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SOME GENERALITIES ON $\mathcal{D}$-MODULES IN POSITIVE CHARACTERISTIC

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After the ring theoretic study of differential operators in positive characteristic by S.U. Chase and S.P. Smith, B. Haastert started investigation of $\mathcal{D}$-modules on smooth varieties in positive characteristic, and the work of R. Bøgvad followed. The purpose of this paper is to complement some basics for further study.

We fix an algebraically closed field $\mathfrak{k}$ of positive characteristic p. All the varieties considered in this paper will be smooth over $\mathfrak{k}$ unless otherwise specified. A celebrated theorem of A. Beilinson and J. Bernstein says that if a variety $\mathfrak{X}$ is $\mathcal{D}$-affine, then the category of $D(\mathfrak{X})$-modules is equivalent to its local version the category of $\mathcal{D}_{\mathfrak{X}}$-modules that are quasicoherent over $\mathcal{O}_{\mathfrak{X}}$. In §1 we note that the converse also holds. In §2 we will verify the base change theorem for the direct image functor of $\mathcal{D}$-modules as in characteristic 0, that will enable us to introduce a structure of $\mathfrak{G}$-equivariant $\mathcal{D}$-module on local cohomology modules. If $\mathfrak{G}$ is an affine algebraic $\mathfrak{k}$-group acting on a variety $\mathfrak{X}$, we give in §3 an infinitesimal criterion for an $\mathcal{O}_{\mathfrak{X}}$-module to be $\mathfrak{G}$-equivariant, introduce two $\mathfrak{G}$-equivariant versions of Haastert's $\mathfrak{X}^{\infty}$-modules, and show that the equivalence in characteristic 0 of the category of Harish-Chandra $(\mathrm{Dist}(\mathfrak{G}), \mathfrak{H})$-modules to the category of quasi-$\mathfrak{G}$-equivariant $\mathcal{D}_{\mathfrak{G}/\mathfrak{H}}$-modules carries over to positive characteristic for a closed subgroup scheme $\mathfrak{H}$ of $\mathfrak{G}$. §4 contains a few applications on the flag variety.

Notations. By $\mathbf{Alg}_{\mathfrak{k}}$ (resp. $\mathbf{Sch}_{\mathfrak{k}}$) we will denote the category of $\mathfrak{k}$-algebras (resp. $\mathfrak{k}$-schemes). The tensor product $\otimes$ without a subscript is always taken over $\mathfrak{k}$. If A is a $\mathfrak{k}$-algebra, $A\mathbf{Mod}$ (resp. $\mathbf{Mod}A$) will denote the category of left (resp. right) A-modules. If A is commutative and if there is no need to distinguish left and right, the category of A-modules is denoted by $\mathbf{Mod}_A$. If there are two $\mathfrak{k}$-algebra homomorphisms from A into C, one making C into a left A-module and the other into a right A-module, we will call C a left (resp. right) A-ring, and denote the category of left (resp. right) A-rings by $A\mathbf{Rng}$ (resp. $\mathbf{Rng}A$). For a $\mathfrak{k}$-variety $\mathfrak{X}$ the category of quasicoherent $\mathcal{O}_{\mathfrak{X}}$-modules is denoted by $\mathbf{qc}_{\mathfrak{X}}$, and the category of sheaves of abelian groups on $\mathfrak{X}$ by $\mathbf{Ab}_{\mathfrak{X}}$. If $\mathcal{A}$ is a sheaf of $\mathfrak{k}$-algebras on $\mathfrak{X}$, $\mathcal{A}\mathbf{Mod}$ will denote the category of left $\mathcal{A}$-modules replacing A by $\mathcal{A}$ above, and define likewise $\mathbf{Mod}\mathcal{A}$, etc. In case $\mathcal{A} = \mathcal{O}_{\mathfrak{X}}$, we will abbreviate $\mathcal{O}_{\mathfrak{X}}$ as $\mathfrak{X}$ and write $\mathfrak{X}$**Mod** for $\mathcal{O}_{\mathfrak{X}}$**Mod**, etc. The tensor product $\otimes_{\mathfrak{X}}$ will be taken over $\mathcal{O}_{\mathfrak{X}}$. The sheaf of the ring of differential operators on $\mathfrak{X}$ is denoted $\mathcal{D}_{\mathfrak{X}}$ with $D(\mathfrak{X})$ the ring of the global sections of $\mathcal{D}_{\mathfrak{X}}$, and $\mathcal{D}_{\mathfrak{X}}$**qc** denotes the category of left $\mathcal{D}_{\mathfrak{X}}$-modules that are quasicoherent over $\mathcal{O}_{\mathfrak{X}}$. For each $r \in \mathbb{N}$ we will denote by $\mathfrak{X}^{(r)}$ the $\mathfrak{k}$-variety such that $\mathfrak{X}^{(r)}(A) = \mathfrak{X}(A^{(-r)})$ for each $\mathfrak{k}$-algebra A, where $A^{(-r)} = A$ as a ring with the $\mathfrak{k}$-algebra structure given by $\xi \mapsto \xi^{p^r}$. Let $\mathfrak{F}^r_{\mathfrak{X}} : \mathfrak{X} \to \mathfrak{X}^{(r)}$ be the Frobenius morphism induced by the $\mathfrak{k}$-algebra homomorphism $A \to A^{(-r)}$ via $a \mapsto a^{p^r}$. If $\mathfrak{G}$ is a $\mathfrak{k}$-group, $\mathfrak{G}$**Mod** will denote the category of $\mathfrak{G}$-modules. There should be no confusion with $\mathcal{O}_{\mathfrak{G}}$**Mod**.

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§1.

In this section $\mathfrak{X}$ will denote a smooth $\mathfrak{k}$-variety of dimension N. We will employ the notations of [**EGAIV**, §16] unless otherwise specified.

1.1. The variety $\mathfrak{X}$ admits a finite affine open cover $(\mathfrak{X}_i)_i$ with each $\mathfrak{X}_i$ étale over the affine N-space $\mathbb{A}^{\mathrm{N}}_{\mathfrak{k}}$. If A is an étale domain over $\mathfrak{k}[\mathbb{A}^{\mathrm{N}}]$, then A is flat over $\mathfrak{k}[\mathbb{A}^{\mathrm{N}}]$ and

(1) $$\Omega^1_{A/\mathfrak{k}} \simeq A \otimes \Omega^1_{\mathfrak{k}[\mathbb{A}^{\mathrm{N}}]/\mathfrak{k}} \qquad \text{in } A\mathbf{Mod}.$$

If $f : \mathfrak{k}[\mathbb{A}^{\mathrm{N}}] \to A$ is the structure homomorphism and if we write $\mathfrak{k}[\mathbb{A}^{\mathrm{N}}] = \mathfrak{k}[t_1, \ldots, t_{\mathrm{N}}]$ with indeterminates t_i, $\Omega^1_{A/\mathfrak{k}} = \coprod_{i=1}^{\mathrm{N}} A \, \mathrm{d}_{A/\mathfrak{k}} \, z_i$ with $z_i = f(t_i)$. We call $\mathbf{z} = (z_1, \ldots, z_{\mathrm{N}})$ a regular system of parameters on A. As A is smooth over $\mathfrak{k}$ [**EGAIV**, 16.10],

$$\begin{aligned} \mathrm{gr}(P_{A/\mathfrak{k}}) &\simeq S^{\cdot}_A(\Omega^1_{A/\mathfrak{k}}) \simeq S^{\cdot}_A(A \otimes_{\mathfrak{k}[\mathbb{A}^{\mathrm{N}}]} \Omega^1_{\mathfrak{k}[\mathbb{A}^{\mathrm{N}}]/\mathfrak{k}}) \\ &\simeq A \otimes_{\mathfrak{k}[\mathbb{A}^{\mathrm{N}}]} S^{\cdot}_{\mathfrak{k}[\mathbb{A}^{\mathrm{N}}]/\mathfrak{k}}(\Omega^1_{\mathfrak{k}[\mathbb{A}^{\mathrm{N}}]/\mathfrak{k}}) \quad \text{as } A \text{ is flat over } \mathfrak{k}[\mathbb{A}^{\mathrm{N}}] \\ &\simeq A \otimes_{\mathfrak{k}[\mathbb{A}^{\mathrm{N}}]} \mathrm{gr}(P_{\mathfrak{k}[\mathbb{A}^{\mathrm{N}}]/\mathfrak{k}}). \end{aligned}$$

Dualizing one obtains [**EGAIV**, 16.11]

(2) $$D(A) \simeq A \otimes_{\mathfrak{k}[\mathbb{A}^{\mathrm{N}}]} D(\mathbb{A}^{\mathrm{N}}_{\mathfrak{k}}) \qquad \text{in } A\mathbf{Mod}.$$

In particular,

(3)
$$D(A) = \coprod_{\mathbf{n} \in \mathbb{N}^{\mathrm{N}}} A\partial^{\mathbf{n}} = \coprod_{\mathbf{n} \in \mathbb{N}^{\mathrm{N}}} \partial^{\mathbf{n}} A \quad \text{with} \quad \partial^{\mathbf{n}}(\mathbf{z}^{\mathbf{m}}) = \binom{\mathbf{m}}{\mathbf{n}} \mathbf{z}^{\mathbf{m}-\mathbf{n}} \quad \forall \mathbf{m} \in \mathbb{N}^{\mathrm{N}},$$

where $\mathbf{z}^{\mathbf{m}} = \prod_{i=1}^{\mathrm{N}} z_i^{m_i}$. Also [**EGA0**, 21.1.7]

$$A = \coprod_{\mathbf{n}\in[0,p^r-1]^{\mathrm{N}}} A^{(r)}\mathbf{z}^{\mathbf{n}} \quad \forall r \in \mathbf{N}, \tag{4}$$

where $A^{(r)} = \{a^{p^r} \mid a \in A\}$. In addition to the standard filtration $\mathcal{D}iff_{\mathfrak{X}}^n$ there is another filtration on $\mathcal{D}_{\mathfrak{X}}$, called the p-filtration, defined by

$$\mathcal{D}_{\mathfrak{X},r} = \mathcal{M}od_{\mathcal{O}_{\mathfrak{X}}^{(r)}}(\mathcal{O}_{\mathfrak{X}}, \mathcal{O}_{\mathfrak{X}}),$$

where $\mathcal{O}_{\mathfrak{X}}^{(r)}$ is the sheaf of $\mathfrak{k}$-algebras such that $\mathcal{O}_{\mathfrak{X}}^{(r)}(\mathfrak{U}) = \mathcal{O}_{\mathfrak{X}}(\mathfrak{U})^{(r)}$ on each open $\mathfrak{U}$ of $\mathfrak{X}$ [**H87**, 1.2]. By direct computation one checks:

Lemma. *If A is an étale algebra over $\mathfrak{k}[\mathbb{A}^1]$ with a regular parameter z, then in $D(A)$ for any m and $r \in \mathbb{N}$*

$$[\partial^m, z^{p^r}] = \begin{cases} \partial^{m-p^r} & \textit{if } m \geq p^r \\ 0 & \textit{otherwise.} \end{cases}$$

1.2. One then obtains

Proposition ([**MN**, 1.2.2])**.** *For each $r \in \mathbb{N}$ the sheaf of $\mathfrak{k}$-algebras $\mathcal{D}_{\mathfrak{X},r}$ is generated by $\mathcal{D}iff_{\mathfrak{X}}^{p^r-1}$ in both* $\mathfrak{X}$**Rng** *and* **Rng**$\mathfrak{X}$.

1.3. Let $\mathfrak{Y}$ be another smooth $\mathfrak{k}$-variety of dimension L, and let $\mathfrak{p}_{\mathfrak{X}}$ (resp. $\mathfrak{p}_{\mathfrak{Y}}$) : $\mathfrak{X} \times_{\mathfrak{k}} \mathfrak{Y} \to \mathfrak{X}$ (resp. $\mathfrak{Y}$) be the natural projection. We will suppress $\mathfrak{k}$ in $\mathfrak{X} \times_{\mathfrak{k}} \mathfrak{Y}$. If $\mathcal{M} \in \mathbf{Mod}_{\mathfrak{X}}$ and $\mathcal{N} \in \mathbf{Mod}_{\mathfrak{Y}}$, we will write $\mathcal{M} \boxtimes \mathcal{N}$ for $(\mathfrak{p}_{\mathfrak{X}}^* \mathcal{M}) \otimes_{\mathfrak{X}\times\mathfrak{Y}} (\mathfrak{p}_{\mathfrak{Y}}^* \mathcal{N})$.

Proposition. *There is a natural isomorphism $\mathcal{D}_{\mathfrak{X}} \boxtimes \mathcal{D}_{\mathfrak{Y}} \to \mathcal{D}_{\mathfrak{X}\times\mathfrak{Y}}$ under which*

$$\sum_{i+j=n+1} \mathcal{D}iff_{\mathfrak{X}}^i \boxtimes \mathcal{D}iff_{\mathfrak{Y}}^j \simeq \mathcal{D}iff_{\mathfrak{X}\times\mathfrak{Y}}^n \;\; \textit{and} \;\; \mathcal{D}_{\mathfrak{X},n} \boxtimes \mathcal{D}_{\mathfrak{Y},n} \simeq \mathcal{D}_{\mathfrak{X}\times\mathfrak{Y},n} \; \forall n \in \mathbb{N},$$

where $\boxtimes$ is taken with respect to the left $\mathcal{O}_{\mathfrak{X}}$ (resp. $\mathcal{O}_{\mathfrak{Y}}$)-module structure on $\mathcal{D}_{\mathfrak{X}}$ (resp. $\mathcal{D}_{\mathfrak{Y}}$).

Proof. If $\mathfrak{U}$ (resp. $\mathfrak{V}$) is an affine open of $\mathfrak{X}$ (resp. $\mathfrak{Y}$), then $\Gamma(\mathfrak{U} \times \mathfrak{V}, \mathcal{D}_{\mathfrak{X},r} \boxtimes \mathcal{D}_{\mathfrak{Y},r}) \simeq D_r(\mathfrak{U}) \otimes D_r(\mathfrak{V})$ and $\Gamma(\mathfrak{U} \times \mathfrak{V}, \mathcal{D}_{\mathfrak{X}\times\mathfrak{Y},r}) \simeq \mathbf{Mod}_{(\mathfrak{k}[\mathfrak{U}]\otimes\mathfrak{k}[\mathfrak{V}])^{(r)}}(\mathfrak{k}[\mathfrak{U}] \otimes \mathfrak{k}[\mathfrak{V}], \mathfrak{k}[\mathfrak{U}] \otimes \mathfrak{k}[\mathfrak{V}])$ for each $r \in \mathbb{N}$. The natural maps $D_r(\mathfrak{U}) \otimes D_r(\mathfrak{V}) \to \mathbf{Mod}_{(\mathfrak{k}[\mathfrak{U}]\otimes\mathfrak{k}[\mathfrak{V}])^{(r)}}(\mathfrak{k}[\mathfrak{U}] \otimes \mathfrak{k}[\mathfrak{V}], \mathfrak{k}[\mathfrak{U}] \otimes \mathfrak{k}[\mathfrak{V}])$ glue together to yield a morphism

$\mathcal{D}_{\mathfrak{X},r} \boxtimes \mathcal{D}_{\mathfrak{Y},r} \to \mathcal{D}_{\mathfrak{X}\times\mathfrak{Y},r}$. Then by taking the direct limits one obtains a morphism in $(\mathfrak{X}\times\mathfrak{Y})\mathbf{Rng}$

$$\mathcal{D}_{\mathfrak{X}} \boxtimes \mathcal{D}_{\mathfrak{Y}} \to \mathcal{D}_{\mathfrak{X}\times\mathfrak{Y}}. \tag{1}$$

To see the assertions about the morphism (1), the question being local we may assume that both $\mathfrak{X}$ and $\mathfrak{Y}$ are affine and étale over $\mathbb{A}^{\mathrm{N}}_{\mathfrak{k}}$ and $\mathbb{A}^{\mathrm{L}}_{\mathfrak{k}}$, respectively, equipped with a regular system of parameters $\mathbf{x}$ on $A = \mathcal{O}_{\mathfrak{X}}(\mathfrak{X})$ and $\mathbf{y}$ on $C = \mathcal{O}_{\mathfrak{Y}}(\mathfrak{Y})$. The assertion about the p-filtrations is immediate from (1.1.4). One has (cf. [**EGAIV**, 16.4.23]) an isomorphism $(\Omega^1_{A/\mathfrak{k}} \otimes B) \oplus (A \otimes \Omega^1_{B/\mathfrak{k}}) \to \Omega^1_{A\otimes B/\mathfrak{k}}$ in $\mathbf{Mod}_{A\otimes B}$ via

$$(\mathrm{d}\,a \otimes b, a' \otimes \mathrm{d}\,b') \longmapsto (1\otimes b)\,\mathrm{d}(a\otimes 1) + (a'\otimes 1)\,\mathrm{d}(1\otimes b').$$

Hence $(\mathbf{x}\otimes 1, 1\otimes \mathbf{y}) = (x_i \otimes 1, 1\otimes y_j)_{i,j}$ forms a regular system of parameters on $A\otimes B$. Then for each $\mathbf{u},\mathbf{v} \in \mathbb{N}^{\mathrm{N}}$ and $\mathbf{p},\mathbf{q}\in\mathbb{N}^{\mathrm{L}}$

$$(\partial^{\mathbf{u}} \otimes \partial^{\mathbf{p}})(\mathbf{x}^{\mathbf{v}} \otimes \mathbf{y}^{\mathbf{q}}) = \binom{\mathbf{v}}{\mathbf{u}}\binom{\mathbf{q}}{\mathbf{p}} \mathbf{x}^{\mathbf{v}-\mathbf{u}} \otimes \mathbf{y}^{\mathbf{q}-\mathbf{p}},$$

hence $\partial^{\mathbf{u}} \otimes \partial^{\mathbf{p}} \mapsto \partial^{(\mathbf{u},\mathbf{p})}$ under the morphism (1). Consequently one obtains the isomorphisms with respect to the standard filtrations.

1.4. By [**H87**, 1.3.3, 5]

$$\mathcal{D}_{\mathfrak{X}} \text{ is not noetherian but coherent in } \mathcal{D}_{\mathfrak{X}}\mathbf{Mod}, \tag{1}$$

hence a left $\mathcal{D}_{\mathfrak{X}}$-module of finite presentation type is coherent over $\mathcal{D}_{\mathfrak{X}}$. As $\mathcal{D}_{\mathfrak{X}}$ is $\mathcal{O}_{\mathfrak{X}}$-quasicoherent, any $\mathcal{D}_{\mathfrak{X}}$-coherent module belongs to $\mathcal{D}_{\mathfrak{X}}\mathbf{qc}$. By [**BVI**, 2.1]

$$\mathcal{D}_{\mathfrak{X}}\mathbf{qc} \text{ has enough injectives.} \tag{2}$$

The following lemma was kindly communicated from Hotta R.

1.5.

Lemma. *An injective of $\mathcal{D}_{\mathfrak{X}}\mathbf{qc}$ remains injective in $\mathfrak{X}\mathbf{Mod}$, hence flasque.*

Proof. Let $\mathcal{I}$ be an injective of $\mathcal{D}_{\mathfrak{X}}\mathbf{qc}$. We first show that $\mathcal{I}$ is injective in $\mathfrak{X}\mathbf{qc}$. Given $f \in \mathbf{Mod}_{\mathfrak{X}}(\mathcal{M},\mathcal{I})$ and a mono $j \in \mathbf{Mod}_{\mathfrak{X}}(\mathcal{M},\mathcal{N})$, $\mathcal{D}_{\mathfrak{X}} \otimes_{\mathfrak{X}} j \in \mathcal{D}_{\mathfrak{X}}\mathbf{Mod}(\mathcal{D}_{\mathfrak{X}} \otimes_{\mathfrak{X}} \mathcal{M}, \mathcal{D}_{\mathfrak{X}} \otimes_{\mathfrak{X}} \mathcal{N})$ remains monic as $\mathcal{D}_{\mathfrak{X}}$ is locally free in $\mathbf{Mod}\mathfrak{X}$. Then there is $\hat{f} \in \mathcal{D}_{\mathfrak{X}}\mathbf{Mod}(\mathcal{D}_{\mathfrak{X}} \otimes_{\mathfrak{X}} \mathcal{N}, \mathcal{I})$ such that $\hat{f} \circ (\mathcal{D}_{\mathfrak{X}} \otimes_{\mathfrak{X}} j) = \mathcal{D}_{\mathfrak{X}} \bar{\otimes}_{\mathfrak{X}} f : \delta \otimes m \mapsto \delta m$. If $\imath_{\mathcal{M}} \in \mathbf{Mod}_{\mathfrak{X}}(\mathcal{M}, \mathcal{D}_{\mathfrak{X}} \otimes_{\mathfrak{X}} \mathcal{M})$ is the natural imbedding and likewise $\jmath_{\mathcal{N}}$, then

$$f = (\mathcal{D}_{\mathfrak{X}} \bar{\otimes}_{\mathfrak{X}} f) \circ \imath_{\mathcal{M}} = \hat{f} \circ (\mathcal{D}_{\mathfrak{X}} \otimes_{\mathfrak{X}} j) \circ \imath_{\mathcal{M}} = \hat{f} \circ \imath_{\mathcal{N}} \circ j,$$

hence $\mathcal{I}$ is injective in $\mathfrak{X}\mathbf{qc}$. Next, as $\mathfrak{X}$ is noetherian, by [**RD**, II.7.18]

(1) there is $\mathcal{J} \in \mathfrak{X}\mathbf{qc}$, that is injective in $\mathfrak{X}\mathbf{Mod}$, and a mono $\imath \in \mathfrak{X}\mathbf{Mod}(\mathcal{I}, \mathcal{J})$.

Then $\imath$ has a left inverse $\imath' \in \mathfrak{X}\mathbf{Mod}(\mathcal{J}, \mathcal{I})$. Given $h \in \mathfrak{X}\mathbf{Mod}(\mathcal{M}', \mathcal{I})$ and a mono $\jmath' \in \mathfrak{X}\mathbf{Mod}(\mathcal{M}', \mathcal{N}')$, there is $h' \in \mathfrak{X}\mathbf{Mod}(\mathcal{N}', \mathcal{J})$ such that $h' \circ \jmath' = \imath \circ h$, hence $\imath' \circ h' \circ \jmath' = \imath' \circ \imath \circ h = h$. Thus $\mathcal{I}$ remains injective in $\mathfrak{X}\mathbf{Mod}$. □

1.6. One says $\mathfrak{X}$ is $\mathcal{D}$-affine iff for each $\mathcal{M} \in \mathcal{D}_{\mathfrak{X}}\mathbf{qc}$ the following two conditions hold: (i) The natural morphism $\mathcal{D}_{\mathfrak{X}} \otimes_{D(\mathfrak{X})} \mathcal{M}(\mathfrak{X}) \to \mathcal{M}$ is epic, and (ii) $\mathrm{H}^i(\mathfrak{X}, \mathcal{M}) = 0 \quad \forall i > 0$. The celebrated Beilinson-Bernstein local-global principle says

(1) if $\mathfrak{X}$ is $\mathcal{D}$-affine, then $\Gamma(\mathfrak{X}, ?) : \mathcal{D}_{\mathfrak{X}}\mathbf{qc} \to D(\mathfrak{X})\mathbf{Mod}$ is an equivalence of categories with quasi-inverse $\mathcal{D}_{\mathfrak{X}} \otimes_{D(\mathfrak{X})} ?$.

The equivalence is called the Beilinson-Bernstein correspondence. Conversely,

Proposition. *If the Beilinson-Bernstein correspondence holds on $\mathfrak{X}$, then $\mathfrak{X}$ is $\mathcal{D}$-affine.*

Proof. Let $\mathcal{M} \in \mathcal{D}_{\mathfrak{X}}\mathbf{qc}$. If $M = \mathcal{M}(\mathfrak{X})$ and $M \to I^{\cdot}$ is an injective resolution in $D(\mathfrak{X})\mathbf{Mod}$, then $\mathcal{D}_{\mathfrak{X}} \otimes_{D(\mathfrak{X})} M \to \mathcal{D}_{\mathfrak{X}} \otimes_{D(\mathfrak{X})} I^{\cdot}$ remains an injective resolution in $\mathcal{D}_{\mathfrak{X}}\mathbf{qc}$, hence

$$\begin{aligned} \mathrm{H}^i(\mathfrak{X}, \mathcal{M}) &\simeq \mathrm{H}^i(\mathfrak{X}, \mathcal{D}_{\mathfrak{X}} \otimes_{D(\mathfrak{X})} M) \simeq \mathrm{H}^i(\Gamma(\mathfrak{X}, \mathcal{D}_{\mathfrak{X}} \otimes_{D(\mathfrak{X})} I^{\cdot})) \quad \text{by (1.5)} \\ &\simeq \mathrm{H}^i(I^{\cdot}) \\ &= 0 \quad \text{for } i > 0. \end{aligned}$$

□

§2.

In this section $\mathfrak{f} \in \mathbf{Sch}_{\mathfrak{k}}(\mathfrak{X}, \mathfrak{Y})$ will denote a morphism from a smooth $\mathfrak{k}$-variety $\mathfrak{X}$ of dimension N to another smooth $\mathfrak{k}$-variety $\mathfrak{Y}$ of dimension L.

2.1. By a theorem of Kleiman [**H**, Ex. III.6.8]

(1) any coherent $\mathcal{O}_{\mathfrak{X}}$-module is the quotient of a locally free $\mathcal{O}_{\mathfrak{X}}$-module of finite rank.

In case $\mathfrak{X}$ is affine, by [**C**, Th. 3.5], [**Sm**, Th. 3.7]

$$\text{gldim} D(\mathfrak{X}) = \dim \mathfrak{X}. \tag{2}$$

As $\mathcal{D}_{\mathfrak{X}}$ is coherent, one obtains as in characteristic 0 (cf. [**TH**, Prop. I.1.4.3]):

Proposition. *The category $\mathcal{D}_{\mathfrak{X}}\mathbf{qc}$ has enough locally free objects. More precisely, each $\mathcal{M} \in \mathcal{D}_{\mathfrak{X}}\mathbf{qc}$ admits a resolution*

$$0 \to \mathcal{M}^{-\mathrm{N}} \to \cdots \mathcal{M}^{-1} \to \mathcal{M}^0 \to \mathcal{M} \to 0$$

in $\mathcal{D}_{\mathfrak{X}}\mathbf{qc}$ with $\mathcal{M}^{-i}$ locally free, $i \in [0, n-1]$, $\mathcal{M}^{-\mathrm{N}}$ locally projective, and $\mathrm{N} = \dim \mathfrak{X}$. If $\mathcal{M}$ is coherent over $\mathcal{D}_{\mathfrak{X}}$, one could take all $\mathcal{M}^i$ coherent over $\mathcal{D}_{\mathfrak{X}}$.

2.2. The inverse image functor $\mathfrak{f}^0 : \mathcal{D}_{\mathfrak{Y}}\mathbf{qc} \to \mathcal{D}_{\mathfrak{X}}\mathbf{qc}$ is defined by $\mathfrak{f}^0 = \mathcal{D}_{\mathfrak{f}\to} \otimes_{\mathfrak{f}^{-1}\mathcal{D}_{\mathfrak{Y}}} \mathfrak{f}^{-1}?$ with $\mathcal{D}_{\mathfrak{f}\to} = \mathfrak{f}^*\mathcal{D}_{\mathfrak{Y}} = \mathcal{O}_{\mathfrak{X}} \otimes_{\mathfrak{f}^{-1}\mathcal{O}_{\mathfrak{Y}}} \mathfrak{f}^{-1}\mathcal{D}_{\mathfrak{Y}} \in \mathcal{D}_{\mathfrak{X}}\mathbf{Mod}\mathfrak{f}^{-1}\mathcal{D}_{\mathfrak{Y}}$. Haastert [**H86**, 3.6.1], however, gave another definition using the $\mathfrak{Y}^\infty$-module structure. If $\mathcal{M} \in \mathcal{D}_{\mathfrak{Y}}\mathbf{qc}$, then $(\mathcal{M}^{(r)})_{r\in\mathbb{N}}$ with

$$\mathcal{M}^{(r)} = \mathcal{M}od_{\mathcal{O}_{\mathfrak{Y}}^{(r)}}(\mathcal{O}_{\mathfrak{Y}}, \mathcal{O}_{\mathfrak{Y}}^{(r)}) \otimes_{\mathcal{D}_{\mathfrak{Y},r}} \mathcal{M}$$

forms a projective system of $\mathfrak{Y}^\infty$-module [**H87**, 2.2.3]. Let $\mathfrak{f}^{(r)} \in \mathbf{Sch}_{\mathfrak{k}}(\mathfrak{X}^{(r)}, \mathfrak{Y}^{(r)})$ with $\mathfrak{f}^{(r)}(A) = \mathfrak{f}(A^{(-r)})$ for each $A \in \mathbf{Alg}_{\mathfrak{k}}$ so that $\mathfrak{f}^{(r)} \circ \mathfrak{F}_{\mathfrak{X}}^r = \mathfrak{F}_{\mathfrak{Y}}^r \circ \mathfrak{f}$. Then

$$(\mathfrak{f}^{(r)*}(\mathcal{M}^{(r)}))_r \text{ forms an } \mathfrak{X}^\infty\text{-module,} \tag{1}$$

hence $\varinjlim_r (\mathfrak{F}_{\mathfrak{X}}^r)^*(\mathfrak{f}^{(r)})^*(\mathcal{M}^{(r)})$ carries a structure of $\mathcal{D}_{\mathfrak{X}}\mathbf{qc}$ such that $\mathcal{D}_{\mathfrak{X},r}$ acts on $(\mathfrak{F}_{\mathfrak{X}}^r)^*(\mathfrak{f}^{(r)})^*(\mathcal{M}^{(r)}) \simeq \mathcal{O}_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}^{(r)}} \mathfrak{f}^{(r)*}(\mathcal{M}^{(r)})$ by the operation on $\mathcal{O}_{\mathfrak{X}}$.

