) = u'' \quad \text{and} \quad \tau_0(f_2) = v''.$$

So $u = u' \oplus u''$ and $v = v' \oplus v''$. Suppose that $sp(\phi(f_2)) = S^1$, then $X \subset S^1 \times S^1$. By Lemma 4.2, there is a norm continuous path of unitaries $\{w_t\}$ in $B(H)/\mathcal{K}$ such that

$$w_0 = u'',\ [w_t, v''] = 0$$

and $C^*(w_1, v'') \cong C(S^1 \times S^1)$. Set $z_t = u' \oplus w_t$. Then

$$z_0 = u,\ [z_t, v] = 0 \quad \text{and} \quad C^*(z_1, v) \cong C(S^1 \times S^1).$$

So we can then apply the case (1). □

**Theorem 4.4.** *Let*

$$0 \to \mathcal{K} \to E' \to C(S^1) \to 0$$

*be a unital essential extension. Suppose that* $\alpha : E' \to E'$ *is an automorphism such that* $\pi \circ \alpha = \pi$*, where* $\pi : E' \to C(S^1)$ *is the quotient map. Then* $\alpha$ *is approximately inner.*

*Proof.* It is well known that there exists a unitary $W \in B(H)$ such that

$$\alpha(a) = W^* a W \quad \text{for all} \ \ a \in E'.$$

Let $u$ be the standard generator of $C(S^1)$. Denote the quotient map from $B(H)$ onto $B(H)/\mathcal{K}$ by $\pi$ too. Since $\pi \circ \alpha = \pi$,

$$\|[\pi(W), u]\| = 0.$$

It follows from Lemma 4.3, there is a continuous path of unitaries $\{v_t : t \in [0,1]\}$ in $B(H)/\mathcal{K}$ such that $v_0 = \pi(W)$, $v_1 = 1$ and $[v_t, u] = 0$. We claim that there are unitaries $W_1, W_2, ..., W_N \in B(H)$ such that

$$\|W_i - W_{i+1}\| < 1/2 \quad \text{and} \quad \pi(W_i) = v_{t_i},$$

where $W_0 = W$, $i = 1, 2, ..., N$ and $0 = t_0 < t_1 < ... < t_N = 1$. This claim is shown by applying the following statement repeatly:

If $Z_1 \in B(H)$ is a unitary, $\pi(Z_1) = z_1$ and $z_2 \in B(H)/\mathcal{K}$ is also a unitary with the property that

$$\|z_1 - z_2\| < 1/4,$$

then there exists a unitary $Z_2 \in B(H)$ such that

$$\|Z_1 - Z_2\| < 8\|z_1 - z_2\| \quad \text{and} \quad \pi(Z_2) = z_2.$$

To prove this, we note that there exist unitary $Z_2' \in B(H)$ with $\pi(Z_2') = z_2$ and $a \in \mathcal{K}$ such that

$$\|Z_1 - (Z_2' + a)\| < 1/4.$$

Set $b = Z_1^*(Z_2' + a)$. Then

$$\|1 - b\| < 1/4.$$

So $b$ is invertible. Let $Z_2 = Z_1 b (b^* b)^{-1/2}$. Note that $\pi(b) = z_1^* z_2$ and $\pi(b^* b) = 1$. It is then easily checked that $Z_2$ meets the requirement.

Since $[\pi(W_i), u] = 0$, $\alpha_i(a) = W_i^* a W_i$ for $a \in E'$ defines an automorphism on $E'$ for $i = 0, 1, ..., N$. Then

$$\|\alpha_i - \alpha_{i+1}\| < 1$$

for $i = 0, 1, ..., N-1$. Therefore

$$\|\alpha_i \circ \alpha_{i+1}^{-1} - \mathrm{id}\,\| < 1, \;\; i = 0, 1, ..., N-1.$$

By [**KR**] (see also Theorem 8.7.7 of [**Pd**]), $\alpha_i \circ \alpha_{i+1}^{-1}$ are derivable automorphisms. We have

$$\alpha = (\alpha \circ \alpha_1^{-1}) \circ \cdots \circ (\alpha_{N-1} \circ \alpha_N^{-1}) \circ \alpha_N.$$

Since $\pi(W_N) = 1$, $W_N \in \tilde{\mathcal{K}}$, the unitization of $\mathcal{K}$. There is a norm continuous path of unitaries in $\tilde{\mathcal{K}}$ which connects $W_N$ with 1. This implies that $\alpha_N$ is contained in the connected component of automorphism group of $E'$ containing the identity automorphism. It follows from [**KR**] (see also 8.7.8 of [**Pd**]) that $\alpha_N$ is a product of derivable automorphisms. Therefore $\alpha$ is a product of derivable automorphisms of $E'$. Since every derivable automorphism is approximately inner (4.2), we conclude that $\alpha$ is approximately inner.

**Remark 4.5.** It is easy to see that not every automorphism $\alpha$ in 4.4 is inner. Examples can be constructed from our proof of 4.4, using an extension of $C(S^1 \times S^1)$. The following is a classical example which is provided to us by Jingbo Xia. Identify $\mathcal{T}_1$ with the $C^*$-algebra generated by Toeplitz operators with continuous symbols. Let $L^\infty(\mathbb{T})$ be the $C^*$-algebra generated by all Toeplitz operators with symbols $\psi$, where $\psi$ are bounded measurable functions on $\mathbb{T}$. Since $\mathcal{T}_1 \neq L^\infty(\mathbb{T})$, there are unitaries $u \in L^\infty(\mathbb{T})$ which are not in $\mathcal{T}_1$. However, it is well known that $ux - xu$ is a compact operator for every $x \in \mathcal{T}_1$. Let $\alpha : \mathcal{T}_1 \to \mathcal{T}_1$ by defining $\alpha(x) = u^* x u$, $x \in \mathcal{T}_1$ for some $u \in L^\infty(\mathbb{T}) \setminus \mathcal{T}_1$. Clearly $\alpha$ is an automorphism on $\mathcal{T}_1$ with $\pi \circ \alpha = \pi$. It is not inner, since otherwise there would be a unitary $v \in \mathcal{T}_1$ such that $vu^* x = xvu^*$ for each $x \in \mathcal{T}_1$ which would imply that $vu^* = 1$. It would then follow that $u = v \in \mathcal{T}_1$. However, it follows from 4.4 that $\alpha$ is approximate inner.

**Remark 4.6.** Before we typed the paper, we learned from Mikael Rørdam that he had proved, a couple of weeks earlier that an automorphism on $E$ which induces an identity map on the quotient $C(S^1)$ is approximately inner, which is essentially the same result as our 4.4. His nice proof used the Berg's technique. Our proof presented here is completely different and contains other information which is needed in this paper.

**Theorem 4.7.** *Let $B$ be a unital $C^*$-subalgebra of $B(H)$ such that $1_B = 1$ and $\mathcal{K} = \mathcal{K}(H) \subset B$. Suppose that $\phi_i : M_m(\mathcal{T}_k) \to B$, $i = 1, 2$ are two unital monomorphisms such that $1 - \phi_i(S_k)\phi_i(S_k)^* \in \mathcal{K}$ and $\phi_1$ and $\phi_2$ induce the same map from $V(M_m(\mathcal{T}_k))$ into $V(B)$. Then, for any $\varepsilon > 0$ and $z_1, z_2, ..., z_n \in M_m(\mathcal{T}_k)$ which contains the generators $\{S_k, a_{ij}\}$ if $k > 0$ or contains $S_0$ if $k = 0$, there exists $\delta > 0$ satisfying the following: If there is a unitary $W \in B$ and $a_1, a_2, ..., a_n \in \mathcal{K}(H)$ such that*

$$\|W^*\phi_1(z_i)W - \phi_2(z_i) + a_i\| < \delta, \quad i = 1, 2, ..., n$$

*then there is a unitary $U \in B$ such that*

$$\|U^*\phi_1(z_i)U - \phi_2(z_i)\| < \varepsilon, \quad i = 1, 2, ..., n.$$

*Furthermore, $\delta$ does not depend on $B$ nor depends on monomorphisms $\phi_i$ as long as they satisfy the above conditions.*

*Proof.* Denote $M_m(\mathcal{T}_k)$ by $E$. We first assume that $m = 1$. To save the notation, we may aslo assume that $W = 1$. For the case that $k > 0$, without loss of generality, we may further assume that $z_1 = S_k$ and $\{z_2, ..., z_n\} = \{a_{ij}\}$ (see 2.1). Let

$$z_1' = (\phi_2(z_1) - a_1)((\phi_2(z_1) - a_1)^*(\phi_2(z_1) - a_1))^{-1/2}$$

(note that if $\delta < 1/2$, $(\phi_2(z_1) - a_1)^*(\phi_2(z_1) - a_1)$ is invertible). Let $p = 1 - z_1'(z_1')^*$. Then

$$\|(1 - \phi_1(z_1)\phi_1(z_1)^*) - p\| < 2\delta.$$

If $\delta < 1/4$, there is a unitary $W' \in B$ such that

$$\|W' - 1\| < 4\delta \quad \text{and} \quad (W')^*(1 - \phi_1(z_1)\phi_1(z_1))W' = p.$$

Define $z_{1+i}' = (W')^*\phi_1(z_{1+i})W'$, $i = 1, 2, ..., n-1$. Then $\{z_1', z_2', ..., z_n'\}$ generates a $C^*$-subalgebra which is isomorphic to $\mathcal{T}_k$ (see 2.1) and

$$\|\phi_1(z_i) - z_i'\| < 8\delta.$$

Define $\phi_3 : \mathcal{T}_k \to B$ by sending $z_i$ to $z_i'$.

