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RIGGED NON-TANGENTIAL MAXIMAL FUNCTION ASSOCIATED WITH TOEPLITZ OPERATORS AND HANKEL OPERATORS

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We give a sufficient condition for the boundedness of products of Toeplitz operators and Hankel operators in terms of a distributional inequality for the symbol functions.

Let T denote the unit circle and dm the normalized Lebesgue measure on T. For $1 \leq p \leq \infty$, L^p stands for $L^p(T, dm)$. As usual, H^p is the Hardy subspace of L^p. Let $P : L^2 \to H^2$ be the orthogonal projection. For $f \in L^2$, the Toeplitz operator T_f and the Hankel operator H_f are defined by the formulas $T_f\varphi = Pf\varphi$ and $H_f\varphi = (1-P)f\varphi$, $\varphi \in H^2$, whenever these expressions make sense. Thus the domains of T_f and H_f contain at least H^∞.

Let D be the open disc $\{z \in \mathbf{C} : |z| < 1\}$. We denote the Poisson kernel associated with $z \in D$ by P_z. That is, $P_z(\tau) = (1-|z|^2)/|1-\bar{\tau}z|^2$. We write $f(z) = \int_T fP_z dm$, $z \in D$, for $f \in L^1$. In particular, for any measurable set $E \subset T$, $\chi_E(z) = \int_E P_z dm$ is the value of the harmonic extension of χ_E at z. The well-known probabilistic interpretation of $\chi_E(z)$ may help put the results stated below in perspective: $\chi_E(z)$ is the probability that a Brownian walker starting at the point z will exit D through E.

In this note we will address a question raised by Sarason in [**5**, **6**], namely when is the product T_gT_f a bounded operator on H^2? This problem, which is non-trivial only if at least one symbol function is unbounded, arose from Sarason's study of exposed points in the unit ball of H^1. He showed that, for outer functions $f, g \in H^2$, a necessary condition for $T_gT_{\bar{f}}$ to be bounded is that

$$\sup_{|z|<1} \int_T |g|^2 P_z dm \int_T |f|^2 P_z dm < \infty.$$

An argument due to Treil (see [**6**]) further shows that this condition is necessary for the boundedness of $T_gT_{\bar{f}}$ whenever $f, g \in H^2$. Sarason observed that a related (but strictly more general) problem is the boundedness of the product of Hankel operators $H_g^*H_f$.

Using the ideas of Axler, Chang and Sarason [**1**], mainly the area-integral function technique, Zheng showed in [**10**] that for f, $g \in L^2$, if there is an

$\epsilon > 0$ such that

$$(1) \qquad N_1 = \sup_{|z|<1} \int_T |f - f(z)|^{2+\epsilon} P_z dm \int_T |g - g(z)|^{2+\epsilon} P_z dm < \infty,$$

then $H_g^* H_f$ is bounded. He also showed that for f, $g \in H^2$, if there is an $\epsilon > 0$ such that

$$(2) \qquad N_2 = \sup_{|z|<1} \int_T |f|^{2+\epsilon} P_z dm \int_T |g|^{2+\epsilon} P_z dm < \infty,$$

then $T_g T_{\bar{f}}$ is bounded. Recall that the theorem of Marcinkiewicz and Zygmund [4] asserts that Lusin's area-integral operator S is bounded on H^p for all $0 < p < \infty$. Because of the properties of the Hilbert transform, it follows immediately that S is bounded on L^p when $p > 1$. This is the key fact on which [10] relied.

This result was generalized by Treil, Volberg and Zheng to the setting of Orlicz spaces and Lorentz spaces [9]. In both [10] and [9], Hölder's inequality was used in an essential way in their estimates for the boundedness of the operators in question. It is well known that estimates relying on Hölder's inequality are usually less than optimal because of the nature of that inequality. In this paper we show that there are ways to get around Hölder's inequality, thereby obtaining sharper results.

The starting point of our investigation is Stein's result that S also has weak-type (1,1) [7]. This suggests to us that the area-integral technique in [1, 9, 10] can be further exploited. Note that, by Hölder's inequality, (1) and (2) respectively *imply* that for any $z \in D$ and for any measurable sets A, $B \subset T$,

$$(1') \quad \int_A |f - f(z)| P_z dm \int_B |g - g(z)| P_z dm \le N_1^{1/(2+\epsilon)} (\chi_A(z)\chi_B(z))^{(1+\epsilon)/(2+\epsilon)},$$

$$(2') \qquad \int_A |f| P_z dm \int_B |g| P_z dm \le N_2^{1/(2+\epsilon)} (\chi_A(z)\chi_B(z))^{(1+\epsilon)/(2+\epsilon)}.$$

We will replace the factor $(\chi_A(z)\chi_B(z))^{(1+\epsilon)/(2+\epsilon)}$ in the above by something much larger and still obtain boundedness for $H_g^* H_f$ and $T_g T_{\bar{f}}$.

