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ESTIMATES ON SCATTERED WAVES

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We derive estimates on solutions $u(k,x)$ to a scattering problem with variable index of refraction in three space dimensions. To be precise, suppose $n(x) \in C^\infty(\mathbb{R}^3)$ is positive and $n(x) = 1$ for $|x| \geq R$. We want to estimate solutions $u(k,x)$ to

$$\big(\Delta + k^2 n(x)^2\big)u = 0, \quad u = e^{ikx\cdot\omega} + u_s, \tag{0.1}$$

where u_s satisfies the radiation condition. Here, $k \in \mathbb{R}$ denotes the frequency. There are two mechanisms that can make $u(k,x)$ large. One is the presence of trapped rays. In this work we assume there are no trapped rays. The other mechanism is the focusing of waves, i.e., the formation of caustics. Our primary goal here is to estimate the effect of this mechanism, without making any hypothesis on the geometrical nature of whatever caustics might arise. We show that

$$\|u(k,\cdot)\|_{L^\infty(\mathbb{R}^3)} \leq C\langle k\rangle, \tag{0.2}$$

where $\langle k\rangle = (1+k^2)^{1/2}$.

Introduction.

First we derive local L^2-estimates in §1. Then in §2 we show how a dilation argument and elliptic regularity yield an L^∞-estimate which is weaker than (0.2), in that the right side is replaced by $C\langle k\rangle^{3/2}$. An effort to sharpen this suggests a look at some Morrey space estimates for solutions to wave equations, in §3. Then in §4 we use these estimates together with some consequences of the global theory of Fourier integral operators to obtain the estimate (0.2). This estimate cannot be improved in general, as simple examples involving perfect focus caustics show. For simplicity we restrict attention to scattering on $\mathbb{R}^3$. Under analogous hypotheses for $\mathbb{R}^n$, we would replace the right side of (0.2) by $C\langle k\rangle^{(n-1)/2}$. The proof is quite similar, for n odd, with a few more details required for n even. Also, one could consider other perturbations of the free-space wave equation. We leave such extensions to the interested reader.

I thank Margaret Cheney for bringing this sort of problem to my attention, and Jeffrey Rauch for some useful comments on tackling caustics.

1. L^2 estimates on scattered waves.

In this section we establish local L^2-estimates for solutions to (0.1). We will assume there are *no trapped rays* and obtain estimates for all $k \in \mathbb{R}$.

To begin, choose $\varphi \in C^\infty(\mathbb{R}^3)$ such that

$$\varphi(x) = 0 \quad \text{for} \quad |x| \le R, \qquad \varphi(x) = 1 \quad \text{for } |x| \ge 2R. \tag{1.1}$$

Thus $n(x) = 1$ on supp φ. We write

$$u = \varphi(x) e^{ikx\cdot\omega} + v, \tag{1.2}$$

where $v = v(k,x)$ satisfies the radiation condition and solves

$$(\Delta + k^2 n(x)^2) v = -\Psi_k(x), \tag{1.3}$$

with

$$\Psi_k(x) = (\Delta + k^2)(\varphi e^{ikx\cdot\omega}) = (2ik\omega\cdot\nabla\varphi + \Delta\varphi) e^{ikx\cdot\omega}. \tag{1.4}$$

Note that there are no terms containing k^2.

We can relate $v(k,x)$ to the solution to the wave equation:

$$(n(x)^2\partial_t^2 - \Delta) w_k(t,x) = \psi_k(t)\Psi_k(x), \quad w_k(t,x) = 0 \text{ for } t < 0, \tag{1.5}$$

where we pick $\psi \in C_0^\infty((0,1))$ and set $\psi_k(t) = e^{-ikt}\psi(t)$. In fact,

$$\hat{w}_k(k,x) = \hat{\psi}(0) v(k,x). \tag{1.6}$$

Let us arrange that $\hat{\psi}(0) = 1$. Thus, sufficiently good estimates on $w_k(t,x)$ can lead to estimates on $v(k,x)$. Using (1.4), we have

$$\psi_k(t)\Psi_k(x) = (2ik\omega\cdot\nabla\varphi + \Delta\varphi) e^{ik(x\cdot\omega - t)}\psi(t). \tag{1.7}$$