From the assumption, we see that $\phi_i(I(E)) \subset \mathcal{K}(H)$, $i = 2, 3$. We view $\phi_3$ and $\phi_2$ as two faithful representations of $E$. The representation $\rho$ of $E$ which maps $I(E)$ onto $\mathcal{K}(H)$ is a faithful irreducible representation of $E$. From representation theory, this is the only faithful irreducible representation (up to unitary equivalence) of $E$ which maps $I(E)$ onto $\mathcal{K}(H)$.

Suppose that $\phi_2(1-S_kS_k^*)$ has rank $l$. Then $\phi_1(1-S_kS_k^*)$ has rank $l$ too. Therefore, if $\delta$ is small enough, $\phi_3(1-S_kS_k^*)$ has rank $l$. We are going to use Voiculescu's Theorem, as stated in [**Ar**] (Theorem 5). Let $H_e$ be the closure of $\phi_2(E)\cap\mathcal{K}(H)(H)$, the essential subspace of $\phi_2$. By standard results in representation theory, $(\phi_2|_{I(E)})|_{H_e}$ is a direct sum of $l$ copies of the faithful irreducible representations of $I(E)\cong\mathcal{K}$. There are $l$ mutually orthogonal rank one projections $d_1,d_2,...,d_l$ in $\mathcal{K}(H_e)$ which commutes with every element in $\phi_2(I(E))|_{H_e}$. Since $\phi_2(I(E))|_{H_e}$ is an essential ideal of $\phi_2(E)|_{H_e}$, $d_i$ commutes with every element in $\phi_2(E)|_{H_e}$. Since now $d_i\phi_2(1-S_kS_k^*)$ has rank one, $d_i\phi_2|_{H_e}$ is irreducible (and also faithful). Thus $\phi_2|_{H_e}$ is unitarily equivalent to $l$ copies of $\rho$. Since $\phi_3(1-S_kS_k^*)$ has also rank $l$, $\phi_3|_{H_e}$ is also unitarily equivalent to $l$ copies of $\rho$. Therefore the essential parts of $\phi_2$ and $\phi_3$ are unitarily equivalent. It is trivial that $\ker(\pi\circ\phi_2)=\ker(\pi\circ\phi_3)$. A result of Voiculescu (see Theorem 5 of [**Ar**]) now implies that, for any $\varepsilon>0$, there is unitary $V\in B(H)$ such that

$$\|V^*\phi_2(z_i)V-\phi_3(z_i)\|<\varepsilon/3 \text{ and } V^*\phi_2(z_i)V-\phi_3(z_i)\in\mathcal{K}(H),\ i=1,2,...,n.$$

In particular $V^*\phi_2(z_i)V\in B$.

Let $E'$ be the $C^*$-subalgebra of $B$ generated by $\phi_2(E)$ and $\mathcal{K}(H)$. Then, since $\pi(E')=\pi(E)$, $E'\cong\mathcal{T}_k$. From the construction, $\phi_2(z_i)-\phi_3(z_i)\in\mathcal{K}(H)$. In fact, since $\{z_2,z_3,...,z_n\}=\{a_{ij}\}$ are in $\mathcal{K}(H)$, $z_i'\in\mathcal{K}(H)$ for $i=2,...,n$. So $\phi_2(z_i)-\phi_3(z_i)\in\mathcal{K}(H)$ for $i=2,3,...,n$. We also (note $a_1\in\mathcal{K}(H)$) have

$$\pi(z_1')=\pi\circ\phi_2(z_1)\pi(((\phi_2(z_1)-a_1)^*(\phi_2(z_1)-a_1))^{1/2})=\pi\circ\phi_2(z_1),$$

where $\pi:B(H)\to B(H)/\mathcal{K}(H)$ is the quotient map. Thus $\phi_2(z_i)-\phi_3(z_i)\in\mathcal{K}(H)$ for *all* $i$. Therefore $Ad(V)$ gives an automorphism on $E'$. Furthermore, $\pi\circ Ad(V)=\pi$ on $E'$. By 4.5, there is a unitary $U\in E'\subset B$ such that

$$\|U\phi_2(z_i)U^*-V^*\phi_2(z_i)V\|<\varepsilon/3,\quad i=1,2,...,n.$$

Thus

$$\begin{aligned}&\|U^*\phi_1(z_i)U-\phi_2(z_i)\|\\ &\le\|U^*\phi_1(z_i)U-U^*\phi_3(z_i)U\|+\|U^*\phi_3(z_i)U-U^*V^*\phi_2(z_i)VU\|\\ &\quad+\|U^*V^*\phi_2(z_i)VU-\phi_2(z_i)\|<8\delta+\varepsilon/3+\varepsilon/3<\varepsilon,\end{aligned}$$

if $8\delta<\varepsilon/3$.

We now consider the case that $k=0$. Although we may not need this in our proof of the classification theorems, it is of interesting on its own. Note that $S_0$ is a unitary and there is an approximate identity $\{p_m\}$ for $I(\mathcal{T}_0)$ consisting of projections such that

$$\|p_mS_0 - S_0p_m\| \to 0.$$

With an error within $\varepsilon/8$, we may assume that

$$z_i \in p_NI(\mathcal{T}_0)p_N, \quad i = 2, 3, ..., n$$

and

$$\|p_NS_0 - S_0p_N\| < \varepsilon/8$$

for some integer $N$. There is a unitary $s \in (1 - p_N)\mathcal{T}_0(1 - p_N)$ such that

$$\|s - (1 - p_N)S_0(1 - p_N)\| < \varepsilon/8.$$

So, without loss of generality, we may assume that, in this case, $z_1 = s \in (1 - p_N)\mathcal{T}_0(1 - p_N)$, $z_2, z_3, ..., z_n \in p_NI(\mathcal{T}_0)p_N$ (we may have to consider $n$ elements in $p_NI(\mathcal{T}_0)p_N$ instead of $(n - 1)$ elements, but we can change notation). Note that $(1 - p_N)\mathcal{T}_0(1 - p_N) \cong \mathcal{T}_0$. Since $\phi_1$ and $\phi_2$ induce the same map on $V(\mathcal{T}_0)$, $\phi_1(p_N)$ is equivalent to $\phi_2(p_N)$. So there is a unitary $W_1 \in B$ such that

$$W_1^*\phi_1(p_N)W_1 = \phi_2(p_N).$$

Without loss of generality, we may assume that $\phi_1(p_N) = \phi_2(p_N)$. The argument used in the case that $k > 0$ shows that there is a unitary $U' \in \phi_1(1 - p_N)B\phi_1(1 - p_N)$ such that

$$\|(U')^*\phi_1(z_1)U' - \phi_2(z_1)\| < \varepsilon/8.$$

Since $z_2, z_3, ..., z_n$ generate a finite dimensional $C^*$-subalgebra of $p_NI(\mathcal{T}_k)p_N$ and $\phi_1$ and $\phi_2$ induce the same map on $V(\mathcal{T}_0)$, there is a unitary $u_1 \in \phi_1(p_N)\mathcal{K}(H)\phi_1(p_N)$ such that

$$u_1^*\phi_1(z_i)u_1 = \phi_2(z_i), \;\; i = 2, 3, ..., n.$$

Now set $U = U' + u_1$. Then

$$\|U^*\phi_1(z_i)U - \phi_2(z_i)\| < \varepsilon, \; i = 1, 2, ..., n.$$

For the case that $m > 0$, We first find a unitary $Z \in B$ such that $Ad(Z) \circ \phi_1$ agrees with $\phi_2$ on $M_m(\mathbf{C})$. So we may wssume that $Z = 1$. Note then that $\phi_1(e_{11})\mathcal{K}\phi_1(e_{11}) \cong \mathcal{K}$. We then consider $\phi_1(e_{11})\phi_1(-)\phi_1(e_{11})$, $\phi_2(e_{11})\phi_2(-)\phi_2(e_{11})$ and $\phi_1(e_{11})B\phi_1(e_{11})$. So the case that $m > 1$ is reduced to the case that $m = 1$.

Finally, from the proof, we see that our $\delta$ is independent of $C^*$-algebra $B$ and $\phi_i$ as long as they satisfy the conditions of the theorem. □

**Corollary 4.8.** *Let $B$ be as in 4.7. Suppose that $\phi_i : M_m(C(S^1)) \to B, i = 1, 2$ are two unital monomorphisms such that $[\phi_1(e_{11})] = [\phi_2(e_{11})]$ in $V(B)$, where $\{e_{ij}\}_{i,j=1}^m$ is a matrix unit for $M_m(\mathbf{C})$. Then, for any $\varepsilon > 0$ there exists $\delta > 0$ satisfying the following: If there is unitary $W \in B$ and $a_0, a_{ij} \in \mathcal{K}$ such that*

$$\|W^*\phi_1(u)W - \phi_2(u) + a_0\| < \delta$$

*and*

$$\|W^*\phi_1(e_{ij})W - \phi_2(e_{ij}) + a_{ij}\| < \delta$$

*then there is unitary $U \in B$ such that*

$$\|U^*\phi_1(u)U - \phi_2(u)\| < \varepsilon$$

*and*

$$\|U^*\phi_1(e_{ij})U - \phi_2(e_{ij})\| < \varepsilon,$$

*where $i, j = 1, 2, ..., n$ and $u$ is the standard generator of $C(S^1)$.*

*Proof.* Let us first consider the case that $m = 1$. Let $D$ be the C*-subalgebra of $B$ generated by $\phi_2(u)$ and $\mathcal{K}$. Then $D$ is isomorphic to $\mathcal{T}_0$ and $\phi_2(u) - a_0 \in D$. As in the proof of 4.7, one shows that there is a unitary $W \in D$ such that $W\phi_2(u)W^*$ and $\phi_2(u) - a_0$ are close. The result then follows.