To state our main results, we need to introduce a class of weight functions. Let $\mathcal{W}$ denote the collection of functions $w : (0, 1] \to (0, 1]$ which are non-decreasing and have the property that

$$\int_0^1 \left(\frac{w(t)}{t^2} \right)^{2/3} dt < \infty.$$

Theorem 1. *Suppose that f, $g \in L^2$ have the following property: There are u, $v \in \mathcal{W}$ and a positive number N such that for any $z \in D$ and for any measurable sets A, $B \subset T$ of positive measure,*

$$\int_A |f - f(z)|P_z dm \int_B |g - g(z)|P_z dm \le Nu(\chi_A(z))v(\chi_B(z)).$$

Then $H_g^* H_f$ is bounded.

Theorem 2. *Suppose that f, $g \in L^2$ have the following property: There are u, $v \in \mathcal{W}$ and a positive number N such that for any $z \in D$ and for any measurable sets A, $B \subset T$ of positive measure,*

$$\int_A |f|P_z dm \int_B |g|P_z dm \le Nu(\chi_A(z))v(\chi_B(z)).$$

Then $T_g T_f$ is bounded.

The above-mentioned boundedness results of Zheng are recovered by applying these theorems to the case where $u(t) = v(t) = t^{(1+\epsilon)/(2+\epsilon)} = t^{1/2} \cdot t^{\epsilon/(4+2\epsilon)}$. What original motivated our investigation were examples such as $u(t) = v(t) = t^{1/2} \cdot (1 - \log t)^{-(3+\epsilon)/2}, \epsilon > 0$.

Our main technical innovation is the introduction of a rigged non-tangential maximal function. Section 1 contains an L^1-boundedness result for this maximal function. The proofs of Theorems 1 and 2 are given in Section 2 after further preparations.

1. Rigged non-tangential maximal function.

For each $\tau \in T$, let $\Gamma_\tau = \{z : |\tau - z| < 2(1 - |z|), 3/4 < |z| < 1\}$. Suppose that F, G are continuous maps from D into L^2 with respect to the norm topology. To avoid confusion with complex-valued functions we denote their values at $z \in D$ by F_z and G_z respectively. In other words, for each $z \in D$, F_z and G_z are themselves functions on T. Given φ, $\psi \in L^2$, we introduce the rigged non-tangential maximal function

$$M_{F,G}(\varphi, \psi)(\tau) = \sup_{z \in \Gamma_\tau} \int_T |\varphi F_z| P_z dm \int_T |\psi G_z| P_z dm, \quad \tau \in T.$$

To simplify notation, the Lebesgue measure of a measurable set $E \subset T$ will be denoted by $|E|$. Also, for a real-valued function f on T and a $\lambda \in \mathbf{R}$, the set $\{\tau \in T : f(\tau) > \lambda\}$ will simply be denoted by $\{f > \lambda\}$. The sets $\{f \le \lambda\}$, $\{\lambda_1 \le f < \lambda_2\}$, etc., are accordingly understood.

For each $\varphi \in L^1$, denote its usual non-tagential maximal function by $M_{\mathrm{nt}}(\varphi)$. That is,

$$M_{\mathrm{nt}}(\varphi)(\tau) = \sup_{z \in \Gamma_\tau} |\varphi(z)|, \quad \tau \in T.$$

Recall that there is an absolute constant $C > 0$ such that $M_{\rm nt}(\varphi) \le CM(\varphi)$, where $M(\varphi)$ is the Hardy-Littlewood maximal function of φ [3, page 24]. This means $M_{\rm nt}$ is of weak-type (1,1). In other words, there is an absolute constant $C_{\rm nt} > 0$ such that $|\{M_{\rm nt}(\varphi) > \lambda\}| \le C_{\rm nt}\|\varphi\|_1/\lambda$ for all $\varphi \in L^1$ and $\lambda > 0$. In particular, if E is a measurable set in T, then

$$|\{M_{\rm nt}(\chi_E) > 2^{-i}\}| \le C_{\rm nt}2^i|E|, \quad i \in \mathbf{N}. \tag{1.1}$$

