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UNIVERSAL LINKS FOR $S^2 \widetilde{\times} S^1$

Víctor Núñez

There exists a five component link $U \subset S^2 \times S^1$ such that every closed, connected, orientable 3-manifold M with $H^1(M) \neq 0$ is a branched covering over $S^2 \times S^1$ with branching set exactly the link U.

There exists a five component link $U \subset S^2 \otimes S^1$ such that every closed, connected, non-orientable 3-manifold M with the Bockstein of the first Stiefel-Whitney class, $\beta w_1(M) = 0$, is a branched covering over $S^2 \otimes S^1$ branched along the link U.

1. Introduction.

Refining a well known result, Hilden and Montesinos ([**2**, **3**]) proved that every orientable 3-manifold is a 3-fold branched covering of the 3-sphere S^3. With this result as a starting point, Thurston ([**5**]) proved the striking fact that there exists a link $U \subset S^3$ such that every orientable 3-manifold is a branched covering of S^3 with branching set exactly the link U. Thurston called *universal* a link with this property.

Berstein and Edmonds proved ([**1**]), among other things, the following characterization theorem

i) An orientable 3-manifold M is a 3-fold branched covering of $S^2 \times S^1$ if and only if $H^1(M) \neq 0$.

ii) A non-orientable 3-manifold M is a 3-fold branched covering of $S^2 \otimes S^1$ if and only if the Bockstein of the first Stiefel-Whitney class, $\beta w_1(M) = 0$.

We call a link $U \subset S^2 \times S^1$ a *universal link for* $S^2 \times S^1$ if every orientable 3-manifold M with $H^1(M) \neq 0$ is a branched covering of $S^2 \times S^1$ with branching set the link U. We call also a link $U \subset S^2 \otimes S^1$ a *universal link for* $S^2 \otimes S^1$ if every non-orientable manifold M with $\beta w_1(M) = 0$ is a branched covering of $S^2 \otimes S^1$ branched along U.

In this work we show the existence of a universal link for $S^2 \times S^1$, and a universal link for $S^2 \otimes S^1$.

The most interesting result of Berstein and Edmonds is that every non-orientable manifold is a branched covering of $P^2 \times S^1$ of at most six sheets. It would be very interesting, as González-Acuña asked, to decide if there exists a universal link for $P^2 \times S^1$.

I thank Fico González-Acuña for useful conversations.

2. Existence of branched coverings.

The symbol $S^2\tilde{\times}S^1$ denotes either the product $S^2\times S^1$, or the non-orientable bundle of 2-spheres over S^1, $S^2\otimes S^1$. A convenient way of regarding $S^2\tilde{\times}S^1$ is as a quotient of $S^2\times I$. Namely, we identify $S^2\times\{0\}$ with $S^2\times\{1\}$ by the identity to obtain $S^2\times S^1$, and we identify $S^2\times\{0\}$ with $S^2\times\{1\}$ by a reflection along the plane of the paper to obtain $S^2\otimes S^1$ (see Figure 1).

An open proper map $\varphi: M^3\to N^3$ is called a *branched covering* if there exists a link $L\subset N^3$ such that the restriction $\varphi| : M^3-\varphi^{-1}(L)\to N-L$ is a finite covering space; this covering space $\varphi|$ is called the *associated covering space of* φ. The map φ is called a *d-fold branched covering* if the associated covering space is d-fold. We call φ a *simple* branched covering if the preimage of each point under φ has at least $d-1$ points, where φ is d-fold.

Let $\beta: H^1(M;Z_2)\to H^2(M;Z)$ be the coboundary Bockstein homomorphism associated to the exact sequence of coefficients $0\to Z\to Z\to Z_2\to 0$.

A link $U\subset S^2\times S^1$ is called a *universal link for* $S^2\times S^1$ if every orientable 3-manifold M with $H^1(M)\neq 0$ is a branched covering of $S^2\times S^1$ with branching set the link U. A link $U\subset S^2\otimes S^1$ is called a *universal link for* $S^2\otimes S^1$ if every non-orientable manifold M with $\beta w_1(M)=0$ is a branched covering of $S^2\otimes S^1$ branched along U. By a link in $S^2\times I$ we understand a collection of disjoint properly embedded arcs and circles. We call a link $U\subset S^2\times I$ a *universal link for* $S^2\tilde{\times}S^1$ if, after closing $S^2\times I$ obtaining $S^2\times S^1$, the resulting link $\overline{U}$ is a universal link for $S^2\times S^1$, and, after closing $S^2\times I$ obtaining $S^2\otimes S^1$, the resulting link $\overline{U}$ is a universal link for $S^2\otimes S^1$

We state our main theorem, but the proof is postponed to Section 4.

