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TOEPLITZ ALGEBRAS ASSOCIATED WITH ENDOMORPHISMS AND PIMSNER-VOICULESCU EXACT SEQUENCES

M. Khoshkam and G. Skandalis

Let A be a C^*-algebra and α a $*$-endomorphism of A. The analogue of Pimsner-Voiculescu exact sequences are obtained for the pair (A, α). We prove that the corresponding Toeplitz algebra remains KK-equivalent to A.We also consider the situation where a semigroup $(\alpha^t)_{t\in\mathbb{R}_+}$ of $*$-endomorphisms is acting on A and formulate similar exact sequences. In this part we use the language of Connes-Higson E-theory.

Introduction.

One of the most celebrated results in the K-theory of C^*-algebras is the exact sequence proved by M. Pimsner and D. Voiculescu ([10]). This sequence allows one to compute the K-groups of a crossed product C^*-algebra $A\rtimes_\alpha\mathbb{Z}$, from a six term exact sequence involving K-groups of A, obtained from the K-theory sequence associated with an extension of $A\rtimes_\alpha\mathbb{Z}$ by $A \otimes K(H)$ where $\alpha \in \mathrm{Aut}\,(A)$. This extension referred to as the generalized Toeplitz extension, is given by a C^*-algebra denoted by $\mathcal{T}_\alpha$, called the Toeplitz algebra associated with the pair (A, α). Pimsner and Voiculescu proved that the natural inclusion of A in $\mathcal{T}_\alpha$ induces an isomorphism at the level of K-groups. This allows one to obtain a six term exact sequence involving only the K-groups of A and the crossed product $A\rtimes_\alpha\mathbb{Z}$. Later, in [5], using a generalization of Connes' "Thom isomorphism", T. Fack and G. Skandalis obtained the same exact sequence for KK-groups.

In this article we are concerned with extending Pimsner Voiculescu Exact Sequence to the situation where α is an Endomorphism. Our first task will be to define an appropriate notion of the Toeplitz algebra and an extension from which the K-theory sequence can be obtained. It is proved that this generalized Toeplitz algebra is still KK-equivalent to A. We then obtain similar results in the case of semigroups (indexed by $\mathbb{R}_+$) of endomorphisms.

While this work was almost finished, we received a remarkable preprint by Mihai Pimsner ([9]), who considers the same Toeplitz algebra and proves the same extension and KK-theory results as ours in a much more general situation than that of a single endomorphism: Pimsner considers a Hilbert A-module E which is 'generating' in that sense that the closed ideal spanned by the scalar products $\langle x, y\rangle,\ x, y \in E$ is A itself and a morphism φ from A into $\mathcal{L}(E)$. An endomorphism is then just the particular case $E = A$ and $\varphi(A) \subset A = \mathcal{K}(A) \subset \mathcal{L}(A) = \mathcal{M}(A)$[1] .

We think however that our paper may help understanding Pimsner's more general and interesting point of view. Moreover, our results may be used to give an alternate proof of Pimsner's (when $\varphi(A) \subset \mathcal{K}(E)$). Indeed, the condition on E means that $\mathcal{K}(E)$ and A are Morita equivalent; hence $\mathcal{K}(H) \otimes \mathcal{K}(E)$ is isomorphic to $\mathcal{K}(H) \otimes A$ (at least in the separable case). We then get a morphism from $\mathcal{K}(H) \otimes A$ into itself which brings us to our case.

The organization of this paper is as follows.

— In Section 1, the Toeplitz algebra $\mathcal{T}_\alpha$ for a pair (A, α) with $\alpha \in \mathrm{End}\,(A)$ is defined and the basic properties are established. In particular, we show that $\mathcal{T}_\alpha$ is a full corner of a crossed product. This will be useful in realization of certain semigroup C^*-algebras.

— In Section 2, we deal with KK-groups and construction of an invertible element in the group $KK(A, \mathcal{T}_\alpha)$.

— Section 3 is concerned with extending our results of Sections (1) and (2) to a semigroup $(\alpha_t)_{t\in\mathbb{R}^+}$ of endomorphisms of a C^*-algebra A. An appropriate notion of Toeplitz algebra is defined and the corresponding extension is formulated. In the continuous case, the Toeplitz algebra is K-Theoretically trivial.

One possible application for these results is in the study of semigroup C^*-algebras ([**3**]). From the basic theory if S is a simple inverse semigroup, then it has a decomposition into a type of semi-direct product (known as Bruck Reilly product) of a group G with the bicyclic semigroup $\mathcal{C}$. The action of $\mathcal{C}$ on G is given by an endomorphism α of G. It can be proved that $C^*(S)$ the C^*-algebra of S is $*$-isomorphic to the Toeplitz algebra associated with the pair $(C^*(G), \alpha)$. These ideas will be pursued elsewhere.

Finally, we point out that in ([**4**]), Ruy Excel obtains a generalization of Pimsner-Voiculescu Exact Sequence. But he considers a different situation dealing with ideals and C^*-algebras equipped with an action of S. The only overlap is that we both obtain Pimsner Voiculescu Exact Sequence as a special case. However, our methods are independent.

[1]Even in that case, our Toeplitz algebra differs slightly from Pimsner's. This will be explained at the end of the first section.

1. The Toeplitz algebra $\mathcal{T}_\alpha$.

Notation. Recall that if A is a C^*-algebra, E, F are Hilbert A-modules, $x \in E$, and $y \in F$, we denote by $\theta_{x,y} : F \to E$ the operator $z \mapsto x\langle y, z\rangle$. An operator from F to E is said to be compact if it belongs to the closure $\mathcal{K}(F, E)$ of the vector space spanned by $\theta_{x,y}$ for $x \in E,\ y \in F$.

Let A be a C^*-algebra and α an endomorphsm of A. Let $\mathcal{H}_A$ be the Hilbert A-module $\ell^2(\mathbb{N}, A)$, i.e., the set of sequences $(x_n)_{n\in\mathbb{N}}$ such that the series $\sum_{n\in\mathbb{N}} x_n^* x_n$ is norm convergent.

Let $S \in \mathcal{L}(\mathcal{H}_A)$ be the forward shift: i.e., $S\big((x_n)_{n\in\mathbb{N}}\big) = (y_n)_{n\in\mathbb{N}} \in \mathcal{H}_A$ where, for $n \neq 0,\ y_n = x_{n-1}$ and $y_0 = 0$.

Define the faithful $*$-representation π_α of A in $\mathcal{H}_A$ setting for $a \in A$ and $(x_n)_{n\in\mathbb{N}} \in \mathcal{H}_A$

$$\pi_\alpha(a)((x_n)_{n\in\mathbb{N}}) = (\alpha^n(a)x_n)_{n\in\mathbb{N}} \in \mathcal{H}_A.$$

For all $a \in A,\ \pi_\alpha(a)S = S\pi_\alpha(\alpha(a))$. It follows that the closed vector span of $\{\, S^n\pi_\alpha(a)S^{*\,m} \,:\, m, n \in \mathbb{N},\ a \in A\}$ is a C^*-subalgebra of $\mathcal{L}(\mathcal{H}_A)$.

Definition 1.1. The C^*-subalgebra of $\mathcal{L}(\mathcal{H}_A)$ generated by $\{\, S^n\pi_\alpha(a)S^{*\,m} : m, n \in \mathbb{N},\ a \in A\}$ is denoted by $\mathcal{T}_\alpha$ and is called the Toeplitz algebra associated with (A, α). We denote by d_α or just d the morpism π_α as a morphism from A to $\mathcal{T}_\alpha$.

If A is unital and $\alpha(1) = 1$, then $\pi_\alpha(1)$ is the identity element of $\mathcal{L}(\mathcal{H}_A)$, thus $S \in \mathcal{T}_\alpha$ and $\mathcal{T}_\alpha$ is the C^*-subalgebra of $\mathcal{L}(\mathcal{H}_A)$ generated by S and $d_\alpha(A)$. In general, let $\tilde{A}$ be the C^*-algebra obtained from A by adjoining an identity. Let $\tilde{\alpha} : \tilde{A} \to \tilde{A}$ be the unital extension of α to $\tilde{A}$. Then $\mathcal{T}_\alpha$ sits in $\mathcal{T}_{\tilde{\alpha}}$ as a two sided ideal.

The construction of the Toeplitz algebra $\mathcal{T}_\alpha$ satisfies the following naturality. Let A, and B be C^*-algebras with endomorphisms α and β respectively. To any $*$-homomorphism $\varphi : A \to B$ such that $\varphi \circ \alpha = \beta \circ \varphi$ there corresponds a $*$-homomorphism $\tau_\varphi : \mathcal{T}_\alpha \to \mathcal{T}_\beta$ given by $\tau_\varphi(S^n d_\alpha(a) S^{*\,m}) = S^n d_\beta(\varphi(a)) S^{*\,m}$.