Proposition 1.1. *Suppose that F, G are continuous maps from D into L^2 which have the following property: There exist an $N > 0$ and u, $v \in \mathcal{W}$ such that for any $z \in D$ and any measurable sets A, $B \subset T$ with $|A| > 0$ and $|B| > 0$,*

$$\int_A |F_z|P_z dm \int_B |G_z|P_z dm \le Nu(\chi_A(z))v(\chi_B(z)). \tag{1.2}$$

Then there is a $K > 0$ such that for any φ, $\psi \in L^2$,

$$\|M_{F,G}(\varphi,\psi)\|_1 \le K\|\varphi\|_2\|\psi\|_2.$$

Proof. Without loss of generality, we may assume that $u(1) = v(1) = 1$. Set $u_i = 2^{i/3}(u(2^{-i}))^{2/3}$ and $v_i = 2^{i/3}(v(2^{-i}))^{2/3}$, $i \in \mathbf{Z}_+$. It follows from the monotonicity of u and v that

$$\begin{aligned} U = \sum_{i=0}^{\infty} u_i &\le 1 + \frac{2^{1/3}}{\log 2}\sum_{i=0}^{\infty}\int_{2^{-i-1}}^{2^{-i}} (u^2(t)/t)^{1/3}\frac{dt}{t} \\ &= 1 + \frac{2^{1/3}}{\log 2}\int_0^1 \left(\frac{u(t)}{t^2}\right)^{2/3} dt < \infty, \\ V = \sum_{i=0}^{\infty} v_i &\le 1 + \frac{2^{1/3}}{\log 2}\sum_{i=0}^{\infty}\int_{2^{-i-1}}^{2^{-i}} (v^2(t)/t)^{1/3}\frac{dt}{t} \\ &= 1 + \frac{2^{1/3}}{\log 2}\int_0^1 \left(\frac{v(t)}{t^2}\right)^{2/3} dt < \infty. \end{aligned}$$

It suffices to consider non-negative φ, $\psi \in L^2$ with $\|\varphi\|_2 = \|\psi\|_2 = 1$. Define

$$\begin{aligned} A_{k,0} &= \{\varphi^2 \le 2^k\}, \quad B_{k,0} = \{\psi^2 \le 2^k\}, \\ A_{k,i} &= \{2^{k+i-1}/u_{i-1} \le \varphi^2 < 2^{k+i}/u_i\}, \\ B_{k,i} &= \{2^{k+i-1}/v_{i-1} \le \psi^2 < 2^{k+i}/v_i\}, \end{aligned}$$

$k, i \in \mathbf{N}$. (Recall that u and v do not vanish on (0,1].) For such a pair of k and i, let

$$X_{k,i} = \{M_{\rm nt}(\chi_{A_{k,i}}) > 2^{-i}\}, \quad Y_{k,i} = \{M_{\rm nt}(\chi_{B_{k,i}}) > 2^{-i}\}.$$

It follows from (1.2) that

$$
\begin{aligned}
&\int_T |F_z\varphi| P_z dm \int_T |G_z\psi| P_z dm \\
&\le \sum_{i,j=0}^{\infty} \left(\frac{2^{2k+i+j}}{u_i v_j}\right)^{1/2} \int_{A_{k,i}} |F_z| P_z dm \int_{B_{k,j}} |G_z| P_z dm \\
&\le N 2^k \sum_{i,j=0}^{\infty} \left(\frac{2^{i+j}}{u_i v_j}\right)^{1/2} u(\chi_{A_{k,i}}(z)) v(\chi_{B_{k,j}}(z)) \\
&= N 2^k \sum_{i=0}^{\infty} \sqrt{\frac{2^i}{u_i}} u(\chi_{A_{k,i}}(z)) \sum_{j=0}^{\infty} \sqrt{\frac{2^j}{v_j}} v(\chi_{B_{k,j}}(z)). \qquad (1.3)
\end{aligned}
$$

