PACIFIC JOURNAL OF MATHEMATICS
Vol. 183, No. 1, 1998

BOUNDARY BEHAVIOUR OF H^p FUNCTIONS ON CONVEX DOMAINS OF FINITE TYPE IN $\mathbb{C}^n$

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We describe the optimal approach regions for a theorem of Fatou type for H^p functions on convex domains of finite type in $\mathbb{C}^n$. Moreover we show that the Nagel-Stein phenomenon also holds in this context.

1. Introduction.

Holomorphic functions of the H^p classes on a pseudoconvex domain of finite type in $\mathbb{C}^2$ have a boundary limit for almost every point in the boundary of the domain, provided the limit is taken inside certain approach regions, whose shape, in the complex tangential direction, reflects the order of contact with tangential complex hypersurfaces, and therefore changes near weakly pseudoconvex points [**N**, **S**, **W81**], [**N**, **S**, **W85**], [**K72**], [**Ko69**], [**S72**].

The major difficulties met in the study of pseudoconvex domains of finite type in $\mathbb{C}^n$ are the following: The behavior is different in the various complex tangential directions and the different directions interact. In the context of convex domains of finite type in $\mathbb{C}^n$, one can restrict the attention to the order of contact with complex lines [**Mc92**], and exploit certain coordinate systems and polydiscs obtained by extremizing the distance along complex lines, as in [**Mc92**], [**Mc94**], [**Mc**, **S94**], [**Mc**, **S96**], which also give some control over the intermediate directions.

In this paper we describe the natural approach regions for convex domains of finite type in $\mathbb{C}^n$, and prove the corresponding theorem of Fatou type for H^p functions. The shape of the natural approach regions depends on the particular complex tangential direction which is being considered; in particular, it reflects the order of contact of the boundary with the complex line in that direction (see the example at the end of Section 3). A suitable *regularization* of the polydiscs studied in [**Mc92**] is used.

The natural approach regions for the unit disc are the nontangential cones [**F06**]. In fact, no rotation invariant family of tangential *curves* is a region of convergence for H^p holomorphic functions [**L27**]. In 1984, A. Nagel and E.M. Stein showed that almost everywhere convergence does indeed hold, for H^p functions, along certain approach regions containing tangential *sequences*—as opposed to tangential curves [**N**, **S84**]. In particular, the *exotic* approach regions constructed in [**N**, **S84**] are not contained in any of the natural approach regions, near the boundary.

We show that the natural approach regions defined here for convex domains of finite type in $\mathbb{C}^n$ also admit exotic approach regions of convergence. Recall that, for the unit disc and Euclidean half spaces, one first constructs an exotic approach region at one point, and then one translates it to nearby points. The same approach works when the boundary is acted upon by a group, or by a pseudogroup of diffeomorphisms, which preserve the relevant family of balls in the boundary [**Su86**], [**A**, **C92**]. These ingredients are lacking for a convex domain of finite type in $\mathbb{C}^n$, since the shape of the balls changes near weakly pseudoconvex points.

We would like to thank J.D. McNeal for all the insights he shared with us.

2. Geometry.

Let $D \subset\subset \mathbb{C}^n$ be a smoothly bounded, convex domain of finite type M defined by $D = \{r < 0\}$. Without loss of generality we may assume that $D_\varepsilon := \{r < \varepsilon\}$ is also convex if $|\varepsilon| < \varepsilon_0$ for some fixed small positive constant ε_0. Let $S^{2n-1} := \{\gamma \in \mathbb{C}^n : |\gamma| = 1\}$. By $\mathrm{dist}_\gamma(q, A)$ we denote the distance from the point q to a set A along the complex line $\{q + \zeta\gamma : \zeta \in \mathbb{C}\}$. We also use the notation $\delta_\gamma(q, \varepsilon) := \mathrm{dist}_\gamma(q, bD_{r(q)+\varepsilon})$. For each point q with $|r(q)| < \varepsilon_0/2$ and each positive number $\varepsilon < \varepsilon_0/2$ McNeal [**Mc94**] constructed special coordinates $z_i^{q\varepsilon}$ centered at q which are related to the standard coordinates in $\mathbb{C}^n$ by a unitary transformation. Moreover he defined certain numbers $\tau_i(q, \varepsilon)$ (see Proposition 2.1 for more details) and the polydiscs

$$P(q, \varepsilon) := \{z : |z_i^{q\varepsilon}| < \tau_i(q, \varepsilon) \quad \text{for all} \quad i\}.$$

Using these polydiscs one can define the quasi-distance $\varrho(q, p) := \inf\{\varepsilon : p \in P(q, \varepsilon)\}$. For each direction γ one may consider the Taylor expansion of r about q in the direction defined by γ

$$r(q + z\gamma) = r(q) + \sum_{\mu+\nu=1}^{M} a^\gamma_{\mu\nu}(q) z^\mu \bar{z}^\nu + O(|z|^{M+1}).$$

Using the quantities

$$A^\gamma_k(q) := \max\{|a^\gamma_{\mu\nu}(q)| : \quad \mu + \nu = k\}$$

we define

$$s_\gamma(q, \varepsilon) := \min\{(\varepsilon / A^k_\gamma(q))^{1/k} : \quad 1 \le k \le M\}.$$

The following statements are proven in [**Mc94**] and [**Mc**, **S96**].