For the case $m > 1$, the proof is the same as that of 4.7. It can be reduced to the case that $m = 1$. □

**4.9.** The following definition is taken from [**EG2**].

**Definition.** Let $A = B \otimes C(S^1)$, where $B$ is a finite dimensional $C^*$-algebra and let $\epsilon$ be a positive number. A finite subset $F \subset A$ is *weakly approximately constant to within* $\epsilon$ if for any $t \in S^1$ there exists a unitary $U(t) \in B$ such that $\mathrm{Ad}(U(t))f(t)$ and $f(1)$ is within $\epsilon$ for every $f \in F$, i.e.

$$\|\,\mathrm{Ad}(U(t))f(t) - f(1)\| < \varepsilon \quad for\ f \in F.$$

Let $A$ and $B$ be two (unital) $C^*$-algebras, let $F \subset A$ be a subset and let $\phi$, $\psi : A \to B$ two homomorphisms. We say $\phi$ and $\psi$ are *approximately unitarily equivalent on $F$ to within* $\varepsilon$ if there is a unitary $u \in B$ such that

$$\|\,\mathrm{Ad}\, u \circ \phi(f) - \psi(f)\| < \varepsilon \quad for\ f \in F.$$

**Theorem 4.10.** *Let $E_1 = M_n(\mathcal{T}_k)$ and $E_2 = M_m(\mathcal{T}_r)$ be two $\mathcal{T}$-algebras, let $\pi_1$ be the quotient map from $E_1$ to $Q(E_1)$, let $F \subset E_1$ be a finite subset such that $\pi_1(F)$ contains the standard generators of $Q(E_1)$, and let $\phi$ and $\psi : E_1 \to E_2$ be two $*$-homomorphisms which induce the same map on $V_*(E_1)$. For given $\epsilon > 0$, there exists $\delta > 0$ only depending on $E_1, F$ and $\epsilon$ such that the followings hold:*

(1) *If $\phi$ and $\psi$ are injective and if the two induced maps $Q(\phi)$ and $Q(\psi)$ from $Q(E_1)$ to $Q(E_2)$ are approximately the same to within $\delta$ on $\pi_1(F) \subset Q(E_1)$, then $\phi$ and $\psi$ are approximately unitarily equivalent on $F$ to within $\epsilon$.*

(2) *If $\phi$ and $\psi$ are not injective but the two maps $\phi_Q$ and $\psi_Q$ induced by $\phi$ and $\psi$ from $Q(E_1)$ to $E_2$ are injective, and if $Q(\phi)$ and $Q(\psi)$ are approximately the same on $\pi_1(F)$ to within $\delta$, then $\phi$ and $\psi$ are approximately unitarily equivalent on $F$ to within $\epsilon$.*

*Proof.* (1) Since $\phi$ and $\psi$ induce the same map on $V_*(E_1)$, there exists a unitary $X \in E_2$ such that $\phi$ and $\operatorname{Ad} X \circ \psi$ agree on $M_n(\mathbf{C}) \subset M_n(\mathcal{T}_k)$. So we may assume that they agree on $M_n(\mathbf{C})$. Denote $\phi(1) = \psi(1) = p$, and denote $B = pE_2p$.

Fix $\epsilon > 0$ and $F$ as above. If $Q(\phi)$ and $Q(\psi)$ are approximately the same on $\pi_1(F)$ to within $\delta$, there exist $W = 1$ and $a_f$ to satisfy the conditions of Theorem 4.7. It follows that there exists a unitary $U \in B$ such that

$$\|U^*\phi(f)U - \psi(f)\| < \epsilon \qquad f \in F.$$

This completes the proof of (1).

(2) Since $\phi$ and $\psi$ are not injective, they vanish on $I(E_1)$. Now $\phi_Q$ and $\psi_Q$ are defined. By the assumption, they are injective. Since $\phi = \phi_Q \circ \pi_1$ and $\psi = \psi_Q \circ \pi_1$ and since $\pi_1$ induces an onto map from $V(E_1)$ to $V(Q(E_1))$, $\phi_Q$ and $\psi_Q$ induce the same map on $V(Q(E_1))$.

Since $Q(\phi)$ and $Q(\psi)$ are approximately the same on $\pi_1(F)$ to within $\delta$, $\phi_Q$ and $\psi_Q$ are approximately unitarily equivalent on $\pi_1(F)$ to within $\epsilon$ (Corollary 4.8). Since $\phi_Q \circ \pi_1 = \phi$ and $\psi_Q \circ \pi_1 = \psi$, $\phi$ and $\psi$ are approximately unitarily equivalent on $F$ to within $\epsilon$.

This completes the proof of the theorem. □

**Theorem 4.11.** *Let $E_1 = M_n(\mathcal{T}_k)$ and $E_2 = M_m(\mathcal{T}_r)$ be two $\mathcal{T}$-algebras, let $\pi_1$ be the quotient map from $E_1$ to $Q(E_1)$, let $\epsilon > 0$, let $F \subset E_1$ be a finite subset such that $\pi_1(F)$ is of weakly approximately constant to within*

*$\epsilon/2$, and let $\phi$ and $\psi : E_1 \to E_2$ be two $*$-homomorphisms which induce the same map on $V_*(E_1)$. Then the followings hold:*

(1) *If $\phi$ and $\psi$ are not injective, the images of $\phi_Q(u)$ and $\psi_Q(u)$ are of finite spectra, where $\phi_Q$ and $\psi_Q$ are the $*$-homomorphisms from $Q(E_1)$ to $E_2$ induced by $\phi$ and $\psi$, respectively, and where $u$ is the canonical generator of $e_{11}M_n(C(S^1))e_{11}$ ($\{e_{ij}\}$ is the matrix unit for $M_n(\mathbf{C})$), then $\phi$ and $\psi$ are approximately unitarily equivalent on $F$ to within $\epsilon$.*

(2) *If $Q(\phi)$ and $Q(\psi)$, the $*$-homomorphisms from $Q(E_1)$ to $Q(E_2)$ induced by $\phi$ and $\psi$, respectively, are zero, then $\phi$ and $\psi$ are approximately unitarily equivalent on $F$ to within $\epsilon$.*

*Proof.* (1) Since $\phi$ and $\psi$ induce the same map on $V_*(E_1)$, by conjugating a unitary, if necessary, we may assume that $\phi$ and $\psi$ agree on $M_n(\mathbf{C})$. In particular, $\phi(1) = \psi(1) = p \in E_2$. Let $B = pE_2p$. By assumption, $\phi_Q(u) = \sum a_t p_t^{(1)}$ where $a_t \in S^1$ and where $p_t^{(1)}$ are finitely many mutually orthogonal projections in $B$. Let $p_t^{(i)} = \phi_Q(e_{i1})p_t^{(1)}\phi_Q(e_{1i})$, $p_k = \sum_{i=1}^m p_k^{(i)}$ and $p = \sum p_k = \phi(1) \in B$, where $\{e_{ij}\}$ is the matrix unit of $M_n(\mathbf{C})$. For any $h \in Q(E_1) \cong M_n(C(S^1))$, we have

$$\phi_Q(h) = \sum_k \phi_Q(h(a_k))p_k.$$

Similarly, we have

$$\psi_Q(h) = \sum_j \psi_Q(h(b_j))q_j$$

where $\sum q_j = p$, and where $b_k \in S^1$.

Since $\pi_1(F)$ is weakly approximately constant to within $\epsilon/2$, there exist unitaries $X_i, Y_i \in M_n(\mathbf{C})$ such that

$$\|X_i\pi_1(f)(a_i)X_i^* - X_1\pi_1(f)(a_1)X_1^*\| < \epsilon/2$$
$$\|Y_i\pi_i(f)(b_i)Y_i^* - Y_1\pi_1(f)(a_1)Y_1^*\| < \epsilon/2.$$

Write

$$U = \sum \phi_Q(X_k^* X_1)p_k.$$

Then $U$ is a unitary in $B$ and

$$\|\phi_Q(\pi_1(f)) - U^*\phi_Q(\pi_1(f)(a_1))U\| < \epsilon/2 \qquad f \in F.$$

Similarly, we have

$$\|\psi_Q(\pi_1(f)) - V^*\psi_Q(\pi_1(f)(a_1))V\| < \epsilon/2 \qquad f \in F,$$

where $V = \sum \psi_Q(Y_j^* Y_1) q_j$ is a unitary in $B$. Note by the assumption at the beginning of this part of the proof, $\phi_Q$ agrees with $\psi_Q$ on $M_n(\mathbf{C})$. Thus

$$\phi_Q(\pi_1(f)(a_1)) = \psi_Q(\pi_1(f)(a_1)).$$

Hence

$$\|VU^*\phi_Q(\pi_1(f))UV^* - \psi_Q(\pi_1(f))\| < \epsilon \qquad f \in F.$$

Notice that $\phi = \phi_Q \circ \pi_1$ and $\psi = \psi_Q \circ \pi_1$, we have

$$\|VU^*\phi(f)UV^* - \psi(f)\| < \epsilon \qquad f \in F.$$

(2) Again, as in (1), we may assume that $\phi$ and $\psi$ agree on $M_n(\mathbf{C})$. We also know that $\phi$ and $\psi$ factor through $Q(E_1)$. Therefore $\phi_Q$ and $\psi_Q$ are two maps from $Q(E_1)$ to $I(E_2)$. Denote $p = \phi_Q(1) = \psi_Q(1)$. We have $pI(E_2)p \cong M_l$ where $l = [p] \in \mathbf{Z}_+$. So $\phi_Q$ and $\psi_Q$ are two unital $*$-homomorphisms from $Q(E_1)$ to $pI(E_2)p \cong M_l$. We may write

$$\phi_Q(h) = \sum_k \phi_Q(h(a_k))p_k \quad \text{and} \quad \psi_Q(h) = \sum_j \psi_Q(h(b_j))q_j,$$

for $h \in M_n(C(S^1))$, where $a_k, b_j \in S^1$, $\{p_k\}$ and $\{q_j\}$ are mutually orthogonal projections in $pI(E_2)p$ and $\sum_k p_k = \sum_j q_j = p$.