Two properties are apparent:

$$\text{supp } \psi_k(t)\Psi_k(x) \subset [0,1] \times B_{2R}, \tag{1.8}$$

where $B_R = \{x : |x| \le R\}$, and

$$\{\psi_k(t)\Psi_k(x) : k \in \mathbb{R}\} \quad \text{bounded in} \quad H^{-1}(\mathbb{R}\times\mathbb{R}^3). \tag{1.9}$$

We deduce that, for any $T < \infty$,

$$\|w_k\|_{L^2([0,T]\times B_{3R})} \le C(T), \tag{1.10}$$

the right side being independent of k.

Now, under the hypothesis that there are no trapped rays, known results on local exponential energy decay (cf. [**M**]) apply to solutions to (1.5). It follows that there exists $T_0 < \infty$ such that, for all $s < \infty$,

$$\|w_k(t,\cdot)\|_{H^s(B_{3R})} \le Ce^{-At}, \quad \text{for} \quad t \ge T_0. \tag{1.11}$$

Here, $A > 0,\ C < \infty$; A and C may depend on s, but not on k. Let us pick $\zeta \in C_0^\infty(\mathbb{R})$ so that $\zeta(t) = 1$ for $|t| \le T_0,\ \zeta(t) = 0$ for $|t| \ge T_0 + 1$, and use (1.6) to write

$$v(k,x) = v_0(k,x) + v_1(k,x), \tag{1.12}$$

with

$$\begin{aligned} v_0(k,x) &= \int w_k(t,x)e^{ikt}\zeta(t)\,dt, \\ v_1(k,x) &= \int_{T_0}^\infty w_k(t,x)e^{ikt}[1-\zeta(t)]\,dt. \end{aligned} \tag{1.13}$$

Then the estimates (1.10)–(1.11) imply

$$\|v(k,\cdot)\|_{L^2(B_{3R})} \le B, \tag{1.14}$$

a bound independent of k. In subsequent sections, we will derive other estimates, using the facts that (by (1.11))

$$\|v_1(k,\cdot)\|_{H^s(B_{3R})} \le C_{s,N}\langle k\rangle^{-N}, \tag{1.15}$$

while $v_0(k,x)$ is accessible to methods of geometrical optics and other tools of microlocal analysis.

Once we have (1.14), since v solves the free space Helmholtz equation for $|x| \ge 2R$, well-known results on Hankel functions (see Exercise 4 in Chapter 9, §9 of [**T1**] or §5 of [**T2**]) imply

$$\|v(k,\cdot)\|_{L^2(\mathfrak{A}_N)} \le B, \quad \forall\ N, \tag{1.16}$$

where

$$\mathfrak{A}_N = \{x \in \mathbb{R}^3 : NR \le |x| \le N(R+1)\}. \tag{1.17}$$

In fact, the following more precise result follows from these Hankel function estimates. For all $S \in [3R, \infty)$,

$$\int\limits_{|x|=S} |v(k,x)|^2\,dS(x) \le B, \tag{1.18}$$

with B independent of both k and S. This estimate will be useful in the last part of the argument in §4.

We remark that, if instead of taking $k \in \mathbb{R}$, we let k run over $\mathbb{C}^+ = \{k \in \mathbb{C} : \operatorname{Im} k \geq 0\}$, then the considerations above apply, with (1.9) replaced by

$$e^{-a \operatorname{Im} k}\ \psi_k(t)\Psi_k(x) \ \text{ bounded in } \ H^{-1}(\mathbb{R}\times\mathbb{R}^3), \tag{1.19}$$

for some $a \in (0,\infty)$. Thus we get, in place of (1.14),

$$\|v(k,\cdot)\|_{L^2(B_{3R})} \leq Be^{a \operatorname{Im} k}. \tag{1.20}$$

2. First L^∞ estimates on scattered waves.

In this section we first show that elliptic estimates plus a dilation argument yields from the L^2-estimates of §1 some L^∞-estimates on the scattered waves. One of the most interesting features of the problem of estimating these waves is that, while this argument is quite natural, the estimate it yields is not sharp. We then begin to set up steps that will be taken in subsequent sections to obtain a sharp estimate.