Now if $\tau_0 \in T \setminus \cup_{i=1}^{\infty} (X_{k,i} \cup Y_{k,i})$, then $\chi_{A_{k,i}}(z) \le 2^{-i}$ and $\chi_{B_{k,j}}(z) \le 2^{-j}$ for any $z \in \Gamma_{\tau_0}$ and $i,\ j \in \mathbf{N}$. The monotonicity of u then implies that $u_i^{3/2} = 2^{i/2} u(2^{-i}) \ge 2^{i/2} u(\chi_{A_{k,i}}(z))$. That is, $(2^i/u_i)^{1/2} u(\chi_{A_{k,i}}(z)) \le u_i$. Similarly $(2^j/v_j)^{1/2} v(\chi_{A_{k,j}}(z)) \le v_j$. Therefore it follows from (1.3) that

$$
\int_T |F_z\varphi| P_z dm \int_T |G_z\psi| P_z dm \le NUV 2^k
$$

if $\tau_0 \in T \setminus \cup_{i=1}^{\infty} (X_{k,i} \cup Y_{k,i})$ and $z \in \Gamma_{\tau_0}$. This implies that

$$
\{M_{F,G}(\varphi,\psi) > NUV 2^k\} \subset \bigcup_{i=1}^{\infty} (X_{k,i} \cup Y_{k,i}).
$$

By (1.1), $|X_{k,i}| \le C_{\mathrm{nt}} 2^i |A_{k,i}|$, $|Y_{k,i}| \le C_{\mathrm{nt}} 2^i |B_{k,i}|$. Thus

$$
|\{M_{F,G}(\varphi,\psi)/NUV > 2^k\}| \le C_{\mathrm{nt}} \sum_{i=1}^{\infty} 2^i (|A_{k,i}| + |B_{k,i}|).
$$

Since $\|f\|_1 \le 2|T| + \sum_{k=1}^{\infty} 2^{k+1} |\{2^k < |f| \le 2^{k+1}\}|$ for every $f \in L^1$, we have

$$
\|M_{F,G}(\varphi,\psi)/NUV\|_1 \le 2 + 2C_{\mathrm{nt}} \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} 2^{k+i} (|A_{k,i}| + |B_{k,i}|). \qquad (1.4)
$$

Because $u \le 1$ on (0,1], $2^{k+i-1}/u_{i-1} \ge 2^{2i/3} \ge 1$ for all $k, i \in \mathbf{N}$. Therefore

$$
\begin{aligned}
&\sum_{k,i=1}^{\infty} 2^{k+i}|A_{k,i}| \\
&\le 2\sum_{k,i=1}^{\infty} 2^{k+i-1}|\{\varphi^2 \ge 2^{k+i-1}/u_{i-1}\}| \\
&= 2\sum_{\ell=0}^{\infty} \sum_{2^\ell \le 2^{k+i-1}/u_{i-1} < 2^{\ell+1}} (2^{k+i-1}/u_{i-1})u_{i-1}|\{\varphi^2 \ge 2^{k+i-1}/u_{i-1}\}| \\
&\le 2\sum_{\ell=0}^{\infty} 2^{\ell+1}|\{\varphi^2 \ge 2^\ell\}| \sum_{2^\ell \le 2^{k+i-1}/u_{i-1} < 2^{\ell+1}} u_{i-1}.
\end{aligned}
\tag{1.5}
$$

Note that for each pair of $i \in \mathbf{N}$ and $\ell \in \mathbf{Z}_+$, there is at most one $k \in \mathbf{N}$ for which the inequality $2^\ell \le 2^{k+i-1}/u_{i-1} < 2^{\ell+1}$ holds. This means that for each fixed ℓ,

$$\sum_{2^\ell \le 2^{k+i-1}/u_{i-1} < 2^{\ell+1}} u_{i-1} \le \sum_{i=1}^{\infty} u_{i-1} = U.$$

Also, since $1+2+\cdots+2^\ell < 2^{\ell+1}$, $\sum_{\ell=0}^{\infty} 2^\ell|\{\varphi^2 \ge 2^\ell\}| \le 2\|\varphi^2\|_1 = 2\|\varphi\|_2^2 = 2$. Hence it follows from (1.5) that

$$\sum_{k,i=1}^{\infty} 2^{k+i}|A_{k,i}| \le 8U.$$

By the same argument,

$$\sum_{k,i=1}^{\infty} 2^{k+i}|B_{k,i}| \le 8V.$$

Combining these with (1.4), we see that

$$\|M_{F,G}(\varphi,\psi)\|_1 \le NUV(2 + 16C_{\mathrm{nt}}(U+V)).$$

This completes the proof. □

2. Area-Integral.

For each $z = |z|e^{i\theta_z} \in D$, define the open arcs

$$I_z = \{e^{i\theta} : |\theta - \theta_z| < (1-|z|)/2\}, \quad J_z = \{e^{i\theta} : |\theta - \theta_z| < 3(1-|z|)/2\}.$$

Because of the normalization $|T| = 1$, we have $|I_z| = (1-|z|)/2\pi$. If $\tau = e^{i\theta}$ is a point in I_z, then $\tau - z = e^{i\theta}(1-|z|) + z(e^{i(\theta-\theta_z)} - 1)$. Therefore $|\tau - z| \le (1-|z|) + (e-1)|\theta - \theta_z| < 2(1-|z|)$. That is, if $3/4 < |z| < 1$ and $\tau \in I_z$, then $z \in \Gamma_\tau$.