Proposition. *If $\mathcal{M} \in \mathcal{D}_{\mathfrak{Y}}\mathbf{qc}$, then $\varinjlim_r (\mathfrak{F}_{\mathfrak{X}}^r)^*(\mathfrak{f}^{(r)})^*(\mathcal{M}^{(r)}) \simeq \mathcal{D}_{\mathfrak{f}\to} \otimes_{\mathfrak{f}^{-1}\mathcal{D}_{\mathfrak{Y}}} \mathfrak{f}^{-1}\mathcal{M}$ in $\mathcal{D}_{\mathfrak{X}}\mathbf{qc}$, hence the two definitions agree.*

Proof. It is enough to show

$$(\mathfrak{F}_{\mathfrak{X}}^r)^*(\mathfrak{f}^{(r)})^*(\mathcal{M}^{(r)}) \simeq \mathcal{D}_{\mathfrak{f}\to,r} \otimes_{\mathfrak{f}^{-1}\mathcal{D}_{\mathfrak{Y},r}} \mathfrak{f}^{-1}\mathcal{M} \quad \text{in } \mathcal{D}_{\mathfrak{X},r}\mathbf{Mod} \tag{2}$$

where $\mathcal{D}_{\mathfrak{f}\to,r} = \mathfrak{f}^*(\mathcal{D}_{\mathfrak{Y},r})$. One has in $\mathfrak{X}\mathbf{Mod}$

$$(\mathfrak{F}_{\mathfrak{X}}^r)^*(\mathfrak{f}^{(r)})^*(\mathcal{M}^{(r)}) \simeq \mathfrak{f}^*(\mathfrak{F}_{\mathfrak{Y}}^r)^*(\mathcal{M}^{(r)}) \simeq \mathfrak{f}^*\mathcal{M} \simeq \mathcal{D}_{f\to,r} \otimes_{\mathfrak{f}^{-1}\mathcal{D}_{\mathfrak{Y},r}} \mathfrak{f}^{-1}\mathcal{M}. \tag{3}$$

To check that the composite isomorphism is $\mathcal{D}_{\mathfrak{Y},r}$-equivariant, we may assume $\mathfrak{X}$ and $\mathfrak{Y}$ are both affine. If $A = \mathcal{O}_{\mathfrak{X}}(\mathfrak{X})$, $C = \mathcal{O}_{\mathfrak{Y}}(\mathfrak{Y})$, and if $M = \mathcal{M}(\mathfrak{X})$, then the isomorphisms (3) read

$$\begin{array}{cc} A \otimes_{A^{(r)}} A^{(r)} \otimes_{C^{(r)}} \mathbf{Mod}_{C^{(r)}}(C, C^{(r)}) \otimes_{D_r(C)} M & \sum_i ac_i \otimes 1 \otimes \chi_i \otimes m \\ \uparrow & \uparrow \\ A \otimes_C M & a \otimes m \\ \downarrow & \downarrow \\ \mathbf{Mod}_{C^{(r)}}(C, A) \otimes_{D_r(C)} M & \mu_a \otimes m, \end{array}$$

where $\sum_i c_i \otimes \chi_i \mapsto \mathrm{id}_C$ under the bijection $C \otimes_{C^{(r)}} \mathbf{Mod}_{C^{(r)}}(C, C^{(r)}) \simeq D_r(C)$, and μ_a is the multplication by a.

If $\delta \in D_r(A)$, then $\delta \cdot \sum_i ac_i \otimes 1 \otimes \chi_i \otimes m = \sum_i \delta(ac_i) \otimes 1 \otimes \chi_i \otimes m$ while $\delta \cdot (\mu_a \otimes m) = (\delta \circ \mu_a) \otimes m$, hence we must check $\sum_i \mu_{\delta(ac_i)} \circ \chi_i = \delta \circ \mu_a$ in $\mathbf{Mod}_{C^{(r)}}(C, A)$. If $c \in C$, then

$$\begin{aligned} \sum_i (\mu_{\delta(ac_i)} \circ \chi_i)(c) = \sum_i \delta(ac_i)\chi_i(c) = \sum_i \delta(ac_i\chi_i(c)) = \delta\left(a \sum_i c_i\chi_i(c)\right) \\ = \delta(ac) = (\delta \circ \mu_a)(c), \end{aligned}$$

as desired. □

2.3. Let $\omega_{\mathfrak{X}} = \wedge^{\mathrm{N}}\Omega^1_{\mathfrak{X}/\mathfrak{k}}$, $\omega_{\mathfrak{Y}} = \wedge^{\mathrm{L}}\Omega^1_{\mathfrak{Y}/\mathfrak{k}}$, and set

$$\mathcal{D}_{\mathfrak{f}\leftarrow} = \mathfrak{f}^*(\mathcal{D}_{\mathfrak{Y}} \otimes_{\mathfrak{Y}} \omega_{\mathfrak{Y}}^{-1}) \otimes_{\mathfrak{X}} \omega_{\mathfrak{X}} \simeq (\mathcal{D}_{\mathfrak{f}\to} \otimes_{\mathfrak{f}^{-1}\mathcal{D}_{\mathfrak{Y}}} \mathfrak{f}^{-1}(\mathcal{D}_{\mathfrak{Y}} \otimes_{\mathfrak{Y}} \omega_{\mathfrak{Y}}^{-1})) \otimes_{\mathfrak{X}} \omega_{\mathfrak{X}}.$$

The module $\mathcal{D}_{\mathfrak{f}\leftarrow}$ carries a structure of $\mathfrak{f}^{-1}\mathcal{D}_{\mathfrak{Y}}\mathbf{Mod}\mathcal{D}_{\mathfrak{X}}$ [**H88**, 7.1]. To describe the structure locally, let $\mathfrak{V}$ (resp. $\mathfrak{U} \subseteq \mathfrak{f}^{-1}\mathfrak{V}$) be an affine open of $\mathfrak{Y}$ (resp. $\mathfrak{X}$) étale over $\mathbb{A}^{\mathrm{L}}_{\mathfrak{k}}$ (resp. $\mathbb{A}^{\mathrm{N}}_{\mathfrak{k}}$), and let $D_{\mathfrak{f}\to} = \mathcal{D}_{\mathfrak{f}\to}(\mathfrak{U})$. If we take $\mathfrak{U}$ and $\mathfrak{V}$ small enough that $\mathcal{D}_{\mathfrak{f}\leftarrow}(\mathfrak{U})$ may be identified with $D_{\mathfrak{f}\to}$ as $\mathfrak{k}$-linear spaces, then

(1) $$\delta_1 \cdot \theta \cdot \delta_2 = \delta_2^*\theta\delta_1^*, \quad \delta_1 \in D(\mathfrak{V}), \delta_2 \in D(\mathfrak{U}), \theta \in D_{\mathfrak{f}\to},$$

using the structure of $\mathcal{D}_{\mathfrak{X}}\mathbf{Mod}\mathfrak{f}^{-1}\mathcal{D}_{\mathfrak{Y}}$ on $\mathcal{D}_{\mathfrak{f}\to}$, where $^* : D(\mathfrak{U}) \to D(\mathfrak{U})$ is an involutive antiautomorphism of $\mathfrak{k}$-algebras such that $\sum a_{\mathbf{n}}\partial^{\mathbf{n}} \mapsto \sum(-1)^{|\mathbf{n}|}\partial^{\mathbf{n}}a_{\mathbf{n}}$, $a_{\mathbf{n}} \in A = \mathcal{O}_{\mathfrak{X}}(\mathfrak{U})$, $|\mathbf{n}| = \sum_{i=1}^{\mathrm{N}} n_i$ [**H88**, 5.5]. More explicitly, if $D_{\mathfrak{f}\to,r} = \mathcal{D}_{\mathfrak{f}\to,r}(\mathfrak{U})$ and $D_{\mathfrak{f}\leftarrow,r} = \mathcal{D}_{\mathfrak{f}\leftarrow,r}(\mathfrak{U})$ with $\mathcal{D}_{\mathfrak{f}\leftarrow,r} = \mathfrak{f}^*(\mathcal{D}_{\mathfrak{Y},r} \otimes_{\mathfrak{Y}} \omega_{\mathfrak{Y}}^{-1}) \otimes_{\mathfrak{X}} \omega_{\mathfrak{X}}$ and if $C = \mathcal{O}_{\mathfrak{Y}}(\mathfrak{V})$, then $D_{\mathfrak{f}\leftarrow,r}$ can be identified with $D_{\mathfrak{f}\to,r} \simeq \mathbf{Mod}_{C^{(r)}}(C, A)$ as $\mathfrak{k}$-linear spaces with the $D_r(C)\mathbf{Mod}D_r(A)$-structure given by

(2) $$\delta_1 \cdot \theta \cdot \delta_2 = \delta_2^* \circ \theta \circ \delta_1^* \quad \forall \delta_1 \in D_r(C), \delta_2 \in D_r(A), \theta \in \mathbf{Mod}_{C^{(r)}}(C, A).$$

Both $D_r(C)$ and $D_r(A)$ are invariant under * by (1.2). Hence if $\mathcal{M} \in \mathcal{D}_{\mathfrak{X}}\mathbf{qc}$, $\mathcal{D}_{\mathfrak{f}\leftarrow} \otimes_{\mathcal{D}_{\mathfrak{X}}} \mathcal{M}$ carries a structure of left $\mathfrak{f}^{-1}\mathcal{D}_{\mathfrak{Y}}$-module. Define a morphism of ringed spaces $\mathfrak{f}_0 : (\mathfrak{X}, \mathfrak{f}^{-1}\mathcal{D}_{\mathfrak{Y}}) \to (\mathfrak{Y}, \mathcal{D}_{\mathfrak{Y}})$ via $\mathcal{N} \mapsto \mathfrak{f}_*\mathcal{N}$. Then the direct image of $\mathcal{M}$ under $\mathfrak{f}$ is defined by $\mathfrak{f}_0(\mathcal{D}_{\mathfrak{f}\leftarrow} \otimes_{\mathcal{D}_{\mathfrak{X}}} \mathcal{M})$. If one defines $\mathfrak{f}_{0,r}$: $(\mathfrak{X}, \mathfrak{f}^{-1}\mathcal{D}_{\mathfrak{Y},r}) \to (\mathfrak{Y}, \mathcal{D}_{\mathfrak{Y},r})$ likewise, then $\mathfrak{f}_0(\mathcal{D}_{\mathfrak{f}\leftarrow} \otimes_{\mathcal{D}_{\mathfrak{X}}} \mathcal{M}) \simeq \varinjlim_r \mathfrak{f}_{0,r}(\mathcal{D}_{\mathfrak{f}\leftarrow,r} \otimes_{\mathcal{D}_{\mathfrak{X},r}} \mathcal{M})$. As each $\mathfrak{f}_{0,r}(\mathcal{D}_{\mathfrak{f}\leftarrow,r} \otimes_{\mathcal{D}_{\mathfrak{X},r}} \mathcal{M})$ is $\mathcal{O}_{\mathfrak{Y}^{(r)}}$-quasicoherent, hence also $\mathcal{O}_{\mathfrak{Y}}$-quasicoherent [**H88**, 3.1],

(3) $$\mathfrak{f}_0(\mathcal{D}_{\mathfrak{f}\leftarrow} \otimes_{\mathcal{D}_{\mathfrak{X}}} \mathcal{M}) \in \mathcal{D}_{\mathfrak{Y}}\mathbf{qc}.$$

One then defines the derived direct image functor by

$$\int_{\mathfrak{f}} = (\mathrm{R}\,\mathfrak{f}_0) \circ \mathrm{L}(\mathcal{D}_{\mathfrak{f}\leftarrow} \otimes_{\mathcal{D}_{\mathfrak{X}}} ?) : \mathrm{D}^b(\mathcal{D}_{\mathfrak{X}}\mathbf{qc}) \longrightarrow \mathrm{D}^b(\mathcal{D}_{\mathfrak{Y}}\mathbf{qc}),$$

where D^b denotes the bounded derived category. There is a simplification due to [**H88**, 1.2] that

(4) $$\mathcal{D}_{\mathfrak{f}\rightarrow} \text{ is flat in } \mathcal{D}_{\mathfrak{X}}\mathbf{Mod},$$

hence also [**H88**, 7.2]

(5) $$\mathcal{D}_{\mathfrak{f}\leftarrow} \text{ is flat in } \mathbf{Mod}\mathcal{D}_{\mathfrak{X}}.$$

Consequently,

(6) $$\int_{\mathfrak{f}} = (\mathrm{R}\,\mathfrak{f}_0) \circ (\mathcal{D}_{\mathfrak{f}\leftarrow} \otimes_{\mathcal{D}_{\mathfrak{X}}} ?).$$

To see that $\int_{\mathfrak{f}}$ actually lands in $\mathrm{D}^b(\mathcal{D}_{\mathfrak{Y}}\mathbf{qc})$, however, seems to the present author to require as in [**BVI**, 5.1] a spectral sequence argument and J. Bernstein's theorem [**BVI**, 2.10] that if $\mathrm{D}^b_{qc}(\mathcal{D}_{\mathfrak{X}}\mathbf{Mod})$ is the derived category of bounded complexes of left $\mathcal{D}_{\mathfrak{X}}$-modules with $\mathcal{O}_{\mathfrak{X}}$-quasicoherent cohomlogies, then

(7) the imbedding of $\mathcal{D}_{\mathfrak{X}}\mathbf{qc}$ into $\mathcal{D}_{\mathfrak{X}}\mathbf{Mod}$ induces an equivalence
$$\mathrm{D}^b(\mathcal{D}_{\mathfrak{X}}\mathbf{qc}) \to \mathrm{D}^b_{qc}(\mathcal{D}_{\mathfrak{X}}\mathbf{Mod}).$$

As we will need the argument in the proof of the base change theorem, let us recall the spectral sequence from [**BVI**, 5.1].

2.4. Let $\mathcal{U} = \{\mathfrak{X}_i \mid i \in I\}$ be a finite affine open covering of $\mathfrak{X}$. We number I, and for each $J = (i_0, \ldots, i_r) \in I^{r+1}$ with $i_0 < \cdots < i_r$ let $\mathfrak{X}_J = \cap_{j=0}^r \mathfrak{X}_{i_j}$, $\mathfrak{i}_J : \mathfrak{X}_J \hookrightarrow \mathfrak{X}$, and $\mathfrak{f}_J = \mathfrak{f} \circ \mathfrak{i}_J$. If $\mathcal{A}$ is a sheaf of abelian groups on $\mathfrak{X}$, let $\mathcal{A} \to \mathcal{F}^{\cdot}$ be a flasque resolution, and for each $t \in \mathbb{N}$ let $\mathcal{F}^t \to \mathcal{C}^{\cdot}(\mathcal{U}, \mathcal{F}^t)$ be the Čech resolution of $\mathcal{F}^t$ [**H**, II.4]. Then from the double complex $\mathfrak{f}_* \mathcal{C}^{\cdot}(\mathcal{U}, \mathcal{F}^{\cdot}) = \coprod_{s,t} \mathfrak{f}_* \mathcal{C}^s(\mathcal{U}, \mathcal{F}^t)$ one obtains:

Lemma. *If $\mathcal{A}$ is a sheaf of abelian groups on $\mathfrak{X}$, there is a spectral sequence*

$$\mathrm{E}_1^{s,t} = \prod_{|J|=s+1} (\mathrm{R}^t (\mathfrak{f}_J)_*)(\mathcal{A} \mid_{\mathfrak{X}_J}) \Rightarrow (\mathrm{R}\, \mathfrak{f}_*)(\mathcal{A}).$$

2.5. Just like on $\mathbf{qc}_{\mathfrak{X}}$ one has (cf. [**TH**, Th. I.1.8.2]):

Proposition (**Base change**). *Given a cartesian square*

$$\begin{array}{ccc} \mathfrak{Z} \times \mathfrak{X} & \xrightarrow{\ \mathfrak{p}_{\mathfrak{X}}\ } & \mathfrak{X} \\ {\scriptstyle \mathfrak{Z}\times\mathfrak{f}}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{f}} \\ \mathfrak{Z} \times \mathfrak{Y} & \xrightarrow[\ \mathfrak{p}_{\mathfrak{Y}}\ ]{} & \mathfrak{Y} \end{array}$$

one has on $\mathrm{D}^b(\mathcal{D}_{\mathfrak{X}}\mathbf{qc})$

$$\mathfrak{p}_{\mathfrak{Y}}^0 \circ \int_{\mathfrak{f}} \simeq (\mathrm{L}\, \mathfrak{p}_{\mathfrak{Y}}^0) \circ \int_{\mathfrak{f}} \simeq \int_{\mathfrak{Z}\times\mathfrak{f}} \circ (\mathrm{L}\, \mathfrak{p}_{\mathfrak{X}}^0) \simeq \int_{\mathfrak{Z}\times\mathfrak{f}} \circ (\mathfrak{p}_{\mathfrak{X}}^0).$$

Proof. As $\mathfrak{p}_{\mathfrak{X}}$ is flat, $\mathrm{L}\, \mathfrak{p}_{\mathfrak{X}}^0 \simeq \mathfrak{p}_{\mathfrak{X}}^0$, and likewise for $\mathfrak{p}_{\mathfrak{Y}}$. Let $\mathcal{M} \in \mathcal{D}_{\mathfrak{X}}\mathbf{qc}$. Take a locally free resolution $\mathcal{L}^{\cdot} \to \mathcal{M}$ in $\mathcal{D}_{\mathfrak{X}}\mathbf{qc}$ and a finite affine open cover $(\mathfrak{X}_i)_i$ of $\mathfrak{X}$ such that $\mathcal{L}^t \mid_{\mathfrak{X}_i}$ is free for all $t \in \{0, -1\}$ and $i \in I$. As $\mathcal{D}_{\mathfrak{f}\leftarrow}$ is flat in $\mathbf{Mod}\mathcal{D}_{\mathfrak{X}}$, the sequence $\mathcal{D}_{\mathfrak{f}\leftarrow} \otimes_{\mathcal{D}_{\mathfrak{X}}} \mathcal{L}^{\cdot} \to \mathcal{D}_{\mathfrak{f}\leftarrow} \otimes_{\mathcal{D}_{\mathfrak{X}}} \mathcal{M}$ remains exact. By the choice of the resolution $(\mathcal{D}_{\mathfrak{f}\leftarrow} \otimes_{\mathcal{D}_{\mathfrak{X}}} \mathcal{L}^t) \mid_{\mathfrak{X}_J} \simeq \coprod \mathcal{D}_{\mathfrak{f}_J\leftarrow}$ for $t \in \{0, -1\}$, hence

(1) $(\mathcal{D}_{\mathfrak{f}\leftarrow} \otimes_{\mathcal{D}_{\mathfrak{X}}} \mathcal{M}) \mid_{\mathfrak{X}_J}$ comes equipped with a structure of right $\mathcal{O}_{\mathfrak{X}}$-quasicoherent module.

As $\mathfrak{f}^{-1}\mathcal{D}_{\mathfrak{Y}}\mathbf{Mod}$ has enough injectives, that are flasque (cf. [**G**, II.7.1]), one can compute $\mathrm{R}\, \mathfrak{f}_0$ with flasques, hence by (1) and Serre's theorem one obtains from (2.4) a spectral sequence

(2)
$$\mathrm{E}_1^{s,t} = \begin{cases} \prod_{|J|=s+1} (\mathfrak{f}_J)_0 ((\mathcal{D}_{\mathfrak{f}\leftarrow} \otimes_{\mathcal{D}_{\mathfrak{X}}} \mathcal{M}) \mid_{\mathfrak{X}_J}) = \prod_J \int^0_{\mathfrak{f}_J} \mathcal{M}_J & \text{if } t = 0 \\ 0 & \text{otherwise} \end{cases}$$
$$\Longrightarrow (\mathrm{R}\, \mathfrak{f}_0)(\mathcal{D}_{\mathfrak{f}\leftarrow} \otimes_{\mathcal{D}_{\mathfrak{X}}} \mathcal{M}),$$

where $\mathcal{M}_J = \mathcal{M} \mid_{\mathfrak{X}_J}$. One then obtains

$$\int_{\mathfrak{f}} \mathcal{M} \simeq \mathrm{H}^{\cdot} \left(\prod_{|J|=s+1} \int_{\mathfrak{f}_J}^0 \mathcal{M}_J \right), \tag{3}$$

and likewise

$$\int_{\mathfrak{Z}\times\mathfrak{f}} \mathfrak{p}_{\mathfrak{X}}^0(\mathcal{M}) \simeq \mathrm{H}^{\cdot} \left(\prod_{|J|=s+1} \int_{\mathfrak{Z}\times\mathfrak{f}_J}^0 \mathfrak{p}_{\mathfrak{X}_J}^0(\mathcal{M}_J) \right). \tag{4}$$

As $\mathfrak{p}_{\mathfrak{Y}}^0$ is exact, the isomorphism (3) induces

$$\mathfrak{p}_{\mathfrak{Y}}^0 \left(\int_{\mathfrak{f}} \mathcal{M} \right) \simeq \mathrm{H}^{\cdot} \left(\prod_{|J|=s+1} \mathfrak{p}_{\mathfrak{Y}}^0 \left(\int_{\mathfrak{f}_J}^0 \mathcal{M}_J \right) \right). \tag{5}$$

Moreover, for $t \in \{0, -1\}$ one has in $\mathbf{Mod}_{\mathfrak{Z}\times\mathfrak{X}_J}$

(6)

$$\begin{aligned} \mathcal{D}_{\mathfrak{Z}\times\mathfrak{f}_J\leftarrow} \otimes_{\mathcal{D}_{\mathfrak{Z}\times\mathfrak{X}_J}} \mathfrak{p}_{\mathfrak{X}_J}^0(\mathcal{L}^t \mid_{\mathfrak{X}_J}) &\simeq \coprod \mathcal{D}_{\mathfrak{Z}\times\mathfrak{f}_J\leftarrow} \otimes_{\mathcal{D}_{\mathfrak{Z}}\boxtimes\mathcal{D}_{\mathfrak{X}_J}} (\mathcal{O}_{\mathfrak{Z}} \boxtimes \mathcal{D}_{\mathfrak{X}_J}) \\ &\simeq \coprod (\mathcal{D}_{\mathfrak{Z}} \boxtimes \mathcal{D}_{\mathfrak{f}_J\leftarrow}) \otimes_{\mathfrak{p}_{\mathfrak{Z}}^*\mathcal{D}_{\mathfrak{Z}}} \mathcal{O}_{\mathfrak{Z}\times\mathfrak{X}_J} \simeq \coprod (\mathcal{O}_{\mathfrak{Z}} \boxtimes \mathcal{D}_{\mathfrak{f}_J\leftarrow}), \end{aligned}$$

using the right $\mathcal{O}_{\mathfrak{X}_J}$-module structure on $\mathcal{D}_{\mathfrak{f}_J\leftarrow}$ to form $\boxtimes\mathcal{D}_{\mathfrak{f}_J\leftarrow}$. Then for each $r \in \mathbb{N}$ one obtains in $\mathcal{D}_{\mathfrak{Z}\times\mathfrak{Y},r}\mathbf{Mod}$

$$(\mathfrak{Z} \times \mathfrak{f}_J)_0 \mathfrak{p}_{\mathfrak{X}_J}^0(\mathcal{D}_{\mathfrak{f}_J\leftarrow,r}) \simeq \mathfrak{p}_{\mathfrak{Y}}^0 (\mathfrak{f}_J)_0(\mathcal{D}_{\mathfrak{f}_J\leftarrow,r}), \tag{7}$$

hence in $\mathcal{D}_{\mathfrak{Z}\times\mathfrak{Y}}\mathbf{Mod}$

$$\begin{aligned} \int_{\mathfrak{Z}\times\mathfrak{f}_J}^0 \mathfrak{p}_{\mathfrak{X}_J}^0(\mathcal{L}^t \mid_{\mathfrak{X}_J}) &\simeq \coprod \varinjlim_r (\mathfrak{Z} \times \mathfrak{f}_J)_0 \mathfrak{p}_{\mathfrak{X}_J}^0(\mathcal{D}_{\mathfrak{f}_J\leftarrow,r}) \\ &\simeq \coprod \varinjlim_r \mathfrak{p}_{\mathfrak{Y}}^0 (\mathfrak{f}_J)_0(\mathcal{D}_{\mathfrak{f}_J\leftarrow,r}) \simeq \mathfrak{p}_{\mathfrak{Y}}^0 \int_{\mathfrak{f}_J}^0 (\mathcal{L}^t \mid_{\mathfrak{X}_J}). \end{aligned} \tag{8}$$

As $\mathcal{D}_{\mathfrak{Z}\times\mathfrak{f}_J\leftarrow}$ is flat in $\mathbf{Mod}\mathcal{D}_{\mathfrak{Z}\times\mathfrak{X}_J}$, $\mathcal{D}_{\mathfrak{Z}\times\mathfrak{f}_J\leftarrow} \otimes_{\mathcal{D}_{\mathfrak{Z}\times\mathfrak{X}_J}} \mathfrak{p}_{\mathfrak{X}_J}^0(\mathcal{L}^{\cdot} \mid_{\mathfrak{X}_J}) \to \mathcal{D}_{\mathfrak{Z}\times\mathfrak{f}_J\leftarrow} \otimes_{\mathcal{D}_{\mathfrak{Z}\times\mathfrak{X}_J}} \mathfrak{p}_{\mathfrak{X}_J}^0(\mathcal{M}_J)$ remains exact, hence $\mathcal{D}_{\mathfrak{Z}\times\mathfrak{f}_J\leftarrow} \otimes_{\mathcal{D}_{\mathfrak{Z}\times\mathfrak{X}_J}} \mathfrak{p}_{\mathfrak{X}_J}^0(\mathcal{M}_J)$ comes equipped with a structure of right $\mathcal{O}_{\mathfrak{Z}\times\mathfrak{X}_J}$-quasicoherent module. Then $\int_{\mathfrak{Z}\times\mathfrak{f}_J}^0 \mathfrak{p}_{\mathfrak{X}_J}^0(\mathcal{L}^{\cdot} \mid_{\mathfrak{X}_J}) \to \int_{\mathfrak{Z}\times\mathfrak{f}_J}^0 \mathfrak{p}_{\mathfrak{X}_J}^0(\mathcal{M}_J)$ is still exact by Serre's theorem, and likewise $\mathfrak{p}_{\mathfrak{Y}}^0 \int_{\mathfrak{f}_J}^0 (\mathcal{L}^{\cdot} \mid_{\mathfrak{X}_J}) \to \mathfrak{p}_{\mathfrak{Y}}^0 \int_{\mathfrak{f}_J}^0 \mathcal{M}_J$. Hence

$$\int_{\mathfrak{Z}\times\mathfrak{f}_J}^0 \mathfrak{p}_{\mathfrak{X}_J}^0(\mathcal{M}_J) \simeq \mathfrak{p}_{\mathfrak{Y}}^0 \int_{\mathfrak{f}_J}^0 \mathcal{M}_J \quad \text{in } \mathcal{D}_{\mathfrak{Z}\times\mathfrak{Y}}\mathbf{Mod}.$$

The assertion now follows from (4) and (5). ☐

2.6. If $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ is an open immersion,

$$(1) \qquad \mathcal{D}_{\mathfrak{f}\to} \simeq \mathcal{D}_{\mathfrak{x}} \quad \text{and} \quad \mathcal{D}_{\mathfrak{f}\leftarrow} \simeq \mathcal{D}'_{\mathfrak{x}} \quad \text{both in } \mathcal{D}_{\mathfrak{X}}\mathbf{Mod}\mathcal{D}_{\mathfrak{X}},$$

where $\mathcal{D}'_{\mathfrak{X}}$ is $\mathcal{D}_{\mathfrak{X}}$ in $\mathbf{Ab}_{\mathfrak{X}}$ but the $\mathcal{D}_{\mathfrak{X}}\mathbf{Mod}\mathcal{D}_{\mathfrak{X}}$-structure given by the formula (2.3.1). Hence:

Proposition. *If $\mathfrak{f}$ is an open immersion, then $\mathrm{L}\,\mathfrak{f}^0 \simeq \mathfrak{f}^{-1}$ and $\int_{\mathfrak{f}} \simeq \mathrm{R}\,\mathfrak{f}_0$.*

2.7. Assume $\mathfrak{f}$ is étale. As an open immersion is étale, for the local study of $\mathfrak{f}$ we may assume both $\mathfrak{X}$ and $\mathfrak{Y}$ are affine. Thus let $A = \mathcal{O}_{\mathfrak{X}}(\mathfrak{X})$ and $C = \mathcal{O}_{\mathfrak{Y}}(\mathfrak{Y})$. If $(z_i)_{1\le i\le \mathrm{N}}$ is a regular system of parameters on C, then $(\mathfrak{f}^{\mathrm{o}}(z_i))_i$ forms one on A and the isomorphism (1.1.2) generalizes to an isomorphism in $A\mathbf{Mod}$

$$(1) \qquad A \otimes_C D(C) \to D(A) \quad \text{via} \quad a \otimes \partial_C^{\mathbf{n}} \longmapsto a\partial_A^{\mathbf{n}},$$

where $\partial_A^{\mathbf{n}} \in D(A)$ (resp. $\partial_C^{\mathbf{n}} \in D(C)$) are as defined in (1.1.3).