– If A and B are unital and $\varphi(1) = 1$, we have an identification $\mathcal{H}_A \otimes_A B \cong \mathcal{H}_B$ thus a morphism $\mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_B)$ which maps $\mathcal{T}_\alpha$ into $\mathcal{T}_\beta$.

– In particular, let $\varepsilon : \tilde{A} \to \mathbb{C}$ be the morphism with kernel A. Then τ_ε is a morphism from $\mathcal{T}_{\tilde{\alpha}}$ to the Toeplitz algebra $\mathcal{T}$ associated with the identity morphism of $\mathbb{C}$ whose kernel is $\mathcal{T}_\alpha$.

– To prove the existance of the morphism τ_φ in the general case, extend φ to a unital morphism $\tilde{\varphi} : \tilde{A} \to \tilde{B}$; the corresponding morphism $\tau_{\tilde{\varphi}} : \mathcal{T}_{\tilde{\alpha}} \to \mathcal{T}_{\tilde{\beta}}$ maps $\mathcal{T}_\alpha \subset \mathcal{T}_{\tilde{\alpha}}$ into $\mathcal{T}_\beta \subset \mathcal{T}_{\tilde{\beta}}$.

Let us explore the structure of the Toeplitz algebra $\mathcal{T}_\alpha$:

Let $a, b \in A$ and $n, m \in \mathbb{N}$; let ξ, $\eta \in \mathcal{H}_A$ be the elements defined by $\xi(n) = a$, $\xi(k) = 0$ for $k \neq n$, $\eta(m) = b$ and $\eta(k) = 0$ for $k \neq m$. Then, $\theta_{\xi,\eta} = S^n d_\alpha(ab^*)(1 - SS^*)S^{*^m} \in \mathcal{T}_\alpha$. Since the elements of the above form span $\mathcal{K}(\mathcal{H}_A)$, it follows that $\mathcal{K}(\mathcal{H}_A) \subset \mathcal{T}_\alpha$. As $\mathcal{T}_\alpha \subset \mathcal{L}(\mathcal{H}_A) = \mathcal{M}(\mathcal{K}(\mathcal{H}_A))$ it follows that $\mathcal{K}(\mathcal{H}_A) = \mathcal{K}(H) \otimes A$ is contained in $\mathcal{T}_\alpha$ as an essential ideal.

We next "compute" the quotient $\mathcal{T}_A/\mathcal{K}(\mathcal{H}_A)$:

Let $(A_n)_{n\in\mathbb{N}}$ be the sequence of C^*-algebras with $A_n = A$ for every $n \in \mathbb{N}$. For $m \geq n$ set $\varphi_{m,n} = \alpha^{m-n} : A_n \to A_m$. Let $A_\infty = \lim A_n$ be the direct limit C^*-algebra. Let $h_n : A \to A_\infty$ be the canonical map from $A = A_n$ to the direct limit. Define $\alpha_\infty : A_\infty \to A_\infty$ by setting $\alpha_\infty(h_n(x)) = h_n(\alpha(x))$ for $x \in A$. This is compatible with $\varphi_{m,n}$'s and extends to A_∞. Since $\alpha_\infty \circ h_n = h_n \circ \alpha = h_n \circ \varphi_{n,n-1} = h_{n-1}$ it follows that α_∞ is an automorphism of A_∞ (and $\alpha_\infty^{-1} \circ h_n = h_{n+1}$).

We set $h = h_0$. The algebra A_∞ admits the following abstract characterization.

Proposition 1.2. *We keep the above notation. Let B be a C^*-algebra, $\sigma : A \to B$ a $*$-homomorphism and β an automorphism of B such that $\sigma \circ \alpha = \beta \circ \sigma$. Then there exists a unique $*$-homomorphism $\sigma_\infty : A_\infty \to B$ such that $\sigma_\infty \circ \alpha_\infty = \beta \circ \sigma_\infty$ and $\sigma_\infty \circ h = \sigma$. Moreover, A_∞ and α_∞ are uniquely determined by these conditions.*

Proof. If $\sigma_\infty : A_\infty \to B$ is a $*$-homomorphism satisfying the above conditions, then $\beta^n \circ \sigma_\infty \circ h_n = \sigma_\infty \circ \alpha_\infty^n \circ h_n = \sigma_\infty \circ h = \sigma$, whence $\sigma_\infty \circ h_n = \beta^{-n} \circ \sigma$, which shows the uniqueness of σ_∞.

Define $\sigma_m = \beta^{-m} \circ \sigma : A \to B$. If $m \geq n$, then

$$\begin{aligned} \sigma_m \circ \varphi_{m,n} &= \sigma_m \circ \alpha^{m-n} = \beta^{-m} \circ \sigma \circ \alpha^{m-n} \\ &= \beta^{-m} \circ \beta^{m-n} \circ \sigma = \beta^{-n} \circ \sigma = \sigma_n. \end{aligned}$$

By the universal property of direct limit there exists a $*$-homomorphism $\sigma_\infty : A_\infty \to B$. Moreover, for all n we have $\sigma_n \circ \alpha = \beta \circ \sigma_n$, hence $\sigma_\infty \circ \alpha_\infty = \beta \circ \sigma_\infty$.

Let D be a C^*-algebra with an automorphism δ, and let $j : A \to D$ be a $*$-homomorphism such that $j \circ \alpha = \delta \circ j$. Assume that if B is a C^*-algebra, $\sigma : A \to B$ a $*$-homomorphism and β an automorphism of B such that $\sigma \circ \alpha = \beta \circ \sigma$, then there exists a unique $*$-homomorphism $\sigma' : D \to B$ such that $\sigma' \circ j = \sigma$ and $\sigma' \circ \delta = \beta \circ \sigma'$. Then there exist (unique) $*$-homomorphisms $I : D \to A_\infty$ and $J : A_\infty \to D$ intertwining δ with α_∞ and such that $h \circ I = j$ and $j \circ J = h$. The uniquness statements imply that $I \circ J = \mathrm{id}_{A_\infty}$ and $J \circ I = \mathrm{id}_D$, whence D is canonically $*$-isomorphic with A_∞. □

It follows from this proposition that the construction of the pair $(A_\infty, \alpha_\infty)$ is functorial: Let B be another C^*-algebra endowed with an endomorphism β. To any $*$-homomorphism $\varphi : A \to B$ such that $\varphi \circ \alpha = \beta \circ \varphi$ there corresponds a $*$-homomorphism $\varphi_\infty : A_\infty \to B_\infty$ such that $\varphi_\infty \circ \alpha_\infty = \beta_\infty \circ \varphi_\infty$. In particular, let $\tilde{\alpha}$ be the unital endomorphism of $\tilde{A}$ extending α. The corresponding inductive limit C^*-algebra is the algebra $\widetilde{A_\infty}$ obtained by adjoining a unit to A_∞ endowed with the unital automorphism $\widetilde{\alpha_\infty}$ extending α_∞.

In what follows, we consider A_∞ as a C^*-subalgebra of $A_\infty \rtimes_{\alpha_\infty} \mathbb{Z}$.

Corollary 1.3. *Assume that A is unital and that $\alpha(1) = 1$. Let B be a unital C^*-algebra, $\sigma : A \to B$ a unital $*$-homomorphism and v a unitary in B such that $\sigma(\alpha(x)) = v\sigma(x)v^*$, for all $x \in A$. Then there exists a unique $*$-homomorphism $\hat{\sigma} : A_\infty \rtimes_{\alpha_\infty} \mathbb{Z} \to B \rtimes_\beta \mathbb{Z}$ such that $\hat{\sigma}(d(x)) = \sigma(x)$ for all $x \in A$, and $\hat{\sigma}(u) = v$ where β is the inner automorphism of B associated with v and u is the unitary of $A_\infty \rtimes_{\alpha_\infty} \mathbb{Z}$ defining the crossed-product.*

Proof. Let $\sigma_\infty : A_\infty \to B$ be the associated $*$-homomorphism (Proposition 1.2). We have $\sigma_\infty(\alpha_\infty(x)) = v\sigma_\infty(x)v^*$, for all $x \in A_\infty$. By the universal property of the crossed product, there exists a unique $*$-homomorphism $\hat{\sigma} : A_\infty \rtimes_{\alpha_\infty} \mathbb{Z} \to B \rtimes_\beta \mathbb{Z}$ such that $\hat{\sigma}(x) = \sigma_\infty(x)$ for all $x \in A_\infty$ and $\hat{\sigma}(u) = v$.