Let $dA(z)$ denote the area measure on D. Recall that for each $\varphi \in L^1$, Lusin's area-integral function is defined by the formula

$$S(\varphi)(\tau) = \left(\int_{\Gamma_\tau} |\nabla\varphi(z)|^2 dA(z)\right)^{1/2}.$$

Using the Calderón-Zygmund decomposition of L^1-functions, Stein proved that S is of weak-type (1,1) [7, Lemma 12]. (Stein's proof was given for half-spaces in $\mathbf{R}^n$, but, with obvious modifications, the proof works in D as well.) That is, there is an absolute constant $C_S > 0$ such that

$$|\{S(\xi) > \lambda\}| \le C_S\|\xi\|_1/\lambda \tag{2.1}$$

for all $\xi \in L^1$ and $\lambda > 0$.

For each $0 < a < 1$ and each $\tau \in T$, let $\Gamma_{\tau,a} = \Gamma_\tau \cap \{z : 1 - a < |z| < 1\}$. We set $\Gamma_{\tau,0} = \emptyset$. Define the truncated area-integral function

$$S_a(\varphi)(\tau) = \left(\int_{\Gamma_{\tau,a}} |\nabla\varphi(z)|^2 dA(z)\right)^{1/2}.$$

Recall that $\partial = ((\partial/\partial x) - i(\partial/\partial y))/2$ and $\bar\partial = ((\partial/\partial x) + i(\partial/\partial y))/2$ in real variables. Thus $|\nabla\varphi(z)|^2 = 2(|\partial\varphi(z)|^2 + |\bar\partial\varphi(z)|^2)$. For $\varphi \in L^2$, we have $|\nabla((1-P)\varphi)(z)|^2 = 2|\bar\partial((1-P)\varphi)(z)|^2 = 2|\bar\partial\varphi(z)|^2 \le |\nabla\varphi(z)|^2$ and $|\nabla(P\varphi)(z)|^2 = 2|\partial(P\varphi)(z)|^2 = 2|\partial\varphi(z)|^2 \le |\nabla\varphi(z)|^2$. Hence for any given $\varphi \in L^2$,

$$S_a((1-P)\varphi) \le S_a(\varphi), \quad S_a(P\varphi) \le S_a(\varphi). \tag{2.2}$$

There is a $C_{2.3} > 0$ such that for any $f \in L^1$, $3/4 < |z| < 1$ and $\tau \in I_z$,

$$S_{2(1-|z|)}(\chi_{T\setminus J_z} f)(\tau) \le C_{2.3}\int_T |f|P_z dm. \tag{2.3}$$

To verify this elementary claim, note that

$$|\bar\partial(\chi_{T\setminus J_z} f)(\zeta)| = \left|\frac{d}{d\bar\zeta}\int_{T\setminus J_z} \frac{\bar\zeta\gamma f(\gamma)}{1-\bar\zeta\gamma} dm(\gamma)\right| \le \int_{T\setminus J_z} \frac{|f(\gamma)|}{|1-\zeta\bar\gamma|^2} dm(\gamma).$$

There is a $C_1 > 0$ such that $C_1|\gamma - \tau| \ge |\gamma - z|$ when $\tau \in I_z$ and $\gamma \in T\setminus J_z$. Also, there is a $C_2 > 0$ such that $C_2 d(\gamma, \Gamma_\tau) \ge |\gamma - \tau| \ge |\gamma - z|/C_1$ for such z, τ, γ. That is, for $\zeta \in \Gamma_\tau$ and $\gamma \in T\setminus J_z$, $C_1C_2|\gamma - \zeta| \ge |\gamma - z|$. Hence

$$|\bar\partial(\chi_{T\setminus J_z} f)(\zeta)| \le (C_1C_2)^2\int_{T\setminus J_z} \frac{|f(\gamma)|}{|1-z\bar\gamma|^2} dm(\gamma)$$

for $\zeta \in \Gamma_\tau$. There is a $C_3 > 0$ such that the area of $\Gamma_{\tau,2(1-|z|)}$ does not exceed $C_3(1-|z|^2)^2$. Squaring the above and integrating over $\Gamma_{\tau,2(1-|z|)}$, we see that

$$\int_{\Gamma_{\tau,2(1-|z|)}} |\bar{\partial}(\chi_{T\setminus J_z} f)(\zeta)|^2 dA(\zeta) \le C_3 \left((C_1C_2)^2 \int_T |f|P_z dm \right)^2.$$

