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MÖBIUS INVARIANT Q_p SPACES ASSOCIATED WITH THE GREEN'S FUNCTION ON THE UNIT BALL OF $\mathbf{C}^n$

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In this paper, function spaces $Q_p(B)$ and $Q_{p,0}(B)$, associated with the Green's function, are defined and studied for the unit ball B of $\mathbf{C}^n$. We prove that $Q_p(B)$ and $Q_{p,0}(B)$ are Möbius invariant Banach spaces and that $Q_p(B) = \mathrm{Bloch}(B), Q_{p,0}(B) = \mathcal{B}_0(B)$ (the little Bloch space) when $1 < p < n/(n-1), Q_1 = \mathrm{BMOA}(\partial B)$ and $Q_{1,0}(B) = \mathrm{VMOA}(\partial B)$. This fact makes it possible for us to deal with BMOA and Bloch space in the same way. And we give necessary and sufficient conditions on boundedness (and compactness) of the Hankel operator with antiholomorphic symbols relative to $Q_p(B)$ (and $Q_{p,0}(B)$). Moreover, other properties about the above spaces and $|\varphi_z(w)|, \varphi_z(w) \in \mathrm{Aut}(B)$, are obtained.

1. Introduction.

As well known, there are several equivalent statements for analytic functions of bounded mean oscillation (BMOA). On the unit disc D of $\mathbf{C}$, the following condition associated with the Green's function $g_D(z,a)$

$$f \in \mathrm{BMOA}(D) \Leftrightarrow \sup_{a\in D} \int_D |f'(z)|^2 g_D(z,a)\,dxdy < \infty, \tag{1.1}$$

yielded by the Littlewood-Paley identity, is an important one of those equivalences, since not only the characterization of Carleson measure for BMOA and the Fefferman's duality theorem were obtained by it [8], but also the version of (1.1) on the Riemann surface R with the Green's function G_R

$$\sup_{\alpha\in R} \int_R |F'(w)|^2 G_R(w,\alpha)\,dw d\bar{w} < \infty$$

is usually regarded as the definition of BMOA on R for convenience [11]. In [1] and [2], the spaces of analytic functions on $D, 0 < p < \infty$,

$$Q_p(D) = \left\{ f \in A(D) : \sup_{a\in D} \int_D |f'(z)|^2 g_D^p(z,a)\,dxdy < \infty \right\}$$

and

$$Q_{p,0}(D) = \left\{ f \in A(D) : \lim_{|a|\to 1} \int_D |f'(z)|^2 g_D^p(z,a)dxdy = 0 \right\}$$

were introduced which are from the variants of condition (1.1) on the degree of g_D. The main results are that for $1 < p < \infty, Q_p(D) =$ the Bloch space, $Q_{p,0}(D)$ =the little Bloch space, for $p = 1, Q_1(D) = \mathrm{BMOA}(D), Q_{1,0}(D) = \mathrm{VMOA}(D)$ and as $0 < p < \infty, \mathrm{AD} \subset Q_{p,0}(D) \subset Q_p(D)$ where AD is the Dirichlet space on D. This shows that the above $Q_p(D)$ and $Q_{p,0}(D)$ are both nontrivial and significant.

For the unit ball B of $\mathbf{C}^n$, we have [12]

$$f \in \mathrm{BMOA}(\partial B) \Leftrightarrow \sup_{a\in B} \int_B |\tilde{\nabla} f(z)|^2 |f(z) - f(a)|^{p-2} G(z,a) d\lambda(z) < \infty,$$

$0 < p < \infty$. Taking $p = 2$ especially then

$$f \in \mathrm{BMOA}(\partial B) \Leftrightarrow \sup_{a\in B} \int_B |\tilde{\nabla} f(z)|^2 G(z,a) d\lambda(z) < \infty, \tag{1.2}$$

where $\tilde{\nabla}, G$ and $d\lambda$ denote the invariant gradient, the invariant Green's function and the invariant volume measure, respectively [Section 2]. So it is natural to ask what is the class Q_p of holomorphic functions satisfying the condition

$$\sup_{a\in B} \int_B |\tilde{\nabla} f(z)|^2 G^p(z,a) d\lambda(z) < \infty, \quad 0 < p < \infty.$$

The main purpose of this paper is on it.

The paper is organized as follows: In Section 2, we explain some notations, concepts and the results used in what follows which can be found in [9], [12], [13], [14] and [17]. In Section 3, we prove that $Q_p(B)$ are Möbius invariant Banach spaces (Theorem 3.3), and as $1 < p < \frac{n}{n-1}, Q_p(B) = \mathrm{Bloch}(B), Q_1(B) = \mathrm{BMOA}(\partial B)$, as $0 < p \le \frac{n-1}{n}$ or $p \ge \frac{n}{n-1}, Q_p(B)$ are trivial (Theorem 3.8). Section 4 contains the results about $Q_{p,0}, \mathcal{B}_0$ and VMOA corresponding to Section 3. In Appendix, we give an elementary proof of a conclusion used in Lemma 4.2 that $\rho(z,w) = |\varphi_z(w)|$ is a Möbius invariant metric in the unit ball. Although $|\varphi_z(w)|$ is widely applied, but the invariance and the triangle inequality about it have not been shown elsewhere. And they seem difficult to be proved by means of a direct extension from the case of one complex variable in [8].

2. Preliminaries.

Let B denote the unit ball in $\mathbf{C}^n$ ($n \geq 2$ throughout this paper), and for $a \in B, \varphi_a(z)$ is the Möbius transformation of B which satisfying $\varphi_a(0) = a, \varphi_a(a) = 0$ and $\varphi_a = \varphi_a^{-1}$. $\varphi_a \in \mathrm{Aut}(B), \mathrm{Aut}(B)$ is the group of biholomorphic automorphisms of B, cf. [13].

$H(B)$ denotes the collection of all holomorphic functions in B. Let $\nabla f = (\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n})$ denote the complex gradient of $f, \mathcal{R}f = \sum_{j=1}^{n} z_j(\frac{\partial f}{\partial z_j})$ denote the radial derivative of f. Let $d\lambda(z) = \frac{dv(z)}{(1-|z|^2)^{n+1}}$ where dv is the normalized volume measure in $\mathbf{C}^n$, then $d\lambda$ is $\mathcal{M}$-invariant [13], which means,

$$\int_B f(z)d\lambda(z) = \int_B f \circ \psi(z)d\lambda(z)$$

for each $f \in L^1(\lambda)$ and $\psi \in \mathrm{Aut}(B)$.

Let $\tilde{\triangle} f(z) = \triangle(f \circ \varphi_z)(0)$ denote the invariant Laplacian of f [13], and $\tilde{\nabla} f(z) = \nabla(f \circ \varphi_z)(0)$ denote the invariant gradient of f [14]. By a direct computation, we get for $f \in H(B)$,

$$|\tilde{\nabla} f(z)|^2 = (1 - |z|^2)(|\nabla f(z)|^2 - |\mathcal{R}f(z)|^2) = \frac{1}{4}\tilde{\triangle}|f|^2(z).$$

In [14] and [17], the invariant Green's function is defined as $G(z,a) = g(\varphi_a(z))$, where

$$g(z) = \frac{n+1}{2n}\int_{|z|}^1 (1-t^2)^{n-1}t^{-2n+1}dt.$$

About g, we have (cf. [12], Lemma 1):

Proposition 2.1. *Let $n \geq 2$ be an integer, then there are positive constants C_1 and C_2 such that for all $z \in B \setminus \{0\}$,*

$$C_1(1-|z|^2)^n|z|^{-2(n-1)} \leq g(z) \leq C_2(1-|z|^2)^n|z|^{-2(n-1)}.$$

Bloch(B) denotes the Bloch space and $\mathcal{B}_0(B)$ the little Bloch space in B. BMOA(∂B) and VMOA(∂B) denote bounded mean oscillation and vanishing mean oscillation on ∂B, respectively.

In [9], for $1 < p < \infty$ the Besov p-spaces $B_p(B)$ were defined as follows:

$$\|f\|_{B_p} = |f(0)| + (p-1)\|Qf\|_{L^p(\lambda)},$$
$$B_p(B) = \{f \in H(B) : \|f\|_{B_p} < \infty\}$$

(cf. Definition 3.1 of [9]). If $p = \infty$, the corresponding Besov space was defined by the Bloch space. $B_p(B)$ are Möbius invariant Banach spaces and for $1 < p \le q \le \infty$, have

$$B_p(B) \subseteq B_q(B) \subseteq B_\infty(B) = \text{Bloch}(B)$$

(cf. Propositions 3.2 and 3.3 of [9]).

For $f \in H(B), 0 < p < \infty, a \in B$, let

$$I_p(f,a) = \int_B |\tilde{\nabla} f(z)|^2 G^p(z,a) d\lambda(z),$$
$$J_p(f,a) = \int_B |\tilde{\nabla} f(z)|^2 (1 - |\varphi_a(z)|^2)^{np} d\lambda(z).$$

Definition 2.2. We define

$$\|f\|_{Q_p}^2 = \sup_{a \in B} I_p(f,a),$$
$$Q_p(B) = \{f \in H(B) : \|f\|_{Q_p} < \infty\},$$

and

$$Q_{p,0}(B) = \left\{ f \in H(B) : \lim_{|a| \to 1} I_p(f,a) = 0 \right\}.$$

Denote $E(a,r) = \{z \in B : |\varphi_a(z)| < r\}$ and $\bar{E}(a,r) = \{z \in B : |\varphi_a(z)| \le r\}$.

Let $L^2(v)$ denote the Hilbert space of square–integrable complex–valued functions in B and $L^2_a(v)$ the Bergman subspace of holomorphic functions in $L^2(v)$. If P denotes the orthogonal projection of $L^2(v)$ onto $L^2_a(v)$, the Hankel operator of symbol $f \in L^2(v)$ is defined in $L^2_a(v)$ by

$$H_f(h) = (I - P)(fh), \quad h \in L^2_a(B).$$

In general H_f may be unbounded.

Throughout this paper, C and C_j are positive constants which are not necessarily the same in each appearence. When there is no danger of confusion, we shall write Bloch, BMOA, B_p and Q_p in place of Bloch (B), BMOA$(\partial B), B_p(B)$ and $Q_p(B)$, etc.

3. Characterizations of Q_p Spaces and Bloch Space.

Lemma 3.1. *There are positive constants $C_1(p)$ and $C_2(p)$ (independent of f and a) such that*

$$I_p(f,a) \ge C_1(p) J_p(f,a) \ge C_2(p) |\tilde{\nabla} f(a)|^2$$

for $0 < p < \infty, f \in H(B)$ *and* $a \in B$.

