PACIFIC JOURNAL OF MATHEMATICS
Vol. 183, No. 1, 1998

CATENOID-LIKE SOLUTIONS FOR THE MINIMAL SURFACE EQUATION

Jenn-Fang Hwang

Let $\Omega \subset \mathbb{R}^2$ be an unbounded domain with width of polynomial growth and let u satisfy the minimal surface equation in Ω. We find out an upper bound function for u and give an example to illustrate that the upper bound function obtained here is approximately optimal. In fact, the graph of the upper bound function is a generalization of a catenoid.

1. Introduction.

Consider the minimal surface equation in $\mathbb{R}^2$

$$\text{div } Tu = 0, \tag{1}$$

where

$$Tu = \frac{Du}{\sqrt{1+|Du|^2}} \qquad \text{and} \qquad Du = (u_x, u_y).$$

In 1965, Nitsche [5] announced the following result: "Let $\Omega_\alpha \subset \mathbb{R}^2$ be a sector with angle $0 < \alpha < \pi$. If u satisfies the minimal surface equation with vanishing boundary value in Ω_α, then $u \equiv 0$". Hwang extends this result in [3], [4] and proves that, in an unbounded domain Ω properly contained in the half plane in $\mathbb{R}^2$, if u satisfies the minimal surface equation, then, the growth property of u is determined completely by the shape of Ω and the boundary value of u. In this respect, the Phragmén-Lindelöf theorem for the minimal surface equation is better than that for the Laplace equation; (indeed, if u satisfies the Laplace equation in an unbounded domain Ω, the growth property of u cannot be determined completely by the shape of Ω and the boundary data of u alone [8]).

One of the results in [4] is the following:

Theorem 1. *Let $\Omega_m = \{(x,y) \in \mathbb{R}^2 | -y^m < x < y^m,\ y > 0\}$, where m is a constant, $m \geq 1$. If u satisfies the minimal surface equation in Ω_m with vanishing boundary value, then*

$$u \leq \sqrt{1 - \frac{1}{m}}\sqrt{y^{2m} - x^2} \qquad in \quad \Omega_m.$$

The estimate in Theorem 1, however, is not optimal. In §2 of this paper, we shall try to find out the optimal upper bound function for solutions of (1) in Ω_m. Then, in §3, we shall give an example to illustrate that the upper bound function obtained in §2 is approximately optimal. The crucial point of this paper is to approximate the solution of (1) in Ω_m with vanishing boundary value by the so-called catenoid-like solutions (which will be introduced in (3) below).

2. Catenoid-like solutions.

Henceforth, we shall denote Ω_m as the domain

$$\{(x,y) \in \mathbb{R}^2 | -y^m < x < y^m,\ y > 0\}$$

in $\mathbb{R}^2$, where m is a constant, $m \geq 1$.

We first observe that the upper bound function in Theorem 1 is

$$u \leq \sqrt{1-\frac{1}{m}}\sqrt{y^{2m}-x^2} = y^m\sqrt{1-\frac{1}{m}}\sqrt{1-\left(\frac{x}{y^m}\right)^2} \qquad \text{in} \quad \Omega_m.$$

This suggests us to consider comparison functions of the following form

$$F = y^m h\left(\frac{x}{y^m}\right), \tag{2}$$

or, even more generally,

$$F = f(y) h\left(\frac{x}{f(y)}\right). \tag{3}$$

For such a function F, each cross-section $y =$ constant has a similar shape and the graph of F is therefore a generalization of a catenoid; thus, we name such a function F as a catenoid-like solution.

We shall proceed to show the following result:

Theorem 2. *Let $f \in C^2(\mathbb{R}^+) \cap C^0(\overline{\mathbb{R}^+})$, where $\mathbb{R}^+ = (0,\infty) \subset \mathbb{R}$, and let $h \in C^2((-1,1)) \cap C^1([-1,1])$. Suppose that $F = f(y)h(\frac{x}{f(y)})$. Then*

$$\begin{aligned}\operatorname{div} TF = &(1+|DF|^2)^{-\frac{3}{2}}\frac{f'^2}{f} \\ &\cdot\left((1-p)(h-h't)(h'^2+1)+h''(h^2+t^2)+\frac{h''}{(f')^2}\right),\end{aligned}$$

where $t = \frac{x}{f(y)}$ *and* $1 - p = \dfrac{ff''}{(f')^2}$.