Since $\pi_1(F)$ is weakly approximately constant to within $\epsilon/2$, the same argument used in the proof of (1) shows that $\phi_Q$ and $\psi_Q$ are approximately unitarily equivalent on $\pi_1(F)$ to within $\epsilon$. Since $\phi = \phi_Q \circ \pi_1$ and $\psi = \psi_1 \circ \pi_1$, $\phi$ and $\psi$ are approximately unitarily equivalent on $F$ to within $\epsilon$. ☐

## 5. The Classification.

The following is the main result of this paper.

**Theorem 5.1.** *Let $E$ and $E'$ be two unital $C^*$-algebras in $\mathcal{T}$. Suppose that $\eta : V_*(E) \longrightarrow V_*(E')$ is an isomorphism such that $\eta([1_E]) = [1_{E'}]$. Then there exists a $*$-isomorphism $\phi : E \longrightarrow E'$ such that $\phi$ induces $\eta$.*

The theorem is a direct consequence of Lemma 5.4 and Proposition 5.7 below.

We note that if $\phi : E \to E'$ is an isomorphism, then $\phi_*$ induces an isomorphism from $V_*(E)$ onto $V_*(E')$. Therefore, together with 3.7, we give a complete classification of unital $C^*$-algebras in $\mathcal{T}$.

Write $E = \varinjlim(E_n, \phi_{n,n+1})$ and $E' = \varinjlim(E'_n, \psi_{n,n+1})$. One would hope that $\eta$ induces an intertwining between the two sequences $\{V_*(E_n)\}$ and

$\{V_*(E'_m)\}$ for each $n$. However, since $V(\mathcal{T}_0)$ is not finitely generated, it may not be possible. To overcome this difficulty, we first impose a condition on the connecting maps $\phi_{n,n+1}$ and $\psi_{n,n+1}$ (see Lemma 5.4 below). We then show that the general case can be reduced to this case.

We will consider unital $C^*$-algebras only. Therefore, in what follows, we will always assume that all connecting maps in an inductive limit system are unital.

**5.2.** The following is essentially Theorem 6.2 of [**Ell2**].

**Theorem.** *Let $A = \lim_{\to} A_n$ be a $C^*$-algebra of real rank zero where each $A_n$ is a finite direct sum of matrix algebras over $C(S^1)$. Then for any $\epsilon > 0$, for any $n$, and for any finite subset $F \subset A_n$, there exists $m > n$ such that the image of $F$ in $A_k$ for $k \geq m$ is weakly approximately constant to within $\epsilon$.*

**5.3.** The following is Theorem 2.29 of [**EG2**]. We only state it for the special case that we will use later.

**Theorem.** *Let $A = \lim_{\to}(A_n, \phi_{n,n+1})$ be a $C^*$-algebra of real rank zero where each $A_n$ is a finite direct sum of matrix algebras over $C(S^1)$, let $\epsilon > 0$ and let $F \subset A_n$ be a finite subset of weakly approximately constant to within $\epsilon$. If $\phi$ and $\psi$ are two $*$-homomorphisms from $A_n$ to $A_{n+1}$ which induce the same $K$-theory maps, then there exists $m > n$ such that $\phi_{n,k} \circ \phi$ and $\phi_{n,k} \circ \psi$ are approximately unitarily equivalent on $F$ to within $80\epsilon$ for all $k \geq m$.*

**Lemma 5.4.** *Let $E$ and $E'$ be two unital $C^*$-algebras in $\mathbf{\mathcal{T}}$. Suppose that each connecting map in the two inductive limit systems satisfies the following: each partial map of a connecting map from $M_n(\mathcal{T}_0)$ vanishes on the ideal $M_n(\mathcal{K}) \subset M_n(\mathcal{T}_0)$. Suppose that $\eta : V_*(E) \longrightarrow V_*(E')$ is an isomorphism such that $\eta([1_E]) = [1_{E'}]$, then there exists a $*$-isomorphism $\phi : E \longrightarrow E'$ such that $\phi$ induces $\eta$.*

*Proof.* Write $E = \lim_{\to}(E_n, \phi_{n,n+1})$ and $E' = \lim_{\to}(E'_n, \psi_{n,n+1})$ where each $E_n$ or $E'_n$ is a finite direct sums of matrix algebras over $\mathcal{T}$-algebras. We will construct an approximate intertwining between these two sequences.

But first we intertwine $V_*$.

After passing to the subsequences and changing the notation, by 2.14, the isomorphism between $V_*(E)$ and $V_*(E')$ gives the following commutative diagram:

$$\begin{array}{ccccccc} V_*(E_1) & \longrightarrow & V_*(E_2) & \longrightarrow & \cdots & \longrightarrow & V_*(E) \\ \downarrow & \nearrow & \downarrow & \nearrow & & & \updownarrow \\ V_*(E'_1) & \longrightarrow & V_*(E'_2) & \longrightarrow & \cdots & \longrightarrow & V_*(E'). \end{array}$$

This diagram induces a commutative diagram

$$\begin{array}{ccccccc} \mathbf{K}(E_1) & \longrightarrow & \mathbf{K}(E_2) & \longrightarrow & \cdots & \longrightarrow & \mathbf{K}(E) \\ \downarrow & \nearrow & \downarrow & \nearrow & & & \updownarrow \\ \mathbf{K}(E_1') & \longrightarrow & \mathbf{K}(E_2') & \longrightarrow & \cdots & \longrightarrow & \mathbf{K}(E') \end{array}$$

(Lemma 2.14). Furthermore, this diagram gives the following commuting diagram:

$$\begin{array}{ccccccc} K_*(Q(E_1)) & \longrightarrow & K_*(Q(E_2)) & \longrightarrow & \cdots & \longrightarrow & K_*(Q(E)) \\ \downarrow & \nearrow & \downarrow & \nearrow & & & \updownarrow \\ K_*(Q(E_1')) & \longrightarrow & K_*(Q(E_2')) & \longrightarrow & \cdots & \longrightarrow & K_*(Q(E')) \end{array}$$

such that each map preserves the order.

By the existence Theorem 3.1, there are unital $*$-homomorphisms $\alpha_k$ and $\beta_k$ from $E_k$ to $E_k'$ and from $E_k'$ to $E_{k+1}$ that induce the above diagrams Notice that the diagram:

$$\begin{array}{ccccccc} E_1 & \longrightarrow & E_2 & \longrightarrow & \cdots & \longrightarrow & E \\ \downarrow & \nearrow & \downarrow & \nearrow & & & \\ E_1' & \longrightarrow & E_2' & \longrightarrow & \cdots & \longrightarrow & E' \end{array}$$

is not necessarily commutative. We will turn this diagram into an approximate intertwining diagram:

$$\begin{array}{ccccccc} E_{n_1} & \longrightarrow & E_{n_2} & \longrightarrow & \cdots & \longrightarrow & E \\ \alpha_1 \downarrow & \beta_1 \nearrow & \alpha_2 \downarrow & \beta_2 \nearrow & & & \\ E_{m_1}' & \longrightarrow & E_{m_2}' & \longrightarrow & \cdots & \longrightarrow & E' \end{array}$$

as follows.

First, some conventions. By the generators of $M_n(C(S^1))$, we mean the set $\{e_{ij}\}$ and the function $z$, where $\{e_{ij}\}$ is a matrix unit for $M_n(\mathbf{C})$ and $z \in C(S^1)$ which is the identity map on $S^1$. Furthermore, we view $z \in e_{11}M_n(C(S^1))e_{11}$.

Let $H_1$ and $H_2$ be two finite direct sums of matrix algebras over $\mathcal{T}$-algebras. Let $\phi$ from $H_1$ to $H_2$ be a $*$-homomorphism. Then $\phi$ induces uniquely a $*$-homomorphism from $Q(H_1)$ to $Q(H_2)$. We will denote this map by $Q(\phi)$. We will denote by $\pi_i$ and $\pi_i'$ the quotient maps from $E_i$ to $Q(E_i)$ and from $E_i'$ to $Q(E_i')$, respectively.

Let $\{\epsilon_n\}$ and $\{\epsilon_n'\}$ be two sequences of decreasing positive numbers with finite summations. Let $\Delta_i = \{f_n^{(i)}\} \subset E_i$ and $\Delta_i' = \{f_n^{(i)'}\} \subset E_i'$ be dense for all $i$. Let $F_1 \subset E_1$ be a finite subset such that $F_1$ contains $f_1^{(1)}$ and $\pi_1(F_1) \subset Q(E_1)$ contains the generators of $Q(E_1)$. For $\epsilon_1 > 0$, by Theorem

5.2, there exists $n_1$ such that $\pi_{n_1}(\phi_{1,n_1}(F_1))$ is weakly approximately constant to within $\epsilon_1/2$. We will denote $\phi_{1,n_1}(F_1) \subset E_{n_1}$ by $F_1$ again.