Moreover, $h_n(x) = u^{-n} h(x) u^n$ for all $x \in A$ and $n \in \mathbb{N}$, so that $A_\infty \rtimes_{\alpha_\infty} \mathbb{Z}$ is generated by $h(A)$ and u; the uniqueness of $\hat{\sigma}$ follows immediately. ∎

Corollary 1.4. *There exists a unique $*$-homomorphism $\Psi : A_\infty \rtimes_{\alpha_\infty} \mathbb{Z} \to \mathcal{T}_\alpha / \mathcal{K}(\mathcal{H}_A)$ such that, for all $a \in A$, $\Psi(h(a))$ is the image of $d(a) \in \mathcal{T}_\alpha$ in the quotient and, for all $x \in A_\infty$, $\Psi(ux) = v\psi(x)$, where u is as above and v is the image of $S^* \in \mathcal{M}(\mathcal{T}_\alpha)$ in $\mathcal{M}(\mathcal{T}_\alpha / \mathcal{K}(\mathcal{H}_A))$.*

Proof. If A is unital and $\alpha(1) = 1$, this is an immediate consequence of Corollary 1.3.

In the non unital case, let $\tilde{\alpha}$ be the unital endomorphism of $\tilde{A}$ extending α. By the unital case, we get a homomorphism $\tilde{\Psi} : \widetilde{A_\infty} \rtimes_{\widetilde{\alpha_\infty}} \mathbb{Z} \to \mathcal{T}_{\tilde{\alpha}} / \mathcal{K}(\mathcal{H}_{\tilde{A}})$. Note that moreover $A_\infty \rtimes_{\alpha_\infty} \mathbb{Z}$ is the kernel of the map $\widetilde{A_\infty} \rtimes_{\widetilde{\alpha_\infty}} \mathbb{Z} \to C^*(\mathbb{Z})$ corresponding to the unital equivariant morphism $\widetilde{A_\infty} \to \mathbb{C}$ and that $\mathcal{T}_\alpha / \mathcal{K}(\mathcal{H}_A)$ is the kernel of the morphism $\mathcal{T}_{\tilde{\alpha}} / \mathcal{K}(\mathcal{H}_{\tilde{A}}) \to \mathcal{T} / \mathcal{K}(\ell^2(\mathbb{N}))$. Since the diagram

$$\begin{array}{ccc} \widetilde{A_\infty} \rtimes_{\alpha_\infty} \mathbb{Z} & \xrightarrow{\tilde{\Psi}} & \mathcal{T}_{\tilde{\alpha}} / \mathcal{K}(\mathcal{H}_{\tilde{A}}) \\ \downarrow & & \downarrow \\ C^*(\mathbb{Z}) & \longrightarrow & \mathcal{T}_i / \mathcal{K}(\mathcal{H}) \end{array}$$

is commutative, it follows that $\tilde{\Psi}(A_\infty \rtimes_{\alpha_\infty}) \subset \mathcal{T}_\alpha/\mathcal{K}(\mathcal{H}_A)$.

Furthermore, any $\Psi : A_\infty \rtimes_{\alpha_\infty} \mathbb{Z} \to \mathcal{T}_\alpha/\mathcal{K}(\mathcal{H}_A)$ satisfying the conditions of the statement, extends to a morphism from $\widetilde{A_\infty} \rtimes_{\widetilde{\alpha_\infty}} \mathbb{Z}$ to $\mathcal{T}_{\tilde{\alpha}}/\mathcal{K}(\mathcal{H}_{\tilde{A}})$ mapping u to v, from which the uniqueness of Ψ follows. □

Theorem 1.5. *The $*$-homomorphism Ψ of Corollary 1.4 is an isomorphism. In other words, we have an exact sequence*

$$0 \to A \otimes \mathcal{K}(\mathcal{H}) \to \mathcal{T}_\alpha \to A_\infty \rtimes_{\alpha_\infty} \mathbb{Z} \to 0.$$

Proof. For $x \in A$, $m, n \in \mathbb{N}$, the image of $S^m d(x) S^{*n}$ in $\mathcal{T}_\alpha/\mathcal{K}(\mathcal{H}_A)$ is $\Psi(u^m h(x) u^{*n})$, whence Ψ is onto.

To show that Ψ is one to one, we may assume that A is unital and $\alpha(1) = 1$. Let $(e_n)_{n\in\mathbb{N}}$ denote the canonical basis of $\ell^2(\mathbb{N})$ and set $b_n = e_n \otimes 1 \in \ell^2(\mathbb{N}, A) = \mathcal{H}_A$. The set of $T \in \mathcal{L}(\mathcal{H}_A)$ such that the sequence $\alpha_\infty^{-n} \circ h(\langle b_n, T b_n\rangle)$ converges in norm in A_∞ is a closed subspace of $\mathcal{L}(\mathcal{H}_A)$. Moreover, for all $x \in A$, $m, n, k \in \mathbb{N}$ we have $\alpha_\infty^{-k}(\langle b_k, S^m d(x) S^{*n} b_k\rangle) = \alpha_\infty^{-m} \circ h(x)$ if $k \geq m = n$ and to 0 otherwise. Consequently, $T \mapsto \lim_{n\to+\infty} \alpha_\infty^{-n} \circ h(\langle b_n, T b_n\rangle)$ is a completely positive map $E : \mathcal{T}_\alpha \to A_\infty$, such that, for all $x \in A$, $m, n \in \mathbb{N}$, $E(S^m d(x) S^{*n}) = 0$ if $m \neq n$ and $E(S^m d(x) S^{*m}) = \alpha_\infty^{-m} \circ h(x)$. Clearly $\lim_{n\to+\infty} \alpha_\infty^{-n} \circ h(\langle b_n, T b_n\rangle) = 0$ for all $T \in \mathcal{K}(\mathcal{H}_A)$, so that E defines a completely positive map $\Phi : \mathcal{T}_\alpha/\mathcal{K}(\mathcal{H}_A) \to A_\infty$. The composition $\Phi \circ \Psi$ is easily seen to be the conditional expectation $A_\infty \rtimes_{\alpha_\infty} \mathbb{Z} \mapsto A_\infty$ which is the identity on A_∞ and maps $u^k x$ to 0 for all $x \in A_\infty$ and $k \neq 0$. As this conditional expectation is faithfull, Ψ is one to one. □

When α is an automorphism of A we see immediately that A_∞ identifies with A; therefore, the exact sequence of Theorem 1.5 is a generalization of the Toeplitz exact sequence of [**10**].

The following theorem characterizes the $*$-representations of the Toeplitz algebra $\mathcal{T}_\alpha$. If π is a non degenerate $*$-representation of $\mathcal{T}_\alpha$, then $\pi \circ d$ is a $*$-representation σ of A and $T = \tilde{\pi}(S)$ is an isometry, where $\tilde{\pi}$ is the extension of π to the multiplier algebra. For all $a \in A$ we have $\sigma(a)T = T\sigma(\alpha(a))$. The converse is also true:

Theorem 1.6. *Let B be a C^*-algebra and H be a Hilbert B-module. Let $\sigma : A \to \mathcal{L}(H)$ be a $*$-representation of A on H and let $T \in \mathcal{L}(H)$ be an isometry such that $\sigma(a)T = T\sigma(\alpha(a))$. Then, there exists a $*$-representation $\pi : \mathcal{T}_\alpha \to \mathcal{L}(H)$ such that for all $x \in A$, $m, n \in \mathbb{N}$, $\pi(S^m d(x) S^{*n}) = T^m \sigma(x) T^{*n}$. Moreover, π is faithful if and only if the restriction of σ to the kernel of T^* is faithful.*

Proof. Up to passing to $\tilde{A}$, we may assume that A is a unital C^*-algebra and that α and σ are unital morphisms. We first treat the case $B = \mathbb{C}$.

Put $H_0 = \ker T^*$ and let H' be the closure in H of the union of $\ker T^{*n}$ ($n \in \mathbb{N}$). As $T^{*n}\sigma(a) = \sigma(\alpha^n(a))T^{*n}$, the subspaces H_0 and H' are invariant under $\sigma(A)$. Denote by σ_0 the restriction of σ to H_0. Moreover, H' admits the orthogonal decomposition $H' = \bigoplus_{n\in\mathbb{N}} T^n H_0$, therefore there exists an isomorphism of Hilbert spaces $U : \mathcal{H}_A \otimes_{\sigma_0} H_0 \to H'$ such that $U((e_n \otimes a) \otimes x) = T^n\sigma_0(a)x = T^n\sigma(a)x$ for all $n \in \mathbb{N}$, $a \in A$, $x \in H_0$ (where $(e_n)_{n\in\mathbb{N}}$ is the canonical basis of $\ell^2(\mathbb{N})$). Also $U(S \otimes 1) = TU$ and for all $a, b \in A$, $n \in \mathbb{N}$, $x \in H_0$

$$\begin{aligned} U(d(a) \otimes 1)((e_n \otimes b) \otimes x) &= U((e_n \otimes \alpha^n(a)b) \otimes x) \\ &= T^n\sigma(\alpha^n(a)b)x \\ &= \sigma(a)T^n\sigma(b)x \\ &= \sigma(a)U((e_n \otimes b) \otimes x). \end{aligned}$$

Since, the restriction of T to $H'^{\perp}$ is a unitary operator v; by Corollary 1.3, there exists a $*$-representation $\pi' : A_\infty \rtimes_{\alpha_\infty} \mathbb{Z} \mapsto \mathcal{L}(H'^{\perp})$ such that $\pi' \circ h$ is the restriction of σ to $H'^{\perp}$ and $\pi'(u) = v$. Then, the $*$-representation $\pi : x \mapsto U(x \otimes 1)U^* + \pi' \circ q(x)$ satisfies the requirements of the theorem, where $q : \mathcal{T}_\alpha \to A_\infty \rtimes_{\alpha_\infty} \mathbb{Z}$ is the composition of the quotient map $\mathcal{T}_\alpha \to \mathcal{T}_\alpha/\mathcal{K}(\mathcal{H}_A)$ with Ψ^{-1} of Corollary 1.4.