Repeating the above argument with ∂ in place of $\bar{\partial}$, we see (2.3) holds with $C_{2.3} = 2C_3^{1/2}(C_1C_2)^2$.

Proposition 2.1. *There is an absolute constant $B_{2.1} > 0$ such that for any continuous maps F, $G : D \to L^2$, φ, $\psi \in L^2$, and any $3/4 < |z| < 1$,*

$$|\{S_{2(1-|z|)}(F_z\varphi)S_{2(1-|z|)}(G_z\psi) \le B_{2.1}M_{F,G}(\varphi,\psi)\} \cap I_z| \ge |I_z|/2.$$

Proof. Since $F_z\varphi = F_z\varphi\chi_{J_z} + F_z\varphi\chi_{T\setminus J_z}$, by the subadditivity of S_a and (2.3),

$$\begin{aligned}\left\{S_{2(1-|z|)}(F_z\varphi) > \frac{12C_S}{|J_z|}\int_{J_z}|F_z\varphi|dm + 2C_{2.3}\int_T |F_z\varphi|P_z dm\right\} \cap I_z \\ \subset \left\{S_{2(1-|z|)}(\chi_{J_z}F_z\varphi) > \frac{12C_S}{|J_z|}\int_{J_z}|F_z\varphi|dm\right\} \cap I_z \\ \subset \left\{S(\chi_{J_z}F_z\varphi) > \frac{12C_S}{|J_z|}\int_{J_z}|F_z\varphi|dm\right\} \cap I_z.\end{aligned}$$

If $\|\chi_{J_z}F_z\varphi\|_1 = 0$, then the above set is empty. If $\|\chi_{J_z}F_z\varphi\|_1 \ne 0$, apply (2.1) to the case where $\xi = \chi_{J_z}F_z\varphi$ and $\lambda = 12C_S\int_{J_z}|F_z\varphi|dm/|J_z| = 12C_S\|\xi\|_1/3|I_z|$. Therefore we have

$$\begin{aligned}\left|\left\{S_{2(1-|z|)}(F_z\varphi) > \frac{12C_S}{|J_z|}\int_{J_z}|F_z\varphi|dm + 2C_{2.3}\int_T |F_z\varphi|P_z dm\right\} \cap I_z\right| \\ \le \left|\left\{S(\chi_{J_z}F_z\varphi) > \frac{12C_S}{|J_z|}\int_{J_z}|F_z\varphi|dm\right\}\right| \le |I_z|/4\end{aligned}$$

in any case. There is a $C_4 > 0$ such that $\chi_{J_z}/|J_z| \le C_4P_z$. Hence

$$\left|\left\{S_{2(1-|z|)}(F_z\varphi) > (12C_4C_S + 2C_{2.3})\int_T |F_z\varphi|P_z dm\right\} \cap I_z\right| \le |I_z|/4.$$

Applying the same argument with G_z, ψ in place of F_z, φ, we obtain

$$\left|\left\{S_{2(1-|z|)}(G_z\psi) > (12C_4C_S + 2C_{2.3})\int_T |G_z\psi|P_z dm\right\} \cap I_z\right| \le |I_z|/4.$$

Let $B_{2.1} = (12C_4C_S + 2C_{2.3})^2$. Then the set E which consists of all $\tau \in I_z$ such that

$$S_{2(1-|z|)}(F_z\varphi)(\tau)S_{2(1-|z|)}(G_z\psi)(\tau) \le B_{2.1}\int_T |F_z\varphi|P_z dm \int_T |G_z\psi|P_z dm$$

has measure at least $|I_z|/2$. Again, $z \in \Gamma_\tau$ if $\tau \in I_z$. Hence the inequality

$$S_{2(1-|z|)}(F_z\varphi)(\tau)S_{2(1-|z|)}(G_z\psi)(\tau) \le B_{2.1}M_{F,G}(\varphi,\psi)(\tau)$$

holds whenever $\tau \in E$. This completes the proof. □

It should be acknowledged that the idea of decomposing $F_z\varphi$ as $F_z\varphi\chi_{J_z} + F_z\varphi\chi_{T\setminus J_z}$ in the above proof can be traced back to [1].