We want to find $m_1 > n_1$ and construct a unital $*$-homomorphism from $E_{n_1}$ to $E'_{m_1}$. The map we are going to construct takes the form $\psi_{k(n_1),m_1} \circ \alpha_{k(n_1)} \circ \phi_{n,k(n_1)}$, where $k(n_1) > n_1$:

$$\begin{array}{ccc} E_{n_1} \longrightarrow & E_{k(n_1)} & \\ & \downarrow & \\ & E'_{k(n_1)} & \longrightarrow E'_{m_1}. \end{array}$$

We now specify the choice of $k(n_1)$ and $m_1$. Let $\Gamma_1$ be a finite subset of $E_{n_1}$ such that $\Gamma_1$ contains $F_1$ and $\pi_{n_1}(\Gamma_1)$ contains the generators of $Q(E_{n_1})$. For $E_{n_1}, \Gamma_1$ and $\epsilon_1/2$, there exists $\delta_1$ to satisfy the conclusions (1) and (2) of Theorem 4.10. Let $G_1 = \pi_{n_1}(\Gamma_1)$. By Theorem 5.2, there exists $k(n_1) > n_1$ such that $Q(\phi_{n_1,k(n_1)})(G_1)$ is of weakly approximately constant to within $\delta_1/160$. Recall that $\alpha_{k(n_1)}$ is a unital $*$-homomorphism from $E_{k(n_1)}$ to $E'_{k(n_1)}$. Let $F \subset E_{n_1}$ be a finite subset which contains the images of the first $n_1$ elements of $\Delta_1, \cdots, \Delta_{(n_1-1)}$ and $\Delta_{n_1}$. We also require that $\pi_{n_1}(F)$ contains the generators of $Q(E_{n_1})$ and the image of $F_1$. By Theorem 5.2 again, there exists $m_1 > k(n_1)$ such that $\pi'_{m_1} \circ \psi_{k(n_1),m_1} \circ \alpha_{k(n_1)} \circ \phi_{n_1,k(n_1)}(F) \subset Q(E'_{m_1})$ is of weakly approximately constant to within $\epsilon'_1/2$. We will denote $\psi_{k(n_1),m_1} \circ \alpha_k(n_1) \circ \phi_{n_1,k(n_1)}(F)$ by $F'_1$. We have a map $\psi_{k(n_1),m_1} \circ \alpha_k(n_1) \circ \phi_{n_1,k(n_1)}$ from $E_{n_1}$ to $E'_{m_1}$.

Next, we will construct a map from $E'_{m_1}$ to $E_{n_2}$ for some $n_2 > m_1$. The map will take the following form:

$$\begin{array}{ccc} & & E_{k(m_1)+1} \longrightarrow E_l \longrightarrow E_{n_2} \\ & \nearrow & \\ E'_{m_1} \longrightarrow E'_{k(m_1)} & & \end{array}$$

where $l > k(m_1)+1$. Let $\Gamma'_1$ be a finite subset of $E'_{m_1}$ such that $\Gamma'_1$ contains $F'_1$ and $\pi'_{m_1}(\Gamma'_1)$ contains the generators of $Q(E'_{m_1})$. For $E'_{m_1}, \Gamma'_1$ and $\epsilon'_1/2$, there exists $\delta'_1$ to satisfy the conclusions (1) and (2) of Theorem 4.10. Let $k(m_1) > m_1$ so that $Q(\psi_{m_1,k(m_1)})(G'_1)$ is of weakly approximately constant to within $\delta'_1/160$. Here $G'_1 = \pi'_{m_1}(\Gamma'_1)$. Recall that there is a unital $*$-homomorphism $\beta_{k(m_1)}$ from $E'_{k(m_1)}$ to $E'_{k(m_1)+1}$. By Theorem 5.3, there exists $l > k(m_1)$ such that $Q(\phi_{k(n_1),l})$ and $Q(\phi_{k(m_1)+1,l} \circ \beta_{k(m_1)} \circ \psi_{k(n_1),k(m_1)} \circ \alpha_{k(n_1)})$ are approximately unitarily equivalent on $Q(\phi_{n_1,k(n_1)})(G_1)$ to within $\delta_1/2$.

We are now in the following situation. There is a not necessarily commutative diagram:

$$(\star) \qquad \begin{array}{ccccccc} E_{n_1} & \longrightarrow & E_{k(n_1)} & \longrightarrow & E_{k(m_1)+1} & \longrightarrow & E_l \\ & & \downarrow & & \uparrow & & \\ & & E'_{m_1} & \longrightarrow & E'_{k(m_1)} & & \end{array}$$

where the down map is $\psi_{k(n_1),m_1} \circ \alpha_{k(n_1)} \circ \phi_{n_1,k(n_1)}$ and the up map is $\beta_{k(m_1)}$. Here $\pi_{n_1}(F_1) \subset Q(E_{n_1})$ is of weakly approximately constant to within $\epsilon_1/2$, $\pi'_{m_1}(F'_1) \subset Q(E'_{m_1})$ is of weakly approximately constant to within $\epsilon'_1/2$ and, by 5.3, the following diagram is approximately commutative, up to an inner automorphism on $Q(E_l)$, on $G_1$ to within $\delta_1/2$:

$$\begin{array}{ccccccc} Q(E_{n_1}) & \longrightarrow & Q(E_{k(n_1)}) & \longrightarrow & Q(E_{k(m_1)+1}) & \longrightarrow & Q(E_l) \\ & & \downarrow & & \uparrow & & \\ & & Q(E'_{m_1}) & \longrightarrow & Q(E'_{k(m_1)}). & & \end{array}$$

Denote by $\beta'_{k(m_1)}$ be the composition of a suitable inner automorphism $\mathrm{Ad}(u)$ on $Q(E_l)$ with the composition of the map from $Q(E'_{k(m_1)})$ to $Q(E_{k(m_1)+1})$ with the map from $Q(E_{k(m_1)+1})$ to $Q(E_l)$ in the above diagram. We choose the inner automorphism so that the following diagram approximately commutative on $G_1$ within $\delta_1/2$:

$$\begin{array}{ccccc} Q(E_{n_1}) & \longrightarrow & Q(E_{k(n_1)}) & \longrightarrow & Q(E_l) \\ & & \downarrow & & \uparrow \beta'_{k(m_1)} \\ & & Q(E'_{m_1}) & \longrightarrow & Q(E'_{k(m_1)}). \end{array}$$

If $\Phi$ is the map from $E'_{k(m_1)}$ to $E_l$ obtained from the diagram marked with ($\star$) in this proof, then

$$\mathrm{Ad}(u) \circ Q(\Phi) = \beta'_{k(m_1)}.$$

By the existence theorem (3.1), there is a map $\Psi : E'_{k(m_1)} \to E_l$ such that $Q(\Psi) = \beta'_{k(m_1)}$ and $\Psi$ and $\Phi$ induces the same map on $V_*(E'_{k(m_1)})$. (The use of $\Psi$ is to avoid to lift $u$.)

We now denote by $\alpha$ the composition of the maps from $E_{n_1}$ to $E_{k(n_1)}$, from $E_{k(n_1)}$ to $E'_{m_1}$, from $E'_{m_1}$ to $E'_{k(m_1)}$ and $\Psi$.

We want to show that there exists $n'_2 > l$ such that $\phi_{n_1,n'_2}$ and $\phi_{l,n'_2} \circ \alpha$ are approximately unitarily equivalent on $F_1$ to within $\epsilon_1$. Note that $\phi_{n_1,l}$ and $\alpha$ induce the same map on $V_*(E_{n_1})$ and $\pi_{n_1}(F_1)$ is weakly approximately constant within $\varepsilon_1/2$.

By conjugating a unitary, we may assume that $\phi_{n_1,l}$ and $\alpha$ agree on the central projections of $E_{n_1}$.

Let $H$ $(= M_k(\mathcal{T}_r))$ be a summand of $E_{n_1}$. Then $\phi_{n_1,l}(1_H) = \alpha(1_H)$, by the above assumption, where $1_H$ is the central projection corresponding to the summand $H$. We also have that $\phi_{n_1,l}|_H$ and $\alpha|_H$ induce the same map on $V_*(H)$. Suppose that $H'$ is a summand of $E_l$ and $P'$ is the corresponding central projection of $E_l$. Then $P'\phi_{n_1}|_H$ and $P'\alpha|_H$ induce the same map on $V_*(H)$.

Let $\gamma_1 = P'\phi_{n_1}|_H$ and $\gamma_2 = P'\alpha|_H$ be a pair of partial maps of $\phi_{n_1,l}$ and $\alpha$, respectively, from $H$ to a summand of $E_l$. Since $\gamma_1$ and $\gamma_2$ induce the same map on $V(I(E_{n_1}))$, $\gamma_1$ and $\gamma_2$ will be both injective or both non-injective. In the case of non-injective, they both factor through $Q(H)$ via $\gamma_{1Q}$ and $\gamma_{2Q} : Q(H) \to E_l$, respectively. Again, since $\gamma_1$ and $\gamma_2$ induce the same map on $V_*(E_{n_1})$, $\gamma_{1Q}$ and $\gamma_{2Q}$ both induce trivial $K_1$-map or both induce non-trivial $K_1$-map. (Notice that the $K_1$-maps are always trivial when the second summand comes from a non-trivial extension. In the case that the second summand does come from a trivial extension, $\pi_l$ induces an isomorphism on the $K_1$-groups.) Finally, it is clear that $\pi_l \circ \gamma_{1Q}$ and $\pi_l \circ \gamma_{2Q}$ are both zero or both are non-zero.

We consider four cases.

*Case* 1: $\gamma_1$ and $\gamma_2$ are injective.

Since $G_1$ contains the generators of $Q(E_{n_1})$ and $\pi_{n_1}(F_1)$, it follows from (1) of the Theorem 4.10 that there exists a unitary $U_\gamma \in \gamma_1(1_H)E_l\gamma_1(1_H)$ such that $\gamma_1$ and $AdU_\gamma \circ \gamma_2$ are within $\epsilon_1$ on $F_1 1_H$. For any $n_2' > l$, define $W_\gamma = \phi_{l,n_2}(U_\gamma)$. Then $\phi_{l,n_2'} \circ \gamma_1$ and $AdW_H \circ \phi_{l,n_2'} \circ \gamma_2$ are within $\epsilon_1$ on $F_1 1_H$.

*Case* 2: $\gamma_1$ and $\gamma_2$ are not injective and $\gamma_{1Q}$ and $\gamma_{2Q}$ induce non-trivial $K_1$-map.