Proposition. *Assume $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ is an étale morphism of affine varieties with $A = \mathcal{O}_{\mathfrak{X}}(\mathfrak{X})$ and $C = \mathcal{O}_{\mathfrak{Y}}(\mathfrak{Y})$.*

(i) *For each $r \in \mathbb{N}$ the bijection (1) induces an A-module isomorphism*

$$A \otimes_C D_r(C) \to D_r(A) \quad \textit{via} \quad a \otimes [(c_{\mathbf{nm}})] \longmapsto a[(\mathfrak{f}^{\mathrm{o}}(c_{\mathbf{nm}}))],$$

where $D_r(A)$ (resp. $D_r(C)$) is identified with the $p^{r\,\mathrm{N}} \times p^{r\,\mathrm{N}}$ matrix algebra over $A^{(r)}$ (resp. $C^{(r)}$). Hence another induced map $D(\mathfrak{f}^{\mathrm{o}}) : D(C) \to D(A)$ such that $c\partial_C^{\mathbf{n}} \mapsto \mathfrak{f}^{\mathrm{o}}(c)\partial_A^{\mathbf{n}}$ is a $\mathfrak{k}$-algebra homomorphism.

(ii) *If $\mathcal{M} \in \mathcal{D}_{\mathfrak{X}}\mathbf{qc}$, then $(\int_{\mathfrak{f}}^0 \mathcal{M})(\mathfrak{Y}) \simeq \mathcal{M}(\mathfrak{X})$ in $D(C)\mathbf{Mod}$ with $D(C)$ acting on $\mathcal{M}(\mathfrak{X})$ via $D(\mathfrak{f}^{\mathrm{o}})$.*

Proof. Define $A\otimes_C D_r(\mathfrak{f}^{\mathrm{o}}) \in A\mathbf{Mod}(A\otimes_C D_r(C), D_r(A))^{\times}$ via $a\otimes[(c_{\mathbf{nm}})] \mapsto a[(\mathfrak{f}^{\mathrm{o}}(c_{\mathbf{nm}}))]$. If $\mathbf{t} \in \mathbb{N}^{\mathrm{N}}$, take $r \gg 0$ that $\partial_C^{\mathbf{t}} \in D_r(C)$. As $\partial_C^{\mathbf{t}}$ is determined by the evaluations at $\mathbf{z}^n$, $\mathbf{n} \le \mathbf{t}$, if $e^C_{\mathbf{nm}} \in D_r(C)$ with $e^C_{\mathbf{nm}}(\mathbf{z}^{\mathbf{l}}) = \delta_{\mathbf{ml}}\mathbf{z}^{\mathbf{n}}$, then

$$(2) \qquad \partial_C^{\mathbf{t}} \in \sum_{\mathbf{n},\mathbf{m}\in[0,p^r-1]^{\mathrm{N}}} \mathfrak{k}[\mathbf{z}]e^C_{\mathbf{nm}}.$$

Then $(A\otimes_C D_r(\mathfrak{f}^{\mathrm{o}}))(\partial_C^{\mathbf{t}}) = \partial_A^{\mathbf{t}}$, hence $\varinjlim_r (A\otimes_C D_r(\mathfrak{f}^{\mathrm{o}}))$ coincides with the map (1), and (i) follows. Also we obtain a bijection $D_{\mathfrak{f}\leftarrow,r} \simeq A\otimes_C D_r(C) \simeq$

$D_r(A)$, under which the $D_r(C)\mathbf{Mod}D_r(A)$-structure on $D_{\mathfrak{f}\leftarrow,r}$ is transferred to $D_r(A)$ such that

$$(3) \qquad (c\partial_C^{\mathbf{n}})\cdot\delta\cdot\delta_A = \delta_A^*\circ\delta\circ(-1)^{|\mathbf{n}|}\partial_A^{\mathbf{n}}\mathfrak{f}^{\mathrm{o}}(c) = \delta_A^*\circ\delta\circ(\mathfrak{f}^{\mathrm{o}}(c)\partial_A^{\mathbf{n}})^*.$$

Then in $D_r(C)\mathbf{Mod}$

$$\left(\int_{\mathfrak{f},r}^0 \mathcal{M}\right)(\mathfrak{Y}) \simeq D_{\mathfrak{f}\leftarrow,r}\otimes_{D_r(A)}\mathcal{M}(\mathfrak{X}) \simeq \mathcal{M}(\mathfrak{X})$$

with $D_r(C)$ acting on $\mathcal{M}(\mathfrak{X})$ on the RHS by $D_r(\mathfrak{f}^{\mathrm{o}})$, hence (ii) by taking the direct limit. □

2.8. Assume in this subsection that $\mathfrak{f}$ is a closed immersion. Put $\mathrm{M} = \mathrm{L}-\mathrm{N} = \dim\mathfrak{Y} - \dim\mathfrak{X}$. We will examine $\mathcal{D}_{\mathfrak{f}\to}$ and $\mathcal{D}_{\mathfrak{f}\leftarrow}$ locally but in an invariant manner using a Koszul complex. Thus let $\mathfrak{V}$ be an affine open of $\mathfrak{Y}$ admitting a regular system of parameters $\mathbf{z} = (z_1,\dots,z_{\mathrm{N}+\mathrm{M}})$ on $C = \mathcal{O}_{\mathfrak{Y}}(\mathfrak{V})$ such that $I = (z_{\mathrm{N}+1},\dots,z_{\mathrm{N}+\mathrm{M}}) \trianglelefteq C$ with $\mathcal{O}_{\mathfrak{X}}(\mathfrak{f}^{-1}\mathfrak{V}) \simeq C/I$, and that $(z_1,\dots,z_{\mathrm{N}})$ induces a regular system of parameters on $A = C/I$. One has then a commutative diagram of short exact sequences

$$(1) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & A\otimes_C (I/I^2) & \longrightarrow & A\otimes_C \Omega^1_{C/\mathfrak{k}} & \longrightarrow & \Omega^1_{A/\mathfrak{k}} & \longrightarrow & 0 \\ & & \wr\downarrow & & \wr\downarrow & & \downarrow\wr & & \\ 0 & \longrightarrow & \coprod_{i=1}^{\mathrm{M}} Az_{\mathrm{N}+i} & \longrightarrow & \coprod_{i=1}^{\mathrm{N}+\mathrm{M}} A\,\mathrm{d}_C\, z_i & \longrightarrow & \coprod_{i=1}^{\mathrm{N}} A\,\mathrm{d}_A\, z_i & \longrightarrow & 0 \end{array}$$

with $z_{\mathrm{N}+i}\mapsto \mathrm{d}_C\, z_{\mathrm{N}+i}$. Let $D_{\mathfrak{f}\to} = \mathcal{D}_{\mathfrak{f}\to}(\mathfrak{f}^{-1}\mathfrak{V}) \simeq A\otimes_C D(C)$. The left $D(A)$-module structure on $D_{\mathfrak{f}\to}$ is given by the A-algebra homomorphism such that

$$(2) \qquad \partial^{\mathbf{n}} \longmapsto 1\otimes\partial^{(\mathbf{n},\mathbf{0})}.$$

As $A\otimes_C D(C) = A\otimes_C \coprod_{\mathbf{n}\in\mathbb{N}^{\mathrm{N}},\mathbf{m}\in\mathbb{N}^{\mathrm{M}}} C\partial^{(\mathbf{n},\mathbf{m})} \simeq D(A)\otimes\coprod_{\mathbf{m}\in\mathbb{N}^{\mathrm{M}}}\mathfrak{k}\partial^{(\mathbf{0},\mathbf{m})}$,

$$(3) \qquad \mathcal{D}_{\mathfrak{f}\to} \text{ is locally free in } \mathcal{D}_{\mathfrak{X}}\mathbf{qc}.$$

The Koszul resolution $\wedge^{\cdot}(\coprod_{i=1}^{\mathrm{M}} C\,\mathrm{d}_C\, z_{\mathrm{N}+i}) \to A$ defined by the C-regular sequence $\mathbf{z} = (z_1,\dots,z_{\mathrm{N}+\mathrm{M}})$ [M, p. 127] induces a free resolution of $D_{\mathfrak{f}\to}$ in $\mathbf{Mod}D(C)$

$$(4) \qquad \left\{\wedge^{\cdot}\left(\coprod_{i=1}^{\mathrm{M}} C\,\mathrm{d}_C\, z_{\mathrm{N}+i}\right)\right\}\otimes_C D(C) \to A\otimes_C D(C).$$

Note that

(5) $$(\mathrm{d}_C\, z_{\mathrm{N}+i})_{1\le i\le \mathrm{M}} \text{ is unique up to } \mathrm{GL}_{\mathrm{M}}(C) \text{ in } \Omega^1_{C/\mathfrak{k}}.$$

If $\mathcal{M} \in \mathcal{D}_{\mathfrak{Y}}\mathbf{qc}$ with $M = \mathcal{M}(\mathfrak{V})$,

(6) $$\Gamma(\mathfrak{f}^{-1}\mathfrak{V}, (\mathrm{L}\,\mathfrak{f}^0)(\mathcal{M})) \simeq D_{\mathfrak{f}\to} \otimes^{\mathrm{L}}_{D(C)} M \simeq \wedge^{\cdot}\left(\coprod_{i=1}^{\mathrm{M}} \mathfrak{k}\,\mathrm{d}_C\, z_{\mathrm{N}+i}\right) \otimes M,$$

that is by (5) independent of the choice of the parameters, hence these can be glued together to give a description of $(\mathrm{L}\,\mathfrak{f}^0)(\mathcal{M})$.

Turning to $D_{\mathfrak{f}\leftarrow} = \mathcal{D}_{\mathfrak{f}\leftarrow}(\mathfrak{f}^{-1}\mathfrak{V})$, that is $D_{\mathfrak{f}\to} \simeq A \otimes_C D(C) \simeq D(A) \otimes \coprod_{\mathbf{m}\in\mathbb{N}^{\mathrm{M}}} \mathfrak{k}\partial^{(\mathbf{0},\mathbf{m})}$ with the $D(C)\mathbf{Mod}D(A)$-structure twisted by *, one has

(7) $$D_{\mathfrak{f}\leftarrow} \simeq \coprod_{\mathbf{m}\in\mathbb{N}^{\mathrm{M}}} \mathfrak{k}\partial^{(\mathbf{0},\mathbf{m})} \otimes D(A) \quad \text{in } \mathbf{Mod}D(A) \text{ via } \theta^* \otimes \partial \leftarrow \partial \otimes \theta,$$

with $D(A)$ acting on the right hand side by the right regular action on $D(A)$. In particular,

(8) $$\mathcal{D}_{\mathfrak{f}\leftarrow} \text{ is locally free in } \mathbf{Mod}\mathcal{D}_{\mathfrak{X}}.$$

Also $D_{\mathfrak{f}\leftarrow}$ admits a Koszul resolution in $D(C)\mathbf{Mod}$

(9) $$\left\{\wedge^{\cdot}\left(\coprod_{i=1}^{\mathrm{M}} \mathfrak{k}\,\mathrm{d}_C\, z_{\mathrm{N}+i}\right)\right\} \otimes D(C) \to D_{\mathfrak{f}\leftarrow}.$$

As $\mathfrak{f}$ is a closed immersion, $\mathfrak{f}_*$ is exact on $\mathbf{Ab}_{\mathfrak{X}}$, hence

(10) $$\int_{\mathfrak{f}} = (\mathrm{R}\,\mathfrak{f}_0) \circ (\mathcal{D}_{\mathfrak{f}\leftarrow} \otimes_{\mathcal{D}_{\mathfrak{X}}} ?) \simeq \mathfrak{f}_0(\mathcal{D}_{\mathfrak{f}\leftarrow} \otimes_{\mathcal{D}_{\mathfrak{X}}} ?), \quad \text{that is exact.}$$

2.9. Assume $\mathfrak{f}$ is still a closed immersion. Define $\mathfrak{f}^+ : \mathcal{D}_{\mathfrak{Y}}\mathbf{qc} \to \mathcal{D}_{\mathfrak{X}}\mathbf{qc}$ via $\mathcal{M} \mapsto (\mathfrak{f}^{-1}\mathcal{D}_{\mathfrak{Y}})\mathcal{M}od(\mathcal{D}_{\mathfrak{f}\leftarrow}, \mathfrak{f}^{-1}\mathcal{M})$ [**H88**, 8.12/8.3]. We have as in [**TH**, Prop. I.1.5.2]:

Proposition. *Assume $\mathfrak{f}$ is a closed immersion.*

(i) [**H87**, 8.4, 8.12] *If $\mathcal{D}^{\mathfrak{X}}_{\mathfrak{Y}}\mathbf{qc}$ is the full subcategory of $\mathcal{D}_{\mathfrak{Y}}\mathbf{qc}$ consisting of the objects with support in $\mathfrak{X}$, then $\mathfrak{f}^+|_{\mathcal{D}^{\mathfrak{X}}_{\mathfrak{Y}}\mathbf{qc}}$ is right adjoint to $\int^0_{\mathfrak{f}}$, hence left exact.*

(ii) *On* $\mathrm{D}^b(\mathcal{D}_{\mathfrak{Y}}\mathbf{qc})$ $\quad \mathrm{R}\,\mathfrak{f}^+ \simeq (\mathrm{L}\,\mathfrak{f}^0)[\dim\mathfrak{X} - \dim\mathfrak{Y}]$.

Proof. (ii) Let $\mathcal{M} \in \mathcal{D}_{\mathfrak{Y}}\mathbf{qc}$. If $\mathcal{M} \to \mathcal{I}^{\cdot}$ is an injective resolution in $\mathcal{D}_{\mathfrak{Y}}\mathbf{qc}$ and if $\mathfrak{f}^{-1}\mathcal{I}^{\cdot} \to \mathcal{J}^{\cdot}$ an injective resolution in $\mathfrak{f}^{-1}\mathcal{D}_{\mathfrak{Y}}\mathbf{Mod}$ [**G**, Th. II.7.1.1], one obtains, as $\mathfrak{f}^{-1}$ is exact, an injective resolution $\mathfrak{f}^{-1}\mathcal{M} \to \mathcal{J}^{\cdot}$.Then in the notation of (2.7), as $\mathcal{J}^{\cdot}|_{\mathfrak{f}^{-1}\mathfrak{V}}$ remains an injective resolution of $\mathfrak{f}^{-1}\mathcal{M}|_{\mathfrak{f}^{-1}\mathfrak{V}}$ in $\mathfrak{f}^{-1}\mathcal{D}_{\mathfrak{Y}}\mathbf{Mod}$,

$$\begin{aligned}\Gamma(\mathfrak{f}^{-1}\mathfrak{V}, (\mathrm{R}\,\mathfrak{f}^{+})(\mathcal{M})) &\simeq (\mathfrak{f}^{-1}\mathcal{D}_{\mathfrak{Y}})\mathbf{Mod}(\mathcal{D}_{\mathfrak{f}\leftarrow}|_{\mathfrak{f}^{-1}\mathfrak{V}}, \mathcal{J}^{\cdot}|_{\mathfrak{f}^{-1}\mathfrak{V}}) \\ &\simeq D(C)\mathbf{Mod}\left(\left\{\wedge^{\cdot}\left(\coprod_{i=1}^{\mathrm{M}} \mathfrak{k}\,\mathrm{d}_C\, z_{\mathrm{N}+i}\right)\right\} \otimes D(C), M\right) \\ &\simeq \mathbf{Mod}_{\mathfrak{k}}\left(\wedge^{\cdot}\left(\coprod_{i=1}^{\mathrm{M}} \mathfrak{k}\,\mathrm{d}_C\, z_{\mathrm{N}+i}\right), M\right) \\ &\simeq \wedge^{\cdot}\left(\coprod_{i=1}^{\mathrm{M}} \mathfrak{k}\,\mathrm{d}_C\, z_{\mathrm{N}+i}\right) \otimes M[\dim\mathfrak{X} - \dim\mathfrak{Y}] \\ &\simeq \Gamma(\mathfrak{f}^{-1}\mathfrak{V}, (\mathrm{L}\,\mathfrak{f}^{0})(\mathcal{M}))[\dim\mathfrak{X} - \dim\mathfrak{Y}].\end{aligned}$$

Taking the cohomology, the end composite isomorphism is invariant under the change of the parameters (2.8.5), hence the assertion. □

2.10. Let $\mathfrak{Z}_1 \supseteq \mathfrak{Z}_2$ be two closed subsets of $\mathfrak{X}$. Define $\Gamma_{\mathfrak{Z}_i} : \mathbf{Ab}_{\mathfrak{X}} \to \mathbf{Ab}_{\mathfrak{X}}$ by $\Gamma_{\mathfrak{Z}_i}(\mathcal{A})(\mathfrak{V}) = \{a \in \mathcal{A}(\mathfrak{V}) \mid \mathrm{supp}(a) \subseteq \mathfrak{Z}_i\}$, $\mathcal{A} \in \mathbf{Ab}_{\mathfrak{X}}$, $\mathfrak{V}$ open of $\mathfrak{X}$, $i = 1, 2$, and set $\Gamma_{\mathfrak{Z}_1/\mathfrak{Z}_2} = \Gamma_{\mathfrak{Z}_1}/\Gamma_{\mathfrak{Z}_2}$, i.e., $\Gamma_{\mathfrak{Z}_1/\mathfrak{Z}_2}(\mathcal{A}) = \Gamma_{\mathfrak{Z}_1}(\mathcal{A})/\Gamma_{\mathfrak{Z}_2}(\mathcal{A})$. Kempf [**Ke**] defined the cohomology sheaf $\mathcal{H}^{\cdot}_{\mathfrak{Z}_1/\mathfrak{Z}_2}(\mathcal{A})$ using the Godement resolution of $\mathcal{A}$. As $\Gamma_{\mathfrak{Z}_1/\mathfrak{Z}_2}$ is exact on flasques [**Ke**, 8.5.c, f] and as the Godement resolution is a flasque resolution, however,

$$\mathcal{H}^{\cdot}_{\mathfrak{Z}_1/\mathfrak{Z}_2} \simeq \mathrm{R}^{\cdot}\,\Gamma_{\mathfrak{Z}_1/\mathfrak{Z}_2} \quad \text{on } \mathbf{Ab}_{\mathfrak{X}}. \tag{1}$$

If $\mathcal{A} \to \mathcal{F}^{\cdot}$ is a flasque resolution, one obtains from a short exact sequence

$$0 \to \Gamma_{\mathfrak{Z}_2}(\mathcal{A}) \to \Gamma_{\mathfrak{Z}_1}(\mathcal{A}) \to \Gamma_{\mathfrak{Z}_1/\mathfrak{Z}_2}(\mathcal{A}) \to 0$$

an exact triangle [**Gr**, 4.10, 5.10]

$$\Gamma_{\mathfrak{Z}_2}(\mathcal{F}^{\cdot}) \to \Gamma_{\mathfrak{Z}_1}(\mathcal{F}^{\cdot}) \to \Gamma_{\mathfrak{Z}_1/\mathfrak{Z}_2}(\mathcal{F}^{\cdot}) \xrightarrow{[1]},$$

hence an exact triangle

$$\mathrm{R}\,\Gamma_{\mathfrak{Z}_2} \to \mathrm{R}\,\Gamma_{\mathfrak{Z}_1} \to \mathrm{R}\,\Gamma_{\mathfrak{Z}_1/\mathfrak{Z}_2} \xrightarrow{[1]} \quad \text{on } \mathrm{D}^b(\mathbf{Ab}_{\mathfrak{X}}). \tag{2}$$

If $\mathfrak{Z}$ is a closed subset of $\mathfrak{X}$ and if $\mathfrak{j} : \mathfrak{X} \setminus \mathfrak{Z} \hookrightarrow \mathfrak{X}$, by [**Ke**, 8.2]

$$\Gamma_{\mathfrak{X}/\mathfrak{Z}} \simeq \mathfrak{j}_*\mathfrak{j}^{-1} \quad \text{on flasques,} \tag{3}$$

hence

$$\mathrm{R}\,\Gamma_{\mathfrak{X}/\mathfrak{Z}} \simeq \mathrm{R}(\mathrm{j}_*\mathrm{j}^{-1}) \simeq (\mathrm{R}\,\mathrm{j}_*)\mathrm{j}^{-1} \tag{4}$$

as j^{-1} sends flasques to flasques, and one obtains an exact triangle

$$\mathrm{R}\,\Gamma_{\mathfrak{Z}} \longrightarrow \mathrm{id} \longrightarrow (\mathrm{R}\,\mathrm{j}_*)\mathrm{j}^{-1} \xrightarrow{[1]} . \tag{5}$$

In particular, $\mathrm{R}\,\Gamma_{\mathfrak{Z}}$ sends $\mathrm{D}^b(\mathbf{Ab}_{\mathfrak{X}})$ to itself by Grothendieck's vanishing theorem applied to the long exact sequence induced by (5), hence also from (2) one obtains

$$\mathrm{R}\,\Gamma_{\mathfrak{Z}_1/\mathfrak{Z}_2} : \mathrm{D}^b(\mathbf{Ab}_{\mathfrak{X}}) \longrightarrow \mathrm{D}^b(\mathbf{Ab}_{\mathfrak{X}}). \tag{6}$$

As an injective of $\mathcal{D}_{\mathfrak{X}}\mathbf{qc}$ is flasque by (1.5) and as $\int_{\mathrm{j}} \simeq \mathrm{R}\,\mathrm{j}_0$ on $\mathrm{D}^b(\mathcal{D}_{\mathfrak{X}}\mathbf{qc})$, one obtains exact triangles on $\mathrm{D}^b(\mathcal{D}_{\mathfrak{X}}\mathbf{qc})$

$$\mathrm{R}\,\Gamma_{\mathfrak{Z}_2} \longrightarrow \mathrm{R}\,\Gamma_{\mathfrak{Z}_1} \longrightarrow \mathrm{R}\,\Gamma_{\mathfrak{Z}_1/\mathfrak{Z}_2} \xrightarrow{[1]} \tag{7}$$

and

$$\mathrm{R}\,\Gamma_{\mathfrak{Z}} \longrightarrow \mathrm{id} \longrightarrow \int_{\mathrm{j}} \circ \mathrm{j}^{-1} \xrightarrow{[1]}, \tag{8}$$

that are compatible with the triangles (2) and (5), respectively, under the forgetful functors.

2.11.

Lemma. *If* $\mathfrak{f}$ *is a closed immersion, one has on* $\mathrm{D}^b(\mathcal{D}_{\mathfrak{X}}\mathbf{qc})$

$$\mathrm{R}\,\Gamma_{\mathfrak{X}} \simeq \int_{\mathfrak{f}} \circ (\mathrm{L}\,\mathfrak{f}^0)[\dim \mathfrak{X} - \dim \mathfrak{Y}].$$

Proof. If $\mathfrak{f}^+_{\mathrm{rgt}} = \mathcal{M}od(\mathfrak{f}^{-1}\mathcal{D}_{\mathfrak{Y}})(\mathcal{D}_{\mathfrak{f}\rightarrow}, \mathfrak{f}^{-1}?) : \mathbf{qc}\mathcal{D}_{\mathfrak{Y}} \rightarrow \mathbf{qc}\mathcal{D}_{\mathfrak{X}}$, one has by [**H88**, 8.10]

$$\mathfrak{f}^+_{\mathrm{rgt}} \simeq \mathfrak{f}^+_{\mathrm{rgt}} \circ \Gamma_{\mathfrak{X}} \quad \text{on } \mathbf{qc}\mathcal{D}_{\mathfrak{Y}}. \tag{1}$$

As $\mathfrak{f}^+ \simeq \{\mathfrak{f}^+_{\mathrm{rgt}}(? \otimes_{\mathfrak{Y}} \omega_{\mathfrak{Y}})\} \otimes_{\mathfrak{X}} \omega_{\mathfrak{X}}^{-1}$ on $\mathcal{D}_{\mathfrak{Y}}\mathbf{qc}$ by [**H88**, 8.12], transferring (1) to $\mathcal{D}_{\mathfrak{Y}}\mathbf{qc}$ reads $\mathfrak{f}^+ \simeq \mathfrak{f}^+ \circ \Gamma_{\mathfrak{X}}$, hence by Kashiwara's equivalence [**H88**, 8.13]

$$\Gamma_{\mathfrak{X}} \simeq \int_{\mathfrak{f}}^0 \circ \mathfrak{f}^+ \circ \Gamma_{\mathfrak{X}} \simeq \int_{\mathfrak{f}}^0 \circ \mathfrak{f}^+ \quad \text{on } \mathcal{D}_{\mathfrak{Y}}\mathbf{qc}.$$

Then on $\mathrm{D}^b(\mathcal{D}_{\mathfrak{Y}}\mathbf{qc})$

$$\begin{aligned}
\mathrm{R}\,\Gamma_{\mathfrak{X}} &\simeq \mathrm{R}\left(\int_{\mathfrak{f}}^{0}\mathfrak{f}^{+}\right)\\
&\simeq \left(\mathrm{R}\int_{\mathfrak{f}}^{0}\right)\circ \mathrm{R}\,\mathfrak{f}^{+} \quad \text{as } \mathfrak{f}^{+} \text{ sends injectives to injectives by (2.8.10, 2.9.i)}\\
&\simeq \int_{\mathfrak{f}}\circ(\mathrm{L}\,\mathfrak{f}^{0})[\dim\mathfrak{X}-\dim\mathfrak{Y}] \quad \text{by (2.9.ii).}
\end{aligned}$$

□

2.12. Together with (2.5) one obtains from (2.11) as in [**TH**, Th. I.1.8.2]:

Theorem (Base change). *Given a cartesian square*

$$\begin{array}{ccc}
\mathfrak{Z}\times_{\mathfrak{Y}}\mathfrak{X} & \xrightarrow{\pi_{\mathfrak{X}}} & \mathfrak{X}\\
{\scriptstyle \pi_{\mathfrak{Z}}}\downarrow & & \downarrow{\scriptstyle \mathfrak{f}}\\
\mathfrak{Z} & \xrightarrow[\mathfrak{g}]{} & \mathfrak{Y}
\end{array}$$

with all $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$, $\mathfrak{Z}\times_{\mathfrak{Y}}\mathfrak{X}$ smooth varieties over $\mathfrak{k}$,

$$(\mathrm{L}\,\mathfrak{g}^{0})\circ\int_{\mathfrak{f}}[\dim\mathfrak{Z}+\dim\mathfrak{X}-\dim\mathfrak{Y}]\simeq\int_{\pi_{\mathfrak{Z}}}\circ(\mathrm{L}\,\pi_{\mathfrak{X}}^{0})[\dim\mathfrak{Z}\times_{\mathfrak{Y}}\mathfrak{X}].$$

2.13.