Proposition 2.1. *Using the notation given above we have:*

(i) $\tau_1(q,\varepsilon) = \mathrm{dist}(q, bD_{r(q)+\varepsilon})$ *and the point where this distance is realized lies on the positive* $x_1^{q\varepsilon}$*-axis where* $z_1^{q\varepsilon} = x^{q\varepsilon} + iy_1^{q\varepsilon}$.

(ii) *For* $2 \leq i \leq n$, $\tau_i(q,\varepsilon) = \mathrm{dist}_{\gamma_i}(q, bD_{r(q)+\varepsilon})$, *where* γ_i *is the direction given by the* $z_i^{q\varepsilon}$*-coordinate.*

(iii) *There exists a constant* c_0 *independent of* q *and* ε *such that*

$$c_0 P(q,\varepsilon) \subset D_{r(q)+\varepsilon}.$$

(iv) (*Vitali-type engulfing property.*) *There exists a constant* C_1 *independent of* p, q *and* ε *such that if* $P(p,\varepsilon) \cap P(q,\varepsilon) \neq \emptyset$ *then*

$$P(q,\varepsilon) \subset C_1 P(p,\varepsilon).$$

(v) (*Doubling property.*) *There exists a constant* C_2 *independent of* q *and* ε *such that*

$$P(q,2\varepsilon) \subset C_2 P(q,\varepsilon).$$

(vi) *The quasi-distance satisfies*

$$\varrho(p,q) \approx \varrho(q,p) \quad \text{and} \quad \varrho(p,q) \lesssim \varrho(p,r) + \varrho(r,q).$$

(vii) *Let* γ_i *be the direction given by the* $z_i^{q\varepsilon}$*-coordinate and write* $\gamma = \sum a_i \gamma_i$ *with* $a_i \in \mathbb{C}$. *Then we have*

$$\delta_\gamma(q,\varepsilon) \approx \left(\sum \frac{|a_i|}{\tau_i(q,\varepsilon)} \right)^{-1}.$$

(viii) *If* $p \in P(q,\varepsilon)$ *then*

$$\tau_i(p,\varepsilon) \approx \tau_i(q,\varepsilon) \quad \text{for} \quad i = 1, ..., n.$$

(ix) *If* σ *is the surface measure on* bD *then*

$$\sigma\left(P(w,\varepsilon) \cap bD\right) \approx \varepsilon^{-1} \Pi_{\nu=1}^n \tau_\nu(w,\varepsilon)^2.$$

Note that statement (*vii*) means that even in the intermediate directions the distance to the boundary of the domain can be controlled by the distance to the boundary of the polydisc in the γ_i directions. This fact holds because of the convexity of the domain. Together with part (*iii*) it implies us that these polydiscs are the largest ones that fit inside the domain.

The following lemma is also implicit in [**Mc94**]. Since it is one of the basic ingredients in most of the other estimates, we will give the proof here.

Lemma 2.2. *We have*

$$\delta_\gamma(q,\varepsilon) \approx s_\gamma(q,\varepsilon).$$

Proof. Consider the following Taylor expansion

$$r(q+z\gamma)=r(q)+\sum_{\mu+\nu=1}^{M}a_{\mu\nu}z^{\mu}\bar{z}^{\nu}+O(|z|^{M+1})=:r(q)+f(z).$$

Now $\delta_\gamma(q,\varepsilon)$ is the supremum of all positive real numbers c such that $f(c\,e^{i\theta})<\varepsilon$ for all θ. If we set $c=(2M(M+1))^{-1}s_\gamma(q,\varepsilon)$ then we have

$$f((2M(M+1))^{-1}s_\gamma(q,\varepsilon)e^{i\theta})\le(1/2)\varepsilon+O(\varepsilon^{\frac{M+1}{M}})\le\varepsilon,$$

for $\varepsilon\le\varepsilon_0$ (after perhaps shrinking ε_0). Therefore $s_\gamma(q,\varepsilon)\lesssim\delta_\gamma(q,\varepsilon)$.