Theorem 2.1. *The link in Figure 1 is a universal link for $S^2\tilde{\times}S^1$.*

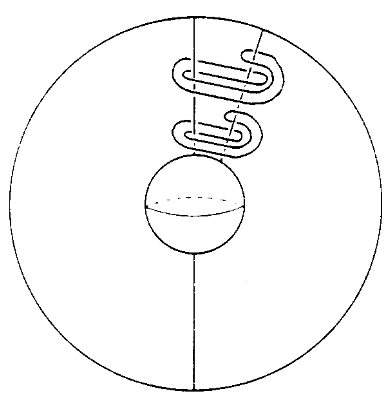

Figure 1.

In this section we review a theorem of Berstein and Edmonds ([1]) on the existence of branched coverings over $S^2\widetilde{\times}S^1$. Our purpose is to find a 'normal' explicit form of the branching set.

Let M be a non-orientable 3-manifold, and let $F \subset M$ be an orientable (connected) surface. Then F is called a *Stiefel-Whitney surface* for M if $M - F$ is orientable.

Remark. If F is a Stiefel-Whitney surface for M, then F represents the Poincaré dual of the first Stiefel-Whitney class, $PDw_1(M) \in H_2(M; Z_2)$.

Lemma 2.1 ([1])**.** *Let M be a non-orientable 3-manifold. Then $\beta w_1(M) = 0$ if and only if there exists $F \subset M$ a two-sided Stiefel-Whitney surface for M.*

We recall some definitions. If c is an oriented simple closed curve in an oriented 2-manifold X^2, then the *Dehn homeomorphism* (*or Dehn twist*) along c is the homeomorphism $t(c) : X^2 \to X^2$ defined as follows: Choose a regular neighbourhood of $A \subset X^2$ of c with a fixed orientation preserving embedding $[-1,1] \times S^1 \cong A$ such that c corresponds to $\{0\} \times S^1$, and the orientation of c agrees with the standard orientation of S^1. Then $t(c)$ is defined on A by $(r, e^{i\theta}) \mapsto (r, e^{i(\theta + r/2 + 1/2)})$, and by the identity outside A. If a is an arc in $intX^2$, then the *arc homeomorphism* (*or disc twist*) along a is the homeomorphism $t(a) : X^2 \to X^2$ defined as follows: Choose a regular neighbourhood $D \subset X^2$ of a, then $t(a)$ is the identity outside D, and define $t(a)$ on D so that a is mapped onto itself with the ends reversed and so that a foliation of $D - a$ by circles is preserved and each circle is mapped to itself by a rotation through an angle which varies from $0°$ on ∂D to $180°$ on circles close to a.

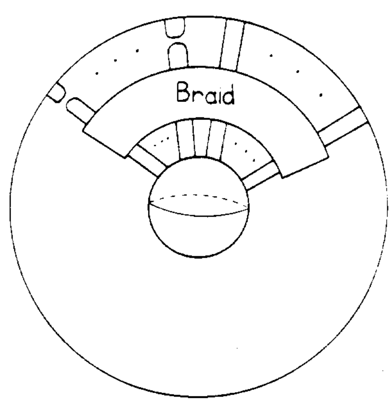

Figure 2.

Lemma 2.2. *Let M be a closed, connected 3-manifold. If M is orientable, suppose $H^1(M;Z) \neq 0$. If M is non-orientable, suppose $\beta w_1(M) = 0$. Then there exists a 3-fold simple branched covering $\varphi : M \to S^2 \widetilde{\times} S^1$ with branching set a link as depicted in Figure 2.*

Remark. In Lemma 2.2 the existence of a three-sheeted branched covering of M over $S^2 \widetilde{\times} S^1$ is a necessary and sufficient condition for $H^1(M;Z) \neq 0$, in case M is orientable, and for $\beta w_1(M) = 0$, in case M is non-orientable (see [**1**]).