Proof. By a direct calculation, we have

$$t_x = \frac{1}{f(y)},$$
$$t_y = \frac{-f'(y)}{f^2(y)}x = -\frac{f'(y)}{f(y)}t,$$

and then

$$F_x = f(y)h'(t)t_x = h'(t),$$
$$F_y = f'(y)h(t) + f(y)h'(t)t_y = f'(y)(h(t) - h'(t)t).$$

Thus

$$\begin{aligned} F_{xx} &= h''(t)t_x = \frac{h''(t)}{f(y)}, \\ F_{xy} &= -\frac{f'(y)}{f(y)}th''(t), \\ F_{yy} &= f''(y)(h(t) - th'(t)) + f'(y)(h'(t) - th''(t) - h'(t))\left(-\frac{f'(y)}{f(y)}t\right) \\ &= f''(y)(h(t) - th'(t)) + \frac{f'^2(y)}{f(y)}t^2h''(t). \end{aligned}$$

Hence

$$\begin{aligned} F_{xx} + F_{yy} &= f''(h - th') + \frac{f'^2}{f}t^2h'' + \frac{h''}{f} \\ &= (1-p)\frac{f'^2}{f}(h - th') + \frac{f'^2}{f}t^2h'' + \frac{h''}{f} \\ &= \frac{f'^2}{f}\left((1-p)(h - th') + t^2h'' + \frac{h''}{f'^2}\right). \end{aligned}$$

And

$$\begin{aligned} &{F_x}^2F_{yy} - 2F_xF_yF_{xy} + {F_y}^2F_{xx} \\ &= h'^2\left(f''(h - th') + \frac{f'^2}{f}t^2h''\right) - 2f'h'(h - th')\left(-\frac{f'}{f}th''\right) + f'^2(h - th')^2\frac{h''}{f} \\ &= (1-p)\frac{f'^2}{f}h'^2(h - th') + \frac{f'^2}{f}t^2h'^2h'' + 2\frac{f'^2}{f}(h - th')th'h'' + \frac{f'^2}{f}h''(h - th')^2 \\ &= \frac{f'^2}{f}((1-p)h'^2(h - th') + h'^2h''t^2 + 2h'(h - th')th'' + h''(h - th')^2) \\ &= \frac{f'^2}{f}((1-p)h'^2(h - th') + h^2h''). \end{aligned}$$

Finally, we obtain, by adding the last two identities,

$$\begin{aligned}\operatorname{div} TF &= (1+|DF|^2)^{-\frac{3}{2}}((1+{F_x}^2)F_{yy} - 2F_xF_yF_{xy} + (1+{F_y}^2)F_{xx})\\ &= (1+|DF|^2)^{-\frac{3}{2}}\frac{f'^2}{f}\left((1-p)(h-th')(1+h'^2) + h''(h^2+t^2) + \frac{h''}{f'^2}\right),\end{aligned}$$

as desired. □

In the special case where

$$f(y) = y^m,\ 1 \le m < \infty,\ m = \text{a constant},$$

we have

$$p = p(f) = \frac{1}{m},$$

which is also a constant; moreover, in the bracket in the expression of $\operatorname{div} TF$, the order of growth of the term $\frac{h''}{f'^2}$ is lower than that of the others. Thus, to obtain an optimal comparison function, we may first consider the equation

$$\left(1-\frac{1}{m}\right)(h_m - th'_m)(1+{h'_m}^2) + h''_m({h_m}^2+t^2) = 0, \tag{4}$$

where the domain of definition is $[-1,1]$, throughout which we require that

$$h_m(t) > 0 \qquad \text{for} \quad t \in (-1,1). \tag{5}$$

We specify one of the initial conditions as

$$h_m(-1) = 0, \tag{6}$$

and now proceed to determine the other initial data $h'_m(-1)$. For this, we note that, (4) yields

$$\frac{h''_m}{1+{h'_m}^2} = \left(1-\frac{1}{m}\right)\frac{th'_m - h_m}{t^2+{h_m}^2};$$

that is,

$$(\tan^{-1} h'_m)' = -\left(1-\frac{1}{m}\right)\left(\tan^{-1}\frac{t}{h_m}\right)',$$

which yields

$$\tan^{-1} h'_m\Big|_0^t = -\left(1-\frac{1}{m}\right)\tan^{-1}\frac{t}{h_m}\Big|_0^t.$$