To prove the other inequality we first have to observe that

$$A_k^\gamma(q)\le(M+1)\max_\theta\left|\sum_{\mu+\nu=k}a_{\mu\nu}(q)e^{i\theta(\mu-\nu)}\right|.$$

Now let $c=(2(M+1)/c_M)s_\gamma(q,\varepsilon)$, where c_M is the constant that appears in Lemma 2.1 in [**B**, **N**, **W88**] and only depends on M. Then

$$\begin{aligned}&f((2(M+1)/c_M)s_\gamma(q,\varepsilon)e^{i\theta})\\&=(a_{10}e^{i\theta}+a_{01}e^{-i\theta})(2(M+1)/c_M)s_\gamma(q,\varepsilon)\\&\quad+\sum_{k=2}^{M}\left(\sum_{\mu+\nu=k}a_{\mu\nu}e^{i\theta(\mu-\nu)}\right)((2(M+1)/c_M)s_\gamma(q,\varepsilon))^k+O(|s_\gamma(q,\varepsilon)|^{M+1}).\end{aligned}$$

Using Lemma 2.1 from [**B**, **N**, **W88**] we get

$$\begin{aligned}&f((2(M+1)/c_M)s_\gamma(q,\varepsilon)e^{i\theta})\\&\ge2\,\mathrm{Re}\,(a_{10}e^{i\theta})(2(M+1)/c_M)s_\gamma(q,\varepsilon)\\&\quad+c_M\sum_{k=2}^{M}\left|\sum_{\mu+\nu=k}a_{\mu\nu}e^{i\theta(\mu-\nu)}\right|\\&\qquad\cdot(((2(M+1)/c_M)s_\gamma(q,\varepsilon))^k)+O(|s_\gamma(q,\varepsilon)|^{M+1}).\end{aligned}$$

Now assume that $s_\gamma(q,\varepsilon)$ is realized by the term $(\varepsilon/A_{k_0}^\gamma(q))^{1/k_0}$ with $k_0>1$ and that the maximum of $\{|\sum_{\mu+\nu=k_0}a_{\mu\nu}(q)e^{i\theta(\mu-\nu)}|\}$ is reached at $\theta=\theta_0$. Note that then $\theta_0+\pi$ also gives this term. So we may assume that $2\,\mathrm{Re}\,(a_{10}e^{i\theta})$ is nonnegative. Now we get

$$\begin{aligned}&f((2(M+1)/c_M)s_\gamma(q,\varepsilon)e^{i\theta_0})\\&\ge c_M\left|\sum_{\mu+\nu=k_0}a_{\mu\nu}e^{i\theta_0(\mu-\nu)}\right|(((2(M+1)/c_M)s_\gamma(q,\varepsilon))^{k_0})+O(|s_\gamma(q,\varepsilon)|^{M+1})\end{aligned}$$

$$\begin{aligned} &\geq c_M \left| \sum_{\mu+\nu=k_0} a_{\mu\nu} e^{i\theta_0(\mu-\nu)} \right| \\ &\qquad \cdot ((2(M+1)/c_M)(\varepsilon/A^\gamma_{k_0}(q))^{1/k_0})^{k_0} + O(\varepsilon^{M+1/M}) \\ &\geq 2\varepsilon + O(\varepsilon^{M+1/M}) \geq \varepsilon \end{aligned}$$

for all $\varepsilon \leq \varepsilon_0$ (after perhaps shrinking ε_0). If $k_0 = 1$ we only have to estimate the first term, that can be treated in the same way. Finally we have $s_\gamma(q,\varepsilon) \gtrsim \delta_\gamma(q,\varepsilon)$. ☐

Together with the numbers $\tau_i(q,\varepsilon)$ defined by McNeal we will also consider the quantities $\tau_\gamma(q,\varepsilon) := \mathrm{dist}_\gamma(q, bP(q,\varepsilon))$. Note that

$$P(q,\varepsilon) = \{p : |p-q| < \tau_\gamma(q,\varepsilon) \quad \text{for} \quad \gamma = (p-q)/|p-q|\}.$$

It turns out that neither the $\tau_i(q,\varepsilon)$ nor the $\tau_\gamma(q,\varepsilon)$ are monotonous in the parameter ε. But instead there is some quasi-monotonicity described in the following lemma:

Lemma 2.3. *There exists a constant C_4 independent of q, ε and γ such that*
(i) $\tau_\gamma(q,\varepsilon) \approx s_\gamma(q,\varepsilon)$.
(ii) *For all C and C' with $C' \geq C_4 \max\{C^M, C\}$ we have $C\tau_\gamma(q,\varepsilon) \leq \tau_\gamma(q, C'\varepsilon)$.*
The inequality in (ii) is strict if $\varepsilon > 0$.