Now, as $\mathcal{K}(\mathcal{H}_A)$ is an essential ideal in $\mathcal{T}_\alpha$, the representation π is faithful if and only if its restriction to $\mathcal{K}(\mathcal{H}_A)$ is faithful, which happens if and only if the representation $a \mapsto \pi(p \otimes a)$ is faithful, where $p \in \mathcal{K}$ is a minimal projection which, by a good choice of p means that $a \mapsto \pi(d(a)(1 - SS^*)) = \sigma(a)(1 - TT^*)$ is faithful.

We finally come to the general case ($B \neq \mathbb{C}$). We may embed $\mathcal{L}(H)$ in some $\mathcal{L}(E)$ where E is a Hilbert space. Then, by the case $B = \mathbb{C}$, there exists a $*$-representation $\pi : \mathcal{T}_\alpha \to \mathcal{L}(E)$ whose image is obviously contained in $\mathcal{L}(H) \subset \mathcal{L}(E)$. □

We end this section with a theorem showing that $\mathcal{T}_\alpha$ is a full corner of a crossed product. Let $C_b(\mathbb{Z}, A)$ be the C^*-algebra of norm bounded sequences $(a_n)_{n\in\mathbb{Z}}$ of elements of A under pointwise operations and infinity norm. For each $p \in \mathbb{Z}$ let $j_p : A \to C_b(\mathbb{Z}, A)$ be the morphism such that $j_p(a)$ is the sequence whose n^{th} term is zero if $n < p$ and $\alpha^{n-p}(a)$ if $n \geq p$. Let D be the C^*-subalgebra of $C_b(\mathbb{Z}, A)$ generated by the elements $j_p(a)$ for $a \in A$ and $p \in \mathbb{Z}$. The shift on $C_b(\mathbb{Z}, A)$ induces an automorphism β of D such that $\beta \circ j_p = j_{p-1}$, so that D is the smallest subalgebra of $C_b(\mathbb{Z}, A)$ containing $j_0(A)$ and invariant under the shift.

Lemma 1.7. *The C^*-subalgebra $C_0(\mathbb{Z}, A)$ of $C_b(\mathbb{Z}, A)$ consisting of the sequences vanishing at infinity is contained in D as an essential ideal. There* *is a $*$-isomorphism $\varphi : D/C_0(\mathbb{Z}, A) \to A_\infty$ such that $\varphi \circ q \circ j_0 = h$ and $\varphi \circ q \circ \beta = \alpha_\infty \circ \varphi \circ q$, where $q : D \to D/C_0(\mathbb{Z}, A)$ is the quotient map.*

Proof. For $a \in A$ and $p \in \mathbb{Z}$, the only nonzero term of the sequence $j_p(a) - j_{p+1}(\alpha(a))$ is a in p^{th} position. Consequentely $C_0(\mathbb{Z}, A) \subset D$ and as $D \subset C_b(\mathbb{Z}, A) = \mathcal{M}(C_0(\mathbb{Z}, A))$, $C_0(\mathbb{Z}, A)$ is contained in D as an essential ideal.

Note that D is the inductive limit of the algebras $D_p = C_0(\mathbb{Z}, A) + j_p(A)$. Therefore, a bounded sequence $(a_n)_{n\in\mathbb{Z}}$ is in D, if and only if, $\lim_{n\to-\infty} \|a_n\| = 0$ and, for every $\varepsilon > 0$, there exists $n \in \mathbb{Z}$ such that, for every $m \in \mathbb{N}$, $\|a_{n+m} - \alpha^m(a_n)\| \le \varepsilon$.

Moreover, for every $p \in \mathbb{Z}$, let $\varphi_p : D \to A_\infty$ be the map $(a_n)_{n\in\mathbb{Z}} \mapsto \alpha_\infty^{-p} \circ h(a_p)$. Clearly $\varphi_p \circ j_k = \alpha_\infty^{-k}$ if $p \ge k$. Therefore, for all $x \in D$ the sequence $\varphi_p(x)$ converges to some element $\varphi(x)$, when $p \to +\infty$. Obviously, φ is a $*$-homomorphism whose kernel contains $C_0(\mathbb{Z}, A)$ and whose image is invariant under α_∞ and contains $h(A)$; therefore ψ is surjective. Let $x = (a_n)_{n\in\mathbb{Z}} \in \ker\varphi$. For every ε, there exists $n \in \mathbb{Z}$ such that for every $m \in \mathbb{N}$, $\|a_{n+m} - \alpha^m(a_n)\| \le \varepsilon$. Then

$$\begin{aligned}\|h(a_n)\| &= \|\alpha_\infty^{-n} \circ h(a_n)\| \\ &= \|\alpha_\infty^{-n} \circ h(a_n) - \varphi(x)\| \\ &= \lim_{m\to+\infty} \|\alpha_\infty^{-n-m} \circ h(\alpha^m(a_n) - a_{n+m})\| \le \varepsilon.\end{aligned}$$

Therefore $\limsup_{m\to+\infty} \|\alpha^m(a_n)\| \le \varepsilon$, whence $\limsup_{m\to+\infty} \|a_{n+m}\| \le 2\varepsilon$. It follows that $\ker\varphi = C_0(\mathbb{Z}, A)$; therefore φ induces the desired isomorphism. $\square$

Theorem 1.8. *Let $v \in \mathcal{L}(\ell^2(\mathbb{Z}, A))$ be the backward shift: i.e., $v\big((x_n)_{n\in\mathbb{Z}}\big) = (y_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}, A)$ where, for $y_n = x_{n+1}$. Moreover let $\rho : D \to \mathcal{L}(\ell^2(\mathbb{Z}, A))$ be the $*$-representation such that $\rho\big((a_n)_{n\in\mathbb{Z}}\big)\big((x_n)_{n\in\mathbb{Z}}\big) = (a_n x_n)_{n\in\mathbb{Z}}$. The pair (ρ, v) is a covariant representation of (D, β) and the corresponding representation of $D \rtimes_\beta \mathbb{Z}$ is faithful. Identify $D \rtimes_\beta \mathbb{Z}$ with its image in $\mathcal{L}(\ell^2(\mathbb{Z}, A))$; the projection P of $\ell^2(\mathbb{Z}, A)$ onto $\ell^2(\mathbb{N}, A)$ is a multiplier of $D \rtimes_\beta \mathbb{Z}$ and $P(D \rtimes_\beta \mathbb{Z})P$ is the Toeplitz algebra $\mathcal{T}_\alpha$; it is a full corner in $D \rtimes_\beta \mathbb{Z}$.*

Proof. It is clear that the pair (ρ, v) is a covariant representation of (D, β). The restriction of the corresponding representation of $D \rtimes_\beta \mathbb{Z}$ to $C_0(\mathbb{Z}, A) \rtimes_\beta \mathbb{Z}$ is the canonical isomorphism of $C_0(\mathbb{Z}, A) \rtimes_\beta \mathbb{Z}$ with the algebra of compact operators in $\ell^2(\mathbb{Z}, A)$. As $C_0(\mathbb{Z}, A)$ is an essential ideal in D, $C_0(\mathbb{Z}, A) \rtimes_\beta \mathbb{Z}$ is an essential ideal in $D \rtimes_\beta \mathbb{Z}$ therefore the representation of $D \rtimes_\beta \mathbb{Z}$ associated with (ρ, v) is faithful.

As P is a multiplier of $\rho(D)$, it is a multiplier of $D \rtimes_\beta \mathbb{Z}$. Moreover, $(1 - P)\rho(D) \subset \mathcal{K}(\ell^2(\mathbb{Z}, A))$ so that $(1 - P)(D \rtimes \mathbb{Z}) \subset \mathcal{K}(\ell^2(\mathbb{Z}, A))$; it follows that

$D\rtimes\mathbb{Z} = \mathcal{T}_\alpha + \mathcal{K}(\ell^2(\mathbb{Z},A))$; as $\mathcal{K}(\ell^2(\mathbb{N},A)) \subset P(D\rtimes_\beta\mathbb{Z})P$ and $\mathcal{K}(\ell^2(\mathbb{N},A))$ is a full corner in $\mathcal{K}(\ell^2(\mathbb{Z},A))$, it follows that $P(D\rtimes_\beta\mathbb{Z})P$ is a full corner in $D\rtimes_\beta\mathbb{Z}$.