Proof of Lemma 2.2. We will follow closely the proof of Berstein and Edmonds. Let M be a closed connected 3-manifold. If M is orientable, choose $F \subset M$ a surface representing the Poincaré dual of a primitive class $v \in H^1(M;Z)$. If M is non-orientable, choose $F \subset M$ a two-sided Stiefel-Whitney surface for M. Let W denote M cut open along F. Then W is a connected, orientable 3-manifold, and ∂W is the disjoint union of two copies, F_0 and F_1, of F. We can recover M from W by identifying F_0 and F_1 with a homeomorphism $h : F_0 \to F_1$. Choosing a handle decomposition for W having only 1-handles and 2-handles, we can write

$$W = V_0^3 \cup (G^2 \times I) \cup V_1^3,$$

where V_i^3 is homeomorphic to the union of $F_i \times I$ plus 1-handles joined along $F_i \times \{1\}$, and $\partial V_i^3 = F_i \cup G^2$, $i = 0, 1$.

We let $\psi : F \to S^2$ be a fixed 3-fold simple branched covering. This induces a (product) 3-fold simple branched covering $\psi_i : F_i \times I \to S^2 \times I$ ($i = 0, 1$) with branching set the disjoint union of $2g(F) + 2$ properly embedded arcs connecting $S^2 \times \{0\}$ with $S^2 \times \{1\}$.

Let $\eta_1 : D^3 \to D^3$ be the 2-fold branched covering with branching set a properly embedded arc in the 3-ball D^3, and let $\eta_2 : D^3 \to D^3$ be a homeomorphism. Then $\eta = \eta_1 \cup \eta_2 : D^3 \cup D^3 \to D^3$ is a 3-fold simple branched covering. Choose $w \subset \partial D^3$ a 2-disk that misses the branching set of η, and choose $w' \subset S^2 \times \{1\}$ a 2-disk that misses the branching set of ψ_i. Glueing $F_i \times I$ with $D^3 \cup D^3$ along the preimages $\eta^{-1}(w)$ and $\psi_i^{-1}(w')$, we obtain the fiber boundary connected sum of ψ_i and η

$$\psi_i \# \eta : (F_i \times I) \cup (D^3 \cup D^3) \to (S^2 \times I) \cup D^3.$$

Of course $(F_i \times I) \cup (D^3 \cup D^3)$ is homeomorphic to $F_i \times I$ plus one 1-handle, and $(S^2 \times I) \cup D^3 \cong S^2 \times I$.

Notice that the branching set of $\psi_i \# \eta$ is the union of the branching set of ψ_i with a separated little arc connecting $S^2 \times \{1\}$. In this way, adding 1-handles by 'fiber boundary connected sums', we can construct a 3-fold simple branched covering $\varphi_i : V_i \to S^2 \times I$ with branching set the union of $2g(F_i) + 2$ long arcs, which join both boundary components of $S^2 \times I$, and the union of several little arcs, one for each 1-handle of V_i, which join $S^2 \times \{1\}$ with itself. See Figure 3.

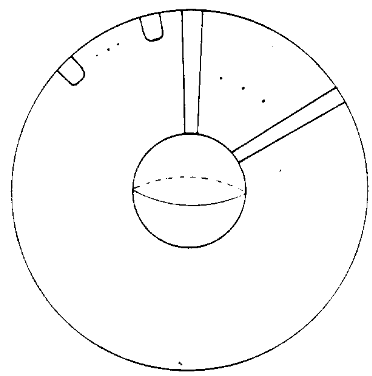

Figure 3.

By construction $\varphi_i|F_i \times \{0\}$ coincides with the covering $\psi_i : F \to S^2$. And $\varphi_0|G \times \{0\}$ is equivalent to $\varphi_1|G \times \{1\}$ ([**1**, Theorem 3.4]). Now if M is orientable, the glueing homeomorphism $h : F_0 \to F_1$ is isotopic to a lifting of a product of arc homeomorphisms of S^2, which permute the branching set of ψ_i; and, in case M is non-orientable, h is isotopic to a lifting of a product of arc homeomorphisms of S^2 as before but followed by a reflection of S^2 along a circle which contains the branching set of ψ_i ([**1**, Theorem 4.1]). One can visualize this last reflection as the result of reflecting Figure 3 along the plane of the paper. Extending these arc homeomorphisms to $S^2 \times I$ we produce a braid in the long arcs of the branching set of φ_0. Thus by glueing V_0 with V_1 along F, we obtain a branched covering $\hat{\varphi} : V_0 \cup_F V_1 \to S^2 \times I \cup_{S^2} S^2 \times I$ with branching set as in Figure 4.