Let us retain the hypotheses made in §1. We then have L^2 bounds on solutions to (0.1) which imply, in particular,

$$\|u(k,\cdot)\|_{L^2(B_1(p))} \leq C, \tag{2.1}$$

for all $p \in \mathbb{R}^3$, where C is independent of k and of p. Let us dilate $u(k,x)$; set

$$u_k(x) = u(k, x/k). \tag{2.2}$$

This solves the PDE

$$(\Delta + n(x/k)^2)u_k = 0, \tag{2.3}$$

and (2.1) implies

$$\|u_k\|_{L^2(B_1(q))} \leq Ck^{3/2}, \tag{2.4}$$

for all $q \in \mathbb{R}^3$, $k \geq 1$. Note that, for $k \geq 1$, $n(x/k)\big|_{B_1(q)}$ is bounded in $C^\infty(B_1(q))$. Hence elliptic estimates yield

$$\|u_k\|_{L^\infty(B_{1/2}(q))} \leq Ck^{3/2}, \tag{2.5}$$

or simply $\|u_k\|_{L^\infty(\mathbb{R}^3)} \leq Ck^{3/2}$, for $k \geq 1$. Estimates on $u(k,x)$ for $0 \leq k \leq 1$ are easy. We thus have

$$\|u(k,\cdot)\|_{L^\infty(\mathbb{R}^3)} \leq C\langle k\rangle^{3/2}. \tag{2.6}$$

As mentioned above, the estimate (2.6) is not sharp. In §4 we will show that

$$\|u(k,\cdot)\|_{L^\infty(\mathbb{R}^3)} \le C\langle k\rangle. \tag{2.7}$$

We can prove (2.7), provided we show that

$$\|u(k,\cdot)\|_{L^2(B_{1/k}(p))} \le C\langle k\rangle^{-1/2}. \tag{2.8}$$

Indeed, once we establish (2.8), we can improve (2.4) to

$$\|u_k\|_{L^2(B_1(q))} \le Ck, \tag{2.9}$$

for $k \ge 1$, hence improve (2.5), and attain (2.7). In view of the estimates of §1, the estimate (2.8) will follow if we can show that the solution w_k to (1.5) satisfies

$$\|w_k(t,\cdot)\|_{L^2(B_{1/k}(p))} \le C\langle k\rangle^{-1/2}, \quad 0 \le t \le T_0. \tag{2.10}$$

Let us make some additional comments on $w_k(t,x)$, which will be relevant for §3. Suppose that actually $n(x) = 1$ for $|x| \ge R-1$. Then, for $0 \le t \le 1$, (1.5) is a free space wave equation on the support of w_k, and a simple asymptotic analysis gives

$$w_k(t,x) = a(t,x,k)e^{ik(x\cdot\omega - t)}, \quad \partial_t w_k(t,x) = b(t,x,k)e^{ik(x\cdot\omega-t)}, \tag{2.11}$$

for $0 \le t \le 1$, where $a(t,x,k),\ b(t,x,k)$ are supported in $|x| \le 2R+1$ and, as $k \to \infty$,

$$\begin{aligned} a(t,x,k) &\sim a_0(t,x) + a_{-1}(t,x)k^{-1} + \cdots, \\ b(t,x,k) &\sim b_1(t,x)k + b_0(t,x) + \cdots. \end{aligned} \tag{2.12}$$

Now, for $1 \le t \le T_0,\ w_k(t,x)$ satisfies the *homogeneous* equation

$$(n(x)^2\partial_t^2 - \Delta)w_k = 0, \quad t \ge 1, \tag{2.13}$$

with Cauchy data

$$w_k(1,x) = a(1,x,k)e^{ik(x\cdot\omega-1)}, \quad \partial_t w_k(1,x) = b(1,x,k)e^{ik(x\cdot\omega-1)}. \tag{2.14}$$

To end this section, we indicate how to produce examples showing that the estimate (2.7) is sharp. Such examples arise when perfect focus caustics occur. One can produce a positive function $n(x) \in C^\infty(\mathbb{R}^3)$, such that $n(x) = 1$ for $|x| \ge R$, having the following properties. First, the simple progressing wave expansion of geometrical optics for the solution to (2.11) is valid for