Now for all $m,n \in \mathbb{N}$, and $a \in A$, we have $v^{*m}\rho\circ j_0(a)v^n = P(v^{*m}\rho\circ j_0(a)v^n)P$ and acts on $\ell^2(\mathbb{N},A)$ as $S^m d(a)S^{*n}$. It follows that $P(D\rtimes_\beta\mathbb{Z})P$ contains $\mathcal{T}_\alpha$. Now $D\rtimes_\beta\mathbb{Z}$ is generated by $v^k\rho\circ j_p(a)$ where $p,k\in\mathbb{Z}$, $a\in A$. Moreover, if $n\in\mathbb{N}$, $\rho(j_{p-n}(a) - j_p(\alpha^n(a))) \in \mathcal{K}(\ell^2(\mathbb{N},A)) \subset \mathcal{T}_\alpha$; it is enough to show that $P(v^k\rho\circ j_p(a))P \in \mathcal{T}_\alpha$ when $p\geq 0$ and $p-k\geq 0$. But $v^k\rho\circ j_p(a) = v^{k-p}\rho\circ j_0(a)v^p$ and the result follows. □

2. KK-Groups.

In ([**10**]) it is proved that, when α is an automorphism, the canonical inclusion of A in $\mathcal{T}_\alpha$ induces an isomorphism at the K-theory level, and deduced a six term exact sequence computing the K-groups of a crossed-product by $\mathbb{Z}$. Here we prove that this holds in general, by showing that the same map considered as an element of the group $KK(A,\mathcal{T}_\alpha)$ is invertible. As a consequence of this fact, we obtain a generalized version of Pimsner-Voiculescu exact sequence for endomorphisms.

Recall (cf. [**6**]) that if A and B are C^*-algebras, an element of $KK(A,B)$ is given by the homotopy class of a triple $(\mathcal{E},\pi,F)$, where $\mathcal{E}$ is a $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert B-module, $\pi : A\to\mathcal{L}(\mathcal{E})$ is a $*$-representation of A on $\mathcal{L}(\mathcal{E})$ as degree zero operators, and $F\in\mathcal{L}(\mathcal{E})$ has degree 1 such that for all $a\in A$, $[\pi(a),F]\in\mathcal{K}(\mathcal{E})$, $\pi(a)(F-F^*)\in\mathcal{K}(\mathcal{E})$ and $\pi(a)(1-F^2)\in\mathcal{K}(\mathcal{E})$.

Given a $*$-homomorphism $\varphi : A\to B$ we denote by $[\varphi]$ the element of $KK(A,B)$ given by the class of $(B,\varphi,0)$.

We keep the notation of the first section. In particular $d : A\to\mathcal{T}_\alpha$ is the embedding of A into $\mathcal{T}_\alpha$. Set $\mathcal{E}^{(0)} = \ell^2(\mathbb{N},A)$ and let $\mathcal{E}^{(1)} = \ell^2(\mathbb{N}\setminus\{0\},A)$ be the subspace of $\mathcal{E}^{(0)}$ with zero in the first coordinate. Let $Q : \mathcal{E}^{(0)}\to\mathcal{E}^{(1)}$ be the orthogonal projection. Let $\mathcal{E}$ denote the $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert A-module $\mathcal{E}^{(0)}\oplus\mathcal{E}^{(1)}$.

By Theorem 1.6, there is a $*$-representation of $\pi^- : \mathcal{T}_{\tilde\alpha}\to\mathcal{L}(\mathcal{E}^{(1)})$ such that $\pi^-\circ d$ is the restriction of d to the invariant subspace $\mathcal{E}^{(1)}$ of $\mathcal{E}^{(0)}$ and $\pi^-(S) = QSQ = SQ$. In fact $\pi^-(x) = S\tau_\alpha(x)S^*$ where $\tau_\alpha : \mathcal{T}_\alpha\to\mathcal{T}_\alpha$ is the map induced by $\alpha : A\to A$. Let $\pi : \mathcal{T}_\alpha\to\mathcal{L}(\mathcal{E})$ be the $*$-representation such that for $x\in\mathcal{T}_\alpha$, $\xi\in\mathcal{E}^{(0)}$ $\eta\in\mathcal{E}^{(1)}$ we have $\pi(x)(\xi,\eta) = (x\xi,\pi^-(x)\eta)$. Let $F\in\mathcal{L}(\mathcal{E})$ be defined for $\xi\in\mathcal{E}^{(0)}$ $\eta\in\mathcal{E}^{(1)}$ by $F(\xi,\eta) = (\eta,Q\xi)$.

Lemma 2.1. *The triple $(\mathcal{E},\pi,F)$ defines an element of $KK(\mathcal{T}_\alpha,A)$.*

Proof. Clearly $F=F^*$ and $1-F^2$ is the projection $(\xi,\eta)\mapsto((1-Q)\xi,\eta)$, so that $(1-F^2)\mathcal{T}_\alpha\subset\mathcal{K}(\mathcal{E})$. If $a\in A$, then $\pi\circ d(a)$ and F commute. Moreover

$(F\pi(S)-\pi(S)F)(\xi,\eta)=(S\eta-S\eta, QS\xi-SQ\xi)=(0,S(1-Q)\xi)$, so that $(F\pi(S)-\pi(S)F)\mathcal{T}_\alpha\subset\mathcal{K}(\mathcal{E})$. □

Definition 2.2. We denote by $[d]$ the class of the morphism d in $KK(A,\mathcal{T}_\alpha)$ and by β the class of the triple $(\mathcal{E},\pi,F)$ in $KK(\mathcal{T}_\alpha,A)$.

Theorem 2.3. *We have $[d]\otimes_{\mathcal{T}\alpha}\beta=1_A\in KK(A,A)$ and $\beta\otimes_A[d]=1_{\mathcal{T}\alpha}\in KK(\mathcal{T}_\alpha,\mathcal{T}_\alpha)$. In particular, the C^*-algebras A and $\mathcal{T}_\alpha$ are KK-equivalent.*

Proof. Here the Kasparov products are easily computed: We have $[d]\otimes_{\mathcal{T}\alpha}\beta=d^*(\beta)$ and $\beta\otimes_A[d]=d_*(\beta)$. Since $\pi\circ d$ commutes with F and F is a self adjoint partial isometry it follows that the class of $(\mathcal{E},\pi\circ d,F)$ coincides with the class of $((1-F^2)\mathcal{E},i,0)$ where i is the restriction of $\pi\circ d$ to $(1-F^2)\mathcal{E}=A$ and is therefore given by the identity map $A\to A=\mathcal{K}(A)$, hence $d^*(\beta)=1_A$.

Now $d_*(\beta)$ is given by $(\mathcal{F},\sigma,G)$ where $\mathcal{F}=\mathcal{E}\otimes_A\mathcal{T}_\alpha$, $G=F\otimes 1$ and, for all $x\in\mathcal{T}_\alpha$, $\sigma(x)=\pi(x)\otimes 1$. Therefore $\mathcal{F}^{(0)}=\ell^2(\mathbb{N},\mathcal{T}_\alpha)$, $\mathcal{F}^{(1)}=\ell^2(\mathbb{N}\setminus\{0\},\mathcal{T}_\alpha)$, $\sigma=\sigma^{(0)}\oplus\sigma^{(1)}$ where $\sigma^{(0)}:\mathcal{T}_\alpha\to\mathcal{L}(\ell^2(\mathbb{N},\mathcal{T}_\alpha))$ and $\sigma^{(1)}:\mathcal{T}_\alpha\to\mathcal{L}(\ell^2(\mathbb{N}\setminus\{0\},\mathcal{T}_\alpha))$ are defined by $\sigma^{(i)}(d(a))\xi(n)=d(\alpha^n(a))\xi(n)$ for $a\in A$ and $\sigma^{(i)}(S)\xi(n)=\xi(n-1)$ if $n>i$ and $\sigma^{(i)}(S)\xi(i)=0$ ($i=0,1$).