Thus, imposing an additional condition that

$$h'_m(0) = 0, \tag{7}$$

we have

$$\tan^{-1} h'_m = -\left(1 - \frac{1}{m}\right) \tan^{-1} \frac{t}{h_m},$$

or, equivalently,

$$h'_m(t) = -\tan\left(\left(1 - \frac{1}{m}\right) \tan^{-1} \frac{t}{h_m}\right). \tag{8}$$

Since (5) holds for all $t \in (-1, 1)$, we note, by (8) and the assumed continuity of h, that

$$h'_m(-1) = -\tan\left(\left(1 - \frac{1}{m}\right)\left(-\frac{\pi}{2}\right)\right) = \tan\left(\left(1 - \frac{1}{m}\right)\frac{\pi}{2}\right), \tag{9}$$

which is the second initial condition of (4). We note that (7) and (8) yield

$$h_m(-t) = h_m(t),$$

for all $t \in (-1, 1)$. Hence,

$$\begin{cases} h_m(1) = 0, & \\ h'_m(t) \geq 0, & \text{for } \; -1 \leq t \leq 0, \\ h'_m(t) \leq 0, & \text{for } \; 0 \leq t \leq 1. \end{cases} \tag{10}$$

From this, it follows that:

Lemma 1. *$h_m(t_2) \leq h_m(t_1)$, whenever $0 \leq |t_1| \leq |t_2| \leq 1$.*

Also, (10) yields

$$h_m - t h'_m \geq 0, \qquad \text{for} \quad -1 \leq t \leq 1,$$

and hence, by virtue of (4),

$$h''_m \leq 0, \qquad \text{for} \quad -1 \leq t \leq 1. \tag{11}$$

We may also note that, for a constant q with $1 \leq q < m < \infty$, since h_m and h_q are both solutions of (4) with

$$h_m(\pm 1) = h_q(\pm 1) = 0$$

while

$$h'_m(-1) = \tan^{-1}\left(\left(1-\frac{1}{m}\right)\frac{\pi}{2}\right) > \tan^{-1}\left(\left(1-\frac{1}{q}\right)\frac{\pi}{2}\right) = h'_q(-1),$$

therefore

$$h_m(t) > h_q(t) \tag{12}$$

for all $t \in (-1,1)$.

In general, we cannot write out explicitly a solution of the equation (8) with initial data (6). However, for some specific m, the solution can be explicitly written out. For example, for $m=2$, we have

$$h_2(t) = \frac{1-t^2}{2}.$$

In case that $f(y)=e^y$ in (3), we have $p(f) = 1 - \frac{ff''}{f'^2} = 0$. Thus, we may formally define $m=\infty$ and $\frac{1}{m}=0$, substituting this into (8) to obtain

$$h_\infty = \sqrt{1-t^2}; \tag{13}$$

since $h_\infty(-1)=0$ and $h'_\infty(-1)=\infty$, we have

$$h_m(t) < h_\infty(t)$$

for every $t \in (-1,1)$ and every constant $1 \le m < +\infty$.

We are now in a position to prove the following Main Theorem of this paper.

Theorem 3. *Let $1 \le m < \infty$ be a constant and let*

$$\Omega \subset \{(x,y) \in \mathbb{R}^2 | -ay^m < x < ay^m,\ y>0\},$$

where a is a positive constant. Let $u \in C^0(\bar{\Omega}) \cap C^2(\Omega)$. Suppose that

$$\begin{cases} \text{(i) div } Tu \ge 0 & \textit{in} \quad \Omega. \\ \text{(ii) } u \le ay^m h_m(\frac{x}{ay^m}) & \textit{on} \quad \partial\Omega. \end{cases}$$

Then

$$u \le ay^m h_m\left(\frac{x}{ay^m}\right) \qquad \textit{in} \quad \Omega.$$

Remark.

(i) In this theorem, no growth condition on u is imposed.