$(t,x) \in [0,T_0] \times \mathbb{R}^3$. Second, there is a ball $B = \{x \in \mathbb{R}^3 : |x - z_0| \le \rho\}$ (with $\rho < R/2$), contained in the region $|x| > 2R$, and an interval $[T_0 - \varepsilon, T_0]$ such that, for $(t,x) \in I \times B$, this expansion has the form

$$a(t,x,k) e^{ik(\alpha + |x - z_0| - t)}, \tag{2.15}$$

for some constant $\alpha \in \mathbb{R}$, and $a(t,x,k)$ of the form (2.12), vanishing on all of B except for a small neighborhood of some boundary point, for $t \in I$. The perfect focus will occur at z_0, for some $t \in (T_0, T_0 + \rho]$, and the simple geometrical optics expansion will break down. However, the Kirchhoff formula for the solution to the free-space wave equation can be applied to analyze the solution to (2.11) for $(t,x) \in [T_0, T_0 + 2\rho] \times B$. One obtains

$$w_k(t, z_0) \sim \big[\beta_1(t)k + \beta_0(t) + \cdots\big] e^{ik(\alpha - T_0)}, \quad t \in J = [T_0, T_0 + 2\rho], \tag{2.16}$$

and β_1 is typically not identically zero on J. In turn, this leads to examples of solutions $u(k,x)$ to (0.1) such that $u(k,z_0) \sim Cke^{ik(\alpha - T_0)}$, and C is typically not zero.

3. L^∞ and Morrey-space estimates on solutions to wave equations.

Our purpose in this section is to discuss the following:

Property A. Let $w(t,x)$ solve the Cauchy problem

$$(n(x)^2 \partial_t^2 - \Delta) w = 0, \quad w(0) = f, \ w_t(0) = g \tag{3.1}$$

on $I \times \mathbb{R}^3$. Assume $f, g \in L^\infty(\mathbb{R}^3)$. Then, for each $t \in I, \ p \in \mathbb{R}^3, \ \rho \in (0,1]$,

$$\|w(t,\cdot)\|_{L^2(B_\rho(p))} \le C\|f\|_{L^\infty} \rho^{1/2} + C\|g\|_{L^\infty} \rho^{3/2}, \tag{3.2}$$

with $C = C(t)$.

If this property holds, with $I = \mathbb{R}$, then we can establish desirable estimates on $w_k(t,x)$, for $t \in [1, T_0]$, using (2.13)–(2.14). The Cauchy data (2.14) satisfy

$$\|f\|_{L^\infty} \le C, \quad \|g\|_{L^\infty} \le C\langle k \rangle, \tag{3.3}$$

and taking $\rho = 1/k$ in (3.2) this yields, for $k \ge 1$,

$$\|w_k(t,\cdot)\|_{L^2(B_{1/k}(p))} \le Ck^{-1/2} + C\langle k \rangle k^{-3/2} \le C' k^{-1/2}, \tag{3.4}$$

as desired in (2.10). Thus, when Property A holds, with $I = \mathbb{R}$, we can sharpen the estimate (2.6) on scattered waves to (2.7).

We next show that Property A does hold, with $I = \mathbb{R}$, if $n(x)$ is identically 1. While this result is not applicable to the problem raised in §2, it is intrinsically interesting, and also has a nontrivial application to a variant of (2.13)–(2.14). We will establish two results; in fact both are quite simple.

Proposition 3.1. *Let $w(t,x)$ solve the Cauchy problem*

$$(\partial_t^2 - \Delta)w = 0, \quad w(0) = 0, \ w_t(0) = g \tag{3.5}$$

on $\mathbb{R} \times \mathbb{R}^3$. If $g \in L^\infty(\mathbb{R}^3)$, then

$$\|w(t,\cdot)\|_{L^\infty} \le C|t| \cdot \|g\|_{L^\infty}. \tag{3.6}$$

Proof. In fact, the Kirchhoff formula for the solution to (3.5) gives

$$w(t,x) = \frac{t}{4\pi} \int\limits_{S^2} g(x - t\omega)\, dS(\omega), \tag{3.7}$$