For each $t\in[0,\frac{\pi}{2}]$ let $T_t\in\mathcal{L}(\mathcal{F}^{(0)})$ be defined by

$$(T_t\xi)(n)=\begin{cases}\xi_{n-1} & \text{if } n\geq 2\\ (\cos t)\xi_0 & \text{if } n=1\\ (\sin t)S\xi_0 & \text{if } n=0.\end{cases}$$

Then,

$$(T_t^*\xi)(n)=\begin{cases}\xi_{n+1} & \text{if } n\geq 1\\ (\cos t)\xi_1+(\sin t)S^*\xi_0 & \text{if } n=0.\end{cases}$$

One checks immediately that T_t is an isometry such that $\sigma^{(0)}(d(a))T_t=T_t\sigma^{(0)}(d(\alpha(a)))$ for every $a\in A$. Hence, by Theorem 1.6, there exists a $*$-representation $\sigma_t^{(0)}:\mathcal{T}_\alpha\to\mathcal{L}(\mathcal{F}^{(0)})$ defined by $\sigma_t(S)=T_t$ and $\sigma_t^{(0)}(d(a))=\sigma^{(0)}(d(a))$. Moreover, for every $x\in\mathcal{T}_\alpha$, $\sigma_t^{(0)}(x)-\sigma^{(0)}\in\mathcal{K}(\mathcal{F}^{(0)})$. Consequently, $(\mathcal{F},\sigma_t^{(0)}\oplus\sigma^{(1)},G)$ is a homotopy connecting the elements $d_*(\beta)$ and $(\mathcal{F},\sigma_{\pi/2}^{(0)}\oplus\sigma^{(1)},G)$.

Now $\mathcal{F}^{(0)}$ admits the decomposition $\mathcal{F}^{(0)}=\mathcal{T}_\alpha\oplus\mathcal{F}^{(1)}$ which is invariant under $\sigma_{\pi/2}^{(0)}$. It follows that $(\mathcal{F},\sigma_{\pi/2}^{(0)}\oplus\sigma^{(1)},G)$ is the sum of $1_{\mathcal{T}_\alpha}$ and a degenerate element. We conclude that $d_*(\beta)=1_{\mathcal{T}_\alpha}$. □

Lemma 2.4. *Let $\theta : A \to \mathcal{T}_\alpha$ be defined by $\theta(a) = d(a)(1 - SS^*)$. Then,* $[\theta] \otimes_{\mathcal{T}_\alpha} \beta = 1_A - [\alpha] \in KK(A, A)$.

Proof. The element $[\theta] \otimes_{\mathcal{T}\alpha} \beta = \theta^*(\beta)$ is defined by $(\mathcal{E}, \pi \circ \theta, F)$. Given $\xi \in \mathcal{E}^{(0)} = \ell^2(\mathbb{N}, A)$ we have $(\pi \circ \theta(a)\xi)(n) = \pi(d(a)(1 - SS^*))\xi(n) = 0$ if $n \neq 0$ and $(\pi \circ \theta(a)\xi)(0) = a\,\xi(0)$. On the other hand, if $\xi \in \mathcal{E}^{(1)} = \ell^2(\mathbb{N} \setminus \{0\}, A)$, then $(\pi \circ \theta(a)\xi)(n) = \pi(d(a)(1 - SS^*))\xi(n) = 0$ if $n \neq 1$ and $(\pi \circ \theta(a)\xi)(1) = \alpha(a)\,\xi(1)$. Hence, up to a degenerate module $\theta^*(\beta)$ is represented by the triple $(\mathcal{E}', \mu, 0)$ where $\mathcal{E}'^{(0)} = \mathcal{E}'^{(1)} = A$ and, for $a \in A$, $\mu(a)$ is given by the matrix $= \begin{pmatrix} a & \sigma \\ 0 & \alpha(a) \end{pmatrix}$. □

Using exactness of Connes-Higson's E-theory ([**2**]), Theorem 2.3 to replace $\mathcal{T}_\alpha$ by A in the exact sequence of E-groups associated with the extension of C^*-algebras of Theorem 1.5 and Lemma 2.4 to compute the map from $E(D, A)$ (resp. $E(A, D)$) into itself, we get:

Theorem 2.5. *Let A, α, and α_∞ be as in 1.6. Then we have exact sequences of Connes-Higson's E-groups*

$$\begin{array}{ccccc} E(D,A) & \xrightarrow{1-\alpha_*} & E(D,A) & \longrightarrow & E(D, A_\infty \rtimes_{\alpha_\infty} \mathbb{Z}) \\ \uparrow & & & & \downarrow \\ E_1(D, A_\infty \rtimes_{\alpha\infty} \mathbb{Z}) & \longleftarrow & E_1(D,A) & \xrightarrow{1-\alpha_*} & E_1(D,A) \end{array}$$

and

$$\begin{array}{ccccc} E(A,D) & \xleftarrow{1-\alpha^*} & E(A,D) & \longleftarrow & E(A_\infty \rtimes_{\alpha_\infty} \mathbb{Z}, D) \\ \downarrow & & & & \uparrow \\ E_1(A_\infty \rtimes_{\alpha\infty} \mathbb{Z}, D) & \longrightarrow & E_1(A,D) & \xrightarrow{1-\alpha^*} & E_1(A,D). \end{array}$$

Remarks 2.6.

(a) When α is an automorphism, we recover Pimsner-Voiculescu's exact sequences ([**10**]).

(b) The same result holds of course with the "KK^{nuc}"-groups of [**11**] instead of E-groups.

(c) We may compare the two Toeplitz extensions comming from α and α_∞. We get a diagram of the form:

$$\begin{array}{ccccccc}
 & & E(D,A) & \xrightarrow{1-\alpha_*} & E(D,A) & & \\
 & \nearrow & & & & \searrow & \\
E_1(D,A_\infty \rtimes_{\alpha\infty}\mathbb{Z}) & & h_*\downarrow & & h_*\downarrow & & E(D,A_\infty \rtimes_{\alpha\infty}\mathbb{Z}) \\
 & \searrow & & & & \nearrow & \\
 & & E(D,A_\infty) & \xrightarrow{1-\alpha_{\infty *}} & E(D,\mathcal{T}_{\alpha_\infty}) & &
\end{array}$$

for which both top and bottom lines are exact. It follows in particular that h_* induces an isomorphism from the kernel of $1-\alpha_*$ onto the kernel of $1-\alpha_{\infty *}$ and from the cokernel of $1-\alpha_*$ onto the cokernel of $1-\alpha_{\infty *}$. Note that when $D=\mathbb{C}$, the group $E(D,A_\infty)=K_0(A_\infty)$ is the inductive limit of $(K_0(A),\alpha_*)$ and it is clear that h_* induces isomorphisms at these kernel and cokernel levels.

3. Semigroup of Endomorphisms.

In this section we define the Topelitz algebra associated with a semigroup of endomorphisms of a C^*-algebra A and formulate the corresponding Toeplitz extension.

By a semigroup of endomorphisms of a C^*-algebra A we mean a morphism $\alpha : t \mapsto \alpha^t$ from the (additive) monoid $\mathbb{R}_+$ to the monoid $End(A)$ of endomorphisms of a A satisfying $\alpha^0 = id_A$ and $t \mapsto \alpha^t(a)$ is continuous for every $a \in A$. As α, is a morphism for all $s,t \in \mathbb{R}_+$, we have $\alpha^{t+s} = \alpha^t \circ \alpha^s$.

Note that we have:

Lemma 3.1. *Let $(\alpha^t)_{t\in\mathbb{R}_+}$ be a semigroup of endomorphism of a C^*-algebra A. If α^t is an automorphism of A for some $t>0$, then $\alpha^s \in \mathrm{Aut}\,(A)$ for every $s \in \mathbb{R}_+$.*

The continuous analogue of the Toeplitz algebra of Section 1 is defined as follows.

Let $\pi_\alpha : A \to \mathcal{L}(L^2(\mathbb{R}_+)\otimes A)$ be defined by $\pi_\alpha(a)\xi(t) = \alpha^t(a)\xi(t)$ for every $\xi \in L^2(\mathbb{R}_+)\otimes A = L^2(\mathbb{R}_+, A)$ and every $a \in A$. Let $S_t \in \mathcal{L}(L^2(\mathbb{R}_+)\otimes A)$ be defined by $(S_t\xi)(s) = \xi(s-t)$ if $s \geq t$ and $(S_t\xi)(s) = 0$ if $s < t$.

Clearly $(S_t)_{t\in\mathbb{R}_+}$ is a semigroup of isometries of $\mathcal{L}(L^2(\mathbb{R}_+))$. Moreover, for every $a \in A$ and every $t \in \mathbb{R}_+$ we have $\pi_\alpha(a)S_t = S_t\pi_\alpha(\alpha^t(a))$.

It follows that the integrals $\int_0^\infty\int_0^\infty S_s\pi_\alpha(a(s,t))S_t^*\,dsdt$ where $(s,t)\mapsto a(s,t)$ is a continuous function from $\mathbb{R}_+\times\mathbb{R}_+$ to A with compact support form a $*$-subalgebra of $\mathcal{L}(L^2(\mathbb{R}_+)\otimes A)$.

Definition 3.2. Let A and α be as above. The associated Toeplitz algebra, denoted by $\mathcal{T}_\alpha$, is the closure in $\mathcal{L}(L^2(\mathbb{R}_+)\otimes A)$ of the algebra formed by the integrals

$$\int_0^\infty\int_0^\infty S_s\pi_\alpha(a(s,t))S_t^*\,dsdt$$

where $(s,t)\mapsto a(s,t)$ from $\mathbb{R}_+\times\mathbb{R}_+$ to A is continuous with compact support.