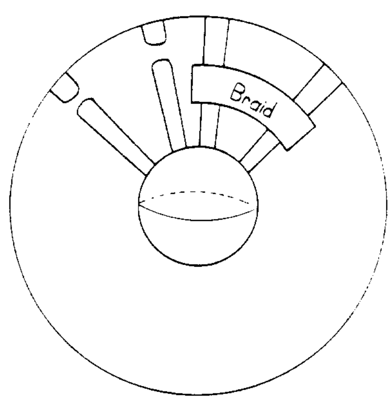

Figure 4.

In a similar way we glue $G \times \{0\}$ and $G \times \{1\}$ with a lifting of a product of arc homeomorphisms of S^2. And we obtain a 3-fold simple branched covering $\varphi : M \to S^2 \widetilde{\times} S^1$ with branching set as shown in Figure 5. This has the required form of Figure 2. $\square$

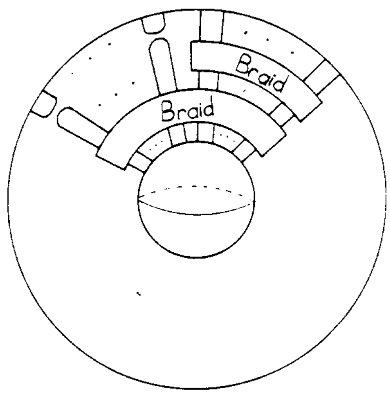

Figure 5.

3. The moves.

Our goal is to find a link $L \subset S^2 \widetilde{\times} S^1$ such that any closed 3-manifold M is a branched covering of $S^2 \widetilde{\times} S^1$ with branching set precisely the link L, where, if M is orientable, it holds $H^1(M; Z) \neq 0$, and if M is non-orientable, it holds $\beta w_1(M) = 0$.

The plan to follow is:

Start with a simple three-sheeted branched covering $\varphi : M \to S^2 \widetilde{\times} S^1$ as in Lemma 2.2. We work with the restriction $\varphi : \widehat{M} \to S^2 \times I$, where $S^2 \times I$ is the result of cutting $S^2 \widetilde{\times} S^1$ along a 2-sphere $S^2 \subset S^2 \widetilde{\times} S^1$, and $\widehat{M}$ denotes M cutted open along $\varphi^{-1}(S^2)$. We will fix this notation throughout this section.

We perform a number of moves in $S^2 \times I$ and in $\widehat{M}$ which, in the interesting case, are surgeries; but we do not change the homeomorphism type of $S^2 \times I$ nor of $\widehat{M}$. Each time we do a move, we construct a new simple 3-fold branched covering $\varphi' : \widehat{M} \to S^2 \times I$ such that we can still recover a branched covering $M \to S^2 \widetilde{\times} S^1$; this is possible because the moves are made far away from the boundaries, and the surgeries are 'equivariant' with respect to φ, that is, we construct the glueing maps for each surgery in such a way that some branched coverings, previously defined in each involved piece, are 'glued' to give rise to a global branched covering. Eventually we will obtain that the branching set of φ' is a link $L' \subset S^2 \times I$, which is very symmetrical.

Then we find a sequence of coverings $S^2\widetilde{\times}S^1 \to S^2\widetilde{\times}S^1$ whose composition is a branched covering and such that the preimage of the link of Figure 1 above, under this composition, is exactly the link L'.

Finally, composing with the covering $M \to S^2\widetilde{\times}S^1$, we obtain that M is a branched covering of $S^2\widetilde{\times}S^1$ with branching set the link U.

Move 0-*th.* Add 'unbranched' components to the branching set. A branched covering $\psi : X \to Y$ is, by definition, a covering space outside a link $L \subset Y$. If we choose $K \subset Y$ any link disjoint from L, then the same map $\psi : X \to Y$ gives us a covering space $\psi| : X - \psi^{-1}(L \cup K) \to Y - (L \cup K)$; of course K is fake branching, and this move will serve only to emphasize some symmetries in certain pictures below.