so (3.6) is obvious. □

Proposition 3.2. *Let $w(t,x)$ solve the Cauchy problem*

$$(\partial_t^2 - \Delta)w = 0, \quad w(0) = f, \ w_t(0) = 0 \tag{3.8}$$

on $\mathbb{R} \times \mathbb{R}^3$. If $f \in L^\infty(\mathbb{R}^3)$, then, for $p \in \mathbb{R}^3$, $\rho \in (0,1]$,

$$\|w(t,\cdot)\|_{L^2(B_\rho(p))} \le C\langle t\rangle \|f\|_{L^\infty} \rho^{1/2}. \tag{3.9}$$

Proof. By the strong Huygens principle, the value of $w(t,x)$ for $x \in B_\rho(p)$ is unaffected if f is replaced by

$$f^\#(x) = \begin{cases} f(x) & \text{if } |t| - 2\rho \le |x-p| \le |t| + 2\rho, \\ 0 & \text{otherwise.} \end{cases} \tag{3.10}$$

Clearly

$$\|f^\#\|_{L^2(\mathbb{R}^3)} \le C\langle t\rangle \rho^{1/2} \|f\|_{L^\infty}, \tag{3.11}$$

so

$$w^\#(t) = \cos t\sqrt{-\Delta}\ f^\# \Longrightarrow \|w^\#(t)\|_{L^2} \le C\langle t\rangle \rho^{1/2} \|f\|_{L^\infty}. \tag{3.12}$$

Since $w(t,x) = w^\#(t,x)$ for $x \in B_\rho(p)$, we have (3.9). □

Note that Proposition 3.2 can be stated in terms of a Morrey space:

$$\cos t\sqrt{-\Delta} : L^\infty(\mathbb{R}^3) \longrightarrow M_2^3(\mathbb{R}^3). \tag{3.13}$$

One simple consequence of Propositions 3.1–3.2 is the following:

Proposition 3.3. *Let $w_k(t,x)$ solve the Cauchy problem*

$$(\partial_t^2 - \Delta)w_k = 0, \quad w_k(0,x) = a(x)e^{ik\theta(x)},\ \partial_t w_k(0,x) = kb(x)e^{ik\theta(x)}, \tag{3.14}$$

where $a, b \in C_0^\infty(\mathbb{R}^3)$ and $\theta \in C^\infty(\mathbb{R}^3)$ is real valued. Then, for $p \in \mathbb{R}^3, k \in \mathbb{R}$,

$$\|w_k(t,\cdot)\|_{L^2(B_{1/k}(p))} \le C(t)\langle k\rangle^{-1/2}. \tag{3.15}$$

Proof. The argument is the same as the derivation of (3.4) from (3.2). □

Note that in Proposition 3.3 we do not need to assume $\nabla\theta \neq 0$ on supp $a \cup$ supp b.

We now show that Property A holds under our hypotheses on $n(x)$ for $I = [-\tau, \tau]$, when τ is sufficiently small. In fact, the solution to (3.1) can be written

$$w(t) = R'(t)f + R(t)g, \tag{3.16}$$

and, for $|t| < \tau$,

$$R(t) = R_0(t) + B(t), \tag{3.17}$$

where $R_0(t)$ and $B(t)$ have the following properties. First,

$$R_0(t)g(x) = t\int_{S^2} a(t,x,\omega)g(\gamma_x(t\omega))\ dS(\omega), \tag{3.18}$$

where a is smooth on $(-\tau,\tau) \times \mathbb{R}^3 \times S^2$ and $\gamma_x : \mathbb{R}^3 \to \mathbb{R}^3$ is the geodesic flow from x, for the Riemannian metric

$$g_{jk}(x) = n(x)^2\delta_{jk}. \tag{3.19}$$

Furthermore, we can assume that γ_x maps $\{v : |v| < \tau\}$ diffeomorphically onto an open set in $\mathbb{R}^3$. Next, for $|t| < \tau$, $B(t)$ is a family of Fourier integral operators (FIOs) of order -2, and $B'(t)$ is a family of FIOs of order -1, having the mapping properties

$$B(t) : H^s(\mathbb{R}^3) \to H^{s+2}(\mathbb{R}^3), \quad B'(t) : H^s(\mathbb{R}^3) \to H^{s+1}(\mathbb{R}^3). \tag{3.20}$$

The representation (3.16)–(3.18) is a special case of the Hadamard parametrix construction; a derivation can be found in Proposition 17.4.3 of [**H**].