Proof. Writing $\gamma = \sum a_i\gamma_i$, where γ_i is the direction given by the $z_i^{q\varepsilon}$-coordinate, we see that

$$\tau_\gamma(q,\varepsilon) = \min\left\{\frac{\tau_i(q,\varepsilon)}{|a_i|}\right\} = \left(\max\left\{\frac{|a_i|}{\tau_i(q,\varepsilon)}\right\}\right)^{-1} \approx \left(\sum\left\{\frac{|a_i|}{\tau_i(q,\varepsilon)}\right\}\right)^{-1}.$$

Part (*i*) now follows from Proposition 2.1 (*vii*) and Lemma 2.2. Note that in the above equation even the a_i depend on ε because the coordinates $z_i^{q\varepsilon}$ do. In the definition of $s_\gamma(q,\varepsilon)$ the terms $A^\gamma_k(w)$ do not depend on ε. Therefore this gives us fairly explicit expression in ε. Part (*ii*) is then a simple application of (*i*). ☐

The polydiscs $P(q,\varepsilon)$ need not be continuous neither in q nor with respect to ε. We would like to use them to define the approach regions but in this case these regions would not be open. Moreover it will be important, in the proof of Theorem 4.1, that the sets $\{w : z \in \mathcal{A}_\alpha(w)\}$ are open. Therefore we use the following two regularizations. Let us define

$$P'(q,\varepsilon) := \bigcup_{\varepsilon' < \varepsilon} P(q,\varepsilon')$$

and

$$P''(q,\varepsilon) := \bigcup_{P'(p,\varepsilon) \ni q} P'(p,\varepsilon).$$

Note that these sets are no longer polydiscs but they are related to the $P(q,\varepsilon)$ polydiscs by the following lemma:

Lemma 2.4

(i) *There exist constants c_5 and C_5 independent of q and ε such that*

$$P(q,c_5\varepsilon) \subset P'(q,\varepsilon) \subset P(q,C_5\varepsilon).$$

(ii) *There exist constants c_6 and C_6 independent of q and ε such that*

$$P'(q,c_6\varepsilon) \subset P''(q,\varepsilon) \subset P'(q,C_6\varepsilon).$$

(iii) $\varrho''(q,p) := \inf\{\varepsilon : p \in P''(q,\varepsilon)\}$ *is a quasi-distance equivalent to $\varrho(q,p)$.*

Proof. Observe that $P'(q,\varepsilon)$ contains $P(q,c_5\varepsilon)$ for every $c_5 < 1$. We know from Lemma 2.3 (*ii*) that there is a constant C_4 such that $P(q,\varepsilon) \subset P(q,C\varepsilon)$ for all $C \geq C_4$. Choosing $C_5 \geq C_4$ we see that $P(q,C_5\varepsilon)$ contains all $P(q,\varepsilon')$ with $\varepsilon' < \varepsilon$ and therefore also $P'(q,\varepsilon)$. This proves (*i*). Since $P'(q,\varepsilon)$ contains q we have $P'(q,c_6\varepsilon) \subset P''(q,\varepsilon)$ for $c_6 = 1$. Using part (*i*), Proposition 2.1 (*iv*) and Lemma 2.3 (*ii*) we see that every $P'(p,\varepsilon)$ which contains q is a subset of $P'(q,C_6\varepsilon)$ for $C_6 = C_4 C_1^M C_5/c_5$. Therefore $P''(q,\varepsilon)$ is also a subset of $P'(q,C_6\varepsilon)$. The third part is a consequence of (*i*) and (*ii*). $\square$

Now we can use the balls $P''(q,\varepsilon)$ to define the family of *admissible* approach regions. For every positive number α and every point $w \in bD$ we set

$$\mathcal{A}_\alpha(w) := \{z \in D : \pi(z) \in P''(w, \alpha|r(z)|)\},$$

where π is the projection which maps every point in a neighborhood of the boundary to the nearest boundary point. Observe that $\mathcal{A}_\alpha(w)$ is a subset of $\mathcal{A}_{\alpha'}(w)$ for $\alpha < \alpha'$.

The set $\mathcal{A}_\alpha(w)$ is open. The proof of this fact is based on the following property of the balls $P''(w,r)$: If $z \in P''(w,r)$, then, for r' close enough to r, the ball $P''(w,r')$ contains a small neighborhood of z.

For the proof of our results we need some more lemmas where the following set plays an important role. For $z \in D$ and a small positive constant k define

$$D_k(z) := P''(z, k|r(z)|).$$

Lemma 2.5. *If k is small enough then*

$$|r(\zeta)| \approx |r(z)|$$

for all $\zeta \in D_k(z)$.