First move. The branched covering $\varphi : \widehat{M} \to S^2 \times I$ is determined by a representation $\rho : \pi_1(S^2 \times I - L) \to \Sigma_3$, where Σ_3 is the symmetric group in three symbols. The meridians of L are sent to transpositions under ρ, for φ is simple. We picture this representation by writing a permutation (ab) near to each bridge of the projection of L, and we will call (ab) the *colour* of the corresponding meridian.

At each crossing of L the meridians satisfy a relation of type $z = yzx^{-1}$ in the group $\pi_1(S^2 \times I - L)$. Since ρ is a homomorphism, it must send x, y and z to, either the same transposition, or to three distinct transpositions.

One can allways guarantee the second case by using the method of 'infiltrating strings' of Hilden, Lozano and Montesinos ([3]), which is an isotopy, as shown in Figure 6; and which is our first basic move.

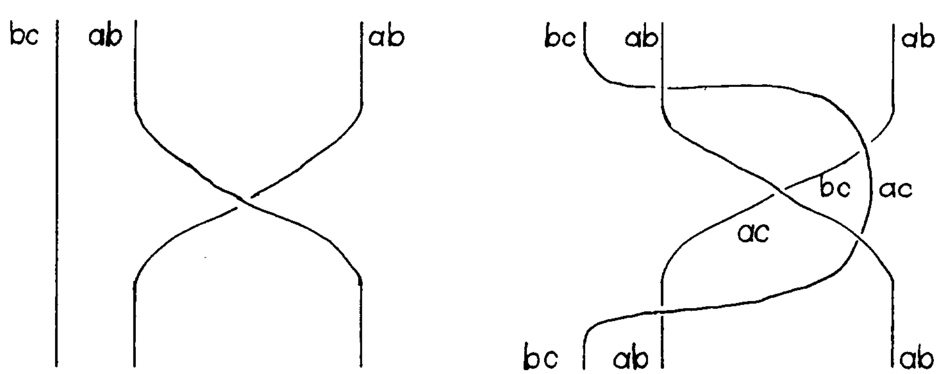

Figure 6.

Second move. We need a lemma from Berstein and Edmonds.

Lemma 3.1 ([**1**, Lemma 4.3])**.** *Let $\psi : D^2 \to D^2$ be a 3-fold simple branched covering with branching set $\{y_1, y_2\} \subset \operatorname{int} D^2$, and let t be a disk twist about a simple arc $a \subset \operatorname{int} D^2$ connecting y_1 and y_2. Then t^3 lifts to a disk twist about the arc $\psi^{-1}(a)$.*

If we choose a ball $B^3 \subset S^2 \times I$ such that $B^3 \cap L$ is a pair of properly embedded arcs in B^3 with different colour, then $\varphi^{-1}(B^3) = \widetilde{B}$ is a 3-ball. Regarding B^3 as a product $D^2 \times I$, then $\varphi| = \psi \times 1 : \widetilde{B} \to B^3$, where ψ is the covering of Lemma 3.1.

Let $s : \partial(D^2 \times I) \to \partial(D^2 \times I)$ be the homeomorphism defined by $t^3 \times \{0\}$ on the bottom $D^2 \times \{0\}$, and by the identity on $\partial(D^2 \times I) - (D^2 \times \{0\})$. Then by Lemma 3.1, we can lift $s : \partial B^3 \to \partial B^3$,under φ, to a homeomorphism $\widetilde{s} : \partial\widetilde{B} \to \partial\widetilde{B}$ such that, again by identifying $\widetilde{B}$ with a product $D^2 \times I$, $\widetilde{s}$ coincides with $t \times \{0\}$ on the bottom $D^2 \times \{0\}$, and $\widetilde{s}$ is the identity on $\partial(D^2 \times I) - (D^2 \times \{0\})$.

Thus if we perform surgery in $S^2 \times I$ along B^3 with the homeomorphism $s : B^3 \to B^3$, and we perform surgery in $\widehat{M}$ along $\widetilde{B}$ with the lifting $\widetilde{s} : \partial\widetilde{B} \to \partial\widetilde{B}$, we obtain a new 3-fold simple branched covering $\widehat{M} \to S^2 \times I$ with branching set the link $L - B^3$ union the picture modified inside B^3 as depicted in Figure 7; which is our second basic move.