We can now prove the following extensions of Propositions 3.1–3.2.

Proposition 3.4. *Let $w(t,x)$ solve the Cauchy problem*

$$(n(x)^2\partial_t^2 - \Delta)w = 0, \quad w(0) = 0, \; w_t(0) = g \tag{3.21}$$

on $\mathbb{R}\times\mathbb{R}^3$. If $g \in L^\infty(\mathbb{R}^3)$, then, for $|t| < \tau$,

$$\|w(t,\cdot)\|_{L^\infty} \le C(t)\|g\|_{L^\infty}. \tag{3.22}$$

Proof. The fact that $R_0(t) : L^\infty(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3)$ for $|t| < \tau$ is clear from (3.18). That $B(t) : L^\infty(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3)$ follows from finite propagation speed plus (3.20), plus the Sobolev imbedding result that $H^2(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3)$. □

Proposition 3.5. *Let $w(t,x)$ solve the Cauchy problem*

$$(n(x)^2\partial_t^2 - \Delta)w = 0, \quad w(0) = f, \; w_t(0) = 0 \tag{3.23}$$

on $\mathbb{R}\times\mathbb{R}^3$. If $f \in L^\infty(\mathbb{R}^3)$, then, for $|t| < \tau,\; p \in \mathbb{R}^3,\; \rho \in (0,1]$,

$$\|w(t,\cdot)\|_{L^2(B_\rho(p))} \le C(t)\|f\|_{L^\infty}\cdot\rho^{1/2}. \tag{3.24}$$

Proof. Defining $f^\#$ as in (3.10), we see that

$$R_0'(t)f = R_0'(t)f^\# \;\text{ for }\; x \in B_\rho(p), \tag{3.25}$$

and since $R_0'(t) : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$, we obtain

$$\|R_0'(t)f\|_{L^2(B_\rho(p))} \le C(t)\|f^\#\|_{L^2} \le C(t)\|f\|_{L^\infty}\rho^{1/2}. \tag{3.26}$$

Meanwhile, by finite propagation speed and (3.20) we have

$$\|B'(t)f\|_{H^1(B_1(p))} \le C(t)\|f\|_{L^\infty}, \tag{3.27}$$

and (3.24) follows from (3.26)–(3.27), since

$$H^1(B_1(p)) \subset L^6(B_1(p)) \subset M_2^3(B_1(p)). \tag{3.28}$$

□

More generally, we can replace $\mathbb{R}^3$ by $\mathbb{R}^n$ for any odd n, and obtain the same result. Also, we can replace L^∞ by $L^p,\ 2 \le p \le \infty$, obtaining

$$\|w(t,\cdot)\|_{L^2(B_\rho(p))} \le C(t)\|f\|_{L^p}\rho^{1/2-1/p}, \tag{3.29}$$

or, equivalently,

$$\|w(t)\|_{M_2^q} \le C(t)\|f\|_{L^p}, \quad q = \frac{2n}{n-1+\frac{2}{p}}. \tag{3.30}$$

In particular,

$$\|w(t)\|_{M_2^{2n/(n-1)}} \le C(t)\|f\|_{L^\infty}. \tag{3.31}$$

For related results, see [**T3**].

4. Sharper estimate on scattered waves.

Given $x_0 \in \mathbb{R}^3,\ t_0 \in [0,T_0]$, we want to estimate $w(t,x)$ in a neighborhood of (t_0,x_0), when w solves (2.13)–(2.14). Now $w(t,x)$ is given by a simple geometrical optics expansion away from a caustic set $\mathcal{C} \subset (1,\infty)\times\mathbb{R}^3$. In particular, as a consequence of the global theory of Fourier integral operators, as presented in [**D**] or [**H**], such a geometrical optics expansion is even valid in a region swept out by rays that have passed through $\mathcal{C}$. Here, "rays" are null geodesics in $\mathbb{R}\times\mathbb{R}^3$, with the Lorentz metric $-dt^2 + n(x)^2\sum_{j=1}^3 dx_j^2$.