Proof. The first inequality can be obtained by Proposition 2.1 and $|\varphi_a(z)| < 1$. The second inequality is proved as follows:

$$\begin{aligned} J_p(f,a) &= \int_B |\tilde{\nabla} f(z)|^2 (1-|\varphi_a(z)|^2)^{np} d\lambda(z) \\ &= \int_B |\tilde{\nabla}(f\circ\varphi_a)(z)|^2(1-|z|^2)^{np-n-1}dv(z) \\ &\geq \int_{B_{1/2}} |\tilde{\nabla}(f\circ\varphi_a)(z)|^2(1-|z|^2)^{np-n+1}(1-|z|^2)^{-2}dv(z). \end{aligned}$$

For $z \in B_{1/2}$, $(1-|z|^2)^{np-n+1} \geq (3/4)^{np-n+1}$ when $np-n+1 \geq 0$ and $(1-|z|^2)^{np-n+1} \geq 1$ when $np-n+1 < 0$, thus

$$J_p(f,a) \geq C(p) \int_{B_{1/2}} |\tilde{\nabla}(f\circ\varphi_a)(z)|^2(1-|z|^2)^{-2}dv(z).$$

Since $\mathcal{R}(f\circ\varphi_a)(z) = \langle \nabla(f\circ\varphi_a)(z), \bar{z}\rangle$, by Schwarz inequality we can get

$$\begin{aligned} |\tilde{\nabla}(f\circ\varphi_a)(z)|^2 &= (1-|z|^2)(|\nabla(f\circ\varphi_a)(z)|^2 - |\mathcal{R}(f\circ\varphi_a)(z)|^2) \\ &\geq (1-|z|^2)^2|\nabla(f\circ\varphi_a)(z)|^2. \end{aligned}$$

It follows from the subharmonicity of $|\nabla(f\circ\varphi_a)(z)|^2$ that

$$\begin{aligned} J_p(f,a) &\geq C(p)\int_{B_{1/2}} |\nabla(f\circ\varphi_a)(z)|^2 dv(z) \\ &\geq C(p)|B_{1/2}||\nabla(f\circ\varphi_a)(0)|^2 \\ &= C_2(p)|\tilde{\nabla} f(a)|^2. \end{aligned}$$

□

Lemma 3.2. *For* $f \in H(B)$, *the following three quantities are equivalent:*

$$\|f\|^2_{\mathcal{B}}, \quad \sup_{z\in B} |\tilde{\nabla} f(z)|^2, \quad \sup_{a\in B} J_{\frac{n+1}{n}}(f,a).$$

Proof. We can get the result by a little modification of the proof of Theorem 2.4 in [6] and using the equality $\tilde{\triangle}|f|^2(z) = 4|\tilde{\nabla} f(z)|^2$. □

Theorem 3.3. Q_p *are Möbius invariant Banach spaces equipped with the norm* $|f(0)| + \|f\|_{Q_p}$.Q_p *are* $\mathcal{M}$*-invariant,which means* $f\circ\varphi \in Q_p$ *and* $\|f\circ\varphi\|_{Q_p} = \|f\|_{Q_p}$ *whenever* $f \in Q_p$ *and* $\varphi \in \mathrm{Aut}(B)$.

Proof. The invariance of Q_p and $\|\cdot\|_{Q_p}$ is obvious because $|\tilde{\nabla} f|, G(z,a)$ and $d\lambda$ are $\mathcal{M}$– invariant.

From the definition of $\tilde{\nabla} f(z) = \nabla(f \circ \varphi_z)(0)$,

$$\begin{aligned}\tilde{\nabla}(f+h)(z) &= \nabla[(f+h)\circ\varphi_z](0)\\ &= \nabla(f\circ\varphi_z)(0) + \nabla(h\circ\varphi_z)(0)\\ &= \tilde{\nabla} f(z) + \tilde{\nabla} h(z).\end{aligned}$$

Therefore

$$|\tilde{\nabla}(f+h)(z)|^2 \le (|\tilde{\nabla} f(z)| + |\tilde{\nabla} h(z)|)^2.$$

$$\begin{aligned}I_p(f+h,a) &= \int_B |\tilde{\nabla}(f+h)(z)|^2 G^p(z,a)\,d\lambda(z)\\ &\le \int_B (|\tilde{\nabla} f(z)| + |\tilde{\nabla} h(z)|)^2 G^p(z,a)\,d\lambda(z)\\ &= I_p(f,a) + I_p(h,a) + 2\int_B |\tilde{\nabla} f(z)|\cdot|\tilde{\nabla} h(z)| G^p(z,a)\,d\lambda(z).\end{aligned}$$

By Hölder inequality,

$$\begin{aligned}&\int_B |\tilde{\nabla} f(z)|\cdot|\tilde{\nabla} h(z)| G^p(z,a)\,d\lambda(z)\\ &\le \left[\int_B |\tilde{\nabla} f(z)|^2 G^p(z,a)\,d\lambda(z)\right]^{1/2}\left[\int_B |\tilde{\nabla} h(z)|^2 G^p(z,a)\,d\lambda(z)\right]^{1/2}\\ &= [I_p(f,a)]^{1/2}[I_p(h,a)]^{1/2}.\end{aligned}$$

Thus

$$\begin{aligned}I_p(f+h,a) &\le I_p(f,a) + I_p(h,a) + 2[I_p(f,a)]^{1/2}[I_p(h,a)]^{1/2},\\ [I_p(f+h,a)]^{1/2} &\le [I_p(f,a)]^{1/2} + [I_p(h,a)]^{1/2}.\end{aligned}$$

Taking $\sup_{a\in B}$ on the two sides, we can get

$$\|f+h\|_{Q_p} \le \|f\|_{Q_p} + \|h\|_{Q_p}.$$

$\|f\|_{Q_p} = 0$, when $f \equiv \text{const}$. On the other hand, suppose $\|f\|_{Q_p} = 0$, then $(1-|z|^2)^2|\nabla f(z)|^2 \le |\tilde{\nabla} f(z)|^2 \equiv 0$ and so $|\nabla f(z)| \equiv 0$ on B, thus $f \equiv$ const. on B.

We have proved that $\|\cdot\|_{Q_p}$ is a seminorm on Q_p. If two functions whose difference is only a constant function are regarded as the same one, then Q_p is a normed linear space with norm $\|\cdot\|_{Q_p}$. In the following we will prove the completeness.

Let $\{f_k\}_{k=1}^{\infty}$ be a Cauchy sequence in Q_p. By Lemma 3.1 and Lemma 3.2, $\|f\|_{\mathcal{B}} \le C\|f\|_{Q_p}$ for all $f \in H(B)$, thus $\{f_k\}$ is also a Cauchy sequence in

Bloch space (in the Bloch norm). According to the proof of Proposition 4.2 of [15], there is a function $f \in H(B)$ such that $f_k \to f$ uniformly on compact subsets of B (as $k \to \infty$), therefore $\frac{\partial f_k(z)}{\partial z_j} \to \frac{\partial f(z)}{\partial z_j}$ as $k \to \infty$, for $j = 1, 2, \cdots, n$ and $z \in B$. So we can get $|\tilde{\nabla}(f_k - f_m)(z)|^2 \to |\tilde{\nabla}(f_m - f)(z)|^2$ as $k \to \infty$. For any $\varepsilon > 0$, there exists a positive integer N, such that

$$\|f_m - f_k\|_{Q_p} < \varepsilon,$$

as $m, k \geq N$. By Fatou's Lemma, as $m \geq N$,

$$\begin{aligned} I_p(f_m - f, a) &= \int_B |\tilde{\nabla}(f_m - f)(z)|^2 G^p(z,a) d\lambda(z) \\ &= \int_B \lim_{k\to\infty} |\tilde{\nabla}(f_m - f_k)(z)|^2 G^p(z,a) d\lambda(z) \\ &\leq \varliminf_{k\to\infty} \int_B |\tilde{\nabla}(f_m - f_k)(z)|^2 G^p(z,a) d\lambda(z) \\ &\leq \lim_{k\to\infty} \|f_m - f_k\|^2_{Q_p} \leq \varepsilon^2. \end{aligned}$$

Taking $\sup\limits_{a\in B}$ and square roots at the two ends above, gives

$$\lim_{m\to\infty} \|f_m - f\|_{Q_p} = 0.$$

By the triangle inequality,

$$\|f\|_{Q_p} \leq \|f_N - f\|_{Q_p} + \|f_N\|_{Q_p} \leq \varepsilon + \|f_N\|_{Q_p} < \infty,$$

hence $f \in Q_p$. □

Proposition 3.4. $f \in Q_p \Leftrightarrow \sup\limits_{a\in B} J_p(f,a) < \infty$ *for* $0 < p \leq 1$.

Proof. Suppose $f \in Q_p$, then $\sup\limits_{a\in B} I_p(f,a) < \infty$. By Lemma 3.1, $J_p(f,a) \leq CI_p(f,a)$, therefore

$$\sup_{a\in B} J_p(f,a) < \infty.$$

On the other hand, suppose $\sup\limits_{a\in B} J_p(f,a) < \infty$, then from $1 - |\varphi_a(z)|^2 \leq 1$ and $0 < p \leq 1$, we have

$$J_1(f,a) \leq J_p(f,a),$$

thus

$$\sup_{a\in B} \int_B |\tilde{\nabla} f(z)|^2 (1 - |\varphi_a(z)|^2)^n d\lambda(z) < \infty.$$

It leads to $f \in$BMOA by Lemma 4.1 and Theorem B of [5]. By (1.2), we have

$$\sup_{a\in B}\int_B |\tilde{\nabla} f(z)|^2 G(z,a)d\lambda(z) < \infty, \tag{3.1}$$

when $f \in$ BMOA. By properties of the Green's function $G(z,a)$, there exists a positive $\delta \in (0,1)$, so that $|G(z,a)| \geq 1$ when $|\varphi_a(z)| < \delta$ and $G(z,a) \leq C(\delta)(1-|\varphi_a(z)|^2)^n$ when $|\varphi_a(z)| \geq \delta$, thus

$$\begin{aligned} I_p(f,a) &= \int_B |\tilde{\nabla} f(z)|^2 G^p(z,a)d\lambda(z) \\ &= \left(\int_{|\varphi_a(z)|<\delta} + \int_{|\varphi_a(z)|\geq\delta}\right) |\tilde{\nabla} f(z)|^2 G^p(z,a)d\lambda(z) \\ &\leq \int_{|\varphi_a(z)|<\delta} |\tilde{\nabla} f(z)|^2 G(z,a)d\lambda(z) \\ &\quad + [C(\delta)]^p \int_{|\varphi_a(z)|\geq\delta} |\tilde{\nabla} f(z)|^2 (1-|\varphi_a(z)|^2)^{np} d\lambda(z) \\ &\leq \int_B |\tilde{\nabla} f(z)|^2 G(z,a)d\lambda(z) + [C(\delta)]^p J_p(f,a). \end{aligned}$$