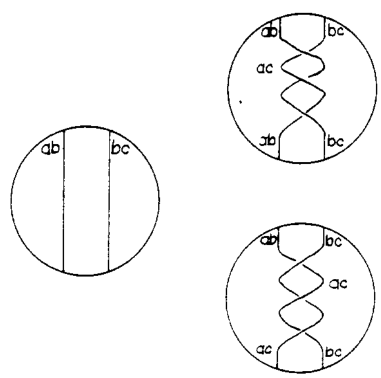

Figure 7.

4. Universal links.

We state again our main theorem.

Theorem 4.1. *Let M be a connected, closed 3-manifold. If M is orientable, suppose $H^1(M; Z) \neq 0$. If M is non-orientable, suppose $\beta w_1(M) = 0$. Then* *there exists a branched covering $M \to S^2\widetilde{\times}S^1$ with branching set the link of Figure 1.*

Proof. Start with a link in $S^2\widetilde{\times}S^1$ such as the one of Figure 2, which is the branching set of a 3-fold simple branched covering $M \to S^2\widetilde{\times}S^1$.

Outside the braid part of Figure 2, we distiguish the 'vertical pieces', which are the arcs joining the sphere $S^2 \times \{1\}$ with the braid part, and the 'clasp pieces', each consisting of two little arcs, one with ends in $S^2 \times \{1\}$, and the other one with ends in the braid part.

We begin with a clasp piece. If the meridians of the arcs of this clasp piece have the same colour, choose a vertical piece with the corresponding meridian coloured with a different colour. And with an isotopy of $S^2 \times I$, obtain the Figure 8.

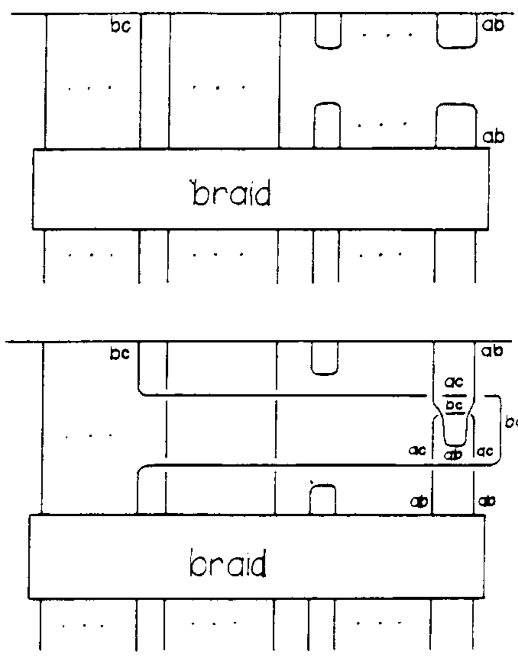

Figure 8.

Applying the basic moves we modify the clasp to obtain two vertical pieces with a little circle around them, as shown in Figure 9 at the bottom: Starting at the top of Figure 9, in the ball suggested by a thin circle, apply the second move; twisting thrice, we end with two crossings as in the middle of Figure 9. Apply the second move again in the middle of Figure 9 inside the ball suggested by the thin circle, and end in the bottom of Figure 9.

With another isotopy, we pull the loop at the bottom of Figure 9 around the new two vertical pieces to the braid part. If the meridians of the arcs of this clasp piece originally had different colour, we don't need to interlace anything, but we just apply the basic moves as in Figure 9. In this way, we obtain the link of Figure 10, which is the branching set of a modified 3-fold simple branched covering $M \to S^2 \widetilde{\times} S^1$.

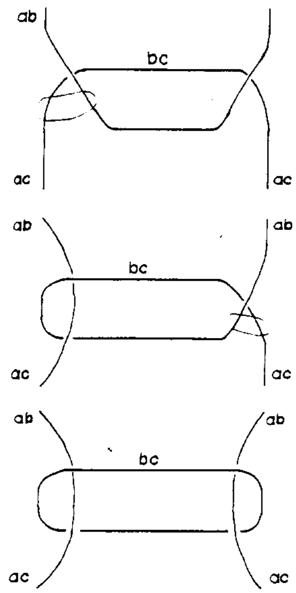

Figure 9.