Example. Assume $\mathfrak{f}$ is a closed immersion, and set

$$\mathcal{B}_{\mathfrak{X}|\mathfrak{Y}}\simeq\int_{\mathfrak{f}}^{0}\mathcal{O}_{\mathfrak{X}}\simeq\mathfrak{f}_{0}(\mathcal{D}_{\mathfrak{f}\leftarrow}\otimes_{\mathcal{D}_{\mathfrak{X}}}\mathcal{O}_{\mathfrak{X}}).$$

Locally, in the notation of (2.7), one has isomorphisms in $D(C)\mathbf{Mod}$

(1)
$$\begin{aligned}
D_{\mathfrak{f}\leftarrow}\otimes_{D(A)}A &\simeq (A\otimes_{C}D(C))\otimes_{D(A)}A\\
&\simeq\{(C/I)\otimes_{C}D(C)\}\otimes_{D(A)}A\\
&\simeq\left\{D(C)/\sum_{i=0}^{\mathrm{M}}z_{\mathrm{N}+i}D(C)\right\}\otimes_{D(A)}\left\{D(A)/\sum_{\mathbf{n}\in\mathbb{N}^{\mathrm{N}}}D(A)\partial^{\mathbf{n}}\right\}\\
&\simeq D(C)/\left\{\sum_{\mathbf{n}}\partial^{(\mathbf{n},\mathbf{0})}D(C)+\sum_{i}z_{\mathrm{N}+i}D(C)\right\}\\
&\simeq D(C)/\left\{\sum_{\mathbf{n}}D(C)\partial^{(\mathbf{n},\mathbf{0})}+\sum_{i}D(C)z_{\mathrm{N}+i}\right\}
\end{aligned}$$

with the last bijection induced from $\theta \mapsto \theta^*$ on $D(C)$, hence the $D(C)$-action on the last term is the one induced by the left regular action.

Consider, more concretely, the case $\mathfrak{f} : \mathbb{A}^{\mathrm{N}}_{\mathfrak{k}} \to \mathbb{A}^{\mathrm{N}+\mathrm{M}}_{\mathfrak{k}}$ defined by the projection $\mathfrak{k}[z_1, \dots, z_{\mathrm{N}+\mathrm{M}}] \to \mathfrak{k}[z_1, \dots, z_{\mathrm{N}}]$, and let $I = (z_{\mathrm{N}+1}, \dots, z_{\mathrm{N}+\mathrm{M}})$. By (2.8.6)

$$\Gamma(\mathbb{A}^{\mathrm{N}}_{\mathfrak{k}}, (\mathrm{L}\,\mathfrak{f}^0)(\mathcal{O}_{\mathbb{A}^{\mathrm{N}+\mathrm{M}}_{\mathfrak{k}}})) \simeq \wedge^{\cdot}\left(\coprod_{i=1}^{\mathrm{M}} \mathfrak{k}[\mathbb{A}^{\mathrm{N}+\mathrm{M}}]\,\mathrm{d}\,z_{\mathrm{N}+i}\right),$$

hence for each $r \in \mathbb{Z}$

$$\Gamma(\mathbb{A}^{\mathrm{N}}_{\mathfrak{k}}, (\mathrm{L}^r\,\mathfrak{f}^0)(\mathcal{O}_{\mathbb{A}^{\mathrm{N}+\mathrm{M}}_{\mathfrak{k}}})) \simeq \begin{cases} \mathfrak{k}[\mathbb{A}^{\mathrm{N}}] & \text{if } r = 0 \\ 0 & \text{otherwise,} \end{cases}$$

that reads in $\mathcal{D}_{\mathbb{A}^{\mathrm{N}+\mathrm{M}}_{\mathfrak{k}}}\mathbf{qc}$

$$(\mathrm{L}^r\,\mathfrak{f}^0)(\mathcal{O}_{\mathbb{A}^{\mathrm{N}+\mathrm{M}}_{\mathfrak{k}}}) \simeq \begin{cases} \mathcal{O}_{\mathbb{A}^{\mathrm{N}}_{\mathfrak{k}}} & \text{if } r = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Then in $\mathcal{D}_{\mathbb{A}^{\mathrm{N}+\mathrm{M}}_{\mathfrak{k}}}\mathbf{qc}$

(3)

$$\begin{aligned}
\mathcal{B}_{\mathbb{A}^{\mathrm{N}}_{\mathfrak{k}}|\mathbb{A}^{\mathrm{N}+\mathrm{M}}_{\mathfrak{k}}} &\simeq \int_{\mathfrak{f}}^0 (\mathrm{L}^0\,\mathfrak{f}^0)(\mathcal{O}_{\mathbb{A}^{\mathrm{N}+\mathrm{M}}_{\mathfrak{k}}}) \\
&\simeq \int_{\mathfrak{f}}^0 (\mathrm{L}^{\mathrm{M}}\,\mathfrak{f}^0)[-\mathrm{M}](\mathcal{O}_{\mathbb{A}^{\mathrm{N}+\mathrm{M}}_{\mathfrak{k}}}) \\
&\simeq \mathrm{H}^{\mathrm{M}}\left(\int_{\mathfrak{f}} (\mathrm{L}\,\mathfrak{f}^0)[-\mathrm{M}](\mathcal{O}_{\mathbb{A}^{\mathrm{N}+\mathrm{M}}_{\mathfrak{k}}})\right) \quad \text{as } \int_{\mathfrak{f}}^0 \text{ is exact and by (2.8.10)} \\
&\simeq \mathrm{H}^{\mathrm{M}}(\mathrm{R}\,\Gamma_{\mathbb{A}^{\mathrm{N}}_{\mathfrak{k}}}(\mathcal{O}_{\mathbb{A}^{\mathrm{N}+\mathrm{M}}_{\mathfrak{k}}})) \quad \text{by (2.11)} \\
&\simeq \mathrm{R}^{\mathrm{M}}\,\Gamma_{\mathbb{A}^{\mathrm{N}}_{\mathfrak{k}}}(\mathcal{O}_{\mathbb{A}^{\mathrm{N}+\mathrm{M}}_{\mathfrak{k}}}) \simeq \mathcal{H}^{\mathrm{M}}_{\mathbb{A}^{\mathrm{N}}_{\mathfrak{k}}}(\mathcal{O}_{\mathbb{A}^{\mathrm{N}+\mathrm{M}}_{\mathfrak{k}}}).
\end{aligned}$$

Likewise from (3)

$$\mathcal{H}^i_{\mathbb{A}^{\mathrm{N}}_{\mathfrak{k}}}(\mathcal{O}_{\mathbb{A}^{\mathrm{N}+\mathrm{M}}_{\mathfrak{k}}}) \simeq \mathrm{R}^i\,\Gamma_{\mathbb{A}^{\mathrm{N}}_{\mathfrak{k}}}(\mathcal{O}_{\mathbb{A}^{\mathrm{N}+\mathrm{M}}_{\mathfrak{k}}}) = 0 \quad \text{unless } i = \mathrm{M}. \tag{4}$$

§3.

In this section $\mathfrak{G}$ will denote an affine algebraic $\mathfrak{k}$-group and $\mathfrak{k}[\mathfrak{G}]$ the associated Hopf algebra with the comultiplication $\Delta_{\mathfrak{G}}$, the augmentation ideal $\mathfrak{m}_{\mathfrak{G}}$ and the antipode $\sigma_{\mathfrak{G}}$. We do not have to assume $\mathfrak{G}$ to be connected.

3.1. Let $\mathfrak{I}_{\mathfrak{G}}$ be the kernel of the multiplication $\mathfrak{k}[\mathfrak{G}] \otimes \mathfrak{k}[\mathfrak{G}] \to \mathfrak{k}[\mathfrak{G}]$, and set $P^n_{\mathfrak{G}} = \mathfrak{k}[\mathfrak{G}] \otimes \mathfrak{k}[\mathfrak{G}]/\mathfrak{I}^{n+1}_{\mathfrak{G}}$, $n \in \mathbb{N}$. We will regard $\mathfrak{k}[\mathfrak{G}] \otimes \mathfrak{k}[\mathfrak{G}]$ as $\mathfrak{k}[\mathfrak{G}]$-algebra via

$a \mapsto a \otimes 1$, and likewise $P^n_{\mathfrak{G}}$. If $\mathfrak{l} \in \mathbf{Sch}_{\mathfrak{k}}(\mathfrak{G} \times \mathfrak{G}, \mathfrak{G} \times \mathfrak{G})^{\times}$ via $(x, y) \mapsto (x, xy)$, then the associated $\mathfrak{k}$-algebra automorphism $\mathfrak{l}^{\circ} : \mathfrak{k}[\mathfrak{G}] \otimes \mathfrak{k}[\mathfrak{G}] \to \mathfrak{k}[\mathfrak{G}] \otimes \mathfrak{k}[\mathfrak{G}]$ is given by $a \otimes b \mapsto a\Delta_{\mathfrak{G}}(b)$. Then (cf. [**Sp**, 3.3.2])

$$\mathfrak{l}^{\circ}(\mathfrak{I}_{\mathfrak{G}}) = \mathfrak{k}[\mathfrak{G}] \otimes \mathfrak{m}_{\mathfrak{G}}, \tag{1}$$

hence $\mathfrak{l}^{\circ}$ induces for each $n \in \mathbb{N}$ a $\mathfrak{k}[\mathfrak{G}]$-algebra isomorphism

$$\mathfrak{l}^{\circ}_n : P^n_{\mathfrak{G}} \to \mathfrak{k}[\mathfrak{G}] \otimes (\mathfrak{k}[\mathfrak{G}]/\mathfrak{m}^{n+1}_{\mathfrak{G}}) \tag{2}$$

such that if $n \geq m$, there is induced a commutative diagram of $\mathfrak{k}[\mathfrak{G}]$-algebras

$$\begin{array}{ccc} P^n_{\mathfrak{G}} & \xrightarrow{\mathfrak{l}^{\circ}_n} & \mathfrak{k}[\mathfrak{G}] \otimes (\mathfrak{k}[\mathfrak{G}]/\mathfrak{m}^{n+1}_{\mathfrak{G}}) \\ \downarrow & & \downarrow \\ P^m_{\mathfrak{G}} & \xrightarrow[\mathfrak{l}^{\circ}_m]{} & \mathfrak{k}[\mathfrak{G}] \otimes (\mathfrak{k}[\mathfrak{G}]/\mathfrak{m}^{m+1}_{\mathfrak{G}}) \end{array}$$

with the natural vertical homomorphisms.

3.2. For each $n \in \mathbb{N}$ let $\mathrm{Dist}_n(\mathfrak{G}) = \{\mu \in \mathfrak{k}[\mathfrak{G}]^* \mid \mu(\mathfrak{m}^{n+1}_{\mathfrak{G}}) = 0\}$ and $\mathrm{Dist}(\mathfrak{G}) = \cup_{n \geq 0} \mathrm{Dist}_n(\mathfrak{G})$ the algebra of distributions of $\mathfrak{G}$. If $\mathfrak{F}_{\mathfrak{G}} : \mathfrak{G} \to \mathfrak{G}^{(r)}$ is the r-th Frobenius morphism on $\mathfrak{G}$, let $\mathfrak{G}_r = \ker(\mathfrak{F}^r_{\mathfrak{G}})$ with $\mathfrak{k}[\mathfrak{G}_r] = \mathfrak{k}[\mathfrak{G}]/(a^{p^r} \mid a \in \mathfrak{m}_{\mathfrak{G}})$. If $\Lambda_n = \mathfrak{k}[\mathfrak{G}]\mathbf{Mod}(\mathfrak{l}^{\circ}_n, \mathfrak{k}[\mathfrak{G}])$ one obtains a commutative diagram of $\mathfrak{k}[\mathfrak{G}]$-modules

$$\begin{array}{ccc} \mathfrak{k}[\mathfrak{G}] \otimes \mathrm{Dist}_n(\mathfrak{G}) & \xrightarrow{\Lambda_n} & \mathrm{Diff}^n_{\mathfrak{G}} \\ \downarrow & & \downarrow \\ \mathfrak{k}[\mathfrak{G}] \otimes \mathrm{Dist}(\mathfrak{G}) & \xrightarrow{\varinjlim_n \Lambda_n} & D(\mathfrak{G}) \\ \uparrow & & \uparrow \\ \mathfrak{k}[\mathfrak{G}] \otimes \mathrm{Dist}(\mathfrak{G}_r) & \longrightarrow & D_r(\mathfrak{G}) \end{array} \tag{1}$$

with the vertical maps being inclusions and the bijective horizontal maps given by $a \otimes \mu \mapsto (a \otimes \mu) \circ \Delta_{\mathfrak{G}}$. Moreover, if one defines a $\mathfrak{G}$-action on $D(\mathfrak{G})$ by $(x\delta)(a) = \delta(ax)x^{-1}$ with $ax = a(x?)$, $x \in \mathfrak{G}$ and $a \in \mathfrak{k}[\mathfrak{G}]$, and if $D^{\mathrm{lft}}(\mathfrak{G}) = \{\delta \in D(\mathfrak{G}) \mid x\delta = \delta \ \forall x \in \mathfrak{G}\}$, then the middle horizontal bijection in (1) induces a $\mathfrak{k}$-algebra isomorphsm (cf. [**DG**, II.4.6.5])

$$\mathrm{Dist}(\mathfrak{G}) \to D^{\mathrm{lft}}(\mathfrak{G}). \tag{2}$$

For our purposes, however, it is more convenient to work with $\mathfrak{r} \in \mathbf{Sch}_{\mathfrak{k}}(\mathfrak{G} \times \mathfrak{G}, \mathfrak{G} \times \mathfrak{G})^{\times}$ via $(x, y) \mapsto (x, yx)$ in place of $\mathfrak{l}$ in (3.1) to obtain:

3.3.

Proposition. *For each $r \in \mathbb{N}$ there is a commutative diagram of $\mathfrak{k}[\mathfrak{G}]$-modules*

$$\begin{array}{ccc} \mathfrak{k}[\mathfrak{G}] \otimes \mathrm{Dist}_r(\mathfrak{G}) & \longrightarrow & \mathrm{Diff}^r_{\mathfrak{G}} \\ \downarrow & & \downarrow \\ \mathfrak{k}[\mathfrak{G}] \otimes \mathrm{Dist}(\mathfrak{G}) & \longrightarrow & D(\mathfrak{G}) \\ \uparrow & & \uparrow \\ \mathfrak{k}[\mathfrak{G}] \otimes \mathrm{Dist}(\mathfrak{G}_r) & \longrightarrow & D_r(\mathfrak{G}) \end{array}$$

with the vertical arrows being inclusions and the horizontal ones bijective given by $a \otimes \mu \mapsto a(\mu \otimes \mathfrak{k}[\mathfrak{G}]) \circ \Delta_{\mathfrak{G}}$. If $D^{\mathrm{rgt}}(\mathfrak{G}) = \{\delta \in D(\mathfrak{G}) \mid \delta x = \delta \ \forall x \in \mathfrak{G}\}$ with $(\delta x)(a) = x^{-1}\delta(xa)$ and $xa = a(?x)\ \forall a \in \mathfrak{k}[\mathfrak{G}]$, then the middle horizontal bijection induces a $\mathfrak{k}$-algebra isomorphism $\mathrm{Dist}(\mathfrak{G})^{\mathrm{op}} \to D^{\mathrm{rgt}}(\mathfrak{G})$.

3.4. Let $\mathfrak{a} \in \mathbf{Sch}_{\mathfrak{k}}(\mathfrak{G} \times \mathfrak{X}, \mathfrak{X})$ be a group action of $\mathfrak{G}$ on a smooth variety $\mathfrak{X}$. We will call such a triple $(\mathfrak{G}, \mathfrak{X}, \mathfrak{a})$ a $\mathfrak{G}$-variety, and denote the category of $\mathfrak{G}$-varieties by $\mathbf{Sch}^{\mathfrak{G}}_{\mathfrak{k}}$. A $\mathfrak{G}$-equivariant $\mathcal{O}_{\mathfrak{X}}$-module is a pair $(\mathcal{M}, \phi)$ of $\mathcal{M} \in \mathbf{qc}_{\mathfrak{X}}$ and $\phi \in \mathbf{Mod}_{\mathfrak{G}\times\mathfrak{X}}(\mathfrak{a}^*\mathcal{M}, \mathfrak{p}^*_{\mathfrak{X}}\mathcal{M})^{\times}$ such that the diagram

$$(1)\qquad \begin{array}{ccc} (\mathfrak{mult} \times \mathfrak{X})^*\mathfrak{a}^*\mathcal{M} & \xrightarrow{(\mathfrak{mult}\times\mathfrak{X})^*\phi} & (\mathfrak{mult} \times \mathfrak{X})^*\mathfrak{p}^*_{\mathfrak{X}}\mathcal{M} \\ \wr\downarrow & & \downarrow\wr \\ (\mathfrak{G} \times \mathfrak{a})^*\mathfrak{a}^*\mathcal{M} & & \mathfrak{p}^*_{\mathfrak{G}\times\mathfrak{X}}\mathfrak{p}^*_{\mathfrak{X}}\mathcal{M} \\ {\scriptstyle (\mathfrak{G}\times\mathfrak{a})^*\phi}\downarrow & & \uparrow{\scriptstyle \mathfrak{p}^*_{\mathfrak{G}\times\mathfrak{X}}\phi} \\ (\mathfrak{G} \times \mathfrak{a})^*\mathfrak{p}^*_{\mathfrak{X}}\mathcal{M} & \underset{\sim}{\longrightarrow} & \mathfrak{p}^*_{\mathfrak{G}\times\mathfrak{X}}\mathfrak{a}^*\mathcal{M} \end{array}$$

commutes, where the $\mathfrak{p}$'s are the projections and $^{\times}$ indicates that ϕ is invertible. A morphism $f : (\mathcal{M}, \phi_{\mathcal{M}}) \to (\mathcal{N}, \phi_{\mathcal{N}})$ of $\mathfrak{G}$-equivariant $\mathcal{O}_{\mathfrak{X}}$-modules is a morphism $f : \mathcal{M} \to \mathcal{N}$ of $\mathcal{O}_{\mathfrak{X}}$-modules such that $\phi_{\mathcal{N}} \circ \mathfrak{a}^* f = \mathfrak{p}^*_{\mathfrak{X}} f \circ \phi_{\mathcal{M}}$. We will denote the category of $\mathfrak{G}$-equivariant $\mathcal{O}_{\mathfrak{X}}$-modules by $(\mathfrak{G}, \mathfrak{X})\mathbf{qc}$.

If $(\mathcal{M}, \phi) \in (\mathfrak{G}, \mathfrak{X})\mathbf{qc}$ and if $\mathfrak{i}_{\mathfrak{X}} : \mathfrak{X} \to \mathfrak{G} \times \mathfrak{X}$ via $x \mapsto (e, x)$, one obtains a commutative diagram

$$(2)\qquad \begin{array}{ccc} \mathfrak{i}^*_{\mathfrak{X}}\mathfrak{a}^*\mathcal{M} & \xrightarrow{\mathfrak{i}^*_{\mathfrak{X}}\phi} & \mathfrak{i}^*_{\mathfrak{X}}\mathfrak{p}^*_{\mathfrak{X}}\mathcal{M}. \\ & {\scriptstyle\sim}\nwarrow \quad \nearrow{\scriptstyle\sim} & \\ & \mathcal{M} & \end{array}$$

If $\mathfrak{a}' \in \mathbf{Sch}_{\mathfrak{k}}(\mathfrak{G} \times \mathfrak{X}, \mathfrak{G} \times \mathfrak{X})^{\times}$ such that $(g, x) \mapsto (g, gx)$, then

$$\mathfrak{a} = \mathfrak{p}_{\mathfrak{X}} \circ \mathfrak{a}'. \tag{3}$$

As $\mathfrak{p}_{\mathfrak{X}}$ and $\mathfrak{p}_{\mathfrak{G}\times\mathfrak{X}}$ are both affine and flat and as $\mathfrak{a}'$ is invertible,

(4) $\mathfrak{a}$, $\mathfrak{mult} \times \mathfrak{X}$, and $\mathfrak{G} \times \mathfrak{a}$ are all affine and flat.

In particular,

(5) all $\mathfrak{p}_{\mathfrak{X}}^*$, $\mathfrak{a}^*$, and $\mathfrak{p}_{\mathfrak{G}\times\mathfrak{X}}^*$, $(\mathfrak{mult} \times \mathfrak{X})^*$, $(\mathfrak{G} \times \mathfrak{a})^*$ are exact on $\mathbf{Mod}_{\mathfrak{X}}$ and $\mathbf{Mod}_{\mathfrak{G}\times\mathfrak{X}}$, respectively.

Then for any $f \in (\mathfrak{G}, \mathfrak{X})\mathbf{qc}$, both $\ker f$ and $\operatorname{coker} f$ belong to $(\mathfrak{G}, \mathfrak{X})\mathbf{qc}$, hence

(6) $(\mathfrak{G}, \mathfrak{X})\mathbf{qc}$ is an abelian category.

If we define the category $\mathrm{D}^b_{\mathfrak{G}}(\mathbf{qc}_{\mathfrak{X}})$ to consist of all pairs $(\mathcal{M}^{\cdot}, \phi)$ of $\mathcal{M}^{\cdot} \in \mathrm{D}^b(\mathbf{qc}_X)$ and $\phi \in \mathrm{D}^b(\mathbf{qc}_{\mathfrak{G}\times\mathfrak{X}})(\mathfrak{a}^*\mathcal{M}^{\cdot}, \mathfrak{p}_{\mathfrak{X}}^*\mathcal{M}^{\cdot})^{\times}$ such that the diagram (1) commutes in $\mathrm{D}^b(\mathbf{qc}_{\mathfrak{G}\times\mathfrak{X}})$ with $\mathcal{M}^{\cdot}$ replacing $\mathcal{M}$. If $(\mathcal{M}^{\cdot}, \phi) \in \mathrm{D}^b_{\mathfrak{G}}(\mathbf{qc}_X)$, each $\mathrm{H}^i(\mathcal{M}^{\cdot})$, $i \in \mathbb{Z}$, carries by (6) a structure of $(\mathfrak{G}, \mathfrak{X})\mathbf{qc}$. Hence

(7) $\mathrm{D}^b((\mathfrak{G}, \mathfrak{X})\mathbf{qc})$ is equivalent to $\mathrm{D}^b_{\mathfrak{G}}(\mathbf{qc}_{\mathfrak{X}})$ under the forgetful functor with quasi-inverse $(\mathcal{M}^{\cdot}, \phi) \mapsto (\mathrm{H}^{\cdot}(\mathcal{M}^{\cdot}), \mathrm{H}^{\cdot}(\phi))$.

3.5. Let $\mathfrak{f} \in \mathbf{Sch}_{\mathfrak{k}}(\mathfrak{X}, \mathfrak{Y})$, $\mathcal{M} \in \mathbf{Mod}_{\mathfrak{X}}$, $\mathcal{N} \in \mathbf{Mod}_{\mathfrak{Y}}$, and $\delta \in \mathbf{Mod}_{\mathfrak{k}}(\mathcal{N}, \mathfrak{f}_*\mathcal{M})$. Slightly extending the definition of [**DG**, II.4.5.1] we say a pair $(\mathfrak{f}, \delta)$ is a $\mathfrak{k}$-deviation of order $\leq n$, $n \in \mathbb{N}$, iff

$$\{\mathrm{ad}(a_0)\dots\mathrm{ad}(a_n)\}\delta(\mathfrak{U}) = 0 \quad \forall a_i \in \mathcal{O}_{\mathfrak{Y}}(\mathfrak{U}),\ \mathfrak{U} \text{ open of } \mathfrak{Y},$$

where $\mathrm{ad}(a_i)\delta(\mathfrak{U}) = a_i\delta(\mathfrak{U}) - \delta(\mathfrak{U})a_i$. In particular (cf. [**DG**, II.4.5.4, 5], [**EGAIV**, 16.8.3]),

(1) the set of $\mathfrak{k}$-deviations of order $\leq n$ for $\mathfrak{f} = \mathrm{id}_{\mathfrak{X}}$ and $\mathcal{N} = \mathcal{M}$ is just $\mathrm{Diff}^n_{\mathfrak{X}}(\mathcal{M}) = \mathrm{Diff}^n_{\mathfrak{X}/\mathfrak{k}}(\mathcal{M}, \mathcal{M})$.

Also if $\varepsilon_{\mathfrak{G}} : \mathfrak{e} \to \mathfrak{G}$ is the unit section, then [**DG**, II.4.6.2]

(2) $\mathrm{Dist}_n(\mathfrak{G})$ may be identified with the set of all deviations $(\varepsilon_{\mathfrak{G}}, \mu)$ with $\mu \in \mathbf{Mod}_{\mathfrak{k}}(\mathcal{O}_{\mathfrak{G}}, (\varepsilon_{\mathfrak{G}})_*\mathcal{O}_{\mathfrak{e}})$ of order $\leq n$.

Now let $(\mathcal{M}, \phi) \in (\mathfrak{G}, \mathfrak{X})\mathbf{qc}$ and $\tilde{\phi} \in \mathbf{Mod}_{\mathfrak{X}}(\mathcal{M}, \mathfrak{a}_* \mathfrak{p}_{\mathfrak{X}}^* \mathcal{M})$ the adjoint of ϕ. Then $(\mathfrak{a}, \tilde{\phi})$ is a deviation of order 0. Let $\mathfrak{i}'_{\mathfrak{X}} \in \mathbf{Sch}_{\mathfrak{k}}(\mathfrak{X}, \mathfrak{e} \times \mathfrak{X})^{\times}$ induced by $\mathfrak{i}_{\mathfrak{X}}$ and let $\mathcal{M}' = (\mathfrak{i}'_{\mathfrak{X}})_* \mathcal{M}$. Then $(\mathfrak{i}'_{\mathfrak{X}}, \mathrm{id}_{\mathcal{M}'} : \mathcal{M}' \to (\mathfrak{i}'_{\mathfrak{X}})_* \mathcal{M})$ is also a deviation of order 0. If $\mu \in \mathrm{Dist}_n(\mathfrak{G})$, define a deviation $(\varepsilon_{\mathfrak{G}} \times \mathfrak{X} : \mathfrak{e} \times \mathfrak{X} \to \mathfrak{G} \times \mathfrak{X}, \mu \otimes \mathcal{M} : \mathfrak{p}_{\mathfrak{X}}^* \mathcal{M} \to (\varepsilon_{\mathfrak{G}} \times \mathfrak{X})_* \mathcal{M}')$ of order 0 [**DG**, II.4.5.11] by the commutative diagram

(3)
$$\begin{array}{ccc} (\mathfrak{p}_{\mathfrak{X}}^* \mathcal{M})(\mathfrak{G} \times \mathfrak{U}) & \xrightarrow{(\mu \otimes \mathcal{M})(\mathfrak{G} \times \mathfrak{U})} & ((\varepsilon_{\mathfrak{G}} \times \mathfrak{X})_* \mathcal{M}')(\mathfrak{G} \times \mathfrak{U}) \\ \wr \downarrow & & \downarrow \wr \\ \mathfrak{k}[\mathfrak{G}] \otimes \mathcal{M}(\mathfrak{U}) & \xrightarrow[\mu \otimes \mathcal{M}(\mathfrak{U})]{} & \mathcal{M}(\mathfrak{U}). \end{array}$$

Then the composite deviation

(4)
$$(\mathrm{id}_{\mathfrak{X}} = \mathfrak{a} \circ (\varepsilon_{\mathfrak{G}} \times \mathfrak{X}) \circ \mathfrak{i}'_{\mathfrak{X}}, \mathfrak{a}_*(\varepsilon_{\mathfrak{G}} \times \mathfrak{X})_*(\mathrm{id}_{\mathcal{M}'}) \circ \mathfrak{a}_*(\mu \otimes \mathcal{M}) \circ \tilde{\phi} : \mathcal{M} \to \mathcal{M})$$

is of order $\leq n$ [**DG**, II.4.6.3]. On the other hand, for $n \in \mathbb{N}$ let $\mathfrak{V}_n(\mathfrak{e}) = \mathfrak{Sp}_{\mathfrak{k}}(\mathfrak{k}[\mathfrak{G}]/\mathfrak{m}_{\mathfrak{G}}^{n+1})$, $\varepsilon_n : \mathfrak{V}_n(\mathfrak{e}) \hookrightarrow \mathfrak{G}$, $\mathfrak{a}_n = \mathfrak{a} \circ (\varepsilon_n \times \mathfrak{X}) : \mathfrak{V}_n(\mathfrak{e}) \times \mathfrak{X} \to \mathfrak{X}$, $\mathfrak{p}_{\mathfrak{X},n} = \mathfrak{p}_{\mathfrak{X}} \circ (\varepsilon_n \times \mathfrak{X}) : \mathfrak{V}_n(\mathfrak{e}) \times \mathfrak{X} \to \mathfrak{X}$ the projection, $\phi_n = (\varepsilon_n \times \mathfrak{X})^* \phi : \mathfrak{a}_n^* \mathcal{M} \to \mathfrak{p}_{\mathfrak{X},n}^* \mathcal{M}$, and $\tilde{\phi}_n : \mathcal{M} \to (\mathfrak{a}_n)_* \mathfrak{p}_{\mathfrak{X},n}^* \mathcal{M}$ the adjoint of ϕ_n. Define $\phi' : \mathrm{Dist}(\mathfrak{G}) \to \mathbf{Mod}_{\mathfrak{k}}(\mathcal{M}, \mathcal{M})$ by the commutative diagram

(5)
$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\phi'(\mu)} & \mathcal{M} \\ \tilde{\phi}_m \downarrow & & \uparrow \mu \otimes \mathcal{M} \\ (\mathfrak{a}_m)_* \mathfrak{p}_{\mathfrak{X},m}^* \mathcal{M} & \xrightarrow[\sim]{} & (\mathfrak{k}[G]/\mathfrak{m}_{\mathfrak{G}}^{m+1}) \otimes \mathcal{M} \end{array}$$

for $\mu \in \mathrm{Dist}_n(\mathfrak{G})$ and $m \geq n$, that is independent of the choice of m. If $\varepsilon_{0,n} : \mathfrak{e} \to \mathfrak{V}_n(\mathfrak{e})$ is the inclusion and if we define a deviation $(\varepsilon_{0,n} \times \mathfrak{X}, \mu \otimes \mathcal{M} : \mathfrak{p}_{\mathfrak{X}}^* \mathcal{M} \to (\varepsilon_{0,n} \times \mathfrak{X})_* \mathcal{M}')$ as in (3), then

(6)
$$\phi'(\mu) = (\mathfrak{a}_n)_*(\mu \otimes \mathcal{M}) \circ \tilde{\phi}_n$$

under the identification of $(\mathfrak{a}_n)_*(\varepsilon_{0,n} \times \mathfrak{X})_* \mathcal{M}'$ with $\mathcal{M}$, and the composite deviation $(\mathrm{id}_{\mathfrak{X}} = \mathfrak{a}_n \circ (\varepsilon_{0,n} \times \mathfrak{X}) \circ \mathfrak{i}'_{\mathfrak{X}}, \phi'(\mu))$ coincides with the deviation (4), hence $\phi'(\mu) \in \mathrm{Diff}_{\mathfrak{X}}^n(\mathcal{M})$.