Proof. It follows from Lemma 2.4 (*i*) and (*ii*), Lemma 2.3 (*ii*) and Proposition 2.1 (*iii*) that $D_k(z) \subset P(z, C_5C_6k|r(z)|) \subset c_0P(z, (1/2)|r(z)|) \subset D_{r(z)+(1/2)|r(z)|}$ if k is small enough. This means that $r(\zeta) < r(z) + (1/2)|r(z)|$ or $|r(\zeta)| > (1/2)|r(z)|$ for all $\zeta \in D_k(z)$. To prove the other estimate we first have to observe that $D_k(z) \subset D \cap P(z, |r(z)|)$. If k is small enough this follows as above. Now we make use of the special coordinates constructed by McNeal for $p = z$ and $\varepsilon = |r(z)|$. According to Proposition 2.1 (*i*) we know that the plane $x_1 = \tau_1(z, |r(z)|)$ is a tangential plane. It follows from the convexity of the domain that $x_1(u) < \tau_1(z, |r(z)|)$ for every $u \in bD \cap P(z, |r(z)|)$. For every $\zeta \in D \cap P(z, |r(z)|)$ we know that $-\tau_1(z, |r(z)|) < x_1(\zeta) < x_1(\tilde{\pi}(\zeta))$ where $\tilde{\pi}$ is the projection to the boundary along the x_1-direction. Using the well known fact that $\tau_1(\zeta, |r(\zeta)|) = \mathrm{dist}(\zeta, bD) \approx |r(\zeta)|$ we now get

$$|r(\zeta)| \lesssim \mathrm{dist}(\zeta, bD) \leq x_1(\tilde{\pi}(\zeta)) - x_1(\zeta) \leq 2\tau_1(z, |r(z)|) \lesssim |r(z)|$$

which proves the lemma. □

Lemma 2.6. *If k is small enough then:*

- (i) *$D_k(z)$ is a subset of $P(\zeta, |r(\zeta)|)$ for every $\zeta \in D_k(z)$. For every $\eta \in \pi(D_k(z))$ the one dimensional measure of $\pi^{-1}(\eta) \cap D_k(z)$ is bounded by $C_7|r(z)|$, where C_7 is independent of k, η and z.*
- (ii) *There exists a constant K independent of z such that $\pi(D_k(z)) \subset P''(\pi(z), K|r(z)|)$ for all z with $|r(z)| < \varepsilon_0/2$.*
- (iii) *For every α there exists an α' such that $z \in \mathcal{A}_\alpha(w)$ and $\zeta \in D_k(z)$ implies $\zeta \in \mathcal{A}_{\alpha'}(w)$. The parameter α' does not depend on w.*

Proof. The first part of (*i*) follows from the fact that the quasi-distance between two points in $D_k(z)$ can be estimated by a multiple of $|r(z)|$ and using Lemma 2.5 also by a multiple of $|r(\zeta)|$. The second part is then simply the fact that $\pi^{-1}(\pi(\zeta))$ is exactly the x_1-axis in the coordinate system constructed with respect to $p = \zeta$ and $\varepsilon = |r(\zeta)|$ using again Lemma 2.5. The second statement follows from the triangle inequality of the quasi distance and the fact that $\varrho''(\pi(\zeta),\zeta)$, $\varrho''(\zeta,z)$ and $\varrho''(z,\pi(z))$ can be estimated by multiples of $|r(z)|$. The proof of (*iii*) is similar. After using the triangle inequality we just have to observe that all of the quasi-distances $\varrho''(w,\pi(z))$, $\varrho''(\pi(z),z)$, $\varrho''(z,\zeta)$ and $\varrho''(\zeta,\pi(\zeta))$ can be estimated by multiples of $|r(\zeta)|$. This completes the proof. $\square$

3. The natural approach regions.

Let D and D_ε be as in Section 2. We say that a function f belongs to $H^p(D)$ if f is holomorphic in D and satisfies

$$\sup_{-\varepsilon_0<\varepsilon<0}\int_{bD_\varepsilon}|f(\zeta)|^p d\sigma_\varepsilon(\zeta)<\infty \quad \text{for} \quad 0<p<\infty$$

$$\sup_{\zeta\in D}|f(\zeta)|<\infty \quad \text{for} \quad p=\infty.$$

In this section we will prove that every $f\in H^p(D)$ has limits along the admissible approach regions defined in Section 2 for almost every boundary point w.