(ii) When $f(y) = e^y$, $m = \infty$ and Theorem 3 still holds with ay^m replaced by e^y.
(Cf. [4])

Proof. For every positive constant ϵ, by a direct computation, we have

$$ay^m \leq a\left(y + \frac{\epsilon}{m}\right)^{m+\epsilon}, \qquad \text{for every} \quad y > 0. \tag{14}$$

Set

$$G_\epsilon = a\left(y + \frac{\epsilon}{m}\right)^{m+\epsilon} h_{m+\epsilon}\left(\frac{x}{a(y+\frac{\epsilon}{m})^{m+\epsilon}}\right).$$

Since

$$\left\{(x,y) \in \mathbb{R}^2 | -a\left(y + \frac{\epsilon}{m}\right)^{m+\epsilon} < x < a\left(y + \frac{\epsilon}{m}\right)^{m+\epsilon},\ y > 0\right\}$$
$$\supset \{(x,y) \in \mathbb{R}^2 | -ay^m < x < ay^m,\ y > 0\},$$

by Theorem 2, (4) and (11), we have

$$\text{div}\, TG_\epsilon \leq 0 \qquad \text{in} \quad \Omega.$$

Also, on the boundary $\partial\Omega$ of Ω,

$$\begin{aligned} u &\leq ay^m h_m\left(\frac{x}{ay^m}\right) \qquad \text{(by assumption)} \\ &\leq a\left(y + \frac{\epsilon}{m}\right)^{m+\epsilon} h_m\left(\frac{x}{a(y+\frac{\epsilon}{m})^{m+\epsilon}}\right) \qquad \text{(by (14) and (10))} \\ &= G_\epsilon. \end{aligned} \tag{15}$$

Moreover, by [4, Theorem 2.10],

$$u \leq \sqrt{1 - \frac{1}{m}} ay^m h_\infty\left(\frac{x}{ay^m}\right) \qquad \text{in} \quad \Omega;$$

hence, by (13),

$$u \leq \sqrt{1 - \frac{1}{m}} ay^m \qquad \text{in} \quad \Omega. \tag{16}$$

Note that

$$\lim_{y\to\infty} \frac{ay^m}{a(y+\frac{\epsilon}{m})^{m+\epsilon}} = 0,$$

it is easy to see that

$$\sqrt{1-\frac{1}{m}}ay^m \le G_\epsilon \qquad \text{for } \; y \; \text{ sufficiently large.} \tag{17}$$

An application of the maximum principle, together with (15), (16), and (17), yields that

$$u \le G_\epsilon \qquad \text{in} \quad \Omega.$$

Letting $\epsilon \to 0$, the theorem follows immediately. □

3. Examples to illustrate that the estimate in Theorem 3 is approximately optimal.

Let m, $1 < m < \infty$, be a constant. In this section, we shall construct a solution u of the minimal surface equation in a domain Ω with

$$\lim_{y\to\infty}\frac{|\Gamma_y|}{y^m} = 2 \qquad (\text{where} \quad \Gamma_{y_0} = \Omega \cap \{y = y_0\}, \quad |\Gamma_y| \text{ is the length of } \; \Gamma_y)$$

such that

$$\lim_{y\to\infty}\left(\max_{\Gamma_y}\frac{u}{y^m}\right) = h_m(0).$$

This shall illustrate that the estimate in Theorem 3 is approximately optimal.

The discussion will be divided into two cases seperately, namely

Case 1: $+\infty > m > 1.5$

Case 2: $1.5 \ge m > 1.$

We consider first:

Case 1. $+\infty > m > 1.5$.

Let $f(y) = y^m + y^{0.5}$ where $y > 0$.

Then

(18)

$$\begin{aligned}
&1 - p(f) \\
&= \frac{ff''}{f'^2} \\
&= \frac{(y^m + y^{0.5})(m(m-1)y^{m-2} - 0.25y^{-1.5})}{(my^{m-1} + 0.5y^{-0.5})^2} \\
&= \left(1 - \frac{1}{m}\right) + (f')^{-2}\left[(m-1.5)(m-0.5)y^{m-1.5} - 0.5\left(1 - \frac{0.5}{m}\right)y^{-1}\right].
\end{aligned}$$