3.6.

Proposition. *If* $(\mathcal{M}, \phi) \in (\mathfrak{G}, \mathfrak{X})\mathbf{qc}$*, the* $\mathfrak{k}$*-linear map* ϕ' *of* (3.5.2) *induces a* $\mathfrak{k}$*-algebra homomorphism* $\mathrm{Dist}(\mathfrak{G})^{\mathrm{op}} \to \mathrm{Diff}_{\mathfrak{X}}(\mathcal{M})$*, still denoted* ϕ'*, under which*

$$\mathrm{Dist}_r(\mathfrak{G})^{\mathrm{op}} \to \mathrm{Diff}_{\mathfrak{X}}^r(\mathcal{M})$$

and

$$\mathrm{Dist}(\mathfrak{G}_r)^{\mathrm{op}} \to D_r(\mathcal{M}) = \mathbf{Mod}_{\mathcal{O}_{\mathfrak{X}}^{(r)}}(\mathcal{M},\mathcal{M}) \quad \forall r \in \mathbb{N}.$$

Proof. The assertion that ϕ' induces a $\mathfrak{k}$-algebra homomorphism follows from the commutative diagrams (3.4.1, 2). To see the last assertion, let $\mu \in \mathrm{Dist}(\mathfrak{G}_r)$, $M = \mathcal{M}(\mathfrak{U})$, $\mathfrak{U}$ an affine open of $\mathfrak{X}$, and $C = \mathcal{O}_{\mathfrak{X}}(\mathfrak{U})$. We must show

$$\phi'(\mu)(\mathfrak{U})(c^{p^r}m) = c^{p^r}\phi'(\mu)(\mathfrak{U})(m) \quad \forall c \in C \text{ and } m \in M.$$

As $\mathfrak{G}_r$ is infinitesimal, the action $\mathfrak{a}$ induces an action $\mathfrak{a}_r : \mathfrak{G}_r \times \mathfrak{U} \to \mathfrak{U}$. If $\mathfrak{i}_r : \mathfrak{G}_r \to \mathfrak{G}$ and $\mathfrak{j} : \mathfrak{U} \to \mathfrak{X}$ are two inclusions and if ψ_r is the adjoint of $(\mathfrak{i}_r \times \mathfrak{j})^*\phi$, then

$$\phi'(\mu)(\mathfrak{U}) = (\mu \otimes M) \circ \psi_r(\mathfrak{U}).$$

Also $(\varepsilon_{\mathfrak{G}_r} \times \mathfrak{U})^{\mathrm{o}} \circ \mathfrak{a}_r^{\mathrm{o}} = \mathrm{id}_{\mathfrak{k}[\mathfrak{U}]}$ with $\varepsilon_{\mathfrak{G}_r}$ the unit section of $\mathfrak{G}_r$, hence one can write

$$\mathfrak{a}_r^{\mathrm{o}}(c) = 1 \otimes c + \sum_i a_i \otimes c_i, \quad a_i \in \mathfrak{m}_{\mathfrak{G}}\mathfrak{k}[\mathfrak{G}_r], c_i \in C.$$

If $\psi_r(\mathfrak{U})(m) = \sum_j b_j \otimes m_j$, $b_j \in \mathfrak{k}[\mathfrak{G}_r]$, $m_j \in M$, then

$$\psi_r(\mathfrak{U})(c^{p^r}m) = ((\mathfrak{i}_r \times \mathfrak{j})^*\phi)(\mathfrak{G}_r \times \mathfrak{U})(\mathfrak{a}_r^{\mathrm{o}}(c^{p^r}) \otimes m) = \sum_j b_j \otimes c^{p^r}m_j$$

as $a_i^{p^r} = 0$, hence $\phi'(\mu)(\mathfrak{U})(c^{p^r}m) = \sum_j \mu(b_j)c^{p^r}m_j = c^{p^r}\phi'(\mu)(\mathfrak{U})(m)$. □

3.7. Conversely,

Lemma (**infinitesimal test**). *Assume $\mathcal{M} \in \mathbf{qc}_{\mathfrak{X}}$ is of finite type or locally free. Let $\phi \in \mathbf{Mod}_{\mathfrak{X}}(\mathfrak{a}^*\mathcal{M}, \mathfrak{p}_{\mathfrak{X}}^*\mathcal{M})^{\times}$. If ϕ' induces a $\mathfrak{k}$-algebra homomorphism* $\mathrm{Dist}(\mathfrak{G})^{\mathrm{op}} \to \mathbf{Mod}_{\mathfrak{k}}(\mathcal{M}_x, \mathcal{M}_x)$ *at each $x \in \mathfrak{X}(\mathfrak{k})$, i.e., each $\mathcal{M}_x$ is a $\mathfrak{G}_r^{\mathrm{op}}$-module for all $r \in \mathbb{N}$ compatibly, then the diagram* (3.4.1) *commutes, hence defines a structure of* $(\mathfrak{G}, \mathfrak{X})\mathbf{qc}$ *on $\mathcal{M}$. If $\mathfrak{G}$ is transitive on $\mathfrak{X}$, one has only to test the criterion at arbitrary one point of $\mathfrak{X}(\mathfrak{k})$.*

Proof. We will show that the diagram (3.4.1) commutes at each point $(g_1, g_2, x) \in \mathfrak{G}(\mathfrak{k}) \times \mathfrak{G}(\mathfrak{k}) \times \mathfrak{X}(\mathfrak{k})$:

(1) $(\mathfrak{p}_2^*\phi)_{(g_1,g_2,x)} = (\mathfrak{p}_3^*\phi)_{(g_1,g_2,x)} \circ (\mathfrak{p}_1^*\phi)_{(g_1,g_2,x)}$:

$$\mathcal{O}_{\mathfrak{G}\times\mathfrak{G}\times\mathfrak{X},(g_1,g_2,x)} \otimes_{\mathcal{O}_{\mathfrak{X},g_1g_2x}} \mathcal{M}_{g_1g_2x} \to \mathcal{O}_{\mathfrak{G}\times\mathfrak{G}\times\mathfrak{X},(g_1,g_2,x)} \otimes_{\mathcal{O}_{\mathfrak{X},x}} \mathcal{M}_x,$$

where $\mathfrak{p}_1 = \mathfrak{G} \times \mathfrak{a}$, $\mathfrak{p}_2 = \mathfrak{mult} \times \mathfrak{X}$, and $\mathfrak{p}_3 = \mathfrak{p}_{\mathfrak{G}\times\mathfrak{X}}$. For that it is enough by Krull's intersection theorem [M, Th. 8.9, 10] to show the equality under the natural surjection

$$\mathcal{O}_{\mathfrak{G}\times\mathfrak{G}\times\mathfrak{X},(g_1,g_2,x)} \otimes_{\mathcal{O}_{\mathfrak{X},x}} \mathcal{M}_x \to \{\mathcal{O}_{\mathfrak{G}\times\mathfrak{G}\times\mathfrak{X},(g_1,g_2,x)}/\mathfrak{m}^{r+1}_{(g_1,g_2,x)}\} \otimes_{\mathcal{O}_{\mathfrak{X},x}} \mathcal{M}_x \ \forall r \in \mathbb{N},$$

where $\mathfrak{m}^{r+1}_{(g_1,g_2,x)}$ is the maximal ideal of $\mathcal{O}_{\mathfrak{G}\times\mathfrak{G}\times\mathfrak{X},(g_1,g_2,x)}$. For simplicity put $A = \mathfrak{k}[\mathfrak{G}]$, and let $\mathfrak{U}$ be an affine open neighbourhood of x in $\mathfrak{X}$, $C = \mathcal{O}_{\mathfrak{X}}(\mathfrak{U})$, $\mathfrak{V}_r(g_i) = \mathfrak{Sp}_{\mathfrak{k}}(A/\mathfrak{m}_i^{r+1})$ with $\mathfrak{m}_i = \ker g_i \in \operatorname{Max}(A)$, and $\bar{\mathfrak{m}}$ the maximal ideal of $\mathcal{O}_{\mathfrak{V}_r(g_1)\times\mathfrak{V}_r(g_2)\times\mathfrak{U},(g_1,g_2,x)}$. Then

$$\mathcal{O}_{\mathfrak{G}\times\mathfrak{G}\times\mathfrak{X},(g_1,g_2,x)}/\mathfrak{m}^{r+1}_{(g_1,g_2,x)} \simeq \mathcal{O}_{\mathfrak{V}_r(g_1)\times\mathfrak{V}_r(g_2)\times\mathfrak{U},(g_1,g_2,x)}/\bar{\mathfrak{m}}^{r+1}.$$

Hence, if $\mathfrak{i}_r^{g_i} : \mathfrak{V}_r(g_i) \to \mathfrak{G}$ and $\mathfrak{j} : \mathfrak{U} \to \mathfrak{X}$ are inclusions, the equality (1) will follow from

$$(2) \quad \{(\mathfrak{i}_r^{g_1} \times \mathfrak{i}_r^{g_2} \times \mathfrak{j})^*\mathfrak{p}_2^*\phi\}_{(g_1,g_2,x)} = \{(\mathfrak{i}_r^{g_1} \times \mathfrak{i}_r^{g_2} \times \mathfrak{j})^*((\mathfrak{p}_3^*\phi) \circ (\mathfrak{p}_1^*\phi))\}_{(g_1,g_2,x)}.$$

But $\mathfrak{i}_r^{g_i} = \mathfrak{i}_r \circ g_i^{-1}$ if $\mathfrak{i}_r = \mathfrak{i}_r^e$ and if $g_i : \mathfrak{V}_r(e) \to \mathfrak{V}_r(g_i)$ is the translation by g_i. As $\mathfrak{i}_r^{g_1} \times \mathfrak{i}_r^{g_2} \times \mathfrak{j} = (\mathfrak{i}_r \times \mathfrak{i}_r \times \mathfrak{j}) \circ (g_1^{-1} \times g_2^{-1} \times \mathfrak{U})$ and as $g_1^{-1} \times g_2^{-1} \times \mathfrak{U} : \mathfrak{V}_r(g_1) \times \mathfrak{V}_r(g_2) \times \mathfrak{U} \to \mathfrak{V}_r(e) \times \mathfrak{V}_r(e) \times \mathfrak{U}$ is invertible, we may assume $g_1 = g_2 = e$ in (2). Let $\mathfrak{a}_{2r} = \mathfrak{a} \circ (\mathfrak{i}_{2r} \times \mathfrak{j})$, $\mathfrak{p}_{\mathfrak{X},2r} = \mathfrak{p}_{\mathfrak{X}} \circ \mathfrak{i}_{2r}$, and $\phi_{2r} = (\mathfrak{i}_{2r} \times \mathfrak{j})^*\phi$. If we define $\mathfrak{p}_{i,r}$, $1 \le i \le 3$, by the commutative diagrams

$$\begin{array}{ccc}
\mathfrak{G} \times \mathfrak{G} \times \mathfrak{X} & \xrightarrow{\mathfrak{p}_i} & \mathfrak{G} \times \mathfrak{X} \\
{\scriptstyle \mathfrak{i}_r \times \mathfrak{i}_r \times \mathfrak{j}} \uparrow & & \uparrow {\scriptstyle \mathfrak{i}_{2r} \times \mathfrak{j}} \\
\mathfrak{V}_r(e) \times \mathfrak{V}_r(e) \times \mathfrak{U} & \xrightarrow[\mathfrak{p}_{i,r}]{} & \mathfrak{V}_{2r}(e) \times \mathfrak{U},
\end{array}$$

then the equality (2) reads

$$(3) \quad (\mathfrak{p}_{2,r}^*\phi_{2r})_{(e,e,x)} = (\mathfrak{p}_{3,r}^*\phi_{2r})_{(e,e,x)} \circ (\mathfrak{p}_{1,r}^*\phi_{2r})_{(e,e,x)} : \\ (\mathfrak{p}_{2,r}^*\mathfrak{a}_{2r}^*\mathcal{M})_{(e,e,x)} \to (\mathfrak{p}_{2,r}^*\mathfrak{p}_{\mathfrak{X},2r}^*\mathcal{M})_{(e,e,x)},$$

that in turn will follow from

$$(4) \quad (\mathfrak{p}_{2,r}^*\phi_{2r})(\mathfrak{V}_r(e) \times \mathfrak{V}_r(e) \times \mathfrak{U}) \\ = (\mathfrak{p}_{3,r}^*\phi_{2r})(\mathfrak{V}_r(e) \times \mathfrak{V}_r(e) \times \mathfrak{U}) \circ (\mathfrak{p}_{1,r}^*\phi_{2r})(\mathfrak{V}_r(e) \times \mathfrak{V}_r(e) \times \mathfrak{U}).$$

As both sides are $\mathfrak{k}[\mathfrak{V}_r(e) \times \mathfrak{V}_r(e) \times \mathfrak{U}]$-linear, we have only to verify (4) after composing with the natural homomorphism

$$(5) \qquad \mathcal{M}(\mathfrak{U}) \to (\mathfrak{p}_{2,r}^*\mathfrak{a}_{2r}^*\mathcal{M})(\mathfrak{V}_r(e) \times \mathfrak{V}_r(e) \times \mathfrak{U}).$$

Let $\psi_r : \mathcal{M} \to (\mathfrak{a}_r)_*\mathfrak{p}^*_{\mathfrak{X},r}\mathcal{M}$ be the adjoint of ϕ_r, $M = \mathcal{M}(\mathfrak{U})$, $\Delta_{M,r} = \psi_r(\mathfrak{U}) : M \to (A/\mathfrak{m}_{\mathfrak{G}}^{r+1}) \otimes M$, and define likewise $\Delta_{M,2r}$ from ϕ_{2r}. If $\mathfrak{i}_{r,2r} : \mathfrak{V}_r(e) \to \mathfrak{V}_{2r}(e)$ is the inclusion and if $\Delta_{\mathfrak{G},r} : A/\mathfrak{m}_{\mathfrak{G}}^{2r+1} \to (A/\mathfrak{m}_{\mathfrak{G}}^{r+1}) \otimes (A/\mathfrak{m}_{\mathfrak{G}}^{r+1})$ is the $\mathfrak{k}$-algebra homomorphism induced by the multiplication $\mathfrak{V}_r(e) \times \mathfrak{V}_r(e) \to \mathfrak{V}_{2r}(e)$, the LHS (resp. RHS) of (4) composed with (5) reads $(\mathfrak{i}^{\mathrm{o}}_{r,2r} \otimes \Delta_{M,r}) \circ \Delta_{M,2r}$ (resp. $(\Delta_{\mathfrak{G},r} \otimes M) \circ \Delta_{M,2r}$). But from the hypothesis one has

$$\{(\mathfrak{i}^{\mathrm{o}}_{r,2r} \otimes \Delta_{M,r}) \circ \Delta_{M,2r}\} \otimes_C C_x = \{(\Delta_{\mathfrak{G},r} \otimes M) \circ \Delta_{M,2r}\} \otimes_C C_x \text{ at each } x \in \mathfrak{X}(\mathfrak{k}),$$

hence $(\mathfrak{i}^{\mathrm{o}}_{r,2r} \otimes \Delta_{M,r}) \circ \Delta_{M,2r} = (\Delta_{\mathfrak{G},r} \otimes M) \circ \Delta_{M,2r}$, as desired.

Assume finally that $\mathfrak{G}$ is transitive on $\mathfrak{X}$. If the hypothesis is verified at $z \in \mathfrak{X}(\mathfrak{k})$ and if $gz = x$, $g \in \mathfrak{G}(\mathfrak{k})$, then the equality (3) will follow from

$$(\mathfrak{V}_r(e) \times \mathfrak{V}_r(e) \times g)^* \mathfrak{p}^*_{2,r}\phi_{2r} = (\mathfrak{V}_r(e) \times \mathfrak{V}_r(e) \times g)^* \{(\mathfrak{p}^*_{3,r}\phi_{2r}) \circ (\mathfrak{p}^*_{1,r}\phi_{2r})\}$$

as the translation is invertible, hence the assertion from the hypothesis at z. □

3.8. In the set-up of (3.4) we say $(\mathcal{M}, \phi)$ is a $\mathfrak{G}$-equivariant $\mathcal{D}_{\mathfrak{X}}$-module iff $\mathcal{M} \in \mathcal{D}_{\mathfrak{X}}\mathbf{qc}$ and $\phi \in \mathcal{D}_{\mathfrak{G}\times\mathfrak{X}}\mathbf{Mod}(\mathfrak{a}^0\mathcal{M}, \mathfrak{p}^0_{\mathfrak{X}}\mathcal{M})^\times$. We will denote the category of $\mathfrak{G}$-equivariant $\mathcal{D}_{\mathfrak{X}}$-modules by $(\mathfrak{G}, \mathcal{D}_{\mathfrak{X}})\mathbf{qc}$. As $\mathfrak{a}$ and $\mathfrak{p}_{\mathfrak{X}}$ are both flat, $\mathrm{L}\,\mathfrak{a}^0 = \mathfrak{a}^0$ and $\mathrm{L}\,\mathfrak{p}^0_{\mathfrak{X}} = \mathfrak{p}^0_{\mathfrak{X}}$. We have as in characteristic 0 (cf. [**Sa**, §1]).

Proposition. *Let* $\mathfrak{f} \in \mathbf{Sch}^{\mathfrak{G}}_{\mathfrak{k}}(\mathfrak{X}, \mathfrak{Y})$.

(i) $\mathrm{L}\,\mathfrak{f}^0 : \mathrm{D}^b(\mathcal{D}_{\mathfrak{Y}}\mathbf{qc}) \to \mathrm{D}^b(\mathcal{D}_{\mathfrak{X}}\mathbf{qc})$ *sends* $\mathrm{D}^b((\mathfrak{G}, \mathcal{D}_{\mathfrak{Y}})\mathbf{qc})$ *to* $\mathrm{D}^b((\mathfrak{G}, \mathcal{D}_{\mathfrak{X}})\mathbf{qc})$.

(ii) $\int_{\mathfrak{f}} : \mathrm{D}^b(\mathcal{D}_{\mathfrak{X}}\mathbf{qc}) \to \mathrm{D}^b(\mathcal{D}_{\mathfrak{Y}}\mathbf{qc})$ *sends* $\mathrm{D}^b((\mathfrak{G}, \mathcal{D}_{\mathfrak{X}})\mathbf{qc})$ *to* $\mathrm{D}^b((\mathfrak{G}, \mathcal{D}_{\mathfrak{Y}})\mathbf{qc})$.

(iii) *If* $\mathfrak{Z}_1 \supseteq \mathfrak{Z}_2$ *are two* $\mathfrak{G}$*-invariant closed subsets of* $\mathfrak{X}$*,* $\mathrm{R}\,\Gamma_{\mathfrak{Z}_1/\mathfrak{Z}_2} : \mathrm{D}^b(\mathcal{D}_{\mathfrak{X}}\mathbf{qc}) \to \mathrm{D}^b(\mathcal{D}_{\mathfrak{X}}\mathbf{qc})$ *preserves* $\mathrm{D}^b((\mathfrak{G}, \mathcal{D}_{\mathfrak{X}})\mathbf{qc})$.

Proof. (ii) An application of the base change theorem (2.12) to the cartesian square

$$\begin{array}{ccc} \mathfrak{G} \times \mathfrak{X} & \xrightarrow{\mathfrak{a}_{\mathfrak{X}}} & \mathfrak{X} \\ {\scriptstyle \mathfrak{G}\times\mathfrak{f}}\downarrow & & \downarrow{\scriptstyle \mathfrak{f}} \\ \mathfrak{G} \times \mathfrak{Y} & \xrightarrow[\mathfrak{a}_{\mathfrak{Y}}]{} & \mathfrak{Y} \end{array}$$

yields $\mathfrak{a}^0_{\mathfrak{Y}} \circ \int_{\mathfrak{f}} \simeq \int_{\mathfrak{G}\times\mathfrak{f}} \circ \mathfrak{a}^0_{\mathfrak{X}}$. Likewise $\mathfrak{p}^0_{\mathfrak{Y}} \circ \int_{\mathfrak{f}} \simeq \int_{\mathfrak{G}\times\mathfrak{f}} \circ \mathfrak{p}^0_{\mathfrak{Y}}$. Hence for $(\mathcal{M}^{\cdot}, \phi) \in \mathrm{D}^b((\mathfrak{G}, \mathfrak{X})\mathbf{qc})$ one can define $\phi_{\int_{\mathfrak{f}}\mathcal{M}^{\cdot}} \in \mathrm{D}^b(\mathcal{D}_{\mathfrak{G}\times\mathfrak{Y}}\mathbf{qc})(\mathfrak{a}^0_{\mathfrak{Y}} \int_{\mathfrak{f}} \mathcal{M}^{\cdot}, \mathfrak{p}^0_{\mathfrak{Y}} \int_{\mathfrak{f}} \mathcal{M}^{\cdot})^\times$ by the commutative diagram

$$(1)\qquad \begin{array}{ccc} \mathfrak{a}^0_{\mathfrak{Y}} \int_{\mathfrak{f}} \mathcal{M}^{\cdot} & \xrightarrow{\phi_{\int_{\mathfrak{f}} \mathcal{M}^{\cdot}}} & \mathfrak{p}^0_{\mathfrak{Y}} \int_{\mathfrak{f}} \mathcal{M}^{\cdot} \\ \wr\downarrow & & \uparrow\wr \\ \int_{\mathfrak{G}\times\mathfrak{f}} \mathfrak{a}^0_{\mathfrak{X}} \mathcal{M}^{\cdot} & \xrightarrow[\int_{\mathfrak{G}\times\mathfrak{f}} \phi_{\mathcal{M}^{\cdot}}]{} & \int_{\mathfrak{G}\times\mathfrak{f}} \mathfrak{p}^0_{\mathfrak{X}} \mathcal{M}^{\cdot}. \end{array}$$

The cocyclicity (3.4.1) of $\phi_{\int_{\mathfrak{f}} \mathcal{M}^{\cdot}}$ follows likewise.

(iii) As $\mathfrak{a}_{\mathfrak{X}} = \mathfrak{p}_{\mathfrak{X}} \circ \mathfrak{a}'_{\mathfrak{X}}$ with $\mathfrak{a}'_{\mathfrak{X}}$ invertible, one has by [**Ke**, 11.5]

$$(2)\qquad \mathfrak{a}^0_{\mathfrak{X}} \circ \Gamma_{\mathfrak{z}_1/\mathfrak{z}_2} \simeq \Gamma_{\mathfrak{G}\times\mathfrak{z}_1/\mathfrak{G}\times\mathfrak{z}_2} \circ \mathfrak{a}^0_{\mathfrak{X}} \quad \text{on } \mathcal{D}_{\mathfrak{X}}\mathbf{qc}.$$

Also from [**Ke**, 11.5]

(3) $\mathfrak{p}^0_{\mathfrak{X}}$, hence also $\mathfrak{a}^0_{\mathfrak{X}}$, sends flasques to $\Gamma_{\mathfrak{G}\times\mathfrak{z}_1/\mathfrak{G}\times\mathfrak{z}_2}$-acyclics.

Hence on $\mathrm{D}^b(\mathcal{D}_{\mathfrak{X}}\mathbf{qc})$

$$\mathfrak{a}^0_{\mathfrak{X}} \circ \mathrm{R}\,\Gamma_{\mathfrak{z}_1/\mathfrak{z}_2} \simeq \mathrm{R}\,\Gamma_{\mathfrak{G}\times\mathfrak{z}_1/\mathfrak{G}\times\mathfrak{z}_2} \circ \mathfrak{a}^0_{\mathfrak{X}} \quad \text{and} \quad \mathfrak{p}^0_{\mathfrak{X}} \circ \mathrm{R}\,\Gamma_{\mathfrak{z}_1/\mathfrak{z}_2} \simeq \mathrm{R}\,\Gamma_{\mathfrak{G}\times\mathfrak{z}_1/\mathfrak{G}\times\mathfrak{z}_2} \circ \mathfrak{p}^0_{\mathfrak{X}},$$

and the assertion follows as in (ii). □

3.9. If $\mathfrak{X}$ is affine, $\mathfrak{X}$ is $\mathcal{D}$-affine by Serre's theorem, and the equivariant version of Beilinson-Bernstein's local-global principle carries over. Thus let $(\mathfrak{G}, D(\mathfrak{X}))\mathbf{Mod}$ be the category whose objects are simultaneously $\mathfrak{G}$-modules and $D(\mathfrak{X})$-modules such that (i) the two $\mathrm{Dist}(\mathfrak{G})^{\mathrm{op}}$-actions on M induced by the $\mathfrak{G}$-module structure and the $D(\mathfrak{X})$-module structure coincide, i.e., if $\Delta_M : M \to M \otimes \mathfrak{k}[\mathfrak{G}]$ is the $\mathfrak{G}$-module structure on M, then with $\phi' : \mathrm{Dist}(\mathfrak{G})^{\mathrm{op}} \to D(\mathfrak{X})$ of (3.6) and transposition $\tau : M \otimes \mathfrak{k}[\mathfrak{G}] \to \mathfrak{k}[\mathfrak{G}] \otimes M$

$$(\mu \otimes M) \circ \tau \circ \Delta_M = \phi'(\mu) \quad \forall \mu \in \mathrm{Dist}(\mathfrak{G}),$$

and that (ii) $\forall \delta \in D(\mathfrak{X})$, $g \in \mathfrak{G}$ and $m \in M$, $g(\delta m) = (g\delta)(gm)$.

Proposition (cf. [**Bø**, Prop. 4.5])**.** *If $\mathfrak{X}$ is affine, there is an equivalence of categories $(\mathfrak{G}, \mathcal{D}_{\mathfrak{X}})\mathbf{qc} \to (\mathfrak{G}, D(\mathfrak{X}))\mathbf{Mod}$ via $\mathcal{M} \mapsto \mathcal{M}(\mathfrak{X})$ with quasi-inverse $M \mapsto (\mathcal{D}_{\mathfrak{X}} \otimes_{D(\mathfrak{X})} M, \phi)$ such that $\phi(\mathfrak{G} \times \mathfrak{X}) \in$*

$$D(\mathfrak{G}\times\mathfrak{X})\mathbf{Mod}((\mathfrak{k}[\mathfrak{G}\times\mathfrak{X}], \mathfrak{a}^{\mathrm{o}}) \otimes_{\mathfrak{k}[\mathfrak{X}]} D(\mathfrak{X}) \otimes_{D(\mathfrak{X})} M,$$
$$(\mathfrak{k}[\mathfrak{G}\times\mathfrak{X}], \mathfrak{p}^{\mathrm{o}}) \otimes_{\mathfrak{k}[\mathfrak{X}]} D(\mathfrak{X}) \otimes_{D(\mathfrak{X})} M)$$

is induced by the left $\mathfrak{k}[\mathfrak{G}]$*-comodule map* $\tau \circ \Delta_M : M \to \mathfrak{k}[\mathfrak{G}] \otimes M$*.*

3.10.