Proposition 2.2. *There is an absolute constant $C_{2.2} > 0$ such that the following hold true:*

(i) *For any f, $g \in L^2$, there is a $C(f,g) > 0$ such that for any φ, $\psi \in H^2$ with the property that $H_f\varphi$, $H_g\psi \in L^2$, we have*

$$|\langle H_f\varphi, H_g\psi\rangle| \le C_{2.2}\int_T M_{F,G}(\varphi,\psi)dm + C(f,g)\|\varphi\|_2\|\psi\|_2,$$

where F, $G : D \to L^2$ are defined by the formulas $F_z = f - f(z)$ and $G_z = g - g(z)$.

(ii) *For any f, $g \in L^2$, there is a $B(f,g) > 0$ such that for any φ, $\psi \in H^2$ with the property that $T_f\varphi$, $T_g\psi \in L^2$, we have*

$$|\langle T_f\varphi, T_g\psi\rangle| \le C_{2.2}\int_T M_{F^0,G^0}(\varphi,\psi)dm + B(f,g)\|\varphi\|_2\|\psi\|_2,$$

where F^0, $G^0 : D \to L^2$ are defined by the formulas $F^0_z = f$ and $G^0_z = g$.

Proof. (i) The harmonic extensions of $H_f\varphi$ and $H_g\psi$ vanish at 0. Hence it follows from the Littlewood-Paley formula that

$$\begin{aligned}|\langle H_f\varphi, H_g\psi\rangle| \le \frac{1}{\pi}\int_{7/8<|z|<1} |\langle\nabla(H_f\varphi)(z), \nabla(H_g\psi)(z)\rangle_{\mathbf{C}^2}|\log\frac{1}{|z|}dA(z)\\ + \frac{1}{\pi}\int_{|z|\le 7/8} |\langle\nabla(H_f\varphi)(z), \nabla(H_g\psi)(z)\rangle_{\mathbf{C}^2}|\log\frac{1}{|z|}dA(z).\end{aligned}$$

It is elementary that there is a $C(f,g) > 0$ such that the second term above is bounded by $C(f,g)\|\varphi\|_2\|\psi\|_2$. Thus it suffices to estimate the first term.

Mimicking the definition of $\rho(w)$ on page 494 of [10], for each $\tau \in T$, let $a(\tau)$ be the largest $a \in [0,1/4]$ such that

$$S_a(H_f\varphi)(\tau)S_a(H_g\psi)(\tau) \le B_{2.1}M_{F,G}(\varphi,\psi)(\tau).$$

(We set $S_0(\xi) = 0$.) We claim that the function

$$\tau \mapsto S_{a(\tau)}(H_f\varphi)(\tau)S_{a(\tau)}(H_g\psi)(\tau)$$

is measurable[1] on T. In fact, for each fixed $\tau \in \{S(H_f\varphi) < \infty\} \cap \{S(H_g\psi) < \infty\}$, since the function $a \mapsto S_a(H_f\varphi)(\tau)S_a(H_g\psi)(\tau)$ is non-decreasing and continuous on $[0, 1/4]$, we have

$$S_{a(\tau)}(H_f\varphi)(\tau)S_{a(\tau)}(H_g\psi)(\tau) = \sup_n \min\{B_{2.1}M_{f,g}(\varphi,\psi)(\tau), S_{a_n}(H_f\varphi)(\tau)S_{a_n}(H_g\psi)(\tau)\},$$

where $\{a_n\}$ is any chosen sequence which is dense in $[0, 1/4]$. And $\{S(H_f\varphi) = \infty\} \cup \{S(H_g\psi) = \infty\}$ is a null set. Similarly $\tau \mapsto a(\tau)$ is also measurable.