Let

$$F(x,y) = f(y)h_m\left(\frac{x}{f(y)}\right)$$

where $y > 0$, $-f(y) < x < f(y)$. Now, setting $t = \frac{x}{f(y)}$, we have

$$\operatorname{div} TF = (1 + |DF|^2)^{-\frac{3}{2}} \frac{(f')^2}{f} \cdot \left((1 - p(f))(h_m - th'_m)(1 + {h'_m}^2) + h''_m(t^2 + {h_m}^2) + \frac{h''_m}{(f')^2} \right).$$

Hence, by (4) and (18), we have

$$\operatorname{div} TF = (1 + |DF|^2)^{-\frac{3}{2}} \frac{(f')^2}{f} \cdot \left(\left((f')^{-2} \left[(m - 1.5)(m - 0.5) y^{m-1.5} - 0.5 \left(1 - \frac{0.5}{m} \right) y^{-1} \right] \right) \cdot (h_m - th'_m)(1 + {h'_m}^2) + \frac{h''_m}{(f')^2} \right).$$

By (4) again, we have

$$(1 + {h'_m}^2)(h_m - th'_m) = -\frac{m}{m-1} h''_m({h_m}^2 + t^2).$$

Thus

$$\operatorname{div} TF = (1 + |DF|^2)^{-\frac{3}{2}} \frac{(-h''_m)}{f} \cdot \left(\frac{m}{m-1} \left((m - 1.5)(m - 0.5) y^{m-1.5} - 0.5 \left(1 - \frac{0.5}{m} \right) y^{-1} \right) ({h_m}^2 + t^2) - 1 \right).$$

Hence, as ${h_m}^2 + t^2$ is bounded below by a positive constant depending only on m, (11) yields a positive number y_1, such that

$$\operatorname{div} TF \geq 0, \qquad \text{for all} \quad y \geq y_1. \tag{19}$$

Next, let

$$G = f(y) h_{m+\epsilon} \left(\frac{x}{f(y)} \right),$$

where ϵ is a positive constant. By the same calculation, we obtain

$$\begin{aligned} \operatorname{div} TG = (1 + |DG|^2)^{-\frac{3}{2}} \frac{(-h''_{m+\epsilon})}{f} \cdot \Bigg(\Bigg(\frac{m+\epsilon}{(m+\epsilon)-1} \Bigg(&(m - 1.5)(m - 0.5) y^{m-1.5} \\ &- 0.5 \left(1 - \frac{0.5}{m} \right) y^{-1} \Bigg) (h^2_{m+\epsilon} + t^2) - 1 \Bigg) \\ &+ \frac{m+\epsilon}{(m+\epsilon)-1} \left(\frac{1}{m+\epsilon} - \frac{1}{m} \right) (h^2_{m+\epsilon} + t^2)(f')^2 \Bigg). \end{aligned}$$

Since $f' = my^{m-1} + 0.5y^{-0.5}$, it is easy to see that there exists a positive constant y_2, determined by on m and ϵ, such that $y_2 > y_1$ and

$$\operatorname{div} TG \leq 0, \qquad \text{for all} \quad y \geq y_2. \tag{20}$$

To summarize up, we have constructed the functions F and G as the lower and upper barriers, respectively; that is, we have

$$\begin{cases} \operatorname{div} TG \leq 0 \leq \operatorname{div} TF & \text{in} \quad \Omega' \\ G = F = 0 & \text{on} \quad \partial\Omega' \backslash \{y = y_2\} \\ G \geq F & \text{in} \quad \Omega', \end{cases}$$

where $\Omega' = \{(x, y) \in \mathbb{R}^2 | -(y^m + y^{0.5}) < x < y^m + y^{0.5},\ y > y_2\}$. As Perron's method can be adopted here (cf. [6, p. 593-600]), this gives us a function u' defined in Ω' such that

$$\begin{cases} \operatorname{div} Tu' = 0 & \text{in} \quad \Omega' \\ G \geq u' \geq F & \text{in} \quad \Omega' \\ u' = 0 & \text{on} \quad \partial\Omega' \backslash \{y = y_2\}. \end{cases}$$

Set

$$u = u' - \max_{\{y = y_2\}} u'$$

and

$$\Omega'' = \left\{ (x, y) \in \Omega' | u'(x, y) > \max_{\{y = y_2\}} u', y > y_2 \right\}.$$