By (3.1) and $\sup_{a\in B} J_p(f,a) < \infty$, we get $f \in Q_p$. □

Proposition 3.5. *For $p \in (0,1]$, when $\frac{2n}{n-(n-1)p} < q < \frac{2}{1-p}$, we have $B_q \subseteq Q_p$. Especially let $p=1$, then when $2n < q < \infty$ have $B_q \subseteq$ BMOA.*

Proof. Applying (2.8) in [9], Hölder inequality and the $\mathcal{M}$-invariance of the Green's function, then

$$\begin{aligned} & I_p(f,a) \\ &= \int_B |\tilde{\nabla} f(z)|^2 G^p(z,a)d\lambda(z) \\ &= \int_B \left(\frac{1}{2}\sqrt{\tilde{\triangle}|f|^2(z)}\right)^2 G^p(z,a)d\lambda(z) = \int_B (Qf(z))^2 G^p(z,a)d\lambda(z) \\ &\leq \left(\int_B (Qf(z))^{2\mu} d\lambda(z)\right)^{1/\mu} \cdot \left(\int_B G^{p\mu/(\mu-1)}(z,a)d\lambda(z)\right)^{(\mu-1)/\mu}, \quad \text{for } \mu > 1 \\ &\leq \frac{1}{(2\mu-1)^2}\|f\|^2_{B_{2\mu}} \left(\int_B g^{p\mu/(\mu-1)}(z)d\lambda(z)\right)^{(\mu-1)/\mu}. \end{aligned}$$

By Proposition 2.1

$I(f,a)$

$$\leq \frac{C}{(2\mu-1)^2}\|f\|^2_{B_{2\mu}}\Big[\int_B (1-|z|^2)^{p\mu n/(\mu-1)}|z|^{-2p\mu(n-1)/(\mu-1)}d\lambda(z)\Big]^{(\mu-1)/\mu}$$
$$\leq \frac{C}{(2\mu-1)^2}\|f\|^2_{B_{2\mu}}\Big[\int_0^1 r^{[-2p\mu(n-1)/(\mu-1)]+2n-1}(1-r)^{[p\mu n/(\mu-1)]-n-1}dr\Big]^{(\mu-1)/\mu}.$$

The integral at the end above is finite if $\frac{n}{n-(n-1)p} < \mu < \frac{1}{1-p}$. Let $2\mu = q$. Then when $\frac{2n}{n-(n-1)p} < q < \frac{2}{1-p}$ we have $I_p(f,a) \leq C\|f\|^2_{B_q}$. Thus

$$\|f\|^2_{Q_p} \leq C\|f\|^2_{B_q},$$

and so $B_q \subseteq Q_p$. Let $p = 1$. Then it follows from (1.2) that

$$B_q \subseteq Q_1 = \text{BMOA} \quad \text{when } \ 2n < q < \infty.$$

□

Remark 1. Proposition 3.3 of [9] said that for $1 < p \leq q \leq \infty$, have

$$B_p \subseteq B_q \subseteq B_\infty = \text{Bloch}\,.$$

So our Proposition 3.5 shows that BMOA is such a space that is inserted between $\{B_q, 2n < q < \infty\}$ and the Bloch space, i.e.

$$\{B_q, 2n < q < \infty\} \subseteq \text{BMOA} \subset \text{Bloch}\,.$$

Remark 2. In fact, provided $q < \frac{2}{1-p}$ one have $B_q \subseteq Q_p$ from the nondecreasing of the Besov space. So the condition $\frac{2n}{n-(n-1)p} < q$ may be dropped.

Remark 3. Suppose that $p = \frac{n-1}{n}$. Then $B_q \subseteq Q_{\frac{n-1}{n}}$ if $q < 2n$. But then Theorem 4.6 of [9] is equivalent to that the Besov space B_q is trivial if and only if $q \leq 2n$. This suggests us to consider whether $Q_{\frac{n-1}{n}}$ is also trivial. In Proposition 3.7 below, we will verify that is true by another way.

Proposition 3.6. *For $f \in H(B)$, the following are equivalent.*

(i) $f \in \text{Bloch}(B)$;

(ii) $f \in Q_p$ *for some* $p \in (1, \frac{n}{n-1})$;

(iii) $f \in Q_p$ *for all* $p \in (1, \frac{n}{n-1})$;

(iv) $\sup_{a\in B} J_q(f,a) < \infty$ *for some* $q \in (1,\infty)$;

(v) $\sup_{a\in B} J_q(f,a) < \infty$ *for all* $q \in (1,\infty)$.

Furthermore $\|f\|_{\mathcal{B}}^2, \|f\|_{Q_p}^2 (1 < p < \frac{n}{n-1})$ *and* $\sup_{a\in B} J_q(f,a)$ $(1 < q < \infty)$ *are equivalent.*

Proof. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) is as follows:

Using Lemma 3.1 and Lemma 3.2, we get

$$\|f\|_{Q_p}^2 = \sup_{a\in B} I_p(f,a) \geq C_1 \sup_{a\in B} |\tilde{\nabla} f(z)|^2 \geq \|f\|_{\mathcal{B}}^2,$$

thus from $f \in Q_p$, we have $f \in \text{Bloch}(B)$.

On the other hand, suppose $f \in \text{Bloch}(B)$, by Lemma 3.2 and Proposition 2.1

$$\begin{aligned} I_p(f,a) &= \int_B |\tilde{\nabla} f(z)|^2 G^p(z,a) d\lambda(z) \\ &= \int_B |\tilde{\nabla}(f\circ\varphi_a)(z)|^2 G^p(z,0) d\lambda(z) \\ &\leq C_1 \|f\circ\varphi_a\|_{\mathcal{B}}^2 \int_B \frac{(1-|z|^2)^{np}}{(1-|z|^2)^{n+1}|z|^{2(n-1)p}} dv(z) \\ &\leq C_2 \|f\|_{\mathcal{B}}^2 \int_0^1 (1-r)^{np-n-1} r^{2n-1-2(n-1)p} dr. \end{aligned}$$

When $1 < p < \frac{n}{n-1}$, we have $np-n-1 > -1$ and $2n-1-2(n-1)p > -1$, hence,

$$\int_0^1 (1-r)^{np-n-1} r^{2n-1-2(n-1)p} dr = M(p) < \infty.$$

It follows that

$$\|f\|_{Q_p}^2 = \sup_{a\in B} I_p(f,a) \leq C_3 \|f\|_{\mathcal{B}}^2.$$

(i) $\Rightarrow$ (v): When $f \in \text{Bloch}(B)$, by Lemma 3.2,

$$\sup_{z\in B} |\tilde{\nabla}(f\circ\varphi_a)(z)|^2 \leq C\|f\circ\varphi_a\|_{\mathcal{B}}^2 = C\|f\|_{\mathcal{B}}^2 < \infty,$$

$$\begin{aligned} J_q(f,a) &= \int_B |\tilde{\nabla} f(z)|^2 (1-|\varphi_a(z)|^2)^{nq} d\lambda(z) \\ &= \int_B |\tilde{\nabla}(f\circ\varphi_a)(z)|^2 (1-|z|^2)^{nq} d\lambda(z) \\ &\leq C\|f\|_{\mathcal{B}}^2 \int_B (1-|z|^2)^{nq-n-1} dv(z) \\ &\leq C\|f\|_{\mathcal{B}}^2/(q-1). \end{aligned}$$

Thus for all $q \in (1,\infty)$ we have

$$\sup_{a\in B} J_q(f,a) \leq C\|f\|_{\mathcal{B}}^2/(q-1) < \infty.$$

(v) $\Rightarrow$ (iv) is obvious.

(iv) $\Rightarrow$ (i): Suppose there exists $q > 1$, so that $\sup_{a\in B} J_q(f,a) < \infty$. By Lemma 3.1,

$$J_q(f,a) \geq C(q)|\tilde{\nabla} f(a)|^2,$$

and by Lemma 3.2,

$$\sup_{a\in B} J_q(f,a) \geq C(q) \sup_{a\in B} |\tilde{\nabla} f(a)|^2 \geq C(q)\|f\|_{\mathcal{B}}^2.$$

Therefore $f \in \mathrm{Bloch}(B)$. □

Remark 4. The main theorem of [6] is (i) $\Leftrightarrow$ (iv) in Proposition 3.6 when $q = \frac{n+1}{n}$.

Proposition 3.7. *When $0 < p \leq \frac{n-1}{n}$ or $p \geq \frac{n}{n-1}$, Q_p contain only the constant functions; when $\frac{n-1}{n} < p < \frac{n}{n-1}$, Q_p contain all polynomials.*

Proof. Let f be a nonconstant function in Q_p. Write

$$f(z) = \sum_{\alpha} a_\alpha z^\alpha,$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ is an n-tuple of nonnegative integers and $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$. Since f is nonconstant, there exists $\alpha_0 \neq 0$ such that $a_{\alpha_0} \neq 0$. Now we come to prove $z^{\alpha_0} \in Q_p$.