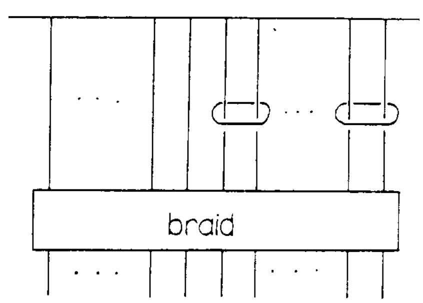

Figure 10.

In this link, we add 'unbranched' circles by move 0th to complete a 'chain' like Figure 11. We call this picture a *Chain of type* I. Notice that this adding of extra 'unbranched' circles, which is Move 0th, corresponds to the assignment of the identity permutation of Σ_3 to each meridian of each new circle. Thus we still have a 3-fold simple branched covering $M \to S^2\widetilde{\times}S^1$.

With the trick of infiltrating arcs, which is the first move, we can guarantee that all crossings in the braid part have exactly three colours. Using the second basic move, if needed, we modify the link in such a way that the braid part is a positive braid (that is, one in which at each crossing the overpass goes from SW to NE). See Figure 12; we are adding three (opposite) crossings to the picture on the left to end with two positive crossings as in the picture on the right.

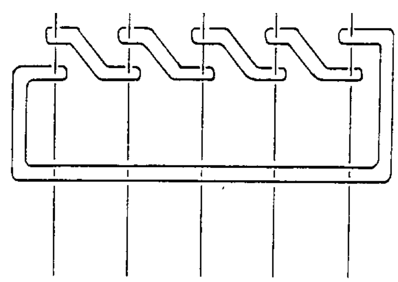

Figure 11.

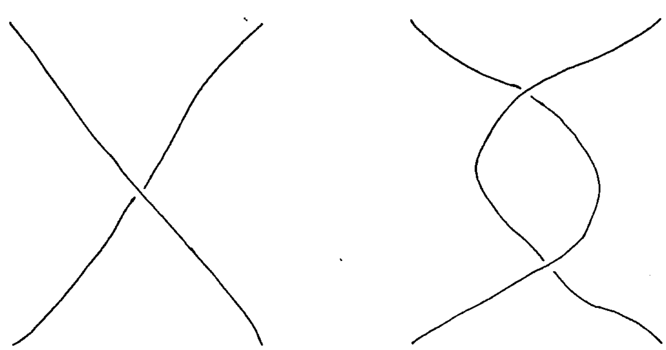

Figure 12.

The braid part of the branching set is now a positive braid. At each crossing of this braid, we modify again the link by succesive applications of the second move, as shown in Figure 13. This modification is another trick of Hilden, Lozano and Montesinos ([**3**]).

Thus the braid part is transformed into vertical pieces with 'horizontal' little circles around them. For each little circle we add extra 'unbranched' circles, using move 0th, to complete a chain like Figure 14. We call this picture a *chain of type* II.

The situation now is as follows: we have a 3-fold simple branched covering $M \to S^2 \widetilde{\times} S^1$ whose branching set consists of vertical pieces only (no clasps nor crossings among these vertical pieces), plus a number of little circles interlaced with the vertical pieces, and which form one chain of type I, and many chains of type II. We add more 'unbranched' circles interlaced with the vertical pieces to form new chains of type I; we place each new one 'unbranched' chain of type I in between each pair of adjacent chains of type II. We obtain then a link like Figure 15, in which, if we traverse a vertical piece, we find chains of little circles by pairs, first one chain of type I and then one chain of type II.

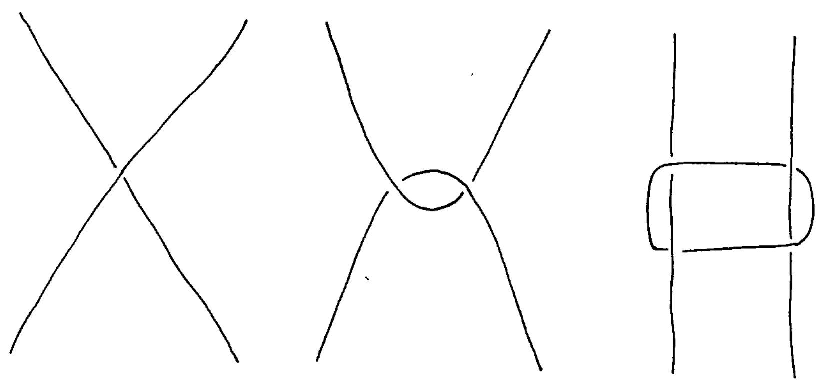

Figure 13.