First we have to give some definitions. If $f\in L^1(bD)$ then by Mf we denote the maximal function of f with respect to Euclidean balls. By $M''f$ we denote the maximal function of f with respect to the balls $P''(w,r)$ defined in Section 2

$$M''f(w):=\sup_{r>0}|P''(w,r)|^{-1}\int_{P''(w,r)}|f(z)|d\sigma(z).$$

Note that instead of the supremum over all balls centered at w one can also take the supremum over all balls which contain w.

The following lemma contains the basic ingredient of the proof of the main theorem (see [**S72**]).

Lemma 3.1. *Let $u\in C(\bar D)$ be a nonnegative, plurisubharmonic function. Define $f:=u|_{bD}$. For each $\alpha>0$ there exists a constant C_α independent of $w\in bD$ such that*

$$\sup_{z\in\mathcal{A}_\alpha(w)}u(z)\le C_\alpha M''(Mf)(w)$$

for all $w\in bD$.

Proof. First note that the subharmonicity of u implies $u(z)\le Pf(z)$, where Pf is the Poisson extension of f. Theorem 3 in [**S72**] shows that

$$u(z)\le Pf(z)\lesssim Mf(\pi(z)).$$

Now we use the fact that the set $D_k(z)$ is a subset of D (see Lemma 2.6 (*iii*)) and that there exists a polydisc $P(z, c_5c_6k|r(z)|)$ which is a subset of $D_k(z)$. Let us denote this polydisc by $d_k(z)$. Since $|u|$ is plurisubharmonic on $d_k(z)$ the submean-value property leads to

$$\begin{aligned}|u(z)| &\leq |d_k(z)|^{-1}\int_{d_k(z)}|u(\zeta)|dV(\zeta)\\ &\lesssim \left(\Pi_{\nu=1}^n\tau_\nu(z, c_5c_6k|r(z)|)^2\right)^{-1}\int_{D_k(z)} Mf(\pi(\zeta))dV(\zeta).\end{aligned}$$

Before we can proceed with this estimate we have to observe a couple of things. First let us introduce the abbreviation $\Delta_C(z) := P(\pi(z), C|r(z)|) \cap bD$. Lemma 2.6 (*ii*) and Lemma 2.4 (*i*), (*ii*) tell us that the projection of $D_k(z)$ belongs to $\Delta_{C_5C_6K}(z)$. Since $z \in \mathcal{A}_\alpha(w)$ we have

$$\pi(z) \in P(w, C_5C_6\alpha|r(z)|).$$

Therefore there exists a constant $C'_\alpha > C_4C_5C_6K$ only depending on α such that $w \in \Delta_{C'_\alpha}(z)$. By Proposition 2.1 (*ix*) the (surface) measure of $\Delta_{C'_\alpha}(z)$ can be estimated from below by the product $c|r(z)|^{-1}\Pi_{\nu=1}^n\tau_\nu(\pi(z), C'_\alpha|r(z)|)^2$ and by Lemma 2.6 (*i*) the (one dimensional) measure of $\pi^{-1}(\pi(\zeta)) \cap D_k(z)$ is bounded by $C_7|r(z)|$ for every $\zeta \in D_k(z)$. Moreover since z belongs to $P(\pi(z), C'_\alpha|r(z)|)$ Proposition 2.1 (*viii*) and Lemma 2.3 tell us that

$$\Pi_{\nu=1}^n\tau_\nu(\pi(z), C'_\alpha|r(z)|)^2 \approx \Pi_{\nu=1}^n\tau_\nu(z, C'_\alpha|r(z)|)^2 \approx C''_\alpha\Pi_{\nu=1}^n\tau_\nu(z, c_5c_6k|r(z)|)^2,$$

where C''_α only depends on α.

Now we can continue the estimate as follows:

$$\begin{aligned}&\left(\prod_{\nu=1}^n\tau_\nu(z, c_5c_6k|r(z)|)^2\right)^{-1}\int_{D_k(z)} Mf(\pi(\zeta))dV(\zeta)\\ &\lesssim \left(\prod_{\nu=1}^n\tau_\nu(z, c_5c_6k|r(z)|)^2\right)^{-1}|r(z)|\int_{\Delta_{C'_\alpha}(z)} Mf(t)d\sigma(t)\\ &\lesssim C_\alpha\left(\prod_{\nu=1}^n\tau_\nu(\pi(z), C'_\alpha|r(z)|)^2\right)^{-1}|r(z)|\int_{\Delta_{C'_\alpha}(z)} Mf(t)d\sigma(t)\\ &\lesssim C_\alpha\frac{1}{|\Delta_{C'_\alpha}(z)|}\int_{\Delta_{C'_\alpha}(z)} Mf(t)d\sigma(t) \quad \lesssim \quad C_\alpha M''(Mf)(w)\end{aligned}$$

which proves the lemma. ∎

Theorem 3.2. *Let $0 < p \leq \infty$, D and $\mathcal{A}_\alpha(w)$ as in Section 2. If $f \in H^p(D)$ then for almost every $w \in bD$ the limit*

$$\lim_{\mathcal{A}_\alpha(w)\ni z\to w} f(z)$$

exists.