Denote $\alpha_0 = (k_1, \dots, k_n)$, then it is easy to know that

(3.2)
$$a_{\alpha_0} z^{\alpha_0} = \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} f(z_1 e^{i\theta_1}, \dots, z_n e^{i\theta_n}) e^{-ik_1\theta_1} \cdots e^{-ik_n\theta_n} d\theta_1 \cdots d\theta_n.$$

Let

$$F(z) = a_{\alpha_0} z^{\alpha_0},$$
$$U_\theta f(z) = f(z_1 e^{i\theta_1}, \dots, z_n e^{i\theta_n}) = f \circ U(z_1, \dots, z_n)$$

where U is diagonal matrix $\mathrm{diag}(e^{i\theta_1}, \dots, e^{i\theta_n})$. Let

$$\tilde{\nabla}_j f(z) = \frac{\partial}{\partial w_j} [(f \circ \varphi_z)(w)]|_{w=0},$$

thus $\tilde{\nabla} = (\tilde{\nabla}_1, \cdots, \tilde{\nabla}_n)$. By (3.2),

$$(F \circ \varphi_z)(w) = \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} (U_\theta f \circ \varphi_z)(w) e^{-ik_1\theta_1} \cdots e^{-ik_n\theta_n} d\theta_1 \cdots d\theta_n,$$

$$\begin{aligned}&\frac{\partial}{\partial w_j}(F\circ\varphi_z)(w)\\&\qquad=\frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi}\frac{\partial}{\partial w_j}(U_\theta f\circ\varphi_z)(w)e^{-ik_1\theta_1}\cdots e^{-ik_n\theta_n}d\theta_1\cdots d\theta_n.\end{aligned}$$

Let $w=0$, then

$$\tilde{\nabla}_jF(z)=\frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi}\tilde{\nabla}_jU_\theta f(z)e^{-ik_1\theta_1}\cdots e^{-ik_n\theta_n}d\theta_1\cdots d\theta_n.$$

By Jensen's Inequality on Convexity (cf. [8]),

$$\begin{aligned}|\tilde{\nabla}_jF(z)|^2&=\left[\frac{1}{(2\pi)^n}\left|\int_0^{2\pi}\cdots\int_0^{2\pi}\tilde{\nabla}_jU_\theta f(z)e^{-ik_1\theta_1}\cdots e^{-ik_n\theta_n}d\theta_1\cdots d\theta_n\right|\right]^2\\&\le\frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi}|\tilde{\nabla}_jU_\theta f(z)|^2d\theta_1\cdots d\theta_n,\end{aligned}$$

thus

$$\begin{aligned}|\tilde{\nabla}F(z)|^2&=\sum_{j=1}^n|\tilde{\nabla}_jF(z)|^2\\&\le\frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi}\left(\sum_{j=1}^n|\tilde{\nabla}_jU_\theta f(z)|^2\right)d\theta_1\cdots d\theta_n\\&=\frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi}|\tilde{\nabla}U_\theta f(z)|^2d\theta_1\cdots d\theta_n.\end{aligned}$$

It follows that

$$\begin{aligned}I_p(F,a)&=\int_B|\tilde{\nabla}F(z)|^2G^p(z,a)d\lambda(z)\\&\le\int_B\left[\frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi}|\tilde{\nabla}U_\theta f(z)|^2d\theta_1\cdots d\theta_n\right]G^p(z,a)d\lambda(z)\\&=\frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi}\left[\int_B|\tilde{\nabla}U_\theta f(z)|^2G^p(z,a)d\lambda(z)\right]d\theta_1\cdots d\theta_n\\&\le\frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi}\|U_\theta f\|_{Q_p}^2d\theta_1\cdots d\theta_n.\end{aligned}$$

Because $U_\theta f=f\circ U$, where $U\in\mathrm{Aut}(B)$, then by Theorem 3.3 we get $\|U_\theta f\|_{Q_p}=\|f\|_{Q_p}$. Therefore

$$I_p(F,a)\le\|f\|_{Q_p}^2\frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi}d\theta_1\cdots d\theta_n$$

$$= \|f\|^2_{Q_p}.$$

Hence

$$\|a_{\alpha_0} z^{\alpha_0}\|^2_{Q_p} = \|F\|^2_{Q_p} = \sup_{a\in B} I_p(F,a) \le \|f\|^2_{Q_p}.$$

So

$$\|z^{\alpha_0}\|_{Q_p} \le \|f\|_{Q_p}/|a_{\alpha_0}| < \infty.$$

This means that

$$z^{\alpha_0} \in Q_p.$$

On the other hand, we will prove that any monomial $z^\alpha \notin Q_p,\ 0 < p \le \frac{n-1}{n}$. Therefore for $0 < p \le \frac{n-1}{n},\ Q_p$ contains only the constant functions.

Let $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$, and $|\alpha| = \alpha_1 + \cdots + \alpha_n \ge 1$. Then

$$\begin{aligned}\nabla(z^\alpha)(z) &= \left(\frac{\partial}{\partial z_1}(z^\alpha)(z), \dots, \frac{\partial}{\partial z_n}(z^\alpha)(z)\right)\\ &= (\alpha_1 z_1^{\alpha_1-1}\cdots z_n^{\alpha_n}, \dots, \alpha_n z_1^{\alpha_1}\cdots z_n^{\alpha_n-1}).\end{aligned}$$

(Here for the sake of unified expression, still denote $\frac{\partial}{\partial z_j}(z^\alpha)(z) = \alpha_j z_1^{\alpha_1}\cdots z_j^{\alpha_j-1}\cdots z_n^{\alpha_n}$ as $\alpha_j = 0$), and

$$\mathcal{R}(z^\alpha)(z) = \alpha_1 z_1^{\alpha_1}\cdots z_n^{\alpha_n} + \cdots + \alpha_n z_1^{\alpha_1}\cdots z_n^{\alpha_n} = |\alpha| z^\alpha.$$

Thus

$$\begin{aligned}&|\tilde\nabla(z^\alpha)(z)|^2\\ &= (1-|z|^2)(|\nabla(z^\alpha)(z)|^2 - |\mathcal{R}(z^\alpha)(z)|^2)\\ &= (1-|z|^2)(\alpha_1^2|z_1^{\alpha_1-1}\cdots z_n^{\alpha_n}|^2 + \cdots + \alpha_n^2|z_1^{\alpha_1}\cdots z_n^{\alpha_n-1}|^2 - |\alpha|^2|z^\alpha|^2)\\ &\triangleq (1-|z|^2)J(z).\end{aligned}$$

Observe that the integral

$$\begin{aligned}&\int_B |\tilde\nabla(z^\alpha)(z)|^2 G^p(z,0)\,d\lambda(z)\\ &\text{(Proposition 2.1)}\\ &\ge C\int_B (1-|z|^2)J(z)(1-|z|^2)^{np}|z|^{-2(n-1)p}(1-|z|^2)^{-n-1}dv(z)\\ &\ge C\int_B (1-|z|^2)^{np-n}J(z)\,dv(z)\\ &\ge C\int_0^1 r^{2n-1}(1-r^2)^{np-n}dr\int_S J(r\zeta)\,d\sigma(\zeta)\end{aligned}$$

$$
\begin{aligned}
&= C\int_0^1 r^{2n-1}(1-r^2)^{np-n}dr \cdot r^{2|\alpha|-2}\int_S(\alpha_1^2|\zeta_1^{\alpha_1-1}\cdots\zeta_n^{\alpha_n}|^2 \\
&\quad + \cdots + \alpha_n^2|\zeta_1^{\alpha_1}\cdots\zeta_n^{\alpha_n-1}|^2 - |\alpha|^2 r^2|\zeta^\alpha|^2)d\sigma(\zeta) \\
&\geq C\int_0^1 r^{2n+2|\alpha|-2-1}(1-r)^{np-n}dr\int_S(\alpha_1^2|\zeta_1^{\alpha_1-1}\cdots\zeta_n^{\alpha_n}|^2 \\
&\quad + \cdots + \alpha_n^2|\zeta_1^{\alpha_1}\cdots\zeta_n^{\alpha_n-1}|^2 - |\alpha|^2|\zeta^\alpha|^2)d\sigma(\zeta)
\end{aligned}
$$

(§1.4.9.(1) of [13])

$$
\begin{aligned}
&= C\Big(\alpha_1\frac{(n-1)!\alpha_1!\cdots\alpha_n!}{(n-1+|\alpha|-1)!} + \cdots + \alpha_n\frac{(n-1)!\alpha_1!\cdots\alpha_n!}{(n-1+|\alpha|-1)!} \\
&\quad - |\alpha|^2\frac{(n-1)!\alpha!}{(n-1+|\alpha|)!}\Big)\cdot\int_0^1 r^{2(n-1+|\alpha|)-1}(1-r)^{np-n}dr \\
&= C\frac{(n-1)!\alpha!}{(n-1+|\alpha|-1)!}|\alpha|\left(1-\frac{|\alpha|}{n-1+|\alpha|}\right)\int_0^1 r^{2(n-1+|\alpha|)-1}(1-r)^{np-n}dr \\
&= C\frac{(n-1)|\alpha|(n-1)!\alpha!}{(n-1+|\alpha|)!}\int_0^1 r^{2(n-1+|\alpha|)-1}(1-r)^{np-n}dr = +\infty,
\end{aligned}
$$

if $n \geq 2$ and $np - n \leq -1$. Thus for any monomial z^α with $|\alpha| \geq 1$, we have

$$z^\alpha \notin Q_p, \quad 0 < p \leq \frac{n-1}{n}.$$

In the following we prove that for $p \geq \frac{n}{n-1}$, Q_p contains only the constant functions as well.

Let $p \geq \frac{n}{n-1}$, suppose that f is not a constant, then there exists a point $a \in B$ and $r, 0 < r < 1$, such that

$$|\tilde{\nabla} f(z)|^2|_{E(a,r)} \geq \delta > 0.$$

Hence

$$
\begin{aligned}
&\int_B |\tilde{\nabla} f(z)|^2 G^p(z,a)d\lambda(z) \\
&\geq C\delta\int_{E(a,r)}(1-|\varphi_a(z)|^2)^{np}|\varphi_a(z)|^{-2(n-1)p}d\lambda(z) \\
&= C\delta\int_{E(0,r)}(1-|w|^2)^{np}|w|^{-2(n-1)p}d\lambda(w) \\
&\geq C(1-r^2)^{np-n-1}\delta\int_{E(0,r)}|w|^{-2(n-1)p}dv(w), \quad np-n-1 \geq p-1 > 0 \\
&= C(r)\delta\int_0^r (t^2)^{n+p-np-1}dt^2 = +\infty.
\end{aligned}
$$

Because $n \geq 2$ and $p \geq \frac{n}{n-1}$. Thus $f \notin Q_p,\ p \geq \frac{n}{n-1}$.

It remains to prove that for $\frac{n-1}{n} < p < \frac{n}{n-1}, Q_p$ contains all polynomials. First, for $1 \le p < \frac{n}{n-1}$, by Propositions 3.5, 3.6 and (1.2)

$$\{B_q, 2n < q < \infty\} \subseteq \text{BMOA} \subset \text{Bloch} = \left\{Q_p, 1 < p < \frac{n}{n-1}\right\}.$$

Checking Theorem 4.6 of [**9**] and its proof, we know that the Besov space B_q is nontrivial if and only if $q > 2n$, and B_q possesses the conditions of Lemma 3 of [**18**]. Therefore for $q > 2n$, B_q contains all polynomials. Thus the conclusion is true for $1 \le p < \frac{n}{n-1}$.