Remark. If $\mathfrak{X}$ is not affine, the proposition implies for each affine open $\mathfrak{U}$ of $\mathfrak{X}$ and for each infinitesimal subgroup $\mathfrak{G}_r$ of $\mathfrak{G}$, $r \in \mathbb{N}$, that there is an equivalence

$$(\mathfrak{G}_r, \mathcal{D}_{\mathfrak{U}})\mathbf{qc} \to (\mathfrak{G}_r, D(\mathfrak{U}))\mathbf{Mod}.$$

3.11. If $\mathfrak{X}$ is a $\mathfrak{G}$-variety, $\mathcal{O}_{\mathfrak{X}}$ is naturally equipped with a structure of $(\mathfrak{G}, \mathcal{D}_{\mathfrak{X}})\mathbf{qc}$, hence also $\mathrm{R}^i\Gamma_{\mathfrak{Z}_1/\mathfrak{Z}_2}(\mathcal{O}_{\mathfrak{X}})$ for each $i \in \mathbb{N}$ and two closed subsets $\mathfrak{Z}_1 \supseteq \mathfrak{Z}_2$ of $\mathfrak{X}$.

Using either the standard filtration or the p-filtration, one can equip $\mathcal{D}_{\mathfrak{X}}$ with a structure of $(\mathfrak{G}, \mathfrak{X})\mathbf{qc}$ as follows. Let $\Delta_{\mathfrak{X}} : \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ be the diagonal immersion, $\mathfrak{I}_{\mathfrak{X}}$ the ideal sheaf of the closed subvariety $\mathrm{im}(\Delta_{\mathfrak{X}})$ in $\mathfrak{X} \times \mathfrak{X}$, $\mathfrak{X}_{\Delta,n} = (|\mathrm{im}\,\Delta_{\mathfrak{X}}|, \mathcal{O}_{\mathfrak{X}\times\mathfrak{X}}/\mathfrak{I}_{\mathfrak{X}}^{n+1})$ the n-th infinitesimal neighbourhood of $\mathrm{im}(\Delta_{\mathfrak{X}})$, and $\mathfrak{p}_{i,n} = \mathfrak{p}_i \circ \mathrm{inc} : \mathfrak{X}_{\Delta,n} \to \mathfrak{X}$, $i = 1, 2$, with $\mathfrak{p}_i : \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$ the i-th projection. By [**EGAIV**, 16.7.1.1]

$$\mathcal{P}^n_{\mathfrak{X}} = (\mathfrak{p}_{1,n})_*\mathfrak{p}^*_{2,n}\mathcal{O}_{\mathfrak{X}}. \tag{1}$$

As $\mathrm{im}(\Delta_{\mathfrak{X}})$ is $\mathfrak{G}$-invariant, so is $\mathfrak{X}_{\Delta,n}$, hence $\mathfrak{p}_{i,n}$ are both $\mathfrak{G}$-equivariant. Then $\mathcal{P}^n_{\mathfrak{X}}$ inherits a structure of $(\mathfrak{G}, \mathfrak{X})\mathbf{qc}$ from $\mathcal{O}_{\mathfrak{X}}$. As $\mathcal{P}^n_{\mathfrak{X}}$ is locally free of finite rank in $\mathfrak{X}\mathbf{Mod}$, one has

$$\mathfrak{a}^*\mathcal{D}iff^n_{\mathfrak{X}} \simeq \mathfrak{a}^*\mathcal{M}od_{\mathfrak{X}}(\mathcal{P}^n_{\mathfrak{X}}, \mathcal{O}_{\mathfrak{X}}) \simeq \mathcal{M}od_{\mathfrak{G}\times\mathfrak{X}}(\mathfrak{a}^*\mathcal{P}^n_{\mathfrak{X}}, \mathfrak{a}^*\mathcal{O}_{\mathfrak{X}})$$

and likewise with $\mathfrak{p}^*_{\mathfrak{X}}$. Hence from the structure morphism $\phi_{\mathcal{P}^n_{\mathfrak{X}}}$ one can define a structure of $(\mathfrak{G}, \mathfrak{X})\mathbf{qc}$ on $\mathcal{D}iff^n_{\mathfrak{X}}$ by the composite

$$\mathcal{M}od_{\mathfrak{G}\times\mathfrak{X}}(\mathfrak{p}^*_{\mathfrak{X}}\mathcal{P}^n_{\mathfrak{X}}, \phi_{\mathcal{O}_{\mathfrak{X}}}) \circ \mathcal{M}od_{\mathfrak{G}\times\mathfrak{X}}(\phi^{-1}_{\mathcal{P}^n_{\mathfrak{X}}}, \mathfrak{a}^*\mathcal{O}_{\mathfrak{X}}) : \\ \mathcal{M}od_{\mathfrak{G}\times\mathfrak{X}}(\mathfrak{a}^*\mathcal{P}^n_{\mathfrak{X}}, \mathfrak{a}^*\mathcal{O}_{\mathfrak{X}}) \to \mathcal{M}od_{\mathfrak{G}\times\mathfrak{X}}(\mathfrak{p}^*_{\mathfrak{X}}\mathcal{P}^n_{\mathfrak{X}}, \mathfrak{p}^*_{\mathfrak{X}}\mathcal{O}_{\mathfrak{X}}).$$

Then $\varinjlim_n \phi_{\mathcal{D}iff^n_{\mathfrak{X}}}$ makes $\mathcal{D}_{\mathfrak{X}}$ into a $\mathfrak{G}$-equivariant $\mathcal{O}_{\mathfrak{X}}$-module. Equivalently, one could define a structure of $(\mathfrak{G}, \mathfrak{X})\mathbf{qc}$ on $\mathcal{D}_{\mathfrak{X},r} \simeq \mathcal{M}od_{\mathfrak{X}}((\mathfrak{F}^r_{\mathfrak{X}})^*(\mathfrak{F}^r_{\mathfrak{X}})_*\mathcal{O}_{\mathfrak{X}}, \mathcal{O}_{\mathfrak{X}})$ [**H87**, 1.2.6] by the composite

$$\mathcal{M}od_{\mathfrak{G}\times\mathfrak{X}}(\mathfrak{p}^*_{\mathfrak{X}}(\mathfrak{F}^r_{\mathfrak{X}})^*(\mathfrak{F}^r_{\mathfrak{X}})_*\mathcal{O}_{\mathfrak{X}}, \phi_{\mathcal{O}_{\mathfrak{X}}}) \circ \mathcal{M}od_{\mathfrak{G}\times\mathfrak{X}}(\phi^{-1}_{(\mathfrak{F}^r_{\mathfrak{X}})^*(\mathfrak{F}^r_{\mathfrak{X}})_*\mathcal{O}_{\mathfrak{X}}}, \mathfrak{a}^*\mathcal{O}_{\mathfrak{X}}),$$

and take $\varinjlim_n \phi_{\mathcal{D}_{\mathfrak{X},r}}$. Unfortunately,

$$\phi_{\mathcal{D}_{\mathfrak{X}}} \text{ is not } \mathcal{D}_{\mathfrak{G}\times\mathfrak{X}}\text{-equivariant, hence } (\mathcal{D}_{\mathfrak{X}}, \phi_{\mathcal{D}_{\mathfrak{X}}}) \notin (\mathfrak{G}, \mathcal{D}_{\mathfrak{X}})\mathbf{qc}. \tag{2}$$

Explicitly, let us consider $\phi_{\mathcal{D}_{\mathfrak{X},r}}$ on $\mathfrak{G}_r \times \mathfrak{U}$, $r \in \mathbb{N}$, $\mathfrak{U}$ affine open of $\mathfrak{X}$. Let $\mathfrak{i} : G_r \hookrightarrow \mathfrak{G}$, $\mathfrak{j} : \mathfrak{U} \hookrightarrow \mathfrak{X}$, $\mathfrak{a}_r = \mathfrak{a} \circ (\mathfrak{i} \times \mathfrak{j})$, $\mathfrak{p}_r = \mathfrak{p}_{\mathfrak{X}} \circ (\mathfrak{i} \times \mathfrak{j})$, $\Phi = ((\mathfrak{i} \times \mathfrak{j})^* \phi_{\mathcal{O}_{\mathfrak{X}}})(\mathfrak{G}_r \times \mathfrak{U})$, $\Phi' = ((\mathfrak{i} \times \mathfrak{j})^* \phi_{(\mathfrak{F}^r_{\mathfrak{X}})^*(\mathfrak{F}^r_{\mathfrak{X}})_* \mathcal{O}_{\mathfrak{X}}})(\mathfrak{G}_r \times \mathfrak{U})$, $(\Phi', \Phi) = ((\mathfrak{i} \times \mathfrak{j})^* \phi_{\mathcal{D}_{\mathfrak{X},r}})(\mathfrak{G}_r \times \mathfrak{U})$, $A = \mathfrak{k}[\mathfrak{G}_r]$, and $C = \mathfrak{k}[\mathfrak{U}]$. Then $\Phi : (A \otimes C, \mathfrak{a}_r^{\mathrm{o}}) \otimes_C C \to A \otimes C$ is given by $a \otimes b \otimes c \longmapsto (a \otimes b)\mathfrak{a}_r^{\mathrm{o}}(c)$, hence $\Phi' : (A \otimes C^{(r)}, \mathfrak{a}_r^{\mathrm{o}}) \otimes_{C^{(r)}} C \to (A \otimes C, \mathfrak{p}_r^{\mathrm{o}}) \otimes_{C^{(r)}} C$ by $a \otimes b \otimes c \longmapsto \sum a a_i \otimes b \otimes c_i$ if $\mathfrak{a}_r^{\mathrm{o}}(c) = \sum a_i \otimes c_i$. Then $(\Phi')^{-1} : a \otimes b \otimes c \longmapsto \sum a\sigma(a_i) \otimes b \otimes c_i$ with σ the antipode of A, hence $(\Phi', \Phi) : (A \otimes C, \mathfrak{a}^{\mathrm{o}}) \otimes_C D_r(C) \to (A \otimes C, \mathfrak{p}^{\mathrm{o}}) \otimes_C D_r(C)$ is given by

$$f \longmapsto (\sigma \bar{\otimes} f) \circ \mathfrak{a}_r^{\mathrm{o}} = (\mathfrak{mult} \otimes C) \circ (\sigma \otimes f) \circ \mathfrak{a}_r^{\mathrm{o}}$$

upon identification of $(A \otimes C, \mathfrak{a}^{\mathrm{o}}) \otimes_C D_r(C)$ (resp. $(A \otimes C, \mathfrak{p}^{\mathrm{o}}) \otimes_C D_r(C)$) with $D_r(C, A \otimes C)$. In particular, the $\mathfrak{G}_r^{\mathrm{op}}$-action on $D_r(C)$ is given by

$$(x \cdot \theta)(c) = x(\theta(x^{-1}c)), \quad x \in \mathfrak{G}_r, c \in C,$$

where the RHS is written with respect to the $\mathfrak{G}_r^{\mathrm{op}}$-action on C given by $xc = (x \otimes C) \circ \mathfrak{a}_r^{\mathrm{o}}(c)$ in $R \otimes C$ if $x \in \mathfrak{G}_r(R)$, $R \in \mathbf{Alg}_{\mathfrak{k}}$. If $\delta_1 \otimes \delta_2 \in D_r(\mathfrak{G}_r) \otimes D_r(C) = D_r(A \otimes C)$, then

$$\{\sigma \bar{\otimes}((\delta_1 \otimes \delta_2) \circ f)\} \circ \mathfrak{a}_r^{\mathrm{o}}(c) = \sum \sigma(a_i)(\delta_1 \otimes \delta_2) f(c_i)$$

while

$$(\delta_1 \otimes \delta_2) \circ (\sigma \bar{\otimes} f) \circ \mathfrak{a}_r^{\mathrm{o}}(c) = \sum (\delta_1 \otimes \delta_2)(\sigma(a_i) f(c_i)).$$

Hence $\phi_{\mathcal{D}_{\mathfrak{X},r}}$ is only $\mathcal{O}_{\mathfrak{G}} \boxtimes \mathcal{D}_{\mathfrak{X},r}$-linear, i.e., $\mathcal{D}_{\mathfrak{X}}$ is a quasi-$\mathfrak{G}$-equivariant $\mathcal{D}_{\mathfrak{X}}$-module [**Ka**, 4.7]. We will denote the category of quasi-$\mathfrak{G}$-equivariant $\mathcal{D}_{\mathfrak{X}}$-modules by $(\mathfrak{G}, \mathcal{D}_{\mathfrak{X}})\mathbf{q}qc$.

Nevertheless, in [**Bø**, Prop. 4.5] it is proved that if $\mathfrak{X} = G/B$ is a flag variety with G semisimple and if M is a $D(\mathfrak{X})$-module such that the induced $\mathrm{Dist}(G)^{\mathrm{op}}$-action on M lifts to make a G^{op}-module, then $\mathcal{D}_{\mathfrak{X}} \otimes_{D(\mathfrak{X})} M$ admits a structure of $(G, \mathcal{D}_{\mathfrak{X}})\mathbf{qc}$.

3.12. Resume the notation of (2.2). We define two equivariant versions of $\mathfrak{X}^\infty$-modules, categories $(\mathfrak{G}, \mathfrak{X}^\infty)\mathbf{qc}$ and $(\mathfrak{G}, \mathfrak{X}^\infty)\mathbf{qqc}$ of $\mathfrak{G}$-equivariant $\mathfrak{X}^\infty$-modules and quasi-$\mathfrak{G}$-equivariant $\mathfrak{X}^\infty$-modules, respectively.

Definition. A $\mathfrak{G}$-equivariant $\mathfrak{X}^\infty$-module is a projective system $(\mathcal{M}^{(r)}, \phi^{(r)})_{r \in \mathbb{N}}$ of $\mathfrak{X}^\infty$-module $(\mathcal{M}^{(r)})_r$ such that

(1) $$(\mathcal{M}^{(r)}, \phi^{(r)}) \in (\mathfrak{G}^{(r)}, \mathfrak{X}^{(r)})\mathbf{qc}$$

and that for each $r \in \mathbb{N}$ the structure morphism $\gamma_{r+1,r} \in \mathbf{Mod}_{\mathcal{O}_{\mathfrak{X}}^{(r+1)}}(\mathcal{M}^{(r+1)}, \mathcal{M}^{(r)})$ of the projective system $(\mathcal{M}^{(r)})_r$ induces an isomorphism

(2) $$\mathcal{O}_{\mathfrak{X}}^{(r)} \otimes_{\mathcal{O}_{\mathfrak{X}}^{(r+1)}} \mathcal{M}^{(r+1)} \to \mathcal{M}^{(r)} \quad \text{in } (\mathfrak{G}^{(r)}, \mathfrak{X}^{(r)})\mathbf{qc},$$

i.e., identifying $\mathfrak{X}^{(r)}$ with $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}^{(r)})$ the composite

$$(\mathfrak{a}^{(r)})^* \left(\mathcal{O}_{\mathfrak{X}}^{(r)} \otimes_{\mathcal{O}_{\mathfrak{X}}^{(r+1)}} \mathcal{M}^{(r+1)}\right) \xrightarrow{(\mathfrak{a}^{(r)})^*(\mathcal{O}_{\mathfrak{X}}^{(r)} \otimes_{\mathcal{O}_{\mathfrak{X}}^{(r+1)}} \gamma_{r+1,r})} (\mathfrak{a}^{(r)})^*(\mathcal{M}^{(r)}) \xrightarrow{\phi^{(r)}} (\mathfrak{p}_{\mathfrak{X}}^{(r)})^*(\mathcal{M}^{(r)})$$

is equal in $\mathbf{Mod}_{(\mathfrak{G}\times\mathfrak{X})^{(r)}}$ to the composite

$$\begin{aligned}
&(\mathfrak{a}^{(r)})^* \left(\mathcal{O}_{\mathfrak{X}}^{(r)} \otimes_{\mathcal{O}_{\mathfrak{X}}^{(r+1)}} \mathcal{M}^{(r+1)}\right) \\
&\xrightarrow{\sim} (\mathfrak{a}^{(r)})^* \mathfrak{F}_{\mathfrak{X}^{(r)}}^* (\mathcal{M}^{(r+1)}) \xrightarrow{\sim} \mathfrak{F}_{(\mathfrak{G}\times\mathfrak{X})^{(r)}}^* (\mathfrak{a}^{(r+1)})^*(\mathcal{M}^{(r+1)}) \\
&\xrightarrow{\mathfrak{F}_{(\mathfrak{G}\times\mathfrak{X})^{(r)}}^*(\phi^{(r+1)})} \mathfrak{F}_{(\mathfrak{G}\times\mathfrak{X})^{(r)}}^* (\mathfrak{p}_{\mathfrak{X}}^{(r+1)})^*(\mathcal{M}^{(r+1)}) \\
&\xrightarrow{\sim} (\mathfrak{p}_{\mathfrak{X}}^{(r)})^* \left(\mathcal{O}_{\mathfrak{X}}^{(r)} \otimes_{\mathcal{O}_{\mathfrak{X}}^{(r+1)}} \mathcal{M}^{(r+1)}\right) \\
&\xrightarrow{(\mathfrak{p}_{\mathfrak{G}\times\mathfrak{X}}^{(r)})^*(\mathcal{O}_{\mathfrak{X}}^{(r)} \otimes_{\mathcal{O}_{\mathfrak{X}}^{(r+1)}} \gamma_{r+1,r})} (\mathfrak{p}_{\mathfrak{X}}^{(r)})^*(\mathcal{M}^{(r)}).
\end{aligned}$$

A quasi-$\mathfrak{G}$-equivariant $\mathfrak{X}^\infty$-module is a projective system $(\mathcal{N}^{(r)}, \phi_r)_{r\in\mathbb{N}}$ of $\mathfrak{X}^\infty$-module $(\mathcal{N}^{(r)})_r$ such that

$$(\mathcal{N}^{(r)}, \phi_r) \in (\mathfrak{G}, \mathfrak{X}^{(r)})\mathbf{qc} \tag{3}$$

and that for each r the structure morphism $\eta_{r+1,r} : \mathcal{N}^{(r+1)} \to \mathcal{N}^{(r)}$ of the projective system $(\mathcal{N}^{(r)})_r$ induces an isomorphism

$$\mathcal{O}_{\mathfrak{X}}^{(r)} \otimes_{\mathcal{O}_{\mathfrak{X}}^{(r+1)}} \mathcal{N}^{(r+1)} \to \mathcal{N}^{(r)} \quad \text{in } (\mathfrak{G}, \mathfrak{X}^{(r)})\mathbf{qc}, \tag{4}$$

i.e., if $\mathfrak{a}_r = \mathfrak{a}^{(r)} \circ (\mathfrak{F}_{\mathfrak{G}}^r \times \mathfrak{X}^{(r)})$ and $\mathfrak{p}_r = \mathfrak{p}_{\mathfrak{X}}^{(r)} \circ (\mathfrak{F}_{\mathfrak{G}}^r \times \mathfrak{X}^{(r)}) : \mathfrak{G} \times \mathfrak{X}^{(r)} \to \mathfrak{X}^{(r)}$, the composite

$$(\mathfrak{a}_r)^* \left(\mathcal{O}_{\mathfrak{X}}^{(r)} \otimes_{\mathcal{O}_{\mathfrak{X}}^{(r+1)}} \mathcal{N}^{(r+1)}\right) \xrightarrow{(\mathfrak{a}_r)^*(\mathcal{O}_{\mathfrak{X}}^{(r)} \otimes_{\mathcal{O}_{\mathfrak{X}}^{(r+1)}} \eta_{r+1,r})} (\mathfrak{a}_r)^*(\mathcal{N}^{(r)}) \xrightarrow{\phi_r} (\mathfrak{p}_r)^*(\mathcal{N}^{(r)})$$

is equal in $\mathbf{Mod}_{\mathfrak{G}\times\mathfrak{X}^{(r)}}$ to the composite

$$\begin{aligned}
(\mathfrak{a}_r)^* \left(\mathcal{O}_{\mathfrak{X}}^{(r)} \otimes_{\mathcal{O}_{\mathfrak{X}}^{(r+1)}} \mathcal{N}^{(r+1)}\right) &\xrightarrow{\sim} (\mathfrak{G} \times \mathfrak{F}_{\mathfrak{X}^{(r)}})^*(\mathfrak{a}_{r+1})^*(\mathcal{N}^{(r+1)}) \\
&\xrightarrow{(\mathfrak{G}\times\mathfrak{F}_{\mathfrak{X}^{(r)}})^*(\phi_{r+1})} (\mathfrak{G} \times \mathfrak{F}_{\mathfrak{X}^{(r)}})^*(\mathfrak{p}_{r+1})^*(\mathcal{N}^{(r+1)}) \\
&\xrightarrow{\sim} (\mathfrak{p}_r)^* \left(\mathcal{O}_{\mathfrak{X}}^{(r)} \otimes_{\mathcal{O}_{\mathfrak{X}}^{(r+1)}} \mathcal{N}^{(r+1)}\right) \\
&\xrightarrow{(\mathfrak{p}_r)^*(\mathcal{O}_{\mathfrak{X}}^{(r)} \otimes_{\mathcal{O}_{\mathfrak{X}}^{(r+1)}} \eta_{r+1,r})} (\mathfrak{p}_r)^*(\mathcal{N}^{(r)}).
\end{aligned}$$

3.13. If $(\mathcal{M}, \phi) \in (\mathfrak{G}, \mathcal{D}_{\mathfrak{X}})\mathbf{qc}$, then by the Cartier-Chase-Smith equivalence [**H87**, 2.2.1] with $\mathcal{M}^{(r)} = \mathcal{M}od_{\mathcal{O}_{\mathfrak{X}}^{(r)}}(\mathcal{O}_{\mathfrak{X}}, \mathcal{O}_{\mathfrak{X}}^{(r)}) \otimes_{\mathcal{D}_{\mathfrak{X},r}} \mathcal{M}$

$$\begin{aligned}(\mathfrak{a}^{(r)})^*(\mathcal{M}^{(r)}) &\simeq \mathcal{M}od_{\mathcal{O}_{\mathfrak{G}\times\mathfrak{X}}^{(r)}}(\mathcal{O}_{\mathfrak{G}\times\mathfrak{X}}, \mathcal{O}_{\mathfrak{G}\times\mathfrak{X}}^{(r)}) \otimes_{\mathcal{D}_{\mathfrak{G}\times\mathfrak{X},r}} (\mathfrak{F}_{\mathfrak{G}\times\mathfrak{X}}^r)^*(\mathfrak{a}^{(r)})^*(\mathcal{M}^{(r)}) \\ &\simeq \mathcal{M}od_{\mathcal{O}_{\mathfrak{G}\times\mathfrak{X}}^{(r)}}(\mathcal{O}_{\mathfrak{G}\times\mathfrak{X}}, \mathcal{O}_{\mathfrak{G}\times\mathfrak{X}}^{(r)}) \otimes_{\mathcal{D}_{\mathfrak{G}\times\mathfrak{X},r}} \mathfrak{a}^*(\mathfrak{F}_{\mathfrak{X}}^r)^*(\mathcal{M}^{(r)}) \\ &\simeq \mathcal{M}od_{\mathcal{O}_{\mathfrak{G}\times\mathfrak{X}}^{(r)}}(\mathcal{O}_{\mathfrak{G}\times\mathfrak{X}}, \mathcal{O}_{\mathfrak{G}\times\mathfrak{X}}^{(r)}) \otimes_{\mathcal{D}_{\mathfrak{G}\times\mathfrak{X},r}} \mathfrak{a}^*\mathcal{M},\end{aligned}$$

and likewise

$$(\mathfrak{p}_{\mathfrak{X}}^{(r)})^*(\mathcal{M}^{(r)}) \simeq \mathcal{M}od_{\mathcal{O}_{\mathfrak{G}\times\mathfrak{X}}^{(r)}}(\mathcal{O}_{\mathfrak{G}\times\mathfrak{X}}, \mathcal{O}_{\mathfrak{G}\times\mathfrak{X}}^{(r)}) \otimes_{\mathcal{D}_{\mathfrak{G}\times\mathfrak{X},r}} \mathfrak{p}_{\mathfrak{X}}^*\mathcal{M}.$$

Under these identifications define

$$\phi^{(r)} \in \mathbf{Mod}_{\mathcal{O}_{\mathfrak{G}\times\mathfrak{X}}^{(r)}}((\mathfrak{a}^{(r)})^*(\mathcal{M}^{(r)}), (\mathfrak{p}_{\mathfrak{X}}^{(r)})^*(\mathcal{M}^{(r)}))^\times$$

to be $\mathcal{M}od_{\mathcal{O}_{\mathfrak{G}\times\mathfrak{X}}^{(r)}}(\mathcal{O}_{\mathfrak{G}\times\mathfrak{X}}, \mathcal{O}_{\mathfrak{G}\times\mathfrak{X}}^{(r)}) \otimes_{\mathcal{D}_{\mathfrak{G}\times\mathfrak{X},r}} \phi$. Then $(\mathcal{M}^{(r)}, \phi^{(r)})_r$ forms a $\mathfrak{G}$-equivariant $\mathfrak{X}^\infty$-module. To check the condition (3.12.2), recall that

$$\begin{aligned}(1)\quad &\mathcal{O}_{\mathfrak{G}\times\mathfrak{X}}^{(r)} \otimes_{\mathcal{O}_{\mathfrak{G}\times\mathfrak{X}}^{(r+1)}} \mathcal{M}od_{\mathcal{O}_{\mathfrak{G}\times\mathfrak{X}}^{(r+1)}}(\mathcal{O}_{\mathfrak{G}\times\mathfrak{X}}, \mathcal{O}_{\mathfrak{G}\times\mathfrak{X}}^{(r+1)}) \otimes_{\mathcal{D}_{\mathfrak{G}\times\mathfrak{X},r+1}} \mathcal{L} \\ &\qquad\simeq \mathcal{M}od_{\mathcal{O}_{\mathfrak{G}\times\mathfrak{X}}^{(r)}}(\mathcal{O}_{\mathfrak{G}\times\mathfrak{X}}, \mathcal{O}_{\mathfrak{G}\times\mathfrak{X}}^{(r)}) \otimes_{\mathcal{D}_{\mathfrak{G}\times\mathfrak{X},r}} \mathcal{L} \quad \forall \mathcal{L} \in \mathcal{D}_{\mathfrak{G}\times\mathfrak{X}}\mathbf{qc}\end{aligned}$$

as applying $\mathcal{O}_{\mathfrak{G}\times\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{G}\times\mathfrak{X}}^{(r)}} ?$ on both sides yields an isomorphism and as $\mathcal{O}_{\mathfrak{G}\times\mathfrak{X}}$ is faithfully flat over $\mathcal{O}_{\mathfrak{G}\times\mathfrak{X}}^{(r)}$. One then obtains a commutative diagram

$$\begin{array}{ccc}
(\mathfrak{a}^{(r)})^*(\mathcal{O}_{\mathfrak{X}}^{(r)} \otimes_{\mathcal{O}_{\mathfrak{X}}^{(r+1)}} \mathcal{M}^{(r+1)}) & \xrightarrow{\sim} & (\mathfrak{a}^{(r)})^*(\mathcal{M}^{(r)}) \\
\wr\downarrow & & \downarrow\wr \\
\mathfrak{F}_{(\mathfrak{G}\times\mathfrak{X})^{(r)}}^*(\mathfrak{a}^{(r+1)})^*(\mathcal{M}^{(r+1)}) & \xrightarrow{\sim} & \mathcal{M}od_{\mathcal{O}_{\mathfrak{G}\times\mathfrak{X}}^{(r)}}(\mathcal{O}_{\mathfrak{G}\times\mathfrak{X}}, \mathcal{O}_{\mathfrak{G}\times\mathfrak{X}}^{(r)}) \otimes_{\mathcal{D}_{\mathfrak{G}\times\mathfrak{X},r}} \mathfrak{a}^*\mathcal{M} \\
\mathfrak{F}_{(\mathfrak{G}\times\mathfrak{X})^{(r)}}^*(\phi^{(r+1)})\downarrow & & \downarrow \mathcal{M}od_{\mathcal{O}_{\mathfrak{G}\times\mathfrak{X}}^{(r)}}(\mathcal{O}_{\mathfrak{G}\times\mathfrak{X}}, \mathcal{O}_{\mathfrak{G}\times\mathfrak{X}}^{(r)}) \otimes_{\mathcal{D}_{\mathfrak{G}\times\mathfrak{X},r}} \phi \\
\mathfrak{F}_{(\mathfrak{G}\times\mathfrak{X})^{(r)}}^*(\mathfrak{p}_{\mathfrak{X}}^{(r+1)})^*(\mathcal{M}^{(r+1)}) & \xrightarrow{\sim} & \mathcal{M}od_{\mathcal{O}_{\mathfrak{G}\times\mathfrak{X}}^{(r)}}(\mathcal{O}_{\mathfrak{G}\times\mathfrak{X}}, \mathcal{O}_{\mathfrak{G}\times\mathfrak{X}}^{(r)}) \otimes_{\mathcal{D}_{\mathfrak{G}\times\mathfrak{X},r}} \mathfrak{p}_{\mathfrak{X}}^*\mathcal{M} \\
\wr\downarrow & & \downarrow\wr \\
(\mathfrak{p}_{\mathfrak{X}}^{(r)})^*(\mathcal{O}_{\mathfrak{X}}^{(r)} \otimes_{\mathcal{O}_{\mathfrak{X}}^{(r+1)}} \mathcal{M}^{(r+1)}) & \xrightarrow[\sim]{} & (\mathfrak{p}_{\mathfrak{X}}^{(r)})^*(\mathcal{M}^{(r)}).
\end{array}$$