Proof. The outline of this proof is exactly the same as in the corresponding statement for the admissible approach regions for strongly pseudoconvex domains [**S72**, pp. 38-40]. The special geometry of the domain is involved only in the previous lemma. ☐

The approach regions $\mathcal{A}_\alpha(w)$ are essentially the greatest *natural* family of approach regions that can be obtained by this method, since they are build using the biggest polydiscs that fit inside the domain. However it will be shown in Section 4 that there are *exotic* approach regions which are not contained in any of the approach regions $\mathcal{A}_\alpha(w)$.

Let us give some more details about the admissible approach regions. In particular we are interested in the shape of $\mathcal{A}_1(w)$ near its vertex. Consider a point $w \in bD$, a tangential direction γ, a small positive parameter b and let z_b be the point in the boundary for which the orthogonal projection to the tangent space $T_w(bD)$ is exactly $w + b\gamma$. Let $h(b)$ be the distance of $\mathcal{A}_1(w)$ to the point z_b along the normal direction at this point. Using the comparability of $P''(q,\varepsilon)$ and $P(q,\varepsilon)$ and that of $\tau_\gamma(q,\varepsilon)$ and $s_\gamma(q,\varepsilon)$ we find that

$$h(b) \approx \max_{1\le k\le M} \left(A_k^\gamma(w) b^k\right).$$

In other words the shape of the approach region reflects exactly the order of contact of the boundary with the complex line in that direction.

If we consider the example $r(z) = |z_1|^2 + |z_2|^4 + |z_3|^6 - 1$ then we get at the point $w_0 = (1,0,0)$

$$\begin{aligned} h &\approx b^4 \quad &&\text{for} \quad \gamma = (0,1,0), \\ h &\approx \max(b^4\cos^4 t, b^6 \sin^6 t) \quad &&\text{for} \quad \gamma = (0,\cos t, \sin t), \\ h &\approx b^6 \quad &&\text{for} \quad \gamma = (0,0,1). \end{aligned}$$

It is also interesting to see how the shape of the approach region changes if the vertex moves, in particular if the vertex moves to a point with higher order of contact. Let us consider the points $w_t := (\sqrt{1-t^6}, 0, t)$. Along the z_2-direction all these points have order of contact 4 and indeed we get

$$h \approx b^4 \quad \text{for} \quad w = w_t \quad \text{and} \quad \gamma = (0,1,0).$$

Now let γ_t be the complex tangent direction orthogonal to $(0,1,0)$. The order of contact along this direction is 2 for every w_t with $t > 0$ and 6 for w_0. If we compute h then we get (modulo higher order terms in t)

$$h \approx \max\left(9t^4b^2,\, 9t^3b^3,\, 9t^2b^4,\, 3tb^5,\, b^6\right) \quad \text{for} \quad w = w_t \quad \text{and} \quad \gamma = \gamma_t.$$

So whenever $t > 0$ the first term will be the maximum if b is small enough. How small b must be depends on t and if $t = 0$ then the last terms will be the maximum.

4. The existence of exotic approach regions.

In this section we prove the following:

Theorem 4.1. *Let D be a convex domain of finite type in $\mathbb{C}^n$. Let $\mathcal{A}_\alpha$ be the admissible approach regions defined in the previous section. Then for each $w \in bD$ there exists a subset $L(w)$ of D, containing w in its closure, such that:* (i) *for almost every $w \in bD$, the set $L(w)$ contains a sequence $z_n(w)$ that converges to w and that, for every $\alpha > 0$, contains a subsequence which is not contained in $\mathcal{A}_\alpha(w)$;* (ii) *for every function $f \in H^p(D)$, the limit*

$$\lim_{L(w) \ni z \to w} f(z)$$

exists for almost every $w \in bD$. The approach region $L(w)$ contains $\mathcal{A}_{\alpha_0}$ for a certain α_0.

The dual ${\mathcal{A}_\alpha}^*$ of $\mathcal{A}_\alpha$, defined by

$${\mathcal{A}_\alpha}^*(z) := \{w \in bD : z \in \mathcal{A}_\alpha(w)\}, \quad z \in D,$$

will play an important role in the following two lemmas.