For $\frac{n-1}{n} < p < 1$, if let $p = \frac{n-1}{n} + \varepsilon < 1$, then by Remark 2 of Proposition 3.5 when $q < \frac{2}{1-(\frac{n-1}{n}+\varepsilon)} = \frac{2n}{1-n\varepsilon}$, have $B_q \subseteq Q_p$. We can choose $\delta \in (0, \frac{2n^2\varepsilon}{1-n\varepsilon})$ so that $2n < 2n + \delta < \frac{2n}{1-n\varepsilon}$. Therefore we have also

$$\{\text{all polynomials}\} \subset B_{2n+\delta} \subseteq Q_p, \quad \text{for } p = \frac{n-1}{n} + \varepsilon.$$

Since $\varepsilon \in (0, \frac{1}{n})$ arbitrary, it follows that $\{\text{all polynomials}\} \subseteq Q_p$, for $\frac{n-1}{n} < p < 1$. The proof of Proposition 3.7 is completed. $\square$

Summarizing the results of Propositions 3.4, 3.6 and 3.7, for the construction of Q_p spaces we have

Theorem 3.8. *Q_p spaces have the following properties:*

(i) *When $0 < p \le \frac{n-1}{n}$ or $p \ge \frac{n}{n-1}, Q_p$ are trivial. i.e. they contain only the constant functions. When $\frac{n-1}{n} < p < \frac{n}{n-1}, Q_p$ are nontrivial, and each Q_p at least contains all polynomials.*

(ii) *$Q_{p_1} \subseteq Q_{p_2}$ for $0 < p_1 \le p_2 \le 1$.*

(iii) *$Q_1 = \text{BMOA}$.*

(iv) *$Q_p = \text{Bloch}$, and $\|\cdot\|_{Q_p}$ is equivalent to $\|\cdot\|_{\mathcal{B}}$ for $1 < p < \frac{n}{n-1}$.*

Proof. (i) and (iv) are Propositions 3.7 and 3.6, respectively. (ii) follows from Proposition 3.4. (iii) is just (1.2). $\square$

Corollary 3.9. *For $f \in H(B), 1 < p < \frac{n}{n-1}, f \in Q_p$ if and only if $H_{\bar f}$ is bounded. Moreover $\|f\|_{Q_p}$ and $\|H_{\bar f}\|$ are equivalent quantities.*

Proof. It follows from Theorem C of [**3**] and (iv) of Theorem 3.8. $\square$

4. Characterizations of $Q_{p,0}$ spaces and $\mathcal{B}_0$ space.

Lemma 4.1. *For every $r \in (0,1)$, if $a_1 \in E(a, \frac{1}{2}r), z \in B$, then there exists $C(r) > 0$, such that*

$$1 - |\varphi_{a_1}(z)|^2 \le C(r)(1 - |\varphi_a(z)|^2).$$

Proof. M. Jévtic proved in [**10**] that if $a_1 \in E(a, \frac{1}{2}r)$, then

$$\frac{1-r/2}{1+r/2} \le \frac{1-|a_1|^2}{1-|a|^2} \le \frac{1+r/2}{1-r/2}.$$

And since

$$1 \ge 1-|\varphi_a(a_1)|^2 = \frac{(1-|a_1|^2)(1-|a|^2)}{|1-\langle a, a_1\rangle|^2} \ge \frac{r^2}{4}.$$

Thus for a fixed r, we have

$$(1-|a_1|^2) \sim (1-|a|^2) \sim |1-\langle a, a_1\rangle| \sim (1-|a_1|), \tag{4.1}$$

where "A ∼ B" means that there exist positive constants C_1 and C_2 so that $C_1 A \le B \le C_2 A$. By the triangle inequality ([**13**], 5.1.2),

$$\begin{aligned} |1-\langle z, a\rangle|^{1/2} &\le |1-\langle z, a_1\rangle|^{1/2} + |1-\langle a, a_1\rangle|^{1/2} \\ &\le |1-\langle z, a_1\rangle|^{1/2} + C(r)(1-|a_1|)^{1/2} \end{aligned}$$

and

$$|1-\langle z, a_1\rangle| \ge 1-|z||a_1| \ge 1-|a_1|.$$

Therefore

$$\begin{aligned} \frac{|1-\langle z, a\rangle|^{1/2}}{|1-\langle z, a_1\rangle|^{1/2}} &\le \frac{|1-\langle z, a_1\rangle|^{1/2} + C(r)(1-|a_1|)^{1/2}}{|1-\langle z, a_1\rangle|^{1/2}} \\ &= 1 + C(r)\left[\frac{1-|a_1|}{|1-\langle z, a_1\rangle|}\right]^{1/2} \le 1 + C(r). \end{aligned}$$

That shows

$$\frac{1}{|1-\langle z, a_1\rangle|^2} \le \frac{C(r)}{|1-\langle z, a\rangle|^2}. \tag{4.2}$$

By (4.1) and (4.2),

$$\begin{aligned} 1-|\varphi_{a_1}(z)|^2 &= \frac{(1-|z|^2)(1-|a_1|^2)}{|1-\langle z, a_1\rangle|^2} \\ &\le C(r)\frac{(1-|z|^2)(1-|a|^2)}{|1-\langle z, a\rangle|^2} = C(r)(1-|\varphi_a(z)|^2). \end{aligned}$$

□

Lemma 4.2. *For $r \in (0,1), a_1 \in E(a, \frac{1}{2}r), z \in B \backslash E(a, r)$, we have*

$$G(z, a_1) \le C(r) G(z, a),$$

where $C(r)$ is a positive constant depending only on r and n (but independent of a, a_1 and z).

Proof. $\rho = |\varphi_a(z)|$ is a metric (cf. Appendix), $|\varphi_a(a_1)| < \frac{r}{2}$ and $|\varphi_a(z)| \geq r$, thus

$$|\varphi_{a_1}(z)| \geq |\varphi_a(z)| - |\varphi_a(a_1)| > |\varphi_a(z)| - \frac{r}{2},$$
$$\frac{|\varphi_{a_1}(z)|}{|\varphi_a(z)|} > 1 - \frac{r}{2|\varphi_a(z)|}.$$

Because $|\varphi_a(z)| \geq r$, we get

$$|\varphi_{a_1}(z)| > \frac{1}{2}|\varphi_a(z)|.$$

By Proposition 2.1 and Lemma 4.1,

$$\begin{aligned} G(z, a_1) &\leq C\frac{(1 - |\varphi_{a_1}(z)|^2)^n}{|\varphi_{a_1}(z)|^{2(n-1)}} \leq C\frac{[C(r)(1 - |\varphi_a(z)|^2)]^n}{2^{-2(n-1)}|\varphi_a(z)|^{2(n-1)}} \\ &= C(r)\frac{(1 - |\varphi_a(z)|^2)^n}{|\varphi_a(z)|^{2(n-1)}} \leq C(r)G(z, a). \end{aligned}$$

□

Proposition 4.3.

(i) *$Q_{p,0}$ are trivial (containing only the constant functions) when $p \geq \frac{n}{n-1}$.*

(ii) *$Q_{p,0} \subset Q_p$ whenever $0 < p < \infty$.*

Proof. (i) suppose $f \in Q_{p,0}$, from the definition of $Q_{p,0}$, there exists a $r_0 \in (0, 1)$, such that

$$I_p(f, a) = \int_B |\tilde{\nabla} f(z)|^2 G^p(z, a) d\lambda(z) \leq 1 \tag{4.3}$$

when $r_0 \leq |a| < 1$, i.e. $a \in B \backslash E(0, r_0)$. Suppose that there exists a $z_0 \in B \backslash E(0, r_0)$ so that $|\tilde{\nabla} f(z)|^2 > 2\varepsilon_0 > 0$. By the continuity of $|\tilde{\nabla} f(z)|^2$, there exists $0 < \delta < \frac{1}{2}$ such that when $z \in E(z_0, \delta), |\tilde{\nabla} f(z)|^2 > \varepsilon_0$. By integral transformation and Proposition 2.1,

$$\begin{aligned} I_p(f, z_0) &= \int_B |\tilde{\nabla} f(z)|^2 G^p(z, z_0) d\lambda(z) \\ &\geq \varepsilon_0 \int_{E(z_0, \delta)} G^p(z, z_0) d\lambda(z) \\ &= \varepsilon_0 \int_{|z| < \delta} g^p(z) d\lambda(z) \end{aligned}$$

$$\geq C\varepsilon_0 \int_{|z|<\delta} \frac{(1-|z|^2)^{np-n-1}}{|z|^{2p(n-1)}} dv(z)$$
$$\geq 2nC\varepsilon_0(1-\delta^2)^{np-n-1} \int_0^\delta r^{2n-1-2p(n-1)} dr.$$

When $p \geq \frac{n}{n-1}$, we have $2n-1-2p(n-1) \leq -1$, thus

$$I_p(f, z_0) = \infty.$$

It contradicts $z_0 \in B \backslash E(0, r_0)$ and (4.3), therefore $|\tilde{\nabla} f(z)|^2 = 0$ for all $z \in B \backslash E(0, r_0)$. Since

$$|\tilde{\nabla} f(z)|^2 = (1-|z|^2)(|\nabla f(z)|^2 - |\mathcal{R} f(z)|^2) \geq (1-|z|^2)^2 |\nabla f(z)|^2,$$

thus $|\nabla f(z)| = 0$ for all $z \in B \backslash E(0, r_0)$. By the subharmonicity of $|\nabla f(z)|$,

$$|\nabla f(z)| = 0 \quad \text{for all} \quad z \in B,$$

so $f \equiv$ const. on B.

(ii) When $p \geq \frac{n}{n-1}$, Q_p are trivial by Theorem 3.8, hence $Q_{p,0} = Q_p$. From now on, we suppose that $0 < p < \frac{n}{n-1}$, and use the idea in [**2**].

Suppose, on the contrary, that there exists $f \in Q_{p,0} \backslash Q_p$. Then

$$\lim_{|a|\to 1} I_p(f, a) = 0, \tag{4.4}$$

$$\sup_{a \in B} I_p(f, a) = \infty. \tag{4.5}$$

By (4.4) there exists $r_0 \in (0, 1)$, so that

$$\sup_{a \in B \backslash \bar{E}(0, r_0)} I_p(f, a) \leq 1,$$

and so, by (4.5) we must have

$$\sup_{a \in \bar{E}(0, r_0)} I_p(f, a) = \infty.$$

There are only two cases: Case 1, there exists at least one point $a \in \bar{E}(0, r_0)$ so that $I_p(f, a) = \infty$; Case 2, $I_p(f, a) < \infty$ for all $a \in \bar{E}(0, r_0)$ but there exists a sequence $\{a_k\} \subset \bar{E}(0, r_0)$ so that $\lim_{k\to\infty} I_p(f, a_k) = \infty$. First we will deal with Case 2.