If $(\mathcal{N}, \psi) \in (\mathfrak{G}, \mathcal{D}_{\mathfrak{X}})\mathbf{qqc}$, then

$$\begin{aligned}&\mathfrak{a}_r^*(\mathcal{N}^{(r)}) \\ &\simeq \mathcal{M}od_{\mathcal{O}_{\mathfrak{G}}\boxtimes\mathcal{O}_{\mathfrak{X}}^{(r)}}\left(\mathcal{O}_{\mathfrak{G}} \boxtimes \mathcal{O}_{\mathfrak{X}}, \mathcal{O}_{\mathfrak{G}} \boxtimes \mathcal{O}_{\mathfrak{X}}^{(r)}\right) \otimes_{\mathcal{O}_{\mathfrak{G}}\boxtimes\mathcal{D}_{\mathfrak{X},r}} (\mathfrak{G} \times \mathfrak{F}_{\mathfrak{X}}^r)^*\mathfrak{a}_r^*(\mathcal{N}^{(r)}) \\ &\simeq \mathcal{M}od_{\mathcal{O}_{\mathfrak{G}}\boxtimes\mathcal{O}_{\mathfrak{X}}^{(r)}}\left(\mathcal{O}_{\mathfrak{G}} \boxtimes \mathcal{O}_{\mathfrak{X}}, \mathcal{O}_{\mathfrak{G}} \boxtimes \mathcal{O}_{\mathfrak{X}}^{(r)}\right) \otimes_{\mathcal{O}_{\mathfrak{G}}\boxtimes\mathcal{D}_{\mathfrak{X},r}} \mathfrak{a}^*\mathcal{N} \\ &\simeq \left\{\mathcal{O}_{\mathfrak{G}} \boxtimes \mathcal{M}od_{\mathcal{O}_{\mathfrak{X}}^{(r)}}\left(\mathcal{O}_{\mathfrak{X}}, \mathcal{O}_{\mathfrak{X}}^{(r)}\right)\right\} \otimes_{\mathcal{O}_{\mathfrak{G}}\boxtimes\mathcal{D}_{\mathfrak{X},r}} \mathfrak{a}^*\mathcal{N},\end{aligned}$$

and likewise

$$\mathfrak{p}_r^*(\mathcal{N}^{(r)}) \simeq \{\mathcal{O}_{\mathfrak{G}} \boxtimes \mathcal{M}od_{\mathcal{O}_{\mathfrak{X}}^{(r)}}(\mathcal{O}_{\mathfrak{X}}, \mathcal{O}_{\mathfrak{X}}^{(r)})\} \otimes_{\mathcal{O}_{\mathfrak{G}} \boxtimes \mathcal{D}_{\mathfrak{X},r}} \mathfrak{p}_{\mathfrak{X}}^* \mathcal{N}.$$

Hence one can define $\psi_r \in \mathbf{Mod}_{\mathcal{O}_{\mathfrak{G}} \boxtimes \mathcal{O}_{\mathfrak{X}}^{(r)}}(\mathfrak{a}_r^*(\mathcal{N}^{(r)}), \mathfrak{p}_r^*(\mathcal{N}^{(r)}))^{\times}$ from ψ. Then $(\mathcal{N}^{(r)}, \psi_r)$ forms a quasi-$\mathfrak{G}$-equivariant $\mathfrak{X}^{(r)}$-module. Hence [**H87**, 2.2.4] is now refined to:

Theorem. *Let $\mathcal{C}(\mathfrak{X}^\infty)$ be the category of $\mathfrak{X}^\infty$-modules. The equivalence of categories $\mathcal{D}_{\mathfrak{X}}\mathbf{qc} \to \mathcal{C}(\mathfrak{X}^\infty)$ via $\mathcal{M} \mapsto (\mathcal{M}^{(r)})_r$ induces upon restriction two equivalences $(\mathfrak{G}, \mathcal{D}_{\mathfrak{X}})\mathbf{qc} \to (\mathfrak{G}, \mathfrak{X}^\infty)\mathbf{qc}$ and $(\mathfrak{G}, \mathcal{D}_{\mathfrak{X}})\mathbf{qqc} \to (\mathfrak{G}, \mathfrak{X}^\infty)\mathbf{qqc}$.*

3.14. Assume $\mathfrak{G}$ is connected and reduced. Let $\mathfrak{H}$ be a closed subgroup scheme of $\mathfrak{G}$ so that the quotient $\mathfrak{G}/\mathfrak{H}$ is a $\mathfrak{G}$-variety via the multiplication from the left. For $M \in \mathfrak{H}\mathbf{Mod}$ let $\mathcal{L}_{\mathfrak{G}/\mathfrak{H}}(M)$ be the $\mathcal{O}_{\mathfrak{G}/\mathfrak{H}}$-module associated to M [**J**, I.5]. It is well-known that

(1) $\mathcal{L}_{\mathfrak{G}/\mathfrak{H}}$ defines a functor $\mathfrak{H}\mathbf{Mod} \to (\mathfrak{G}, \mathfrak{G}/\mathfrak{H})\mathbf{qc}$, that is an equivalence with quasi-inverse $\mathcal{M} \mapsto \mathcal{M}(e) = \mathfrak{k} \otimes_{\mathcal{O}_{\mathfrak{G}/\mathfrak{H},e}} \mathcal{M}_e$.

Let $(\mathrm{Dist}(\mathfrak{G}), \mathfrak{H})\mathbf{Mod}$ be the category of Harish-Chandra $(\mathrm{Dist}(\mathfrak{G}), \mathfrak{H})$-modules [**H87**, 4.3.8], that are the same as $\mathfrak{k}$-linear spaces equipped with a structure of compatible $\mathfrak{G}_r\mathfrak{H}$-modules, $r \in \mathbb{N}$. We can generalize [**H87**, 4.3.7/9] as in characteristic 0 [**Ka**, Th. 4.10.2] to:

Theorem. *The functor $\mathcal{L}_{\mathfrak{G}/\mathfrak{H}}$ induces upon restriction an equivalence of categories*

$$(\mathrm{Dist}(\mathfrak{G}), \mathfrak{H})\mathbf{Mod} \to (\mathfrak{G}, \mathcal{D}_{\mathfrak{G}/\mathfrak{H}})\mathbf{qqc}.$$

In particular, $\quad \mathcal{L}_{\mathfrak{G}/\mathfrak{H}}(\mathrm{Dist}(\mathfrak{G}) \otimes_{\mathrm{Dist}(\mathfrak{H})} \mathfrak{k}) \simeq \mathcal{D}_{\mathfrak{G}/\mathfrak{H}} \quad$ *in* $(\mathfrak{G}, \mathcal{D}_{\mathfrak{G}/\mathfrak{H}})\mathbf{qqc}$.

Proof. Put $\mathfrak{X} = \mathfrak{G}/\mathfrak{H}$, $\mathcal{L} = \mathcal{L}_{\mathfrak{G}/\mathfrak{H}}$, and denote the $\mathfrak{G}$-action on $\mathfrak{X}$ by $\mathfrak{a}$. Let $\mathfrak{a}_r = \mathfrak{a}^{(r)} \circ (\mathfrak{F}_{\mathfrak{G}}^r \times \mathfrak{X}^{(r)})$ and $\mathfrak{p}_r = \mathfrak{p}_{\mathfrak{X}}^{(r)} \circ (\mathfrak{F}_{\mathfrak{G}}^r \times \mathfrak{X}^{(r)}) : \mathfrak{G} \times \mathfrak{X}^{(r)} \to \mathfrak{X}^{(r)}$. We will identify $\mathfrak{X}^{(r)}$ with $\mathfrak{G}/\mathfrak{G}_r\mathfrak{H}$ via the commutative diagram

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{\mathfrak{F}_{\mathfrak{X}}^r} & \mathfrak{X}^{(r)} \\ \| & & \uparrow \wr \\ \mathfrak{G}/\mathfrak{H} & \xrightarrow[\mathfrak{q}_r]{} & \mathfrak{G}/\mathfrak{G}_r\mathfrak{H}, \end{array}$$

where $\mathfrak{q}_r : \mathfrak{G}/\mathfrak{H} \to \mathfrak{G}/\mathfrak{G}_r\mathfrak{H}$ is the quotient morphism.

If $M \in (\mathrm{Dist}(\mathfrak{G}), \mathfrak{H})\mathbf{Mod}$, let $\phi_r : \mathfrak{a}_r^* \mathcal{L}_{\mathfrak{G}/\mathfrak{G}_r\mathfrak{H}}(M) \to \mathfrak{p}_r^* \mathcal{L}_{\mathfrak{G}/\mathfrak{G}_r\mathfrak{H}}(M)$ be the structure morphism of $\mathcal{L}_{\mathfrak{G}/\mathfrak{G}_r\mathfrak{H}}(M)$ in $(\mathfrak{G}, \mathfrak{X}^{(r)})\mathbf{qc}$. Then $(\mathcal{L}_{\mathfrak{G}/\mathfrak{G}_r\mathfrak{H}}(M), \phi_r)_r \in (\mathfrak{G}, \mathfrak{X}^\infty)\mathbf{qqc}$.

On the other hand, if $\mathcal{M} \in (\mathfrak{G}, \mathcal{D}_{\mathfrak{X}})\mathbf{qqc}$ with a structure morphism ϕ, then $(\mathcal{M}^{(r)}, \phi_r)_r \in (\mathfrak{G}, \mathfrak{X}^\infty)\mathbf{qc}$ by (3.13). Hence for each $r \in \mathbb{N}$ one has in $\mathbf{Mod}_{\mathfrak{k}}$

$$\mathfrak{k} \otimes_{\mathcal{O}_{\mathfrak{X},e}} \mathcal{M}_e \simeq \mathfrak{k} \otimes_{\mathcal{O}_{\mathfrak{X},e}} \mathcal{O}_{\mathfrak{X},e} \otimes_{\mathcal{O}^{(r)}_{\mathfrak{X},e}} (\mathcal{M}^{(r)})_e \simeq \mathfrak{k} \otimes_{\mathcal{O}^{(r)}_{\mathfrak{X},e}} (\mathcal{M}^{(r)})_e$$
$$\simeq \mathfrak{k} \otimes_{\mathcal{O}^{(r)}_{\mathfrak{X},e}} \mathcal{O}^{(r)}_{\mathfrak{X},e} \otimes_{\mathcal{O}^{(r+1)}_{\mathfrak{X},e}} (\mathcal{M}^{(r+1)})_e.$$

We may regard $(\mathcal{M}^{(r)}, \phi_r)_r \in (\mathfrak{G}, \mathfrak{G}/\mathfrak{G}_r\mathfrak{H})\mathbf{qc}$. If $\bar{\varepsilon}_r = \mathfrak{q}_r \circ \varepsilon_{\mathfrak{G}} : \mathfrak{e} \to \mathfrak{G}/\mathfrak{G}_r\mathfrak{H}$ and if we let $\mathfrak{G}_r\mathfrak{H}$ act on $\mathfrak{e}$ trivially, then $(\bar{\varepsilon}_r)^*(\mathcal{M}^{(r)}) \simeq \mathfrak{k} \otimes_{\mathcal{O}^{(r)}_{\mathfrak{X},e}} (\mathcal{M}^{(r)})_e$ admits a structure of $\mathfrak{G}_r\mathfrak{H}$-module as $\bar{\varepsilon}_r$ is $\mathfrak{G}_r\mathfrak{H}$-equivariant. Moreover, $\bar{\varepsilon}_{r+1} = \mathfrak{q}_{r,r+1} \circ \bar{\varepsilon}_r$ with $\mathfrak{q}_{r,r+1} : \mathfrak{G}/\mathfrak{G}_r\mathfrak{H} \to \mathfrak{G}/\mathfrak{G}_{r+1}\mathfrak{H}$ the quotient morphism, hence

$$(\bar{\varepsilon}_{r+1})^*(\mathcal{M}^{(r+1)}) \simeq (\bar{\varepsilon}_r)^*(\mathcal{M}^{(r)}) \quad \text{in } \mathfrak{G}_r\mathfrak{H}\mathbf{Mod}.$$

Then $\mathcal{M}(e) = \mathfrak{k} \otimes_{\mathcal{O}_{\mathfrak{X},e}} \mathcal{M}_e$ comes equipped with a structure of $\mathfrak{G}_r\mathfrak{H}$-module compatibly with respect to $r \in \mathbb{N}$, and the first assertion follows.

To see the second assertion, recall that

$$\mathrm{Dist}(\mathfrak{G}) \otimes_{\mathrm{Dist}(\mathfrak{H})} \mathfrak{k} \simeq \varinjlim_r \mathrm{Dist}(\mathfrak{G}_r) \otimes_{\mathrm{Dist}(\mathfrak{H}_r)} \mathfrak{k} \quad \text{in } \mathrm{Dist}(\mathfrak{G})\mathbf{Mod}$$

and that (cf. [J, I.8.15, 8.20])

$$\mathrm{Dist}(\mathfrak{G}_r) \otimes_{\mathrm{Dist}(\mathfrak{H}_r)} \mathfrak{k} \simeq (\mathrm{ind}_{\mathfrak{H}}^{\mathfrak{G}_r\mathfrak{H}} \mathfrak{k})^* \quad \text{in } \mathfrak{G}_r\mathfrak{H}\mathbf{Mod}. \tag{2}$$

We have isomorphisms in $\mathbf{Mod}_{\mathfrak{X}^{(r)}}$

$$\begin{aligned} \mathcal{D}_{\mathfrak{X},r} &\simeq \mathcal{M}od_{\mathcal{O}^{(r)}_{\mathfrak{X}}}(\mathcal{O}_{\mathfrak{X}}, \mathcal{O}_{\mathfrak{X}}) \simeq \mathcal{M}od_{\mathfrak{X}}((\mathfrak{F}^r_{\mathfrak{X}})^*(\mathfrak{F}^r_{\mathfrak{X}})_*\mathcal{O}_{\mathfrak{X}}, \mathcal{O}_{\mathfrak{X}}) \\ &\simeq \mathcal{M}od_{\mathfrak{X}}(\mathfrak{q}_r^*(\mathfrak{q}_r)_*\mathcal{O}_{\mathfrak{X}}, \mathcal{O}_{\mathfrak{X}}) \simeq \mathcal{M}od_{\mathfrak{X}}(\mathcal{L}(\mathrm{ind}_{\mathfrak{H}}^{\mathfrak{G}_r\mathfrak{H}} \mathfrak{k}), \mathcal{O}_{\mathfrak{X}}) \\ &\simeq \mathcal{L}((\mathrm{ind}_{\mathfrak{H}}^{\mathfrak{G}_r\mathfrak{H}} \mathfrak{k})^*). \end{aligned} \tag{3}$$

It is enough to show that the composite isomorphism of (3) belongs to $\mathcal{D}_{\mathfrak{X},r}\mathbf{Mod}$ and also to $(\mathfrak{G}, \mathfrak{X})\mathbf{qc}$. The questions being local, let $\mathfrak{U}$ be an affine open of $\mathfrak{G}/\mathfrak{G}_r\mathfrak{H}$, $\mathfrak{V} = \mathfrak{q}_r^{-1}\mathfrak{U}$ and $\mathfrak{U}' = \pi_r^{-1}\mathfrak{U}$ with $\pi_r : \mathfrak{G} \to \mathfrak{G}/\mathfrak{G}_r\mathfrak{H}$ the quotient morphism. Both $\mathfrak{U}'$ and $\mathfrak{V}$ remain affine. Recall an isomorphism (cf. [J, I.5.18.5])

$$\Gamma(\mathfrak{U}, (\mathfrak{q}_r)_*\mathcal{O}_{\mathfrak{X}}) = \mathfrak{k}[\mathfrak{U}']^{\mathfrak{H}} \to \mathbf{Sch}_{\mathfrak{k}}(\mathfrak{U}', \mathrm{ind}_{\mathfrak{H}}^{\mathfrak{G}_r\mathfrak{H}}(\mathfrak{k}))^{\mathfrak{G}_r\mathfrak{H}}$$
$$= \Gamma(\mathfrak{U}, \mathcal{L}_{\mathfrak{G}/\mathfrak{G}_r\mathfrak{H}}(\mathrm{ind}_{\mathfrak{H}}^{\mathfrak{G}_r\mathfrak{H}}(\mathfrak{k})))$$

via $a \mapsto \hat{a}$ such that

$$\hat{a}(x) = a(x?) \in \mathbf{Sch}_{\mathfrak{k}}(\mathfrak{G}_r\mathfrak{H}, \mathfrak{k})^{\mathfrak{H}} = \mathrm{ind}_{\mathfrak{H}}^{\mathfrak{G}_r\mathfrak{H}}(\mathfrak{k}), \quad x \in \mathfrak{U}', \tag{4}$$

and another (cf. [**K95**, 1.8]) for each $\mathfrak{G}_r\mathfrak{H}$-module Q

$$\begin{aligned} \Gamma(\mathfrak{V}, \mathfrak{q}_r^*\mathcal{L}_{\mathfrak{G}/\mathfrak{G}_r\mathfrak{H}}(Q)) &= \mathfrak{k}[\mathfrak{U}']^{\mathfrak{H}} \otimes_{\mathfrak{k}[\mathfrak{U}']^{\mathfrak{G}_r\mathfrak{H}}} \mathbf{Sch}_{\mathfrak{k}}(\mathfrak{U}', Q)^{\mathfrak{G}_r\mathfrak{H}} \\ &\to \mathbf{Sch}_{\mathfrak{k}}(\mathfrak{U}', Q)^{\mathfrak{H}} = \Gamma(\mathfrak{V}, \mathcal{L}(Q)) \end{aligned} \tag{5}$$

via $a \otimes f \mapsto af$. Hence on $\mathfrak{V}$ the composite (3) sends

$$\theta \in \mathbf{Sch}_{\mathfrak{k}}(\mathfrak{U}', \mathrm{ind}_{\mathfrak{H}}^{\mathfrak{G}_r\mathfrak{H}}(\mathfrak{k})^*)^{\mathfrak{H}} = \mathcal{L}((\mathrm{ind}_{\mathfrak{H}}^{\mathfrak{G}_r\mathfrak{H}}\, \mathfrak{k})^*)(\mathfrak{V})$$

to $\tilde{\theta} = \langle \theta, \hat{?} \rangle \in D_r(\mathfrak{k}[\mathfrak{U}']^{\mathfrak{H}})$ with $\langle \theta, \hat{a} \rangle \in \mathfrak{k}[\mathfrak{U}']^{\mathfrak{H}}$, $a \in \mathfrak{k}[\mathfrak{U}']^{\mathfrak{H}}$, such that

$$\langle \theta, \hat{a} \rangle(x) = \langle \theta(x), \hat{a}(x) \rangle, \quad x \in \mathfrak{U}'. \tag{6}$$

The $\mathcal{D}_{\mathfrak{X},r}$-module structure on $\mathcal{L}((\mathrm{ind}_{\mathfrak{H}}^{\mathfrak{G}_r\mathfrak{H}}\, \mathfrak{k})^*)$ is given on $\mathfrak{V}$ by

$$a \otimes f \mapsto \delta(a) \otimes f, \quad a \in \mathfrak{k}[\mathfrak{U}']^{\mathfrak{H}}, f \in \mathbf{Sch}_{\mathfrak{k}}(\mathfrak{U}', (\mathrm{ind}_{\mathfrak{H}}^{\mathfrak{G}_r\mathfrak{H}}\, \mathfrak{k})^*)^{\mathfrak{G}_r\mathfrak{H}}, \delta \in D_r(\mathfrak{k}[\mathfrak{U}']^{\mathfrak{H}})$$

under the identification (5), hence by $af \mapsto \delta(a)f$. Then we must show

$$(\delta(a)f)^{\sim} = \delta \circ \widetilde{af} \quad \text{in } D_r(\mathfrak{k}[\mathfrak{U}']^{\mathfrak{H}}). \tag{7}$$

If $b \in \mathfrak{k}[\mathfrak{U}']^{\mathfrak{H}}$ and $x \in \mathfrak{U}'$, then

$$\begin{aligned} \{(\delta(a)f)^{\sim}(b)\}(x) &= \langle \delta(a)f, \hat{b} \rangle(x) = \langle (\delta(a)f)(x), \hat{b}(x) \rangle \\ &= \langle (\delta(a))(x)f(x), b(x?) \rangle = (\delta(a))(x)\langle f(x), b(x?) \rangle \end{aligned}$$

while

$$\begin{aligned} \{(\delta \circ \widetilde{af})(b)\}(x) &= \{\delta(a\langle f, \hat{b} \rangle)\}(x) \\ &= \{\delta(a)\langle f, \hat{b} \rangle\}(x) \quad \text{as } \langle f, \hat{b} \rangle \in \mathfrak{k}[\mathfrak{U}']^{\mathfrak{G}_r\mathfrak{H}} \simeq (\mathfrak{k}[\mathfrak{U}']^{\mathfrak{H}})^{(r)} \\ &= (\delta(a))(x)\langle f(x), \hat{b}(x) \rangle, \end{aligned}$$

hence (7), and the composite (3) belongs to $\mathcal{D}_{\mathfrak{X},r}\mathbf{Mod}$.

To see next that the composite belongs to $(\mathfrak{G}, \mathfrak{X})\mathbf{qc}$, it suffices to show by (3.7) that the composite is $\mathrm{Dist}(\mathfrak{G}_s)^{\mathrm{op}}$-equivariant on $\mathfrak{V}$ for each $s \in \mathbb{N}$, i.e., the composite yields

$$\mathcal{L}((\mathrm{ind}_{\mathfrak{H}}^{\mathfrak{G}_r\mathfrak{H}}\, \mathfrak{k})^*)(\mathfrak{V}) \simeq \mathcal{D}_{\mathfrak{X},r}(\mathfrak{V}) \quad \text{in } \mathfrak{G}_s^{\mathrm{op}}\mathbf{Mod}. \tag{8}$$

The assertion will follow from:

3.15.

Lemma. *Let* $M \in \mathfrak{H}\mathbf{Mod}$.

(i) $\mathcal{L}_{\mathfrak{G}/\mathfrak{G}_r\mathfrak{H}}(\mathrm{ind}_{\mathfrak{H}}^{\mathfrak{G}_r\mathfrak{H}}\, M) \simeq (\mathfrak{q}_r)_*\mathcal{L}_{\mathfrak{G}/\mathfrak{H}}(M)$ *in* $(\mathfrak{G}, \mathfrak{G}/\mathfrak{G}_r\mathfrak{H})\mathbf{qc}$.

(ii) $\mathcal{L}_{\mathfrak{G}/\mathfrak{H}}(\mathrm{ind}_{\mathfrak{H}}^{\mathfrak{G}_r\mathfrak{H}} M) \simeq \mathfrak{q}_r^*(\mathfrak{q}_r)_*\mathcal{L}_{\mathfrak{G}/\mathfrak{H}}(M)$ *in* $(\mathfrak{G}, \mathfrak{G}/\mathfrak{H})\mathbf{qc}$.

Proof. We carry over the notations of the previous section. By taking direct limit we may assume M is finite dimensional. Then the question being local, we have by (3.7) only to show for all $s \in \mathbb{N}$

(1) $$\Gamma(\mathfrak{U}, \mathcal{L}_{\mathfrak{G}/\mathfrak{G}_r\mathfrak{H}}(\mathrm{ind}_{\mathfrak{H}}^{\mathfrak{G}_r\mathfrak{H}}(M))) \simeq \Gamma(\mathfrak{U}, (\mathfrak{q}_r)_*\mathcal{L}(M)) \quad \text{in } \mathfrak{G}_s^{\mathrm{op}}\mathbf{Mod}$$

and

(2) $$\Gamma(\mathfrak{V}, \mathcal{L}(\mathrm{ind}_{\mathfrak{H}}^{\mathfrak{G}_r\mathfrak{H}}(M))) \simeq \Gamma(\mathfrak{V}, \mathfrak{q}_r^*(\mathfrak{q}_r)_*\mathcal{L}(M)) \quad \text{in } \mathfrak{G}_s^{\mathrm{op}}\mathbf{Mod}.$$

But in (1) we have an isomorphism in $\mathfrak{G}_s^{\mathrm{op}}\mathbf{Mod}$

$$\begin{aligned}\Gamma(\mathfrak{U}, \mathcal{L}_{\mathfrak{G}/\mathfrak{G}_r\mathfrak{H}}(\mathrm{ind}_{\mathfrak{H}}^{\mathfrak{G}_r\mathfrak{H}}(M))) &= \mathbf{Sch}_{\mathfrak{k}}(\mathfrak{U}', \mathrm{ind}_{\mathfrak{H}}^{\mathfrak{G}_r\mathfrak{H}}(M))^{\mathfrak{G}_r\mathfrak{H}} \\ &\to \mathbf{Sch}_{\mathfrak{k}}(\mathfrak{U}', M)^{\mathfrak{H}} = \Gamma(\mathfrak{U}, (\mathfrak{q}_r)_*\mathcal{L}(M))\end{aligned}$$

via $f \mapsto \mathrm{ev}_M \circ f$ with $\mathrm{ev}_M : \mathrm{ind}_{\mathfrak{H}}^{\mathfrak{G}_r\mathfrak{H}}(M) \to M$ the evaluation at e. Likewise (2) using (1). □

§4.

In this section $\mathfrak{X}$ will denote a flag variety G/B with G a simply connected semisimple $\mathfrak{k}$-group and B a Borel subgroup of G. Let T be a maximal torus of B, $W = \mathrm{N}_G(T)/T$ the Weyl group, B^+ the Borel subgroup of G opposite to B, and U^+ the unipotent radical of B^+. For all other unexplained standard notations we refer to [J].

4.1. Fix $w \in W$. In $\mathfrak{X}$ let $\mathfrak{X}_w = U^+wB/B$, $\overline{\mathfrak{X}_w}$ the closure of $\mathfrak{X}_w$, $\partial\mathfrak{X}_w = \overline{\mathfrak{X}_w} \setminus \mathfrak{X}_w$, $\mathfrak{N}_w = wU^+B/B$, $\mathfrak{i}_w : \mathfrak{X}_w \hookrightarrow \mathfrak{N}_w$, and $\mathfrak{j}_w : \mathfrak{N}_w \hookrightarrow \mathfrak{X}$. Then $\mathfrak{N}_w$ is an affne open of $\mathfrak{X}$, and $\mathfrak{X}_w = \overline{\mathfrak{X}_w} \cap \mathfrak{N}_w$ of codimension $\ell(w)$, the length of w, in $\mathfrak{N}_w$. If ${}^wU^+ = wU^+w^{-1}$ and $U_w^+ = {}^wU^+ \cap U^+$, there is a commutative diagram of $\mathfrak{k}$-varieties

$$\begin{array}{ccc} {}^wU^+ & \longrightarrow & \mathfrak{N}_w \\ \uparrow & & \uparrow \\ U_w^+ & \longrightarrow & \mathfrak{X}_w \end{array}$$

with the vertical arrows being inclusions and the horizontal ones ivertible given by $x \mapsto xwB$. Let $\mathcal{I}_w$ be the ideal sheaf of $\mathfrak{X}_w$ in $\mathfrak{N}_w$. We have as in characteristic 0:

Proposition. *Let* $\ell = \ell(w)$ *and* $B_w^+ = (wB^+w^{-1}) \cap B^+$.

(i) *On* $\mathrm{D}^b(\mathcal{D}_{\mathfrak{X}}\mathbf{qc})$ $\mathrm{R}\,\Gamma_{\overline{\mathfrak{X}_w}/\partial\mathfrak{X}_w} \simeq (\mathfrak{j}_w)_0 \circ \mathrm{R}\,\Gamma_{\mathfrak{X}_w} \circ \mathfrak{j}_w^{-1} \simeq \int_{\mathfrak{j}_w\circ\mathfrak{i}_w} \mathrm{L}(\mathfrak{j}_w \circ \mathfrak{i}_w)^0[-\ell]$. *In particular, if* $\mathcal{M}^{\cdot} \in \mathrm{D}^b((B^+, \mathcal{D}_{\mathfrak{X}})\mathbf{qc})$, $\mathrm{R}\,\Gamma_{\overline{\mathfrak{X}_w}/\partial\mathfrak{X}_w}(\mathcal{M}^{\cdot}) \simeq \int_{\mathfrak{j}_w\circ\mathfrak{i}_w} \mathrm{L}(\mathfrak{j}_w \circ \mathfrak{i}_w)^0(\mathcal{M}^{\cdot})[-\ell]$ *in* $\mathrm{D}^b((B^+, \mathcal{D}_{\mathfrak{X}})\mathbf{qc})$.