First, $\gamma_{1Q}$ and $\gamma_{2Q}$ must be injective. (Here we note that if $\gamma$ is a noninjective homomorphism from $H$ to a $C^*$-algebra $A$, $\gamma$ factor through $\gamma_Q : Q(H) \to A$. So $\gamma_{iQ}$ are two homomorphisms from $M_k(C(S^1))$ to $E_l$, if $H = M_k(\mathcal{T}_r)$, $i = 1, 2$.) Apply (2) of Theorem 4.10 and the above argument, we conclude that, for any $n_2' > l$ there exists a unitary $W_\gamma \in \phi_{l,n_2'} \circ \gamma_1(1_H)E_{n_2'}\phi_{l,n_2'} \circ \gamma_1(1_H)$ such that $\phi_{l,n_2'} \circ \gamma_1$ and $\mathrm{Ad}\, W_\gamma \circ \phi_{l,n_2'} \circ \gamma_2$ are within $\epsilon_1$ on $F_1 1_H$.

*Case* 3: $\gamma_1$ and $\gamma_2$ are not injective, and $\gamma_{1Q}$ and $\gamma_{2Q}$ induce trivial $K_1$-map.

We claim that there are an integer $n_2(\gamma) > l$ and two homomorphisms $\psi_1$ and $\psi_2$ from $Q(E_{n_1})$ into $Q(E_{n_2(\gamma)})$ with finite dimensional ranges such that

$$\|\psi_1(f) - [\phi_{l,n_2(\gamma)} \circ \gamma_1]_Q(f)\| < \varepsilon_1/4 \quad \text{and} \quad \|\psi_2(f) - [\phi_{l,n_2(\gamma)} \circ \gamma_2]_Q(f)\| < \varepsilon_1/4$$

for every $f \in \pi_{n_1}(F_1)1_H$. We note that since $\gamma_i$ is not injective, $\phi_{l,n_2(\gamma)} \circ \gamma_i$ factors through $[\phi_{l,n_2(\gamma)} \circ \gamma_i]_Q : Q(H) \to E_l$.

Now we prove the claim. We let $\{e_{ij}\}$ and $z$ be the generators of $Q(H)$. Note that the $C^*$-algebra $(\phi_{l,\infty})_Q(e_{11})Q(E)(\phi_{1,\infty})_Q(e_{11})$ has real rank zero ([**BP**]). Since $\gamma_{iQ}$ induces trivial map on $K_1(H)$, for any $\eta > 0$, by [**Ln1**], $[\phi_{l,\infty} \circ \gamma_i]_Q(z)$ is approximated by a unitary with finite spectrum, say $w_i \in (\phi_{l,\infty})_Q(e_{11})E(\phi_{l,\infty})_Q(e_{11})$, within $\eta/2$. Suppose that

$$w_i = \sum_{j=1}^{m_i} \lambda_j^{(i)} p_j^{(i)},$$

where $\{\lambda_j^{(i)}\}_{j=1}^{m_i} \in S^1$ and $\{p_j^{(i)}\}$ are mutually orthogonal projections (with sum $(\phi_{l\infty})_Q(e_{11})$) in $(\phi_{l\infty})_Q(e_{11})E(\phi_{l,\infty})_Q(e_{11})$, $i = 1, 2$. By A8 in [**Eff**], we may assume, with additional error of $\eta/2$,

$$w_i \in [\phi_{l,n_2(\gamma)}]_Q(e_{11})E_{n_2(\gamma)}[\phi_{l,n_2(\gamma)}]_Q(e_{11})$$

for some $n_2(\gamma) > l$.

Define the homomorphism $\psi_i : Q(H) \to E_{n_2(H)}$ by sending $e_{kj}$ to $\phi_{n_1,n_2(\gamma)}(e_{kj})$ and $z$ to $w_i$, $i = 1, 2$. By choosing a sufficient small $\eta$, we can have inequalities in the claim.

Let $\{P_j\}$ be the (finite) set of central projections of $E_{n_2(\gamma)}$. By applying (1) of Theorem 4.11 to each $P_j\psi_i$, we obtain a unitary $U_\gamma \in \psi_1(1_H)E_{n_2(\gamma)}\psi_1(1_H)$ such that $\psi_1$ and $\operatorname{Ad} U_\gamma \circ \psi_2$ are within $\epsilon_1/2$ on $\pi_{n_1}(F_1 1_H)$. For $f \in F_1 1_H$,

$$\begin{aligned}
&\|\gamma_1(f) - \operatorname{Ad} U_\gamma \circ \gamma_2(f)\| \\
&= \|\gamma_1(f) - \psi_1(f) + \psi_1(f) - \operatorname{Ad} U_\gamma \gamma_2(f) - \operatorname{Ad} U_\gamma \circ \psi_2(f) + \operatorname{Ad} U_\gamma \circ \psi_2(f)\| \\
&< \epsilon/4 + \epsilon_1/2 + \epsilon_1/4 = \epsilon_1.
\end{aligned}$$

Thus, for any $n_2' \geq n_2(\gamma) > l$, let $W_\gamma = \phi_{n_2(\gamma),n_2'}(U_\gamma)$, we have $\phi_{l,n_2'} \circ \gamma_1$ and $\operatorname{Ad} W_\gamma \circ \phi_{l,n_2'} \circ \gamma_2$ are within $\varepsilon_1$ on $F_1 1_H$.

*Case* 4: $\pi_l \circ \gamma_{1Q}$ and $\pi_l \circ \gamma_{2Q}$ are trivial.

One applies (2) of Theorem 4.11. There exists a unitary $U_H \in \gamma_{1Q}(1_H)E_l\gamma_{1Q}(1_H)$ so that $\gamma_1$ and $\operatorname{Ad} U_H \circ \gamma_2$ are within $\epsilon_1$ on $F_1 1_H$. For any $n_2' > l$, let $W_\gamma = \phi_{l,n_2'}(U_\gamma)$. Then $\phi_{l,n_2'} \circ \gamma_1$ and $\operatorname{Ad} W_H \circ \phi_{l,n_2'} \circ \gamma_2$ are within $\varepsilon_1$ on $F_1 1_H$.

Now let $n_2' = \max\{n_2(\gamma), l+1\}$, where maximum is taking over all summands $H$ of $E_{n_1}$ and central projections $P'$ of $E_l$. Define

$$W = \sum W_\gamma,$$

where the sum is taking over all of summands $H$ of $E_{n_1}$ and the central projections $P'$ of $E_l$. $W$ is a unitary in $E_{n_2'}$. From above, a direct and simple computation shows that $\phi_{n_1,n_2'}$ and $\operatorname{Ad} W \circ \phi_{l,n_2'} \circ \alpha$ are within $\epsilon_1$ on $F_1$. We denote $\operatorname{Ad} W \circ \phi_{l,n_2'} \circ \alpha$ by $\sigma_1$.

Let $F_2 \subset E_{n_2'}$ be a finite subset consisting of the images of the first $m_1$ elements of $\Delta_{(i)}, \Delta_{(j)'}$, $i = 1, 2, \cdots, n_2'$ and $j = 1, 2, \cdots, m_1'$ along all possible routes. We also require that $\pi_{n_2'}(F_2)$ contains the images of the generators of $Q(E_{m_1}')$. By Theorem 5.2, there exists $n_2 > n_2'$ such that $\pi_{n_2}(\phi_{n_2',n_2}(F_2))$ is of weakly approximately constant to within $\epsilon_2/2$. We will denote $\phi_{n_2',n_2}(F_2) \subset E_{n_2}$ by $F_2$ again. We remark that $F_2$ contains the images of $F_1$ and $F_1'$ along all possible routes.

In summary, we have the following diagram:

$$\begin{array}{ccccc} E_{n_1} & \longrightarrow & E_{k(n_1)} & \longrightarrow & E_{n_2} \\ & & \downarrow & & \uparrow \\ & & E'_{m_1} & \longrightarrow & E'_{k(m_1)} \end{array}$$

(1) approximately commutes on $F_1$ to within $\epsilon_1$,

(2) $\pi'_{m_1}(F'_1) \subset Q(E'_{m_1})$ is of weakly approximately constant to within $\epsilon'_1/2$,

(3) $Q(\psi_{m_1,k(m_1)})(G'_1)$ is of weakly approximately constant to within $\delta'_1/160$, and

(4) $\pi_{n_2}(F_2) \subset Q(E_{n_2})$ is of weakly approximately constant to within $\epsilon_2/2$.

Notice that the above diagram can be written as:

$$\begin{array}{ccc} E_{n_1} & \longrightarrow & E_{n_2} \\ \downarrow & \nearrow & \\ E'_{m_1}. & & \end{array}$$

Next, we are going to repeat the above process. We want to find $m_2 > m_1$ and a map from $E_{n_2}$ to $E'_{m_2}$ so that the diagram:

$$\begin{array}{ccc} E_{n_2} & & \\ \uparrow & \searrow & \\ E'_{m_1} & \longrightarrow & E'_{m_2} \end{array}$$

commutes on $F'_1$ to within $\epsilon'_1$. The map will take the form:

$$\begin{array}{cccc} E_{n_2} & \longrightarrow & E_{k(n_2)} & \\ & & \downarrow & \\ & & E'_{k(n_2)} & \longrightarrow E'_{m_2} \end{array}$$

for some $k(n_2) > n_2$. Let $\Gamma_2 \subset E_{n_2}$ be a finite subset such that $\Gamma_2$ contains $F_2$ and $\pi_{n_2}(\Gamma_2)$ contains the generators of $Q(E_{n_2})$. Denote $\pi_{n_2}(\Gamma_2)$ by $G_2$. For fixed $E_{n_2}, \Gamma_2$ and $\epsilon_2/2$, there exists $\delta_2$ as in Theorem 4.10. Let $k(n_2) > n_2$ so that $Q(\phi_{n_2,k(n_2)})(G_2)$ is of weakly approximately constant to within $\delta_2/160$. Recall that $\alpha_{k(n_2)}$ is a $*$-homomorphism from $E_{k(n_2)}$ to $E'_{k(n_2)}$. Now $Q(\psi_{m_1,k(n_2)})$ and $Q(\alpha_{k(n_2)} \circ \phi_{n_2,k(n_2)} \circ \eta'_1 \circ \psi_{m_1,k(m_1)})$ induce the same K-theory map. So, by 5.2, there exists $l > k(n_2)$ such that, up to an inner automorphism in $Q(E'_l)$, the diagram:

$$\begin{array}{ccccccc} & & Q(E_{n_2}) & \longrightarrow & Q(E_{k(n_2)}) & & \\ & & \uparrow & & \downarrow & & \\ Q(E'_{m_1}) & \longrightarrow & Q(E'_{k(m_1)}) & \longrightarrow & Q(E'_{k(n_2)}) & \longrightarrow & Q(E'_l) \end{array}$$

commutes on $G_1'$ to within $\delta_1'/2$ . This is because that the image of $G_1' \subset Q(E'_{k(m_1)})$ is already of approximately constant to within $\delta_1'/160$.