Case 2. $I_p(f, a) < \infty$ for all $a \in \bar{E}(0, r_0)$ but there exists a sequence $\{a_k\}_{k=1}^\infty \subset \bar{E}(0, r_0)$ so that

$$\lim_{k\to\infty} I_p(f, a_k) = \infty. \tag{4.6}$$

Since $\bar{E}(0, r_0)$ is closed and bounded, and so compact, we can suppose that $\lim_{k\to\infty} a_k = a_0 \in \bar{E}(0, r_0)$. We can choose a ball $E(a_0, s)$ small enough, $s \le \frac{1}{3}, r_0 + s \le r_1 < 1$, so that $a_k \in E(a_0, \frac{1}{2}s)$ for $k \ge k_0$, where k_0 is a positive integer. By Lemma 4.2,

$$\int_{B\setminus E(a_0,s)} |\tilde{\nabla} f(z)|^2 G^p(z, a_k) d\lambda(z) \le C^p \int_B |\tilde{\nabla} f(z)|^2 G^p(z, a_0) d\lambda(z) = C_1 < \infty.$$

On the other hand, let $M_1 = \sup\{|\tilde{\nabla} f(z)|^2, z \in \bar{E}(a_0, s)\}, M_2 = \max\{1, (\frac{3}{4})^{np-n-1}\}$, by Proposition 2.1 and $\rho(z, a_k) \le \rho(z, a_0) + \rho(a_0, a_k)$ (cf. Appendix),

$$\begin{aligned}
\int_{E(a_0,s)} |\tilde{\nabla} f(z)|^2 G^p(z, a_k) d\lambda(z) &\le M_1 \int_{E(a_k,\frac{3}{2}s)} G^p(z, a_k) d\lambda(z) \\
&\le CM_1 \int_{|z|<\frac{3}{2}s} \frac{(1-|z|^2)^{np-n-1}}{|z|^{2(n-1)p}} dv(z) \\
&\le CM_1 \int_{|z|<\frac{1}{2}} \frac{(1-|z|^2)^{np-n-1}}{|z|^{2(n-1)p}} dv(z) \qquad (4.7)\\
&\le 2nCM_1M_2 \int_0^{\frac{1}{2}} r^{2n-1-2(n-1)p} dr \\
&= \frac{2nCM_1M_2}{2n-2(n-1)p} \left(\frac{1}{2}\right)^{2n-2(n-1)p} \\
&= C_2 < \infty. \qquad (4.8)
\end{aligned}$$

Here $r_0 + s \le r_1 < 1$ is used, so that $\bar{E}(a_0, s) \subset \bar{E}(0, r_1)$, and so $M_1 < \infty; p < \frac{n}{n-1}$ is also used, so that $2n - 1 - 2(n-1)p > -1$.

For $k \ge k_0$, we have

$$\begin{aligned}
&I_p(f, a_k) \\
&= \int_{E(a_0,s)} |\tilde{\nabla} f(z)|^2 G^p(z, a_k) d\lambda(z) + \int_{B\setminus E(a_0,s)} |\tilde{\nabla} f(z)|^2 G^p(z, a_k) d\lambda(z) \\
&\le C_1 + C_2 < \infty,
\end{aligned}$$

which contradicts (4.6).

Case 1. There exists at least one point $a \in \bar{E}(0, r_0)$ so that $I_p(f, a) = \infty$. That means

$$A = \{a \in \bar{E}(0, r_0) : I_p(f, a) = \infty\} \ne \phi.$$

Let $t = \sup\{|a| : a \in A\}$, then $t \le r_0$ obviously.

(1) When $t = r_0$, for any given $\varepsilon > 0$, there exists $a \in A$ so that $|a| \geq r_0 - \varepsilon$. Therefore we can take $a_0 \in A$ and a small positive number s, satisfying $s < \frac{1}{3}$ and $\frac{2}{3}s + r_0 \leq r_1 < 1$, so that $E(a_0, \frac{1}{2}s) \bigcap [B \backslash \bar{E}(0, r_0)] \neq \phi$. Let $a_1 \in E(a_0, \frac{1}{2}s) \bigcap [B \backslash \bar{E}(0, r_0)]$, then

$$I_p(f, a_1) \leq 1.$$

Since $|\varphi_{a_1}(a_0)| = |\varphi_{a_0}(a_1)|, a_0 \in E(a_1, \frac{1}{2}s)$, then by Lemma 4.2,

$$\begin{aligned}
&\int_{B\backslash E(a_1,s)} |\tilde{\nabla} f(z)|^2 G^p(z, a_0) d\lambda(z) \\
&\leq [C(s)]^p \int_{B\backslash E(a_1,s)} |\tilde{\nabla} f(z)|^2 G^p(z, a_1) d\lambda(z) \\
&\leq [C(s)]^p I_p(f, a_1) < \infty. \qquad (4.9)
\end{aligned}$$

On the other hand, let $M_1 = \sup_{z \in \bar{E}(a_1,s)} |\tilde{\nabla} f(z)|^2$. Since $\frac{3}{2}s + r_0 \leq r_1$ leads to $\bar{E}(a_1, s) \subset \bar{E}(0, r_1)$, thus $M_1 < \infty$. By Proposition 2.1,

$$\begin{aligned}
\int_{E(a_1,s)} |\tilde{\nabla} f(z)|^2 G^p(z, a_0) d\lambda(z) &\leq M_1 \int_{E(a_0, \frac{3}{2}s)} G^p(z, a_0) d\lambda(z) \\
&\leq C M_1 \int_{|z| \leq \frac{1}{2}} \frac{(1-|z|^2)^{np-n-1}}{|z|^{2(n-1)p}} dv(z).
\end{aligned}$$

Repeating the argument from (4.7) to (4.8), we get

$$\int_{E(a_1,s)} |\tilde{\nabla} f(z)|^2 G^p(z, a_0) < \infty. \qquad (4.10)$$

By (4.9) and (4.10),

$$I_p(f, a_0) = \left(\int_{B\backslash E(a_1,s)} + \int_{E(a_1,s)} \right) |\tilde{\nabla} f(z)|^2 G^p(z, a_0) d\lambda(z) < \infty,$$

which contradicts $a_0 \in A$.

(2) When $t < r_0, I_p(f, a) < \infty$ for all $a \in B \backslash \bar{E}(0, t)$. Substituting r_0 in (1) by t, taking $a_0 \in A, s < \frac{1}{3}$ and $\frac{3}{2}s + t \leq r_1 < 1$, so that $E(a_0, \frac{1}{2}s) \bigcap [B \backslash \bar{E}(0, t)] \neq \phi$, repeating the argument in (1), we get

$$I_p(f, a_0) < \infty.$$

That also contradicts $a_0 \in A$.

Summarizing Case 1 and Case 2, we see (ii) is true. □

Lemma 4.4. *$f(z) = z_1 \in Q_{p,0}$ when $\frac{n-1}{n} < p < \frac{n}{n-1}$.*

Proof. When $\frac{n-1}{n} < p < \frac{n}{n-1}$, by Proposition 2.1,

$$\begin{aligned} I_p(f,a) &= \int_B |\tilde{\nabla} f(z)|^2 G^p(z,a)d\lambda(z) \\ &= \int_B (1-|z|^2)(1-|z_1|^2)G^p(z,a)d\lambda(z) \\ &\leq \int_B (1-|z|^2)G^p(z,a)d\lambda(z) \\ &= \int_B (1-|\varphi_a(z)|^2)G^p(z,0)d\lambda(z) \\ &\leq C\int_B \frac{1-|\varphi_a(z)|^2}{(1-|z|^2)^{n+1-np}|z|^{2(n-1)p}}dv(z). \end{aligned} \tag{4.11}$$

Fixed $r_0 \in (0,1)$, when $|z| \leq r_0$, we have

$$\begin{aligned} 1-|\varphi_a(z)|^2 &= \frac{(1-|a|^2)(1-|z|^2)}{|1-\langle z,a\rangle|^2} \\ &\leq \frac{(1-|a|^2)(1-|z|^2)}{(1-|z|)^2} \leq \frac{2(1-|a|^2)}{1-|z|} \\ &\leq \frac{4}{1-r_0^2}(1-|a|^2). \end{aligned}$$

Since $p < \frac{n}{n-1}, 2n-1-2(n-1)p > -1$, let $M = \max\{(1-r_0^2)^{-(n+1-np)}, 1\}$, we get

$$\begin{aligned} &\int_{|z|\leq r_0} \frac{1-|\varphi_a(z)|^2}{(1-|z|^2)^{n+1-np}|z|^{2(n-1)p}}dv(z) \\ &\leq \frac{4(1-|a|^2)}{(1-r_0^2)}M\int_{|z|\leq r_0}\frac{dv(z)}{|z|^{2(n-1)p}} \\ &= \frac{8n(1-|a|^2)}{1-r_0^2}M\int_0^{r_0} r^{2n-1-2(n-1)p}dr \\ &= M_1(1-|a|^2) < \infty. \end{aligned} \tag{4.12}$$

When $n \geq 2, p > \frac{n-1}{n}$, $np-n > -1$ and $1-np < 0$, and

$$\begin{aligned} &\int_B \frac{1-|\varphi_a(z)|^2}{(1-|z|^2)^{n+1-np}}dv(z) \\ &= \int_B \frac{(1-|z|^2)^{np-n}(1-|a|^2)}{|1-\langle a,z\rangle|^2}dv(z) \end{aligned}$$

$$= (1-|a|^2)\int_B \frac{(1-|z|^2)^{np-n}}{|1-\langle a,z\rangle|^{n+1+(np-n)+(1-np)}}dv(z),$$

thus by Proposition 1.4.10 of [13],

$$\int_B \frac{1-|\varphi_a(z)|^2}{(1-|z|^2)^{n+1-np}}dv(z) \le (1-|a|^2)M_2, \tag{4.13}$$

where M_2 is a positive constant.

By (4.11), (4.12) and (4.13)

$$\begin{aligned} I_p(f,a) &\le C\left(\int_{|z|\le r_0} + \int_{|z|>r_0}\right)\frac{1-|\varphi_a(z)|^2}{(1-|z|^2)^{n+1-np}|z|^{2(n-1)p}}dv(z) \\ &\le CM_1(1-|a|^2) + C\int_{|z|>r_0}\frac{1-|\varphi_a(z)|^2}{(1-|z|^2)^{n+1-np}|z|^{2(n-1)p}}dv(z) \\ &\le CM_1(1-|a|^2) + \frac{C}{r_0^{2(n-1)p}}\int_B \frac{(1-|\varphi_a(z)|^2)}{(1-|z|^2)^{n+1-np}}dv(z) \\ &\le C(M_1+M_2)(1-|a|^2). \end{aligned} \tag{4.14}$$

Let $|a|\to 1$ in (4.14), then we can get the conclusion. □

Lemma 4.5 (**Lemma 3 of** [18])**.** *Suppose X is a linear space of holomorphic functions in B with a complete seminorm $\|\cdot\|$. Assume that X satisfies the following conditions:*

(1) *X contains a nonconstant function;*

(2) *$f\circ\varphi \in X$ and $\|f\circ\varphi\| = \|f\|$ whenever $f\in X$ and $\varphi\in \mathrm{Aut}(B)$, where $\mathrm{Aut}(B)$ is the group of biholomorphic mappings of B;*

(3) *$(\theta_1,\dots,\theta_n)\mapsto f(z_1e^{i\theta_1},\dots,z_ne^{i\theta_n}) : [0,2\pi]^n \to X$ is continuous for each f in X.*

Then X contains all polynomials.