(ii) *On* $\mathrm{D}^b(\mathbf{Ab}_{\mathfrak{X}})$ $\mathrm{R}\,\Gamma_{\mathfrak{d}\mathfrak{X}_w} \circ \mathrm{R}\,\Gamma_{\overline{\mathfrak{X}_w}/\mathfrak{d}\mathfrak{X}_w} = 0.$

(iii) *In* $(B^+, \mathcal{D}_{\mathfrak{X}})\mathbf{qc}$ $\forall s \in \mathbb{N}$,

$$\mathcal{H}^s_{\overline{\mathfrak{X}_w}/\mathfrak{d}\mathfrak{X}_w}(\mathcal{O}_{\mathfrak{X}}) \simeq \begin{cases} \int^0_{\mathrm{j}_w \circ \mathrm{i}_w} \mathcal{O}_{\mathfrak{X}_w} & \text{if } s = \ell \\ 0 & \text{otherwise.} \end{cases}$$

(iv) (*cf.* [**Bø**, Prop. 4.7].) *In* $(B^+_w, \mathcal{D}_{\mathfrak{X}})\mathbf{qc}$

$$\mathcal{H}^\ell_{\overline{\mathfrak{X}_w}/\mathfrak{d}\mathfrak{X}_w}(\mathcal{O}_{\mathfrak{X}})\mid_{\mathfrak{N}_w} \simeq \mathcal{H}^\ell_{\mathfrak{X}_w}(\mathcal{O}_{\mathfrak{N}_w}) \simeq \mathcal{B}_{\mathfrak{X}_w|\mathfrak{N}_w},$$

and in $(B^+_w, D(\mathfrak{N}_w))\mathbf{Mod}$

$$\begin{aligned}\mathcal{B}_{\mathfrak{X}_w|\mathfrak{N}_w}(\mathfrak{N}_w) &\simeq \{D(\mathfrak{N}_w)/(D(\mathfrak{N}_w)\,\mathrm{Dist}^+(U^+_w) + D(\mathfrak{N}_w)\mathcal{I}_w(\mathfrak{N}_w))\} \otimes \mathfrak{k}_{w\cdot 0} \\ &\simeq \mathfrak{k}[U^+_w] \otimes \mathrm{Dist}(U^-_w) \otimes \mathfrak{k}_{w\cdot 0}.\end{aligned}$$

In particular,

$$\mathrm{ch}\ \mathrm{H}^\ell_{\overline{\mathfrak{X}_w}/\mathfrak{d}\mathfrak{X}_w}(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) = e(w\cdot 0) \prod_{\alpha \in R^+} \frac{1}{1 - e(-\alpha)},$$

where ch *is the formal character of the* T*-module in question obtained from the* T^{op}*-module by inversion on* T*.*

(v) (*cf.* [**Bø**, Prop. 4.7].) *If* $\mathcal{L}(w) = \mathcal{D}_{\mathfrak{X}}\{\mathrm{H}^\ell_{\overline{\mathfrak{X}_w}/\mathfrak{d}\mathfrak{X}_w}(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})_{w\cdot 0}\}$*, then*

$$\mathrm{supp}\left(\mathcal{H}^\ell_{\overline{\mathfrak{X}_w}/\mathfrak{d}\mathfrak{X}_w}(\mathcal{O}_{\mathfrak{X}})\right) = \overline{\mathfrak{X}_w} = \mathrm{supp}(\mathcal{L}(w)) \qquad \text{and}$$
$$\mathrm{supp}\left(\mathcal{H}^\ell_{\overline{\mathfrak{X}_w}/\mathfrak{d}\mathfrak{X}_w}(\mathcal{O}_{\mathfrak{X}})/\mathcal{L}(w)\right) \subseteq \mathfrak{d}\mathfrak{X}_w.$$

Proof. Let $\mathrm{j}_{\mathfrak{d}} : \mathfrak{N}_w \to \mathfrak{X} \setminus \mathfrak{d}\mathfrak{X}_w$ and $\mathrm{j}'_{\mathfrak{d}} : \mathfrak{X} \setminus \mathfrak{d}\mathfrak{X}_w \to \mathfrak{X}$ be two inclusions.
(i) On flasques of $\mathbf{Ab}_{\mathfrak{X}}$

(1)
$$\begin{aligned}\Gamma_{\overline{\mathfrak{X}_w}/\mathfrak{d}\mathfrak{X}_w} &\simeq (\mathrm{j}'_{\mathfrak{d}})_* \circ \Gamma_{\mathfrak{X}_w} \circ (\mathrm{j}'_{\mathfrak{d}})^{-1} \quad \text{by [\textbf{Ke}, 8.3]} \\ &\simeq (\mathrm{j}_w)_* \circ \Gamma_{\mathfrak{X}_w} \circ (\mathrm{j}_w)^{-1} \quad \text{by excision [\textbf{Ke}, 7.9]},\end{aligned}$$

hence on $\mathrm{D}^b(\mathcal{D}_{\mathfrak{X}}\mathbf{qc})$

$$\begin{aligned}&\mathrm{R}\,\Gamma_{\overline{\mathfrak{X}_w}/\mathfrak{d}\mathfrak{X}_w} \\ &\simeq \mathrm{R}(\mathrm{j}_w)_0 \circ \mathrm{R}\,\Gamma_{\mathfrak{X}_w} \circ \mathrm{R}(\mathrm{j}_w^{-1}) \quad \text{as both } \Gamma_{\mathfrak{X}_w} \text{ and } \mathrm{j}_w^{-1} \text{ send flasques to flasques} \\ &\simeq (\mathrm{j}_w)_0 \circ \mathrm{R}\,\Gamma_{\mathfrak{X}_w} \circ \mathrm{j}_w^{-1} \quad \text{by Serre's theorem as } \mathrm{j}_w \text{ is affine} \\ &\simeq (\mathrm{j}_w)_0 \circ \int_{\mathrm{i}_w} \mathrm{L}(\mathrm{i}^0_w)[-\ell] \circ \mathrm{j}_w^{-1} \quad \text{by (2.11)} \\ &\simeq \int_{\mathrm{j}_w \circ \mathrm{i}_w} \mathrm{L}(\mathrm{j}_w \circ \mathrm{i}_w)^0[-\ell] \quad \text{by (2.6)}.\end{aligned}$$

(ii) By (i), as $(\mathrm{j}'_{\mathfrak{d}})_* \circ \Gamma_{\mathfrak{X}_w} \circ (\mathrm{j}'_{\mathfrak{d}})^{-1}$ sends flasques to flasques,

$$\mathrm{R}\,\Gamma_{\mathfrak{d}\mathfrak{X}_w} \circ \mathrm{R}\,\Gamma_{\overline{\mathfrak{X}_w}/\mathfrak{d}\mathfrak{X}_w} \simeq \mathrm{R}(\Gamma_{\mathfrak{d}\mathfrak{X}_w} \circ (\mathrm{j}'_{\mathfrak{d}})_* \circ \Gamma_{\mathfrak{X}_w} \circ (\mathrm{j}'_{\mathfrak{d}})^{-1}).$$

As $\Gamma_{\mathfrak{d}\mathfrak{X}_w} \circ (\mathrm{j}'_{\mathfrak{d}})_* \circ \Gamma_{\mathfrak{X}_w} \circ (\mathrm{j}'_{\mathfrak{d}})^{-1}$ is left exact and as any sheaf of abelian groups can be imbedded in a flasque sheaf, it is enough to show that $\Gamma_{\mathfrak{d}\mathfrak{X}_w} \circ (\mathrm{j}'_{\mathfrak{d}})_* \circ \Gamma_{\mathfrak{X}_w} \circ (\mathrm{j}'_{\mathfrak{d}})^{-1} = 0$ on flasques. But on flasques the LHS is $\Gamma_{\mathfrak{X}_w} \circ \Gamma_{\overline{\mathfrak{X}_w}/\mathfrak{d}\mathfrak{X}_w}$ by (1). If $\mathcal{F}$ is a flasque sheaf, the short exact sequence

$$0 \longrightarrow \Gamma_{\mathfrak{d}\mathfrak{X}_w}(\mathcal{F}) \longrightarrow \Gamma_{\overline{\mathfrak{X}_w}}(\mathcal{F}) \longrightarrow \Gamma_{\overline{\mathfrak{X}_w}/\mathfrak{d}\mathfrak{X}_w}(\mathcal{F}) \longrightarrow 0$$

remains exact after applying $\Gamma_{\mathfrak{d}\mathfrak{X}_w}$ as $\Gamma_{\mathfrak{d}\mathfrak{X}_w}(\mathcal{F})$ is flasque. Then

$$\Gamma_{\mathfrak{d}\mathfrak{X}_w} \circ \Gamma_{\overline{\mathfrak{X}_w}/\mathfrak{d}\mathfrak{X}_w}(\mathcal{F}) \simeq \Gamma_{\mathfrak{d}\mathfrak{X}_w}(\Gamma_{\overline{\mathfrak{X}_w}}(\mathcal{F}))/\Gamma_{\mathfrak{d}\mathfrak{X}_w}(\Gamma_{\mathfrak{d}\mathfrak{X}_w}(\mathcal{F})) = 0.$$

(iii) As j_w is affine and as i_w is a closed immersion, both $(\mathrm{j}_w)_0$ and $(\mathrm{i}_w)_0$ are exact, hence $\int_{\mathrm{j}_w \circ \mathrm{i}_w} \simeq (\mathrm{j}_w)_0 \circ (\mathrm{i}_w)_0(\mathcal{D}_{\mathrm{i}_w \leftarrow} \otimes_{\mathcal{D}_{\mathfrak{X}_w}} ?)$ is exact. Then

$$\mathcal{H}^s_{\overline{\mathfrak{X}_w}/\mathfrak{d}\mathfrak{X}_w}(\mathcal{O}_{\mathfrak{X}}) \simeq \int_{\mathrm{j}_w \circ \mathrm{i}_w} \mathrm{L}^{s-\ell}((\mathrm{j}_w \circ \mathrm{i}_w)^0)(\mathcal{O}_{\mathfrak{X}}) \simeq \int^0_{\mathrm{j}_w \circ \mathrm{i}_w} \mathrm{L}^{s-\ell}((\mathrm{j}_w \circ \mathrm{i}_w)^0)(\mathcal{O}_{\mathfrak{X}}).$$

By (2.13)

$$\mathrm{L}^{s-\ell}((\mathrm{j}_w \circ \mathrm{i}_w)^0)(\mathcal{O}_{\mathfrak{X}}) \simeq \mathrm{L}^{s-\ell}(\mathrm{i}^0_w)(\mathcal{O}_{\mathfrak{N}_w}) \simeq \begin{cases} \mathcal{O}_{\mathfrak{X}_w} & \text{if } s = \ell \\ 0 & \text{otherwise.} \end{cases}$$

(iv) By (i), (iii) and as j_w is affine, one has in $(B^+_w, \mathcal{D}_{\mathfrak{X}})\mathbf{qc}$

$$\mathcal{H}^\ell_{\overline{\mathfrak{X}_w}/\mathfrak{d}\mathfrak{X}_w}(\mathcal{O}_{\mathfrak{X}}) \simeq (\mathrm{j}_w)_0 \circ \mathcal{H}^\ell_{\mathfrak{X}_w}(\mathcal{O}_{\mathfrak{X}}) \simeq (\mathrm{j}_w)_0 \circ \int^0_{\mathrm{i}_w} \mathcal{O}_{\mathfrak{X}_w} \simeq (\mathrm{j}_w)_0(\mathcal{B}_{\mathfrak{X}_w|\mathfrak{N}_w}).$$

Identifying $D(\mathfrak{N}_w)$ with $D(U^+_w) \otimes D(U^-_w) \simeq \mathfrak{k}[U^+_w] \otimes \mathrm{Dist}(U^+_w) \otimes \mathfrak{k}[U^-_w] \otimes \mathrm{Dist}(U^-_w)$ one has in $D(\mathfrak{N}_w)\mathbf{Mod}$

$$\begin{aligned} \mathcal{B}_{\mathfrak{X}_w|\mathfrak{N}_w}(\mathfrak{N}_w) &\simeq D(\mathfrak{N}_w)/\{D(\mathfrak{N}_w)\,\mathrm{Dist}^+(U^+_w) + D(\mathfrak{N}_w)\mathcal{I}_w(\mathfrak{N}_w)\} \quad \text{by (2.13)} \\ &\simeq \mathfrak{k}[U^+_w] \otimes \mathrm{Dist}(U^-_w). \end{aligned}$$

On the other hand, we know from [**Ke**, Lem. 12.8]

$$\mathrm{ch}\ \mathrm{H}^\ell_{\overline{\mathfrak{X}_w}/\mathfrak{d}\mathfrak{X}_w}(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) = e(w \cdot 0) \prod_{\alpha \in R^+} \frac{1}{1 - e(-\alpha)}.$$

As $\mathrm{ch}\,\mathfrak{k}[U^+] \otimes \mathrm{Dist}(U^-_w) = \prod_{\alpha \in R^+} \frac{1}{1-e(-\alpha)}$, we must have by the affine version of the Beilinson-Bernstein correspondence (3.9)

$$\mathcal{B}_{\mathfrak{X}_w|\mathfrak{N}_w}(\mathfrak{N}_w) \simeq \mathfrak{k}[U^+_w] \otimes \mathrm{Dist}(U^-_w) \otimes (w \cdot 0) \quad \text{in } (B^+_w, D(\mathfrak{N}_w))\mathbf{Mod}.$$

(v) Consider the image of $\mathrm{id}_{\mathfrak{N}_w}$ in

$$D(\mathfrak{N}_w)/\{D(\mathfrak{N}_w)\operatorname{Dist}^+(U_w^+)+D(\mathfrak{N}_w)\mathcal{I}_w(\mathfrak{N}_w)\}\simeq \mathrm{H}^{\ell}_{\overline{\mathfrak{X}_w}/\mathfrak{d}\mathfrak{X}_w}(\mathfrak{X},\mathcal{O}_{\mathfrak{X}}).$$

As $\mathcal{D}_{\mathfrak{N}_w}/\{\mathcal{D}_{\mathfrak{N}_w}\operatorname{Dist}^+(U_w^+)+\mathcal{D}_{\mathfrak{N}_w}\mathcal{I}_w\}\;|_{\mathfrak{X}_w}=\mathcal{D}_{\mathfrak{N}_w}/\mathcal{D}_{\mathfrak{N}_w}\operatorname{Dist}^+(U_w^+)\;|_{\mathfrak{X}_w}$ and as $\mathrm{id}_{\mathfrak{N}_w}$ is a global section of $\mathcal{H}^{\ell}_{\overline{\mathfrak{X}_w}/\mathfrak{d}\mathfrak{X}_w}(\mathcal{O}_{\mathfrak{X}})$,

$$\operatorname{supp}\left(\mathcal{H}^{\ell}_{\overline{\mathfrak{X}_w}/\mathfrak{d}\mathfrak{X}_w}(\mathcal{O}_{\mathfrak{X}})\right)\supseteq\overline{\mathfrak{X}_w},$$

hence $\operatorname{supp}(\mathcal{H}^{\ell}_{\overline{\mathfrak{X}_w}/\mathfrak{d}\mathfrak{X}_w}(\mathcal{O}_{\mathfrak{X}}))=\overline{\mathfrak{X}_w}$ by [**Ke**, Lem. 9.3]. As $\mathrm{id}_{\mathfrak{N}_w}$ has weight $w\cdot 0$ and as $\dim \mathrm{H}^{\ell}_{\overline{\mathfrak{X}_w}/\mathfrak{d}\mathfrak{X}_w}(\mathfrak{X},\mathcal{O}_{\mathfrak{X}})_{w\cdot 0}=1$, $\mathrm{id}_{\mathfrak{N}_w}\in\Gamma(\mathfrak{X},\mathcal{L}(w))$, hence $\operatorname{supp}(\mathcal{L}(w))=\overline{\mathfrak{X}_w}$ also. Finally, as

$$\mathcal{L}(w)\Big|_{\mathfrak{N}_w}=\mathcal{H}^{\ell}_{\overline{\mathfrak{X}_w}/\mathfrak{d}\mathfrak{X}_w}(\mathcal{O}_{\mathfrak{X}})\Big|_{\mathfrak{N}_w},\ \left\{\mathcal{H}^{\ell}_{\overline{\mathfrak{X}_w}/\mathfrak{d}\mathfrak{X}_w}(\mathcal{O}_{\mathfrak{X}})/\mathcal{L}(w)\right\}\Big|_{\mathfrak{N}_w}=0,$$

hence

$$\operatorname{supp}\left(\mathcal{H}^{\ell}_{\overline{\mathfrak{X}_w}/\mathfrak{d}\mathfrak{X}_w}(\mathcal{O}_{\mathfrak{X}})/\mathcal{L}(w)\right)\subseteq\overline{\mathfrak{X}_w}\setminus\mathfrak{N}_w=\mathfrak{d}\mathfrak{X}_w.$$

□

4.2.

Remark. In [**Bø**, Th. 4.6] Bøgvad goes on to show

$$\mathcal{L}(w)=\operatorname{soc}_{(B^+,\mathcal{D}_X)\mathbf{qc}}\mathcal{H}^{\ell}_{\overline{\mathfrak{X}_w}/\mathfrak{d}\mathfrak{X}_w}(\mathcal{O}_{\mathfrak{X}})$$

and that any simple of $(B^+,\mathcal{D}_X)\mathbf{qc}$ with support $\overline{\mathfrak{X}_w}$ is isomorphic to $\mathcal{L}(w)$. Hence [**BBIII**, Prop. 2.7] carries over to positive characteristic except that (cf. [**K90**])

$$\mathrm{H}^{\ell}_{\overline{\mathfrak{X}_w}/\mathfrak{d}\mathfrak{X}_w}(\mathfrak{X},\mathcal{O}_{\mathfrak{X}})\not\simeq\{\operatorname{Dist}(G)\otimes_{\operatorname{Dist}(B)}(-(w\cdot 0))\}^{\star}\quad\text{in }\operatorname{Dist}(G)\mathbf{Mod},$$

where $\star$ denotes the weightwise dual.

4.3. From [**H87**, 4.4.1] we know

(1) any $\mathcal{M}\in\mathcal{D}_{\mathfrak{X}}\mathbf{qc}$ is generated by the global sections over $\mathcal{O}_{\mathfrak{X}}$.

Then (cf. [**Ka**, Th. 1.4.1]) the following three statements are equivalent:

(i) $\mathfrak{X}$ is $\mathcal{D}$-affine,

(ii) for any ample $\mathcal{L}\in\mathbf{Mod}_{\mathfrak{X}}$ if $r\gg 0$, the natural morphism $\mathcal{D}_{\mathfrak{X}}\otimes_{\mathfrak{X}}(\mathcal{L}^{\otimes -r}\otimes\mathcal{L}^{\otimes r}(\mathfrak{X}))\to\mathcal{D}_{\mathfrak{X}}$ splits in $\mathbf{Ab}_{\mathfrak{X}}$,

(iii) there is an ample $\mathcal{L}\in\mathbf{Mod}_{\mathfrak{X}}$ such that if $r\gg 0$, the morphism of (ii) splits in $\mathbf{Ab}_{\mathfrak{X}}$.

If we write $\mathcal{L} = \mathcal{L}(\lambda) = \mathcal{L}_{G/B}(\lambda)$, $\lambda \in \mathbf{Grp}_{\Bbbk}(B, \mathrm{GL}_1)$, the morphism of (ii) is obtained from the commutative diagram in $(G, \mathcal{D}_{\mathfrak{X}})\mathbf{qqc}$

$$\begin{array}{ccc} \mathcal{D}_{\mathfrak{X}} \otimes_{\mathfrak{X}} \{\mathcal{L}(-r\lambda) \otimes \mathrm{H}^0(r\lambda)\} & \longrightarrow & \mathcal{D}_{\mathfrak{X}} \\ \wr\downarrow & & \downarrow\wr \\ \mathcal{L}(\mathrm{Dist}(G) \otimes_{\mathrm{Dist}(B)} -r\lambda \otimes \mathrm{H}^0(r\lambda)) & \longrightarrow & \mathcal{L}(\mathrm{Dist}(G) \otimes_{\mathrm{Dist}(B)} \Bbbk), \end{array} \tag{2}$$

where $\mathrm{H}^0(r\lambda) = \mathcal{L}(r\lambda)(\mathfrak{X})$ and the bottom horizontal morphism is induced from the B-homomorphism $-r\lambda \otimes \mathrm{ev}_{r\lambda} : -r\lambda \otimes \mathrm{H}^0(r\lambda) \to \Bbbk$ with $\mathrm{ev}_{r\lambda} : \mathrm{H}^0(r\lambda) \to r\lambda$ the evaluation at e.

In characteristic 0 [**BB**] finds (cf. [**Ka**, Th. 6.3.1]) that in (2)

(3) $\mathrm{Dist}(G) \otimes_{\mathrm{Dist}(B)} -r\lambda \otimes \mathrm{H}^0(r\lambda) \to \mathrm{Dist}(G) \otimes_{\mathrm{Dist}(B)} \Bbbk$ splits in $B\mathbf{Mod}$,

hence (ii, iii) hold and the $\mathcal{D}$-affinity of $\mathfrak{X}$ follows. In positive characteristic the statement (ii) is equivalent, given $\mathcal{L}$ and r, to the statement that

(4) if $s \gg 0$, the natural morphism $\mathcal{D}_{\mathfrak{X},s} \otimes_{\mathfrak{X}} \{\mathcal{L}^{\otimes -r} \otimes \mathcal{L}^{\otimes r}(\mathfrak{X})\} \to \mathcal{D}_{\mathfrak{X},s}$ splits in $\mathbf{Ab}_{\mathfrak{X}}$.

If $\mathcal{L} = \mathcal{L}(\lambda)$, the morphism is obtained from the commutative diagram in $(G, \mathcal{D}_{\mathfrak{X},s})\mathbf{qqc}$

$$\begin{array}{ccc} \mathcal{D}_{\mathfrak{X},s} \otimes_{\mathfrak{X}} \mathcal{L}(-r\lambda) \otimes \mathrm{H}^0(r\lambda) & \longrightarrow & \mathcal{D}_{\mathfrak{X},s} \\ \wr\downarrow & & \downarrow\wr \\ \mathcal{L}(\hat{Z}_s(r\lambda)^* \otimes \mathrm{H}^0(r\lambda)) & & \mathcal{L}(\hat{Z}_s(\Bbbk)^*) \\ \wr\downarrow & & \downarrow\wr \\ \mathcal{L}(\hat{Z}_s((2(p^s-1)\rho - r\lambda) \otimes \mathrm{H}^0(r\lambda))) & \longrightarrow & \mathcal{L}(\hat{Z}_s(2(p^s-1)\rho)), \end{array} \tag{5}$$

where $\hat{Z}_s = \mathrm{ind}_B^{G_sB}$ and the bottom horizontal morphism is induced from the G_sB-homomorphism $\theta(s, r\lambda) : \hat{Z}_s((2(p^s-1)\rho - r\lambda) \otimes \mathrm{H}^0(r\lambda)) \to \hat{Z}_s(2(p^s - 1)\rho)$, that in turn is induced from the B-homomorphism $(2(p^s-1)\rho - r\lambda) \otimes \mathrm{ev}_{r\lambda} : (2(p^s-1)\rho - r\lambda) \otimes \mathrm{H}^0(r\lambda) \to 2(p^s-1)\rho$.

Unfortunately, we find already in SL_2 that

$$\theta(s, (p^r-1)\rho),\ s \in \mathbb{N}, \text{ does not split in } B\mathbf{Mod}. \tag{6}$$

Nevertheless, we can show

$$\mathrm{id}_{\mathcal{O}_{\mathfrak{X}}} \in \Gamma(\mathfrak{X}, \mathcal{L}(\theta(r+1, (p^r-1)\rho))), \tag{7}$$

and hence G/B is $\mathcal{D}$-affine in SL_2. More generally, Haastert has proved by different arguments [**H87**, 3.2, 4.5.4] that all $\mathbb{P}^n_{\mathfrak{k}}$, $n \in \mathbb{N}$, and the flag variety in SL_3 are $\mathcal{D}$-affine.

References

[BB] A. Beilinson and J. Bernstein, *Localisation de $\mathfrak{g}$-modules*, C. R. Acad. Sci. Paris, **292** (1981), 15-18.

[Bø] R. Bøgvad, *Some results on $\mathcal{D}$-modules on Borel varieties in characteristic $p > 0$*, J. Alg., **173** (1995), 638-667.

[BVI] A. Borel, *Operations on Algebraic $\mathcal{D}$-modules*, 207-270, in A. Borel (ed.) 'Algebraic D-Modules', Academic Press, 1987.

[BBIII] W. Borho and J.-L. Brylinski, *Differential operators on homogeneous spaces*, III, Inv. Math., **80** (1985), 1-68.

[C] S.U. Chase, *On the homological dimension of algebras of differential operators*, Comm. Alg., **1** (1974), 351-363.

[DG] M. Demazure and P. Gabriel, *Groupes Algébriques*, North-Holland, 1970.

[G] R. Godement, *Théorie des Faisceaux*, Hermann, 1964.

[Gr] P.-P. Grivel, *Categories derivees et foncteurs derives*, 1-108, in A. Borel (ed.) 'Algebraic D-Modules', Academic Press, 1987.

[EGA0] A. Grothendieck and J. Dieudonné, *Éléments de Géométrie Algébrique* IV, Pub. Math., **20**, IHES, 1964.

[EGAIV] ———, *Éléments de Géométrie Algébrique* IV, Pub. Math., **32**, IHES, 1967.

[H86] B. Haastert, *Über Differentialoperatoren und $\mathbb{D}$-Moduln in positiver Charakteristik*, Dissertation, Univ. Hamburg, 1986

[H87] ———, *Über Differentialoperatoren und $\mathbb{D}$-Moduln in positiver Charakteristik*, Manusc. Math., **58** (1987), 385-415.

[H88] ———, *On direct and inverse images of $\mathcal{D}$-modules in prime charakteristic*, Manusc. Math., **62** (1988), 341-354.

[RD] R. Hartshorne, *Residues and Duality*, LNM 20, Springer-Verlag, 1966.

[H] ———, *Algebraic Geometry*, GTM 52, Springer-Verlag, 1977.

[J] J.C. Jantzen, *Representations of Algebraic Groups*, Academic Press, 1987.

[K90] M. Kaneda, *A note on the Grothendieck-Cousin complex on the flag variety in positive characteristic*, Nihonkai Math. J., **1** (1990), 229-251.

[K95] ———, *The Frobenius morphism of Schubert schemes*, J. Alg., **174** (1995), 473-488.

[Ka] M. Kashiwara, *Representation theory and $\mathcal{D}$-modules on flag varieties*, Astérisque, **173-174** (1989), 55-109.

[Ke] G. Kempf, *The Grothendieck-Cousin complex of an induced representation*, Adv. Math., **29** (1978), 310-396.

[M] H. Matsumura, *Commutative Ring Theory*, Camb. Univ. Press, 1990.

[MN] Z. Mebkhout and L. Narvaez-Macarro, *Sur les coefficients de Rham-Grothendieck des variétés algébriques*, 267-308, in F. Baldassarri, S. Bosch and B. Dwork (ed.) 'p-adic Analysis', LNM 1454, Springer-Verlag, 1990.

[Sa] M. Saito, *A localization theorem for $\mathcal{D}$-modules*, Tôhoku Math. J., **43** (1991), 213-234.

[Sm] S.P. Smith, *The global homological dimension of the ring of differential operators on a nonsingular variety over a field of positive characteristic*, J. Alg., **107** (1987), 98-105.

[Sp] T. Springer, *Linear Algebraic Groups*, Birkhäuser, 1981.

[TH] T. Tanisaki and R. Hotta, *$\mathcal{D}$-kagun to Daisuugun*, Springer-Verlag, 1995.

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Note: The equation numbers in Sections 2.1, 3.1, and 4.1 of the paper version were incorrect. In addition, references to (2.7.x) have been changed to (2.8.x) for $x \geq 5$, (2.8.i) has been changed to (2.9.i) in line -2 of (2.11), and corrections have been made in line 1 of (2.4), in the paragraph after 2.5.8, in line 2 of (2.7) Proposition, in the second term of (2.7.3), and in (2.9.i).