PACIFIC JOURNAL OF MATHEMATICS
Vol. 183, No. 2, 1998

QUASIMINIMAL SURFACES OF CODIMENSION 1 AND JOHN DOMAINS

Guy David and Stephen Semmes

We study codimension 1 quasiminimizing surfaces in $\mathbb{R}^n$, and establish uniform rectifiability and other geometric properties of these surfaces. For instance, their complementary components must be John domains. In fact we give a complete characterization of quasiminimizers. As an application we show that sets which are not too large and which separate points in a definite way must have a large uniformly rectifiable piece. In this way we use area quasiminimizers to solve a problem in geometric measure theory.

1. Introduction.

One of the main goals of this paper is to prove that quasiminimal surfaces of codimension 1 in $\mathbb{R}^n$ (as defined in terms of functions of bounded variation) are Ahlfors-regular sets that bound exactly two domains in $\mathbb{R}^n$, each of which is a John domain. In particular they enjoy quantitative rectifiability properties.

Let us first describe what we mean by quasiminimizers. Let $BV = BV(\mathbb{R}^n)$ denote the space of functions of bounded variation on $\mathbb{R}^n$, i.e., the space of real-valued locally integrable functions f on $\mathbb{R}^n$ such that each of the distributional first derivatives $\dfrac{\partial f}{\partial x_j}$ of f is a finite (signed) measure. This is equivalent to requiring that

$$(1.1)\quad N(f) = \sup\left\{ \left|\int_{\mathbb{R}^n} f(x)\ \mathrm{div}\,\varphi(x)dx\right| : \varphi : \mathbb{R}^n \to \mathbb{R}^n \text{ is } C^1, \right.$$
$$\left. \text{compactly supported, and such that } \|\varphi\|_\infty \le 1 \right\}$$

be finite.

Let Q_0 and Q_1 be fixed closed cubes in $\mathbb{R}^n$, with $Q_0 \subset \mathrm{int}(Q_1)$. Set

$$(1.2)\quad \mathcal{F} = \Big\{ V \subset \mathbb{R}^n : V \text{ is Lebesgue-measurable,}$$
$$\mathrm{int}(Q_0) \subset V \subset Q_1 \text{ and } \mathbb{1}_V \in BV \Big\}.$$

In effect, we shall not be directly interested in the set V, but its equivalence class modulo equality almost-everywhere.

Definition 1.3. Let Q_0, Q_1 and $\mathcal{F}$ be as above. We say that $W \in \mathcal{F}$ is a quasiminimizer for $N(\cdot)$ if there is a constant $\alpha \in (0,1)$ so that

$$N(1\!\!1_W) \leq N(1\!\!1_V) + \alpha N(1\!\!1_V - 1\!\!1_W) \tag{1.4}$$

for all $V \in \mathcal{F}$.

Notice that (1.4) contains no information when $\alpha = 1$. At the other end of the spectrum, $\alpha = 0$ corresponds to (true) minimizers. This use of BV norms is quite standard in the context of minimal surfaces in codimension 1. It is useful for getting existence results. We shall return to this point later.

Before we state our main result on the structure of quasiminimizers we need to state some more definitions.

Definition 1.5. Let E be a compact subset of $\mathbb{R}^n$, and suppose that $0 < d \leq n$. We say that E is Ahlfors-regular with dimension d if there exists a constant $C_0 > 0$ so that

$$C_0^{-1} r^d \leq H^d(E \cap B(x,r)) \leq C_0 r^d \text{ for all } x \in E \text{ and } 0 < r \leq \operatorname{diam} E.$$

Here H^d denotes d-dimensional Hausdorff measure, and $\operatorname{diam} E$ is the diameter of E. In the present paper, the dimension d will always equal $n-1$.

Definition 1.6. Let E be a compact, Ahlfors-regular set of dimension $n-1$ in $\mathbb{R}^n$. We say that E satisfies condition B if there is a constant C_1 such that, for each $x \in E$ and each radius $0 < r \leq \operatorname{diam} E$, we can find two balls B_1, B_2 of radius $C_1^{-1} r$ that are contained in $B(x,r) \backslash E$ and lie in different connected components of $\mathbb{R}^n \backslash E$.