Lemma 4.2. *The sets ${\mathcal{A}_\alpha}^*(z) \subset bD$ are open and, for every $\alpha > 0$, there are positive constants $c_8(\alpha)$ and $C_8(\alpha)$ such that for every $z \in D$ close enough to the boundary, there is a ball $P''(w_z, r_z) \subset bD$ such that*

$$P''(w_z, c_8(\alpha)\, r_z) \subset {\mathcal{A}_\alpha}^*(z) \subset P''(w_z, C_8(\alpha)\, r_z).$$

In fact, $w_z = \pi(z)$ and $r_z = |r(z)|$.

Proof. The first assertion follows from the regularity properties of P' and P''. As for the second, let C be the constant such that $w \in P''(u, r)$ implies that $u \in P''(w, C\, r)$. Then

$$P''\left(\pi(z), \frac{\alpha}{C}|r(z)|\right) \subset \{w \in bD :\ z \in \mathcal{A}_\alpha(w)\} \subset P''(\pi(z), C\, \alpha |r(z)|).$$

□

We now apply [**Ch90**, Theorem 11, p. 6-7], that yields a decomposition of dyadic type of bD, i.e. a sequence of nested partitions of bD in open sets $\{Q_x\}_x$ that are uniformly comparable to balls $\{P''_x\}_x$ of geometrically decreasing radius. In particular, let T be the tree which encodes the inclusion relations between the open sets $\{Q_x\}_{x \in T}$ of the decomposition, and let $\{P''_x\}_{x\in T}$ be the corresponding family of balls $P''_x \subset bD$. For $x \in T$ let $|x|$ denote the *generation* to which x belongs. Then there is a constant h, $0 < h < 1$ such that

$$P''_x = P''(\varphi(x), h^{|x|})\,,$$

for certain points $\varphi(x) \in bD$. Moreover, there are positive constants a_0, a_1, a_2, η such that

$$P''(\varphi(x), a_0\, h^{|x|}) \subset Q_x \subset P''(\varphi(x), a_1\, h^{|x|})$$

and

$$\sigma(\{w \in Q_x :\ \varrho''(w, bD \setminus Q_x) \le t\, h^{|x|}\}) \le a_2 t^\eta \sigma(Q_x)\,, t > 0\,, x \in T.$$

Lemma 4.3. *Fix a dyadic type decomposition of bD, as above. Then, for $x \in T$ with $|x|$ large enough, there is a point $z_x \in D$, such that for each $\alpha > 0$ there are constants $c_9(\alpha)$ and $C_9(\alpha)$, with the property that $c_9(1) = a_1$ and, for all $x \in T$ and $\alpha > 0$,*

$$P''(\varphi(x), c_9(\alpha)\, h^{|x|}) \subset \mathcal{A}_\alpha{}^*(z_x) \subset P''(\varphi(x), C_9(\alpha)\, h^{|x|})\,.$$

Moreover, the point z_x is close to the center of P''_x, in the sense that the Euclidean distance between z_x and $\varphi(x)$ tends to zero as $|x| \to \infty$.

Proof. Let ν_w be the inner unit normal to bD at $w \in bD$. Recall that there are constants $\bar{c}, \bar{C}$ such that $\bar{c}\, R \le |r(w + R\nu_w)| \le \bar{C}\, R$, for all $w \in bD$ and $R > 0$. Define, for $|x|$ large enough,

$$z_x := \varphi(x) + \frac{a_1\, h^{|x|}}{\bar{c}\, c_8(1)} \nu_{\varphi(x)}.$$

In particular, $\pi(z_x) = \varphi(x)$ and $\frac{a_1}{c_8(1)} h^{|x|} < |r(z_x)| < \frac{\bar{C} a_1}{\bar{c}\, c_8(1)} h^{|x|}$. Then, by Lemma 4.2,

$$P''\left(\varphi(x), \frac{c_8(\alpha)\, a_1}{c_8(1)} h^{|x|}\right) \subset \mathcal{A}_\alpha{}^*(z_x) \subset P''\left(\varphi(x), \frac{C_8(\alpha)\, \bar{C}\, a_1}{\bar{c}\, c_8(1)} h^{|x|}\right).$$

$\square$

Proof of Theorem 4.1. We can apply [**DB95**, Theorem 5.32], by Lemma 4.2 and Lemma 4.3. In fact, Lemma 4.2 says that the approach regions $\mathcal{A}_\alpha$ form a *natural one parameter family of approach regions*, while Lemma 4.3 says that the embedding $x \in T \to z_x \in D$ is *admissible*. □

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Received July 24, 1996. The first author was supported by C.N.R. grant 203.01.63. The second author was partially supported by Alexander von Humboldt Foundation.

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