Denote by $\alpha'_{k(n_2)}$ the composition of a suitable inner automorphism $\mathrm{Ad}(u')$ on $Q(E_l')$ with the composition of the map from $Q(E_{k(n_2)})$ to $Q(E_{k(n_2)})$ with the map from $Q(E_{k(n_2)})$ to $Q(E_l)$ in the above diagram. We choose the inner automorphism so that the following diagram approximately commutative on $G_1$ within $\delta_1'/2$:

$$\begin{array}{ccccc} & & Q(E_{n_2}) & \longrightarrow & Q(E_{k(n_2)}) \\ & & \uparrow & & \downarrow \alpha'_{k(n_2)} \\ Q(E'_{m_1}) \longrightarrow & Q(E'_{k(m_1)}) & Q(E'_{k(n_2)}) & \longrightarrow & Q(E_l'). \end{array}$$

If $\Phi'$ is the map from $E_{k(n_2)}$ to $E_l'$ obtained from the above diagram, then

$$\mathrm{Ad}(u') \circ Q(\Phi') = \alpha'_{k(n_2)}.$$

By the existence Theorem (3.1), there is a map $\Psi' : E_{k(n_2)} \to E_l'$ such that $Q(\Psi') = \alpha'_{k(n_2)}$ and $\Psi'$ and $\Phi'$ induces the same map on $V_*(E'_{k(n_2)})$.

By repeating the above argument, applying 4.10 and 4.11, we obtain an integer $m_2' > l$ and a homomorphism from $E_{k(n_2)}$ into $E'_{m_2'}$ such that the following diagram is approximately commutative to within $\epsilon'$ on $F_1'$:

$$\begin{array}{ccc} E_{n_2} & \longrightarrow & E_{k(n_2)} \\ \uparrow & & \downarrow \\ E'_{k(m_1)} & \longrightarrow & E'_{m_2'}. \end{array}$$

Let $F_2' \subset E_{m_2'}$ consist of the images of $F_1$, $F_1'$, $F_2$ and the images of the first $n_2$ elements of $\Delta_{(1)}, \dots, \Delta_{(n_2)}$ and $\Delta_{(1)'}, \dots, \Delta_{(m_1)'}$ from all possible routes. We also require that $\pi_{m_2'}(F)$ contains the generators of $Q(E_{m_2'})$. Let $m_2 > m_2'$ such that $\pi'_{m_2} \circ \psi_{m_2', m_2}(F')$ is of weakly approximately constant to within $\epsilon_2'/2$. We will denote $\psi_{m_2',m_2}(F')$ by $F_2'$ again. Hence we have a diagram

$$\begin{array}{ccccccc} E_{n_1} & \longrightarrow & E_{k(n_1)} & \longrightarrow & E_{n_2} & \longrightarrow & E_{k(n_2)} \\ & & \downarrow & & \uparrow & & \downarrow \\ & & E'_{m_1} & \longrightarrow & E'_{k(m_1)} & \longrightarrow & E'_{m_2} \end{array}$$

such that

(1) the first square commutes on $F_1$ to within $\epsilon_1$,

(2) the second square commutes on $F_1'$ to within $\epsilon_1'$,

(3) $Q(\phi_{n_2,k(n_2)})(G_2)$ is of weakly approximately constant to within $\delta_2/160$,

(4) $\pi_{n_2}(F_2) \subset Q(E_{n_2})$ is of wearly approximately constant to within $\epsilon_2/2$,

(5) $\pi'_{m_2}(F_2') \subset Q(E'_{m_2})$ is of weakly approximately constant to within $\epsilon_2'/2$.

Notice that if we combine these together and compress some of the algebras, we have

$$\begin{array}{ccccc} E_{n_1} & \longrightarrow & E_{n_2} & \longrightarrow & E_{k(n_2)} \\ \downarrow & \nearrow & \downarrow & & \downarrow \\ E'_{m_1} & \longrightarrow & E'_{m_2} & = & E'_{m_2}. \end{array}$$

The first triangle is commutative on $F_1$ to within $\epsilon_1$ and the second is commutative on $F'_1$ to within $\epsilon'_1$.

By repeating the above process, we get two subsequences $\{E_{n_k}\}$ and $\{E'_{m_k}\}$, two subsequences of finite sets $\{F_k \subset E_{n_k}\}$ and $\{F'_k \subset E'_{m_k}\}$ and the following not necessarily commutative diagram:

$$\begin{array}{ccccccc} E_{n_1} & \longrightarrow & E_{n_2} & \longrightarrow & \cdots & \longrightarrow & E \\ \downarrow & \nearrow & \downarrow & \nearrow & & & \\ E'_{m_1} & \longrightarrow & E'_{m_2} & \longrightarrow & \cdots & \longrightarrow & E'. \end{array}$$

Here the up triangle in the $k$th square is commutative on $F_k$ to within $\epsilon_k$ and the lower triangle is commutative on $F'_k$ to within $\epsilon'_k$. The images of the sets $\{F_k\}$ and $\{F'_k\}$ in $E$ and $E'$ are dense since they contain the images of all $\Delta_{(k)}$ and $\Delta'_{(i)}$. This approximately intertwining satisfies the conditions of Theorem 2.2 [**Ell2**]. Hence it induces a $*$-isomorphism from $E$ to $E'$. Clearly, $\phi$ induces $\eta$. □

**5.5.** For $\mathcal{T}_1$-algebra, there is no trivial extension appears in the inductive limit systems $E = \lim_{\rightarrow}(E_n, \phi_{n,n+1})$. By Corollary 2.11, a homomorphism from $V(E)$ to $V(E')$ extends uniquely to a homomorphism from $V_*(E)$ to $V_*(E')$. As a consequence of Lemma 5.4, we have the following:

**Theorem.** *Let $E = \lim_{\rightarrow}(E_n, \phi_{n,n+1})$ and $E' = \lim_{\rightarrow}(E'_n, \psi_{n,n+1})$ be two $\mathcal{T}_1$-algebras. Suppose that $\eta : V(E) \to V(E')$ is an isomorphism such that $\eta([1_E]) = [1_{E'}]$. Then there exists a $*$-isomorphism $\phi : E \to E'$ such that $\phi$ induces $\eta$.*

**5.6.** Next, we will show that any $A\mathcal{T}$-algebra can be written as an inductive limit of finite direct sums of matrix algebras over $\mathcal{T}$-algebras with the additional condition on the connecting maps. Namely, it satisfies the requirement imposed in Lemma 5.4.

**Lemma.** *For any subsequence $\{n_k\}$ of $\mathbf{N}$, there exists an increasing sequence $B_k \subset \mathcal{T}_0$ such that $\cup_k^\infty B_k$ is dense in $\mathcal{T}_0$, where $B_k \cong M_{n_k} \oplus C^*(U_k)$, and where $U_k \in \mathcal{T}_0$ is a normal partial isometry with its essential spectrum of the whole circle $S^1$.*

*Proof.* (Note that $\cup_{k=1}^\infty M_n(B_k)$ is dense in $M_n(\mathcal{T}_0)$.)

*Proof.* Fix a separable (infinite dimensional) Hilbert space $H$. Let $\{p_i\}$ be rank one projections in $B(H)$ with $\sum p_i = 1$. Then $\mathcal{T}_0$ is generated by $\mathcal{K}$, the compact operators, and a unitary

$$U = \sum \lambda_i p_i$$

where $\{\lambda_i\}_{i=k}^{\infty}$ is dense in $S^1$ for each $k$.

Let $\{e_{ij}\}$ be a matrix unit for $\mathcal{K}$. Let $M_{n_k}$ be the algebra generated by $\{e_{ij}\}_{i,j\le n_k}$. Denote $U_k = \sum_{i>n_k} \lambda_i p_i$. Then $M_{n_k}$ is orthogonal to $C^*(U_k)$ and

$$B_k = M_{n_k} \oplus C^*(U_k) \hookrightarrow \mathcal{T}_0.$$

We now show that $B_k \subset B_{k+1}$. Write

$$U_k = \sum_{n_k < i \le n_{k+1}} \lambda_i p_i + U_{k+1}.$$

It is clear that $U_k$ is then an element of $B_{k+1}$. $M_{n_k} \subset M_{n_{k+1}}$. So $B_k \hookrightarrow B_{k+1}$.

To show that $\mathcal{T}_0 = \varinjlim B_k$, we notice that $U \in B_k$ for all $k$ and $\{v_{ij}\} \subset \bigcup B_k$. ☐

**Proposition 5.7.** *Let $E = \varinjlim(E_n, \phi_n)$ be an inductive limit $C^*$-algebra where each $E_n$ is a finite direct sum of matrix algebras over $\mathcal{T}$-algebras. Then $E$ is isomorphic to $E' = \varinjlim E'_n$ where each $E'_n$ is a finite direct sum of matrix algebras over $\mathcal{T}$-algebras and where each connecting map from $E'_n$ to $E'_{n+1}$ satisfies the following: If $M_k(\mathcal{T}_0)$ is a summand of $E'_n$, then the connecting map restricted to this block vanishes on $M_k(\mathcal{K})$.*

*Proof.* We construct $E' = \varinjlim E'_n$ in several steps.