Remarks.

(a) By density of continuous functions with compact support in L^1-functions, for every $a\in L^1(\mathbb{R}_+\times\mathbb{R}_+;A)$, $\int_0^\infty\int_0^\infty S_s^*\pi_\alpha(a(s,t))S_t\,dsdt\in\mathcal{T}_\alpha$.

(b) Let $b:t\mapsto b(t)$ be a continuous function from $\mathbb{R}_+$ to A with compact support; for $s,t\in\mathbb{R}^+$ set $a(s,t)=\alpha^t(b(s-t))$ when $t\le\inf(1,s)$ and $a(s,t)=0$ otherwise. Then

$$\begin{aligned}\int_0^\infty\int_0^\infty S_s\pi_\alpha(a(s,t))S_t^*\,dsdt &= \int_0^1 dt\left(\int_s^\infty S_sS_t^*\pi_\alpha(b(s-t))\,ds\right)\\ &= \int_0^\infty S_s\pi_\alpha(b(s))\,ds.\end{aligned}$$

It follows that $\int_0^\infty S_s\pi_\alpha(b(s))\,ds\in\mathcal{T}_\alpha$. Clearly $\mathcal{T}_\alpha$ is the C^*-subalgebra of $\mathcal{L}(L^2(\mathbb{R}_+)\otimes A)$ generated by these elements.

As in the case of a single endomorphism we have:

Proposition 3.3. *The Toeplitz algebra $\mathcal{T}_\alpha$ contains the ideal of compact operators of $L^2(\mathbb{R}_+)\otimes A$.*

Proof. Set $V=1-2\int_0^\infty e^{-t}S_t\,dt$. It is an isometry and the kernel of V^{*n} is formed by the functions $t\mapsto e^{-t}P$ where P is a polynomial of degree less than n. It follows that $\mathcal{T}_\alpha$ contains the elements $k\pi_\alpha(a)k'$ for every $k,k'\in\mathcal{K}(L^2(\mathbb{R}_+))$ and $a\in A$. Let $k\in\mathcal{K}(L^2(\mathbb{R}_+))$ and $a\in A$; by continuity of the mapping $t\mapsto\alpha^t(a)$ given $\varepsilon>0$ there exists $\eta>0$ such that $\|\alpha^t(a)-a\|<\varepsilon$ whenever $t\le\eta$. But we can choose $x,y\in\mathcal{K}(L^2(\mathbb{R}_+))$ such that $k=xy$ and x has support in $[0,\eta]$. It follows that $\|x\pi_\alpha(a)y-k\otimes a\|<\varepsilon$, whence $k\otimes a\in\mathcal{T}_\alpha$. □

Next we show that $\mathcal{T}_\alpha$ is a full corner of an appropriate crossed product.

Let A_∞ be the C^*-algebra as defined in section 1 corresponding to the endomorphism α^1 of A, and let $h:A\to A_\infty$ be the canonical map. Then α^1 induces an automorphism on A_∞ which we denote by α^1_∞. Since $\alpha^1\circ\alpha^t=\alpha^t\circ\alpha^1$ each α^t induces an endomorphism, α^t_∞ of the algebra A_∞. Hence, by Lemma 3.1 we obtain an action of $\mathbb{R}$ on A_∞ corresponding to the family $(\alpha^t_\infty)_{t\in\mathbb{R}_+}$ which will be denoted by α_∞.

Let $C_b(\mathbb{R},A)$ be the C^*-algebra of bounded functions from $\mathbb{R}$ to A. Let $D\subset C_b(\mathbb{R};A)$ be the subalgebra of elements $a\in C_b(\mathbb{R},A)$ such that $\lim_{t\to-\infty}\|a(t)\|=0$, and for every $\varepsilon>0$, there exists $t\in\mathbb{R}$ such that for every $s>0$, $\|a(s+t)-\alpha^s(a(t))\|\le\varepsilon$. Let $\beta:\mathbb{R}\to\mathrm{Aut}\,(D)$ be defined by $(\beta^tf)(s)=f(s-t)$. Clearly D contains $C_\circ(\mathbb{R},D)$ as an ideal.

Lemma 3.4. *There exists a $*$-isomorphism $\varphi : D/C_0(\mathbb{R}, A) \to A_\infty$ such that for every $a \in D$ we have $\varphi \circ q(a) = \lim_{t\to+\infty} \alpha_\infty^{-t} \circ h(a(t))$, where $q : D \to D/C_0(\mathbb{R}, A)$ is the quotient map.*

Proof. It is easy to see that for every $a \in D$, the function $t \mapsto \alpha_\infty^{-t} \circ h(a(t))$ admits a limit when $t \to +\infty$. It follows that φ is well defined on the quotient. For each $a \in A$ let $\hat{a}(t) = \alpha^t(a)$. Then, $\hat{a} \in D$ and $\lim_{t\to+\infty} \alpha_\infty^{-t} \circ h(\hat{a}(t)) = h(a)$. It follows that φ is surjective.

Moreover, if $a \in \ker \varphi \circ q$, for every ε, there exists $t \in \mathbb{R}$ such that for every $s \in \mathbb{R}_+$, $\|a(s + t) - \alpha^s(a(t))\| \le \varepsilon$. Choose t such that $\|h(a(t))\| = \|\alpha_\infty^{-t} \circ h(a(t)) - \varphi \circ h(a)\| \le \varepsilon$. Therefore $\limsup_{s\to+\infty} \|\alpha^s(a(t))\| \le \varepsilon$, whence $\limsup_{s\to+\infty} \|a(s + t)\| \le 2\varepsilon$. It follows that $\ker \varphi \circ q = C_0(\mathbb{R}, A)$. Hence φ is an isomorphism. ☐

Theorem 3.5. *Let $v_t \in \mathcal{L}(L^2(\mathbb{R}, A))$ be defined by $(v_t\xi)(s) = \xi(s+t)$. Moreover let $\rho : D \to \mathcal{L}(L^2(\mathbb{R}, A))$ be the $*$-representation such that $(\rho(a)\xi)(s) = a_s\xi_s$. The pair (ρ, v) is a covariant representation of (D, β) and the corresponding representation of $D\rtimes_\beta\mathbb{R}$ is faithful. Identify $D\rtimes_\beta\mathbb{R}$ with its image in $\mathcal{L}(L^2(\mathbb{R}, A))$; the projection P of $L^2(\mathbb{R}, A)$onto $L^2(\mathbb{R}_+, A)$ is a multiplier of $D\rtimes_\beta\mathbb{Z}$ and $P(D\rtimes_\beta\mathbb{R})P$ is the Toeplitz algebra $\mathcal{T}_\alpha$; it is a full corner in $D\rtimes_\beta\mathbb{R}$.*

Proof. It is clear that the pair (ρ, v) is a covariant representation of (D, β). The restriction of the corresponding representation of $D\rtimes_\beta\mathbb{R}$ to $C_0(\mathbb{R}, A)\rtimes_\beta\mathbb{R}$ is the canonical isomorphism of $C_0(\mathbb{R}, A)\rtimes_\beta\mathbb{R}$ with the algebra of compact operators in $L^2(\mathbb{R}, A)$. As $C_0(\mathbb{R}, A)$ is an essential ideal in D, $C_0(\mathbb{R}, A)\rtimes_\beta\mathbb{R}$ is an essential ideal in $D\rtimes_\beta\mathbb{R}$ therefore the representation of $D\rtimes_\beta\mathbb{R}$ associated with (ρ, v) is faithful.

Let f be a continuous function on R such that $f(t) = 1$ if $t < 0$ and $f(t) = 0$ if $t > 1$. As f is a multiplier of D and $fD \subset C_0(\mathbb{R}, A)$, f defines a multiplier of $D\rtimes_\beta\mathbb{R}$ and $f\, D\rtimes_\beta\mathbb{R} \subset \mathcal{K}(L^2(\mathbb{R}, A))$. As $(1 - P)$ is a multiplier of $\mathcal{K}(L^2(\mathbb{R}, A))$ and $(1 - P) = (1 - P)f$, it follows that P is a multiplier of $D\rtimes_\beta\mathbb{R}$ and $(1 - P)D\rtimes_\beta\mathbb{R} \subset \mathcal{K}(L^2(\mathbb{R}, A))$. As $\mathcal{K}(L^2(\mathbb{R}_+, A)) \subset P(D\rtimes_\beta\mathbb{R})P$, it follows that $P(D\rtimes_\beta\mathbb{R})P$ is a full corner in $D\rtimes_\beta\mathbb{R}$.