We claim that $\int_{I_z} \chi_{\Gamma_{\tau,a(\tau)}}(z)dm(\tau) \geq (1 - |z|)/4\pi$ when $7/8 < |z| < 1$. Indeed if

$$S_{2(1-|z|)}((f-f(z))\varphi)(\gamma)S_{2(1-|z|)}((g-g(z))\psi)(\gamma) \leq B_{2.1}M_{F,G}(\varphi,\psi)(\gamma), \tag{2.4}$$

then, because $H_f\varphi = H_{f-f(z)}\varphi$, $H_f\varphi = H_{f-f(z)}\varphi$, and because of (2.2), we have

$$S_{2(1-|z|)}(H_f\varphi)(\gamma)S_{2(1-|z|)}(H_g\psi)(\gamma) \leq B_{2.1}M_{F,G}(\varphi,\psi)(\gamma).$$

That is, if $\gamma \in I_z$ is such that (2.4) holds, then $a(\gamma) \geq 2(1 - |z|) > 1 - |z|$, consequently $|z| > 1 - a(\gamma)$. This implies $z \in \Gamma_{\gamma,a(\gamma)}$ since $z \in \Gamma_\gamma$ whenever $\gamma \in I_z$. But Proposition 2.1 tells us that the set of γ's in I_z for which (2.4) holds has measure at least $|I_z|/2$. In other words, when $7/8 < |z| < 1$, the function $\tau \mapsto \chi_{\Gamma_{\tau,a(\tau)}}(z)$ equals 1 on a subset of I_z with measure at least $|I_z|/2 = (1 - |z|)/4\pi$. This verifies our claim. Borrowing an idea from [**10**], we have

$$\begin{aligned}
&\frac{1}{\pi}\int_{7/8<|z|<1} (1 - |z|)|\langle\nabla(H_f\varphi)(z), \nabla(H_g\psi)(z)\rangle|dA(z) \\
&\leq 4\int_{7/8<|z|<1} \left(\int_{I_z} \chi_{\Gamma_{\tau,a(\tau)}}(z)dm(\tau)\right) |\langle\nabla(H_f\varphi)(z), \nabla(H_g\psi)(z)\rangle_{\mathbf{C}^2}|dA(z) \\
&= 4\int_T \int_{7/8<|z|<1} \chi_{\Gamma_{\tau,a(\tau)}}(z)|\langle\nabla(H_f\varphi)(z), \nabla(H_g\psi)(z)\rangle_{\mathbf{C}^2}|dA(z)dm(\tau) \\
&\leq 4\int_T S_{a(\tau)}(H_f\varphi)(\tau)S_{a(\tau)}(H_g\psi)(\tau)dm(\tau) \leq 4B_{2.1}\int_T M_{f,g}(\varphi,\psi)dm.
\end{aligned}$$

But there is a $C_{2.2.1} > 0$ such that $\log(1/|z|) < C_{2.2.1}(1 - |z|)$ whenever $7/8 < |z| < 1$. Thus $C_{2.2} = 4C_{2.2.1}B_{2.1}$ satisfies our requirement.

(ii) Note that $\langle T_f\varphi, T_g\psi\rangle = \langle Pf\varphi, g\psi\rangle = \langle P(f\varphi-(f\varphi)(0)), g\psi-(g\psi)(0)\rangle+$ garbage. Now there is a $C'(f,g) > 0$ such that $C'(f,g)\|\varphi\|_2\|\psi\|_2$ dominates

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[1] Note that a similar measurability issue was overlooked on page 494 of [**10**].

the garbage term. Applying the Littlewood-Paley formula to the other term,

$$\langle P(f\varphi - (f\varphi)(0)), g\psi - (g\psi)(0)\rangle = \frac{1}{\pi}\int_D \langle \nabla(Pf\varphi)(z), \nabla(g\psi)(z)\rangle_{\mathbf{C}^2} \log\frac{1}{|z|} dA(z),$$

the rest of the proof proceeds as in (i) with the only modification that $F_z^0 = f$ and $G_z^0 = g$ now replace $F_z = f - f(z)$ and $G_z = g - g(z)$ respectively. □

With the foregoing preparation, we are now ready to prove our main results.

Proof of Theorem 1 (*resp.* 2). By Proposition 2.2(i) (resp. 2.2(ii)), it suffices to show that there is a $K > 0$ such that $\|M_{F,G}(\varphi,\psi)\|_1 \le K\|\varphi\|_2\|\psi\|_2$ when $F_z = f - f(z)$ and $G_z = g - g(z)$ (resp. $F_z = f$ and $G_z = g$). But with this notation, the assumption of the theorem now reads

$$\int_A |F_z| P_z dm \int_B |G_z| P_z dm \le Nu(\chi_A(z))v(\chi_B(z)).$$

Proposition 1.1 asserts that such a K exists under this condition. □

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