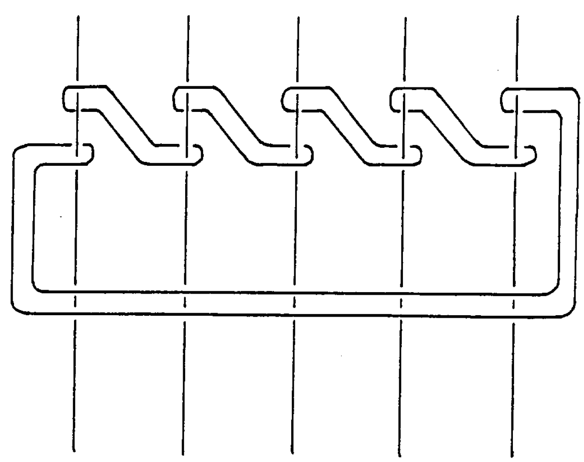

Figure 14.

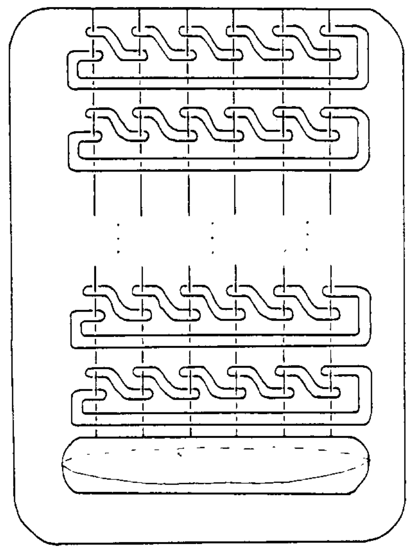

Figure 15.

Figure 15 is a n-fold covering space of Figure 16. That is, there is a covering space $S^2\widetilde{\times}S^1 \to S^2\widetilde{\times}S^1$ of n sheets, one sheet for each pair of chains, such that the preimage of Figure 16 is exactly Figure 15. Adding extra 'unbranched' components, if necessary, we guarantee that n is odd in case M is non-orientable.

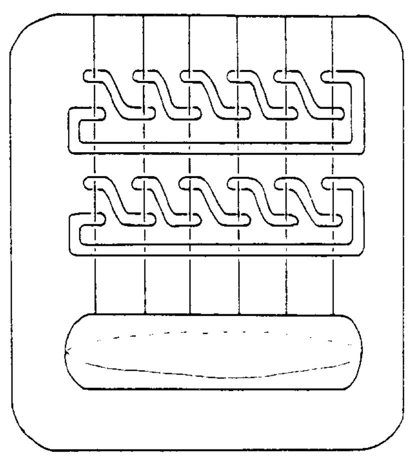

Figure 16.

A better picture for the link in Figure 16 is the link of Figure 17. Clearly the link of Figure 17 is contained in the preimage of the link of Figure 1

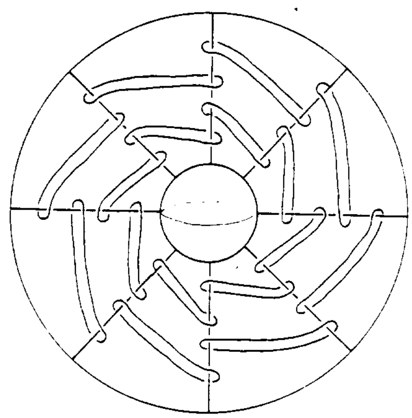

Figure 17.

under a m-fold cyclic branched covering of $S^2\tilde{\times}S^1$ (m is the number of vertical pieces) branched along two extra vertical pieces. These new vertical pieces correspond to the axis of rotational simmetry of Figure 17. We can conclude that the link of Figure 1 is a universal link. $\square$

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CIMAT
A.P. 402
Guanajuato 36000, Gto.
MÉXICO
E-mail address: victor@fractal.cimat.mx

Note: Theorem 4.1 appeared in the paper version as Theorem 5.1.