Let $D_0 \subset D$ be the set of $b \in D$ such that for all $u \ge 0$, $b(u) = \alpha^u(b(0))$. Let $(s, t) \mapsto a(s, t)$ be a continuous function from $\mathbb{R}_+ \times \mathbb{R}_+$ to A with compact support. Let $b : \mathbb{R}_+ \times \mathbb{R}_+ \to D_0$ be a function such that for every $s, t \in \mathbb{R}_+$, $b(s, t)(0) = a(s, t)$. Then

$$\int_0^\infty\int_0^\infty v_s^* P b(s, t) v_t \, ds dt \in P(D\rtimes_\beta\mathbb{R})P$$

and acts on $L^2(\mathbb{R}_+, A)$ as

$$\int_0^\infty\int_0^\infty S_s\pi_\alpha(a(s,t))S_t^*\,dsdt.$$

It follows that $P(D\rtimes_\beta\mathbb{R})P$ contains $\mathcal{T}_\alpha$.

Now $D\rtimes_\beta\mathbb{R}$ is generated by integrals over s,t of terms of the form $v_s\beta^t(a(s,t)) = v_{s-t}a(s,t)v_t$, where for $s,t\in\mathbb{R}$, $a(s,t)\in D_0$. Moreover, since $\cup\beta_t(D_\circ)$ is dense in D and $\beta_t(D_0)$ increases with t, we may assume $t>0$ and $t\geq s$. Moreover $\int_0^\infty\int_{-\infty}^t v_{s-t}(1-P)b(s,t)v_t\,dsdt\in\mathcal{K}(L^2(\mathbb{R},A))$ and hence $P\left(\int_0^\infty\int_{-\infty}^t v_{s-t}b(s,t)v_t\,dsdt\right)P$ is the sum of

$$P\left(\int_0^\infty\int_{-\infty}^t v_{s-t}(1-P)b(s,t)v_t\,dsdt\right)P\in\mathcal{K}(L^2(\mathbb{R}_+,A))$$

and

$$\int_0^\infty\int_{-\infty}^t v_{s-t}Pb(s,t)v_t\,dsdt\in\mathcal{T}_\alpha$$

and the result follows. □

Remark. Note that any isomorphism of $L^2(\mathbb{R},A)$ with $L^2(\mathbb{R}_+,A)$ which is the identity on $L^2((k,+\infty);A)$ (for k large enough) obviously induces an isomorphism between $D\rtimes_\beta\mathbb{R}$ and $\mathcal{T}_\alpha$.

Corollary 3.6. *The quotient algebra $\mathcal{T}_\alpha/\mathcal{K}(L^2(\mathbb{R}_+,A))$ is naturally isomorphic with $A_\infty\rtimes_{\alpha\infty}\mathbb{R}$. In other words, there is an exact sequence*

$$0\to\mathcal{K}(L^2(\mathbb{R}_+,A))\to\mathcal{T}_\alpha\to A_\infty\rtimes_{\alpha\infty}\mathbb{R}\to 0.$$

Proof. By Theorem 3.5, since $(1-P)D\rtimes_\beta\mathbb{R}$ is contained in $\mathcal{K}(L^2(\mathbb{R},A))$ it follows that $\mathcal{T}_\alpha+\mathcal{K}(L^2(\mathbb{R},A)) = D\rtimes_\beta\mathbb{R}$. Hence, $\mathcal{T}_\alpha/\mathcal{K}(L^2(\mathbb{R}_+,A))$ is canonically isomorphic to $D\rtimes_\beta\mathbb{R}/\mathcal{K}(L^2(\mathbb{R},A)) = D\rtimes_\beta\mathbb{R}/C_0(\mathbb{R},A)\rtimes_\beta\mathbb{R}$; it is therefore isomorphic to $(D/C_0(\mathbb{R},A))\rtimes_{\alpha_\infty}\mathbb{R}$, i.e., to $A_\infty\rtimes_{\alpha\infty}\mathbb{R}$ (see Lemma 3.4). □

Let us now come to K-theoretic considerations.

Theorem 3.7. *The morphism $h : A\to A_\infty$ is an isomorphism in E-theory. The C^*-algebras D and $\mathcal{T}_\alpha$ are contractible in E-theory, i.e., for any C^*-algebra B the groups $E(\mathcal{T}_\alpha,B)$, $E(D,B), E(B,\mathcal{T}_\alpha)$ and $E(B,D)$ are trivial.*

Proof. Set $D_+ = D/C_0((-\infty,0),A)$. The exact sequence $0\to C_0(\mathbb{R}_+,A)\to D_+\to A_\infty\to 0$ is an asymptotic morphism φ from A_∞ to A.

Note that for every C^*-algebra B, the identity element of the ring $E(B,B)$ is given by the asymptotic morphism associated with the exact sequence $0 \to C_0(\mathbb{R}_+, B) \to C(\mathbb{R}_+ \cup \{+\infty\}, B) \to B \to 0$.

We have a commuting diagram

$$\begin{array}{ccccccccc}
0 \to & C_0(\mathbb{R}_+, A) & \to & C(\mathbb{R}_+ \cup \{+\infty\}, A) & \to & A & \to 0 \\
& {\scriptstyle\beta}\downarrow & & \downarrow & & {\scriptstyle h}\downarrow & \\
0 \to & C_0(\mathbb{R}_+, A) & \to & D_+ & \to & A_\infty & \to 0 \\
& {\scriptstyle h'}\downarrow & & \downarrow & & {\scriptstyle \mathrm{id}}\downarrow & \\
0 \to & C_0(\mathbb{R}_+, A_\infty) & \to & C(\mathbb{R}_+ \cup \{+\infty\}, A_\infty) & \to & A_\infty & \to 0
\end{array}$$

where $\beta : C_0(\mathbb{R}_+, A) \to C_0(\mathbb{R}_+, A)$ is given by $(\beta(f))(t) = \alpha_t(f(t))$ for every continuous function $f : \mathbb{R}_+ \to A$ and $h' : C_0(\mathbb{R}_+, A) \to C_0(\mathbb{R}_+, A_\infty)$ is given by $(h'(f))(t) = h_t(f(t))$ for every continuous function $f : \mathbb{R}_+ \to A$ (recall that $h_t = \alpha_\infty^{-t} \circ h$). As β is homotopic to the identity among $C_0(\mathbb{R}_+)$-linear endomorphisms of $C_0(\mathbb{R}_+, A)$, the compositions $h^*(\varphi)$ defines the identity element of $E(A, A)$; as h' is homotopic to the map $f \mapsto h \circ f$ among $C_0(\mathbb{R}_+)$-linear homomorphisms of $C_0(\mathbb{R}_+, A)$ into $C_0(\mathbb{R}_+, A_\infty)$, $h_*(\varphi)$ defines the identity element of $E(A_\infty, A_\infty)$.

It follows from the six term exact sequence of E-theory that D is E-contractible. By Connes' analogue of the Thom isomorphism it follows that $D \rtimes \mathbb{R}$ is E-contractible and by Theorem 3.5, $\mathcal{T}_\alpha$ is also E-contractible. □

References

[1] A. Connes, *An Analogue of the Thom Isomorphism for crossed product by an action of* $\mathbb{R}$, Advances in Math., **39** (1981), 31-55.

[2] A. Connes and N. Higgson, *Deformation, morphismes asymptotiques et K-Théorie bivariante*, C.R. Acad. Sci. Paris, **311(I)** (1990), 101-106.

[3] Duncan and Barnes, *C^*-Algebras of Certain Inverse Semigroups.*

[4] R. Exel, *Circle Actions on C^*-algebras, Partial Automorphism and a Generalized Pimsner-Voiculescu Exact Sequence*, to appear.

[5] T. Fack and G. Skandalis, *Connes' Analogue of the Thom Isomorphism for the Kasparov Groups*, Invert. Math., **64** (1981), 7-14.

[6] G.G. Kasparov, *The operator K-functor and extensions of C^*-algebras*, Math. U.S.S.R. Izv, **16** (1981), 513-572.

[7] ———, *Hilbert C^*-modules, Theorem of Steinspring and Voiculescu*, J. Operator Theory, **4** (1980), 133-150.

[8] G. Peterson, *C^*-Algebras and their Automorphism Groups*, Academic Press, London, 1979.

[9] M. Pimsner, *A class of C^*-algebras generalizing both Cuntz-Krieger algebras and crossed products by $\mathbb{Z}$*, Fields Inst. Comm., **12**, Amer. Math. Soc., Providence, RI, 1997.

[10] M. Pimsner and D. Voiculescu, *Exact Sequences for K-groups and Ext-groups of certain crossed product C^*-Algebras*, J. Operator Theory, **4** (1980), 93-118.

[11] G. Skandalis, *Une notion de nucléarite en K-Théorie*, K-Theory, (1988), 549-574.

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University of Saskatchewan
Saskatoon SK S7N 5E6, Canada
and
equipe d'Alain Connes
College de France
75005 Paris, France