*Step* 1. We first replace those blocks that are isomorphic to matrix algebras over $\mathcal{T}_0$ by direct sums of matrix algebras and matrix algebras over $C(S^1)$.

Take $n_k = k$ in Lemma 5.6. Let $B_k = M_k \oplus C^*(U_k)$. Then $\mathcal{T}_0 = \varinjlim B_k$.

For each $E_n$ we can write $E_n = E_n^{(1)} \oplus E_n^{(2)}$ where $E_n^{(2)} = L_n \otimes \mathcal{T}_0$, $L_n$ is a finite dimensional $C^*$-algebra and there is no summand in $E_n^{(1)}$ which is a matrix algebra over $\mathcal{T}_0$. Clearly,

$$E_n = E_n^{(1)} \oplus \varinjlim L_n \otimes B_k.$$

Let $f^{(n)} = \{f_k^{(n)}\}$ be dense in $E_n$ and let $\epsilon_n = 1/2^n$. Let $G_1 = \{f_1^{(1)}\} \subset E_1$. There exists $n_1$ such that $\mathrm{dist}(f_1^{(1)}, E^{(1)} \oplus L_1 \otimes B_{n_1}) < \varepsilon_1$. Fix $E_1^0 = E_1^{(1)} \oplus B_{n_1}$. Let $h^{(1)} = \{h_i^{(1)}\}$ be dense in $E_1^0$. Let $H_1$ contain $h_1^{(1)}$ and the generators of $E_1^0$. $H_1$ is finite.

Denote by $\phi_1$ the connecting map form $E_1$ to $E_2$. Let $p$ be the projection in $E_2$ corresponding to the summand $E_2^{(2)}$. By 2.4 (f), $\phi_1 \cdot p$ vanishes on $I(E_1^{(1)})$. Thus $\phi_1 \cdot p|_{E_1^{(1)} \oplus L_1 \otimes B_{n_1}}$ factors through $Q(E_1^{(1)}) \oplus L_1 \otimes B_{n_1}$, via, say $\bar{\psi}_1'$. Denote by $q_1$ the quotient map from $E_1^0$ to $Q(E_1^{(1)}) \oplus L_1 \otimes B_{n_1}$. By Lemma 4.2 [**Ell2**], there exists $n_2 > n_1$ and there exists a unital $*$-homomorphism $\psi_1'$ from $Q(E_1^{(1)}) \oplus L_1 \otimes B_{n_1}$ to $E_2^0 = E_2^{(1)} \oplus L_2 \otimes B_{n_2}$ such that $\bar{\psi}_1'$ and $\psi_1'$ are within $\epsilon_1$ on $q_1(H_1)$. Let $\psi_1 = \psi_1 \cdot (1-p) \oplus \psi_1' \circ q_1$, then

$$\begin{array}{ccc} E_1 & \xrightarrow{\phi_1} & E_2 \\ \uparrow & & \uparrow \\ E_1^0 & \xrightarrow{\psi_1} & E_2^0 \end{array}$$

commutes on $H_1$ to within $\epsilon_1$. We may take $n_2$ large so that $E_2^{(1)} \oplus L_2 \otimes B_{n_2}$ contains $G_2 = \{\phi_1(G_1), \phi_1(f_2^{(1)}), f_1^{(2)}, f_2^{(2)}\}$ to within $\epsilon_2$.

Fix $E_2^0 = E_2^{(1)} \oplus B_{n_2}$. Let $h^{(2)} = \{h_i^{(2)}\}$ be dense in $E_2^0$. Let $H_2$ contains

$$\{\psi_1(H_1), \psi_1(h_2^{(2)}), h_1^{(2)}, h_2^{(2)}\}$$

together with the generators of $q_2(E_2^{(1)}) \oplus B_{n_2}$ where $q_2$ is the quotient map from $E_2^{(1)}$ to $Q(E_2^{(1)})$. As above, there exists $n_3 > n_2$ and there exists a unital $*$-homomorphism $\psi_2$ form $E_2^0$ to $E_3^0 = E_3^{(1)} \oplus B_{n_3}$ such that

$$\begin{array}{ccc} E_2 & \longrightarrow & E_3 \\ \uparrow & & \uparrow \\ E_2^0 & \longrightarrow & E_3^0 \end{array}$$

commutes on $H_2$ to within $\epsilon_2$ and $dist(g, E_3^0) < \varepsilon_3$ for every $g \in G_3$, where $G_3$ contains $\phi_2 \circ \phi_1(G_1), \phi_2(G_2), \phi_2 \circ \phi_1\{f_2^{(1)}, f_3^{(1)}\}, \phi_2\{f_2^{(2)}, f_3^{(2)}\}, f_1^{(3)}, f_2^{(3)}, f_3^{(3)}$ to within $\epsilon_3$.

Continuing this way, we have

$$\begin{array}{ccccccccc} E_1 & \longrightarrow & E_2 & \longrightarrow & E_3 & \longrightarrow & \cdots & \longrightarrow & E \\ \uparrow & & \uparrow & & \uparrow & & & & \\ E_1^0 & \longrightarrow & E_2^0 & \longrightarrow & E_3^0 & \longrightarrow & \cdots & \longrightarrow & E^0. \end{array}$$

The above is an one-sided approximately intertwining in the sense of 2.3 [**Ell2**]. Hence it induces a unital $*$-homomorphism $\psi$ from $E^0$ to $E$. Furthermore, $\psi$ is surjective. This follows from the fact that the image of $E^0$ contains a dense set of $E$. To see that $\psi$ is injective, we note that each vertical map in the above diagram is injective. Hence, the induced map is injective.

*Step* 2. Replace these finite dimensional summands in each $E_n^0$ by matrix algebras over circle algebras.

Write $E_n^0 = D_n \oplus L_n$ where $L_n$ is finite dimensional and where $D_n$ is a finite direct sum of matrix algebras over $\mathcal{T}_k$ $(k > 0)$ or $C(S^1)$. Let $j_n$: $L_n \hookrightarrow L_n \otimes C(S^1)$ be the canonical embedding and let $d_n : L_n \otimes C(S^1) \to L_n$ be evaluation at 1. Notice $d_n \circ j_n = 1$. Denote by $\psi_n : E_n^0 \to E_{n+1}^0$. Then $\beta_n = j_{n+1} \circ \psi_n \circ d_n$ is a unital $*$-homomorphism from $D_n \oplus L_n \otimes C(S^1)$ to $D_{n+1} \oplus L_{n+1} \otimes C(S^1)$. Trivially, we have

$$\begin{array}{ccccc} D_n \oplus L_n & \xrightarrow{\psi_n} & D_{n+1} \oplus L_{n+1} & \xrightarrow{\psi_{n+1}} & D_{n+2} \oplus L_{n+2} \\ \uparrow d_n & & j_{n+1} \downarrow & & d_{n+2} \uparrow \\ D_n \oplus L_n \otimes C(S^1) & \xrightarrow{\beta_n} & D_{n+1} \oplus L_{n+1} \otimes C(S^1) & \xrightarrow{\beta_{n+1}} & D_{n+2} \oplus L_{n+2} \otimes C(S^1) \end{array}$$

commutes. Hence

$$\varinjlim (D_n \oplus L_n \otimes C(S^1), \beta_n) = E.$$

Now $E$ is an inductive limit of finite direct sums of matrix algebras over $\mathcal{T}_k$ $(k > 0)$ or $C(S^1)$. We write $D_n \oplus L_n \otimes C(S^1) = R_n \oplus K_n$ where $K_n = L'_n \otimes C(S^1)$ with $L'_n$ finite dimensional and where $R_n$ is a finite direct sum of matrix algebra over $\mathcal{T}_k$ $(k > 0)$.

*Step* 3. To replace $K_n$ by $L'_n \otimes \mathcal{T}_0$ with desired condition on the connecting maps.

Let $E'_n = R_n \oplus L'_n \otimes \mathcal{T}_0$. Then we have trivial extension

$$0 \longrightarrow L'_n \otimes \mathcal{K} \longrightarrow L'_n \otimes \mathcal{T}_0 \longrightarrow K_n \longrightarrow 0.$$

Let $\pi_n : L'_n \otimes \mathcal{T}_0 \longrightarrow K_n$ be the quotient map and let $i_n : K_n \longrightarrow L'_n \otimes \mathcal{T}_0$ be the splitting such that $\pi_n \circ i_n = 1$. Set $\pi'_n = id_{R_n} \oplus \pi_n$ and $i'_n = id_{R_n} \oplus i_n$. Then

$$E = \varinjlim (R_n \oplus L'_n \otimes \mathcal{T}_0, i'_{n+1} \circ \beta_n \circ \pi'_n).$$

Now $E'_n = R_n \oplus L'_n \otimes \mathcal{T}_0$ is a finite direct sum of matrix algebras over $\mathcal{T}$-algebras. Furthermore, $\alpha_n = i'_{n+1} \circ \beta_n \circ \pi'_n$ vanishes on $I(L'_n \otimes \mathcal{T}_0)$. □

**Remark 5.8.** We would like to point out that a $C^*$-algebra $E \in \mathbb{\mathcal{T}}$ is an $A\mathbb{T}$-algebra if and only if the third variable in $V_*(E)$ is zero, i.e., $d([u]) = 0$ for every $[u] \in k(E)_+$. Write $0 \to I(E) \to E \to Q(E) \to 0$. Then the above statement follows immediately from Theorem 5 in [**LR**], since the map $\delta_0$ there will be zero, if $d = 0$, and $\delta_1 = 0$ since $E$ has real rank zero. Therefore $E$ is an $AF$-algebra if and only $k(E)_+ = \{0\}$, since $k(E)_+ = \{0\}$ implies that $d = 0$ and $K_1(E) = \{0\}$.

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Note: The reference to Lemma 2.14 on page 127 appeared as "Corollary 2.14" in the printed version.