Proposition 4.6. *About $\mathcal{B}_0$, the following are equivalent:*

(i) $f\in\mathcal{B}_0$;

(ii) $f\in Q_{p,0}$, *for* $1<p<\frac{n}{n-1}$;

(iii) $\lim\limits_{|a|\to 1} J_p(f,a) = 0$, *for* $1<p<\infty$;

(iv) $\lim\limits_{|z|\to 1} |\tilde\nabla f(z)| = 0$.

Proof. By Proposition 2.1 of [9]

$$Qf(z) = \frac{1}{2}\sqrt{\tilde\triangle |f|^2(z)}.$$

Noting $\tilde{\triangle}|f|^2(z) = 4|\tilde{\nabla} f(z)|^2$ and checking the statement in Definition 3.1 of [9],

$$f \in \mathcal{B}_0 \Leftrightarrow \lim_{|z|\to 1} Qf(z) = 0,$$

we get (i) $\Leftrightarrow$ (iv).

(i) $\Leftrightarrow$ (ii). In order to utilize Lemma 4.5, let $X = Q_{p,0}, \|\cdot\| = \|\cdot\|_{Q_p}$. By Theorem 3.3, it is easy to see that $\|\cdot\|_{Q_p}$ is a $\mathcal{M}$-invariant seminorm on $Q_{p,0}$. Let us come to prove the completeness as follows.

Supposing $\{f_n\} \subset Q_{p,0}$ and $\{f_n\}$ is a Cauchy sequence in $\|\cdot\|_{Q_p}$, Theorem 3.3 asserts that there exists $f \in Q_p$ so that

$$\lim_{n\to\infty} \|f - f_n\|_{Q_p} = 0. \tag{4.15}$$

For any given $\varepsilon > 0$, there exists a positive integer N so that when $n \geq N$, have

$$I_p(f_n - f, a) < \varepsilon,$$

for all $a \in B$. By the triangle inequality in the proof of Theorem 3.3,

$$\begin{aligned}[I_p(f,a)]^{1/2} &\leq [I_p(f_N,a)]^{1/2} + [I_p(f - f_N,a)]^{1/2} \\ &< [I_p(f_N,a)]^{1/2} + \varepsilon^{1/2}.\end{aligned}$$

Since $f_N \in Q_{p,0}$, $\lim_{|a|\to 1} I_p(f_N,a) = 0$, i.e. for the above $\varepsilon > 0$, there is a $r < 1$, such that when $|a| \geq r$, have $I_p(f_N,a) < \varepsilon$. Therefore when $|a| > r$,

$$[I_p(f,a)]^{1/2} \leq \varepsilon^{1/2} + \varepsilon^{1/2} = 2\varepsilon^{1/2}.$$

Since ε is arbitrary, we get $\lim_{|a|\to 1} I_p(f,a) = 0$, and so $f \in Q_{p,0}$. Hence $\|\cdot\|_{Q_p}$ is a complete seminorm on $Q_{p,0}$.

Condition (1) in Lemma 4.5 is given by Lemma 4.4, and Condition (2) is easy to verify. Now let us verify Condition (3). Let

$$U_\theta f = f(z_1 e^{i\theta_1}, \ldots, z_n e^{i\theta_n}).$$

First we prove that for $f \in \mathcal{B}_0, \sup_{z\in B}(1 - |z|^2)|\nabla_z(U_\theta f - f)| \to 0$ when $\theta \to 0$. For any given $\varepsilon > 0$, since $f \in \mathcal{B}_0$, there exists $r_0 \in (0,1)$, such that

$$(1 - |z|^2)|\nabla_z f| < \varepsilon, \tag{4.16}$$

when $|z| > r_0$. Denoting $z^\theta = (z_1 e^{i\theta_1}, \cdots, z_n e^{i\theta_n}) = (z_1^\theta, \cdots, z_n^\theta)$, since

$$\frac{\partial}{\partial z_j} U_\theta f(z) = e^{i\theta_j} \frac{\partial f}{\partial z_j^\theta},$$

hence

$$|\nabla_z(U_\theta f)| = |\nabla_{z^\theta} f|.$$

For all $\theta \in [0, 2\pi]^n$, $|z^\theta| = |z|$, therefore

$$(1-|z|^2)|\nabla_z(U_\theta f)| = (1-|z^\theta|^2)|\nabla_{z^\theta} f| < \varepsilon, \tag{4.17}$$

for $|z| > r_0$. By (4.16) and (4.17),

$$\begin{aligned} &\sup_{|z|>r_0}(1-|z|^2)|\nabla_z(U_\theta f - f)| \\ &\le \sup_{|z|>r_0}(1-|z|^2)(|\nabla_z(U_\theta f)| + |\nabla_z f|) \\ &< 2\varepsilon, \quad \forall \theta \in [0, 2\pi]^n. \end{aligned} \tag{4.18}$$

When $|z| \le r_0, U_\theta f - f$ uniformly converges to 0 (when $\theta \to 0$), and so $|\nabla_z(U_\theta f - f)|$ uniformly converges to 0 (when $\theta \to 0$), which means that for any given $\varepsilon > 0$, there exists $\delta > 0$ so that when $|\theta_j| < \delta$, we have

$$|\nabla_z(U_\theta f - f)| < \varepsilon, \quad \forall z \in \bar{E}(0, r_0). \tag{4.19}$$

Therefore when $|\theta_j| < \delta$, by (4.18) and (4.19),

$$\sup_{z\in B}(1-|z|^2)|\nabla_z(U_\theta f - f)| < 2\varepsilon,$$

which leads to

$$\lim_{\theta\to 0}\sup_{z\in B}(1-|z|^2)|\nabla_z(U_\theta f - f)| = 0.$$

Proposition 3.6 asserts that when $1 < p < \frac{n}{n-1}$, $\|f\|_{Q_p}$ is equivalent to $\sup_{z\in B}(1-|z|^2)|\nabla_z f|$, thus

$$\lim_{\theta\to 0}\|U_\theta f - f\|_{Q_p} = 0.$$

It means that Condition (3) in Lemma 4.5 is satisfied.

By Lemma 4.4 and Lemma 4.5, $Q_{p,0}$ contains all polynomials. From the completeness of $\|\cdot\|_{Q_p}$ on $Q_{p,0}$ we know that $Q_{p,0}$ contains the closure of polynomials in $\|\cdot\|_{Q_p}$. In [**16**], It was proved that $\mathcal{B}_0$ is just the closure of polynomials in $\|\cdot\|_{\mathcal{B}}$. Because of the equivalence of $\|\cdot\|_{\mathcal{B}}$ and $\|\cdot\|_{Q_p}$ (when $1 < p < \frac{n}{n-1}$), we get $Q_{p,0} \supset \mathcal{B}_0$, which shows (i) $\Rightarrow$ (ii).

By Lemma 3.1, (ii) $\Rightarrow$ (iv). By (i) $\Leftrightarrow$ (iv), we get (i) $\Leftrightarrow$ (ii). Using the same method it can be proved that (i) $\Leftrightarrow$ (iii). $\square$

Lemma 4.7.

(i) $f \in \mathrm{VMOA} \Leftrightarrow \lim_{|a|\to 1} J_1(f, a) = 0;$

(ii) $Q_{1,0} = \text{VMOA}$.

Proof. (i) Theorem 5.1 of [4] asserts that for $f \in H^2, f \in \text{VMOA} \Leftrightarrow \mu_f$ is a vanishing Carleson measure. And Theorem 2.1 of [4] asserts that

$$\mu_f \ \text{ is a vanishing Carleson measure } \ \Leftrightarrow \lim_{|a|\to 1} M(\mu_f, a) = 0,$$

where

$$\begin{aligned} M(\mu_f, a) &= \int_B \frac{(1-|a|^2)^n}{|1-\langle a,z\rangle|^{2n}} (|\nabla f(z)|^2 - |\mathcal{R}f(z)|^2) dv(z) \\ &= \int_B |\tilde{\nabla} f(z)|^2 (1-|\varphi_a(z)|^2)^n d\lambda(z). \end{aligned}$$

Therefore

$$f \in \text{VMOA} \Leftrightarrow \lim_{|a|\to 1} \int_B |\tilde{\nabla} f(z)|^2 (1-|\varphi_a(z)|^2)^n d\lambda(z) = 0.$$

(ii) By Theorem A of [5], we know that for $f \in H^2$

$$\frac{4}{n+1} \int_B |\tilde{\nabla} f(z)|^2 G(z,0) d\lambda(z) = \int_S |f - f(0)|^2 d\sigma.$$

By Lemma 4.3 of [5], for $f \in H^2$, there exist positive C_1 and C_2, so that

$$C_1 \|f\|^2_{LP} \le \int_S |f - f(0)|^2 d\sigma \le C_2 \|f\|^2_{LP},$$

where $\|f\|^2_{LP} = \frac{4}{n+1} \int_B |\tilde{\nabla} f(z)|^2 (1-|z|^2)^n d\lambda(z)$. Hence

$$\begin{aligned} C_1 \int_B |\tilde{\nabla} f(z)|^2 (1-|z|^2)^n d\lambda(z) &\le \int_B |\tilde{\nabla} f(z)|^2 G(z,0) d\lambda(z) \\ &\le C_2 \int_B |\tilde{\nabla} f(z)|^2 (1-|z|^2)^n d\lambda(z). \end{aligned}$$

Substituting f in the above by $f \circ \varphi_a$, and using the $\mathcal{M}$-invariance of $|\tilde{\nabla} f(z)|^2$ and $d\lambda$, we get

$$C_1 J_1(f,a) \le I_1(f,a) \le C_2 J_1(f,a).$$

By (i),

$$f \in \text{VMOA} \Leftrightarrow \lim_{|a|\to 1} I_1(f,a) = 0.$$

That is $Q_{1,0}$=VMOA. ∎

Proposition 4.8. *For* $0 < p \le 1, f \in Q_{p,0} \Leftrightarrow \lim_{|a|\to 1} J_p(f,a) = 0.$

Proof. By Lemma 3.1,

$$J_p(f,a) \le C I_p(f,a),$$

thus

$$f \in Q_{p,0} \Rightarrow \lim_{|a|\to 1} I_p(f,a) = 0 \Rightarrow \lim_{|a|\to 1} J_p(f,a) = 0.$$