It is convenient to describe this in terms of the following Lagrangian manifold $\Omega \subset T^*(\mathbb{R}\times\mathbb{R}^3)\setminus 0$. For t close to 1, Ω is the graph of the phase function $\theta(t,x) = x\cdot\omega - t$. For larger t, it is the flow-out of this graph under the geodesic flow on the Lorentz manifold $\mathbb{R}\times\mathbb{R}^3$ described above. The orbits of this flow are the lifts to $T^*(\mathbb{R}\times\mathbb{R}^3)$ of null geodesics, called null bicharacteristics. Note that $\Omega \subset T^*(\mathbb{R}\times\mathbb{R}^3)\setminus 0$. The caustic set $\mathcal{C}$ is the image of that part Σ of Ω where the projection to $\mathbb{R}\times\mathbb{R}^3$ is singular. Over the complement of $\mathcal{C}$, Ω is a finite union of graphs of gradients of functions arising as phases in the geometrical optics expansion of $w(t,x)$.

If $(t_0,x_0) \notin \mathcal{C}$, then $w(t,x)$ is bounded uniformly on a neighborhood of (t_0,x_0). We now consider the case where $(t_0,x_0)\in\mathcal{C}$. Let $\Gamma(t_0,x_0)$ denote the subset of Σ lying over (t_0,x_0); this is a compact subset of $T^*(\mathbb{R}\times\mathbb{R}^3)\setminus 0$.

Take a point $p_0 \in \Gamma(t_0,x_0)$. Methods of the Morse theory of conjugate points (cf. [**C**]) imply that the null bicharacteristic through p_0 intersects Σ in a discrete set. Using a covering argument and partitions of unity, we have the following. (Here, τ is as in Proposition 3.4.)

Lemma 4.1. *There exist $\sigma_j \in (0,\tau),\ 1 \le j \le N$, and Cauchy data*

$$W_j(t_0-\sigma_j,x) = a_j(x,k)e^{ik\theta_j(x)}, \quad W_{jt}(t_0-\sigma_j,x) = b_j(x,k)e^{ik\theta_j(x)}, \tag{4.1}$$

with

$$a_j(x,k) \sim a_{j0}(x)+a_{j1}(x)k^{-1}+\cdots, \quad b_j(x,k) \sim b_{j1}(x)k+b_{j0}(x)+\cdots, \tag{4.2}$$

such that

$$\sum_{j=1}^{N} S(t, t_0 - \sigma_j)(W_j, W_{jt}) \tag{4.3}$$

agrees with $w(t, x)$, *mod* C^∞, *on a neighborhood of* (t_0, x_0) *where* $S(t, s)$ *is the solution operator to the wave equation* (2.13), *with Cauchy data at time* s.

Now, Propositions 3.4–3.5 apply to (4.3), so we have an estimate of the form (3.4) at (t_0, x_0), i.e.,

$$\|w(t_0, \cdot)\|_{L^2(B_{1/k}(x_0))} \le C k^{-1/2}. \tag{4.4}$$

This estimate is seen to hold uniformly for $t_0 \in [1, T_0]$, where T_0 is as in (1.11), and for x_0 in any compact $K \subset \mathbb{R}^3$. Now we are ready to establish our main result:

Theorem 4.2. *Under the hypothesis of no trapped rays, the scattering solution to* (0.1) *satisfies the estimate*

$$|u(k, x)| \le C\langle k \rangle, \tag{4.5}$$

for all $x \in \mathbb{R}^3$.

Proof. From (4.4) and (1.12)–(1.13) we have the estimate

$$\|u(k, \cdot)\|_{L^2(B_{1/k}(p))} \le C k^{-1/2}, \tag{4.6}$$

valid uniformly for p in any compact set, e.g., for $|p| \le 3R$. On the other hand, the estimate (1.18) implies that (4.6) holds uniformly for $|p| \ge 3R$. From here, the argument given in (2.8)–(2.10) establishes (4.5). □

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