Then $u = 0$ on $\partial\Omega''$. It is easy to see thet Ω'' contains $(0, y)$ for all sufficiently large y, and Ω'' contains $\{(x, y)| - y^m < x < y^m, \quad y > y_0\}$ where y_0 is sufficiently large. To see that u is our desired optimal solution, it remains to show that the behaviour of u near infinity is asymptotic to that of the function $y^m h_m(\frac{x}{y^m})$. To do so, we may define

$$H(x, y) = (y + 1)^m h_m \left(\frac{x}{(y + 1)^m} \right) + \max_{\{y = y_2\}} u'(x, y);$$

then, as

$$(y + 1)^m \geq y^m + y^{0.5},$$

we have $H(x, y)$ defined in Ω' and $H \geq u'$ on $\partial\Omega'$. Hence, by Theorem 3,

$$H \geq u' \qquad \text{in} \quad \Omega'.$$

This, together with the inequality that

$$u' \geq F \qquad \text{in} \quad \Omega'$$

gives us the desired estimate of the behaviour of u' and u near the infinity.

Next, we consider

Case 2. $1.5 \geq m > 1$.

Let $f(y) = y^m - y^\beta$, for all $y > 0$, where $\beta = \frac{m+1}{2}$. Then

$$
\begin{aligned}
&1 - p(f) \\
&= 1 - \frac{1}{m} + (f')^{-2}\left(-(m-\beta-1)(m-\beta)y^{m+\beta-2} - \beta\left(1 - \frac{\beta}{m}\right)y^{2\beta-2}\right),
\end{aligned}
$$

in which we may note that $-(m-\beta-1)(m-\beta) > 0$ and $m+\beta-2 > 0$.

Let

$$F(x,y) = f(y)h_m\left(\frac{x}{f(y)}\right),$$

and

$$G(x,y) = f(y)h_{m+\epsilon}\left(\frac{x}{f(y)}\right),$$

where ϵ is a positive constant. By the same reasoning as in Case 1, there exists a positive constant y_3 such that, in

$$\Omega_3 = \{(x,y) \in \mathbb{R}^2 | -f(y) < x < f(y),\ y > y_3\},$$

we have

$$\operatorname{div} TF \geq 0$$

and

$$\operatorname{div} TG \leq 0.$$

These again give us a function u, defined in Ω_3, such that

$$
\begin{cases}
\operatorname{div} Tu = 0 & \text{in} \quad \Omega_3 \\
F \leq u \leq G & \text{in} \quad \Omega_3 \\
u = 0 & \text{on} \quad \partial\Omega_3 \backslash \{y = y_3\}.
\end{cases}
$$

Moreover, since $y^m > y^m - y^\beta = f(y)$, an upper bound for u is

$$y^m h_m\left(\frac{x}{y^m}\right) + \max_{\{y=y_3\}} u.$$

Thus, the function

$$u' = u - \max_{\{y=y_3\}} u$$

is what looked for.

Acknowledgments. The author would like to express his gratitude to Professor Jin-Tzu Chen for bringing this problem to his attention, and he would like to thank the referee for many helpful comments and suggestions.

References

[1] R. Finn, *Equilibrium capillary surfaces*, Springer-Verlag, New York-Berlin-Heidelberg-Tokyo, 1986.

[2] D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order*, (second edition), Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1983.

[3] J. Hwang, *Phragmén-Lindelöf theorem for the minimal surface equation*, Proc. Amer. Math. Soc., **104** (1988), 825-828.

[4] ———, *Growth property for the minimal surface equation in unbounded domains*, Proc. Amer. Math. Soc., **121** (1994), 1027-1037.

[5] J.C.C. Nitsche, *On new results in the theory of minimal surface*, Bull. Amer. Math. Soc., **71** (1965), 195-270.

[6] ———, *Vorlesungen über Minimalflächen*, Springer-Verlag, Berlin-Heidelberg-New York, 1975.

[7] R. Osserman, *A survey of minimal surfaces*, Van Nostrand-Reinhold, New York, 1969.

[8] M.H. Protter and H.F. Weinberger, *Maximum principles in differential equations*, Prentice-Hall, Englewood Cliffs, N.J., 1967.

Received January 15, 1996 and revised October 9, 1996. This research was partially supported by Grant NSC84-2121-M-001-013.

Academia Sinica
Taipei 11529
Taiwan, R.O.C.
E-mail address: MAJFH@ccvax.sinica.edu.tw