On the other hand, suppose $\lim_{|a|\to 1} J_p(f,a) = 0$, then

$$1 - |\varphi_a(z)|^2 < 1 \ \text{ and } \ 0 < p \le 1 \Rightarrow J_1(f,a) \le J_p(f,a),$$

thus

$$\lim_{|a|\to 1} J_p(f,a) = 0 \Rightarrow \lim_{|a|\to 1} J_1(f,a) = 0.$$

By Lemma 4.7,

$$\lim_{|a|\to 1} I_1(f,a) = 0. \tag{4.20}$$

By the property of $G(z,a)$, there exists $\delta \in (0,1)$ so that $G(z,a) \ge 1$ for $|\varphi_a(z)| < \delta$, and $G(z,a) \le C(\delta)(1-|\varphi_a(z)|^2)^n$ for $|\varphi_a(z)| \ge \delta$. Therefore for $0 < p \le 1$, we have

$$\begin{aligned} I_p(f,a) &= \int_B |\tilde{\nabla} f(z)|^2 G^p(z,a) d\lambda(z) \\ &= \int_{|\varphi_a(z)|<\delta} |\tilde{\nabla} f(z)|^2 G^p(z,a) d\lambda(z) \\ &\quad + \int_{|\varphi_a(z)|\ge\delta} |\tilde{\nabla} f(z)|^2 G^p(z,a) d\lambda(z) \\ &\le \int_{|\varphi_a(z)|<\delta} |\tilde{\nabla} f(z)|^2 G(z,a) d\lambda(z) \\ &\quad + C(\delta) \int_{|\varphi_a(z)|\ge\delta} |\tilde{\nabla} f(z)|^2 (1-|\varphi_a(z)|^2)^{np} d\lambda(z) \\ &\le I_1(f,a) + C(\delta) J_p(f,a). \end{aligned}$$

By (4.20) and the hypothesis, we get

$$\lim_{|a|\to 1} I_p(f,a) = 0.$$

□

The conclusions about $Q_{p,0}$ are summarized as follows:

Theorem 4.9. $Q_{p,0}$ *have the following properties:*

(i) *When* $0 < p \le \frac{n-1}{n}$ *or* $p \ge \frac{n}{n-1}, Q_{p,0}$ *are trivial. When* $\frac{n-1}{n} < p < \frac{n}{n-1}, Q_{p,0}$ *are nontrivial (containing at least one nonconstant function).*

(ii) $Q_{p_1,0} \subseteq Q_{p_2,0}$ *for* $0 < p_1 \le p_2 \le 1$.

(iii) $Q_{1,0} = \mathrm{VMOA}$.

(iv) $Q_{p,0} = \mathcal{B}_0$ *for* $1 < p < \frac{n}{n-1}$.

Proof. (i) can be obtained by Lemma 4.4, Proposition 4.3 and (i) of Theorem 3.8.

(ii) can be proved by Proposition 4.8.

(iii) is just Lemma 4.7.

(iv) follows from Proposition 4.6. □

Corollary 4.10. *For* $f \in H(B), 1 < p < \frac{n}{n-1}, f \in Q_{p,0}$ *if and only if* $H_{\bar{f}}$ *is compact.*

Proof. It follows from Theorem D of [**3**] and (iv) of Theorem 4.9. □

5. Appendix.

In the unit ball B, we define $\rho(z,w) = |\varphi_z(w)|$ for $z, w \in B$, where $\varphi_z \in \mathrm{Aut}(B)$, cf. Section 2.2 of [**13**].

Property.

(1) $\rho(\psi(z), \psi(w)) = \rho(z,w)$, whenever $\psi \in \mathrm{Aut}(B)$, and $z, w \in B$;

(2) $\rho(z,w)$ is a metric in B.

Proof. (1) Supposing $a = \psi^{-1}(0)$, by Theorem 2.2.5 of [**13**] we know that there exists a unitary matrix U so that $\psi = U\varphi_a$. Thus

$$\begin{aligned}1 - [\rho(\psi(z), \psi(w))]^2 &= 1 - |\varphi_{\psi(z)}(\psi(w))|^2 \\ &= \frac{(1 - |\psi(z)|^2)(1 - |\psi(w)|^2)}{|1 - \langle \psi(z), \psi(w)\rangle|^2}.\end{aligned}$$

Since

$$1 - \langle \varphi_a(z), \varphi_a(w)\rangle = \frac{(1 - \langle a,a\rangle)(1 - \langle z,w\rangle)}{(1 - \langle z,a\rangle)(1 - \langle a,w\rangle)},$$

thus

$$\begin{aligned}&1 - [\rho(\psi(z), \psi(w))]^2 \\ &= \frac{(1-|a|^2)(1-|z|^2)}{|1-\langle z,a\rangle|^2} \cdot \frac{(1-|a|^2)(1-|w|^2)}{|1-\langle w,a\rangle|^2} \cdot \frac{|1-\langle z,a\rangle|^2 \cdot |1-\langle a,w\rangle|^2}{|1-\langle a,a\rangle|^2 \cdot |1-\langle z,w\rangle|^2} \\ &= \frac{(1-|z|^2)(1-|w|^2)}{|1-\langle z,w\rangle|^2} = 1 - |\varphi_z(w)|^2 \\ &= 1 - [\rho(z,w)]^2.\end{aligned}$$

Therefore

$$\rho(\psi(z), \psi(w)) = \rho(z, w). \tag{5.1}$$

(2) By the definition of $\varphi_z(w)$, we get $|\varphi_z(w)| = |\varphi_w(z)|$, then $\rho(z, w) = \rho(w, z)$. We can also get

$$|\varphi_z(w)| = 0 \Leftrightarrow \varphi_z(w) = 0 \Leftrightarrow z = w,$$

thus

$$\rho(z, w) = 0 \Leftrightarrow z = w.$$

It is clear that $\rho(z, w) \geq 0$. From now on we are going to prove the triangle inequality, which means

$$\rho(x, y) \leq \rho(x, z) + \rho(z, y) \tag{5.2}$$

whenever $x, y, z \in B$.

First we prove (5.2) for a special case: $z = 0$ and $x = (r, 0, \dots, 0)$. Letting $|y| = t, y = (y_1, \dots, y_n), y_1 = se^{i\theta}$, then (5.2) becomes

$$\rho(x, y) \leq \rho(x, 0) + \rho(0, y) = |x| + |y| = r + t. \tag{5.3}$$

(5.3) is equivalent to

$$1 - [\rho(x, y)]^2 = \frac{(1 - t^2)(1 - r^2)}{|1 - rse^{i\theta}|^2} \geq 1 - (r + t)^2,$$

where

$$|1 - rse^{i\theta}|^2 = 1 + r^2 s^2 - 2rs\cos\theta.$$

Then (5.3) is equivalent to

$$rt^2 + 2t + rs^2[(r + t)^2 - 1] - 2s[(r + t)^2 - 1]\cos\theta \geq 0. \tag{5.4}$$

Denoting the left side of (5.4) as $f(r, t, s, \cos\theta)$, then (5.3) is equivalent to

$$f(r, t, s, \cos\theta) \geq 0, \quad \forall r, t \in [0, 1),\ s \in [0, t],\ \cos\theta \in [-1, 1]. \tag{5.5}$$

Case 1. $(r + t)^2 \geq 1$. In this case, we get $f'_{\cos\theta} \leq 0$. Then

$$f(r, t, s, \cos\theta) \geq f(r, t, s, 1), \tag{5.6}$$

where

$$f(r, t, s, 1) = rs^2[(r + t)^2 - 1] - 2s[(r + t)^2 - 1] + rt^2 + 2t.$$

If r and t are regarded as parameters and s as a variable, then $f(r,t,s,1)$ is a parabola opening upwards whose symmetric axes is $-\frac{-2[(r+t)^2-1]}{2r[(r+t)^2-1]} = \frac{1}{r} > 1$. Thus for $s \in [0,t] \subset [0,1]$, $f(r,t,s,1)$ is monotone decreasing in s, and this fact means

$$\begin{aligned} f(r,t,s,1) &\geq f(r,t,t,1) \\ &= t[4-2(r+t)^2+rt(r+t)^2]. \end{aligned} \tag{5.7}$$

Let $h(r,t) = 4-2(r+t)^2+rt(r+t)^2$, then for $r,t \in [0,1)$, we have

$$\begin{aligned} h'_r(r,t) &= (r+t)(t^2+3rt-4) \leq 0, \\ h'_t(r,t) &= (r+t)(r^2+3rt-4) \leq 0. \end{aligned}$$

Therefore

$$h(r,t) \geq h(1,t) \geq h(1,1) = 0.$$

By the above expression, (5.7) and (5.6) we know (5.5) is true for $(r+t)^2 \geq 1$.

Case 2. $(r+t)^2 < 1$. In this case, we can get $f'_{\cos\theta} > 0$. Then

$$f(r,t,s,\cos\theta) > f(r,t,s,-1) \tag{5.8}$$

where

$$f(r,t,s,-1) = rs^2[(r+t)^2-1] + 2s[(r+t)^2-1] + rt^2 + 2t.$$

If r and t are regarded as parameters and s as a variable, then $f(r,t,s,-1)$ is a parabola opening downwards whose symmetric axes is $-\frac{2[(r+t)^2-1]}{2r[(r+t)^2-1]} = -\frac{1}{r} < 0$. Thus for $s \in [0,t] \subset [0,1)$, $f(r,t,s,-1)$ is monotone decreasing in s, which leads to

$$\begin{aligned} f(r,t,s,-1) &\geq f(r,t,t,-1) \\ &= rt^2[(r+t)^2-1] + 2t[(r+t)^2-1] + rt^2 + 2t \\ &= (rt^2+2t)(r+t)^2 > 0. \end{aligned} \tag{5.9}$$

By (5.8) and (5.9) we know (5.5) is also true for $(r+t)^2 < 1$.

Combining Case 1 and Case 2, (5.5) is always true, and so (5.3) is true for $x = (r,0,\cdots,0)$. Given any $x \in B$, there exists a unitary matrix U so that $U_x = (r,0,\cdots,0)$. By (5.1) and (5.3),

$$\begin{aligned} \rho(x,y) = \rho(Ux,Uy) &= \rho((r,0,\cdots,0),Uy) \\ &\leq r + |Uy| = |x| + |y|. \end{aligned}$$

For any $z \in B$, by the above expression and (5.1), we get

$$\begin{aligned} \rho(x,y) &= \rho(\varphi_z(x), \varphi_z(y)) \\ &\leq |\varphi_z(x)| + |\varphi_z(y)| \\ &= \rho(x,z) + \rho(z,y). \end{aligned}$$

□

Remark 5. The same conclusion in one complex variable may be found in [8], but the method in [8] is difficult to extend to several variables. The proof in this appendix utilizes efficiently the $\mathcal{M}$-invariance of ρ.

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