Bilipschitz images of the unit sphere in $\mathbb{R}^n$ satisfy condition B using a theorem of Väisälä [**V**], but there are other examples. Condition B sets have fairly good rectifiability properties: they "contain big pieces of Lipschitz graphs", and hence are uniformly rectifiable. See [**D**] for the original result, [**DS1**] and [**DJ**] for simpler proofs, and [**DS2**] for general information about uniform rectifiability.

Definition 1.7. (John domains). Let $\mathcal{U}$ be an open subset of $\mathbb{R}^n$, and let $z_0 \in \mathcal{U}$. If $\mathcal{U}$ is bounded, we say that $\mathcal{U}$ is a John domain with center z_0 and constant C_3 if for each $x \in \mathcal{U}$ there is a C_3-Lipschitz mapping $\alpha : [0, |x - z_0|] \to \mathcal{U}$ such that $\alpha(0) = x$, $\alpha(|x - z_0|) = z_0$, and $\operatorname{dist}(\alpha(t), \mathbb{R}^n \backslash \mathcal{U}) \geq C_3^{-1} t$ for $0 \leq t \leq |x - z_0|$.

If $\mathcal{U}$ is unbounded, we say that $\mathcal{U}$ is a John domain with center z_0 and constant C_3 if there is a ball B such that $\mathcal{U} \supset \mathbb{R}^n \backslash B$ and for each $x \in \mathcal{U} \cap 2B$ there is a C_3-Lipschitz mapping with the same properties as above.

This condition means that each point in the domain can be accessed "well" from the center. It holds when the boundary is smooth and for cubes, for instance. It fails to hold when the domain has an outward-pointing cusp. A bubble with a small neck forces the John constant to be large.

Theorem 1.8. *Let Q_0, Q_1 be closed cubes in $\mathbb{R}^n$, with $Q_0 \subset \mathrm{int}(Q_1)$, and let $W \in \mathcal{F}$ be a quasiminimizer for $N(\cdot)$. Then there is a unique open set $W_0 \in \mathcal{F}$ with the following properties: the two functions $1\!\!1_W(x)$ and $1\!\!1_{W_0}(x)$ coincide almost everywhere; the boundary ∂W_0 is an Ahlfors-regular set of dimension $n-1$ that satisfies Condition B; and $\mathbb{R}^n \backslash \partial W_0$ has exactly two connected components (namely, W_0 and $\mathbb{R}^n \backslash \overline{W}_0$). Each of these two components is a John domain, and we also have that $\partial W_0 = \partial \left(\mathbb{R}^n \backslash \overline{W}_0 \right)$.*

Moreover, the properties above are satisfied with constants C_1, C_2 and C_3 that depend only on n, Q_0, Q_1, and α.

One should not pay too much attention to the cubes Q_0, Q_1. They provide an obstacle problem for the definition of quasiminimizers which prevents trivialities. This obstacle problem is not special, it was chosen with an eye to an application (Theorem 1.15).

Theorem 1.8 has an analog where the BV norm $N(1\!\!1_W)$ is replaced with the Hausdorff measure $H^{n-1}(\partial W)$ of the boundary. The same basic arguments apply as we shall explain in Section 9. In the context of Hausdorff measure, ∂W is what Almgren calls a (γ, δ)-restricted set. This notion still makes sense in higher codimension, and Almgren showed that such a set is Ahlfors-regular and rectifiable [**Al**]. In a forthcoming paper, we intend to show that it is also uniformly rectifiable, with big pieces of Lipschitz graphs.

One of the main advantages of BV is that its compactness properties permit us to find quasiminimizers very easily. Let us say a few words about this.

For each bounded nonnegative lower semi-continuous function $g : \mathbb{R}^n \to \mathbb{R}^+$, one can define a variant of the norm $N(f)$ on BV by

$$N_g(f) = \int_{\mathbb{R}^n} g \, |\nabla f| \, . \tag{1.9}$$

[See the beginning of Section 2 for a more precise definition.] Examples of quasiminimizers for $N(\cdot)$ will be provided for us by minimizers of $N_g(\cdot)$, through the following result.

Proposition 1.10. *Let Q_0, Q_1 be closed cubes in $\mathbb{R}^n$ with $Q_0 \subset \mathrm{int} Q_1$. Let $g : \mathbb{R}^n \to \mathbb{R}$ be a lower semi-continuous function, and suppose that there exists constants $0 < m \leq M < +\infty$ such that $m \leq g(x) \leq M$ for all $x \in \mathbb{R}^n$. Then there is a set $W \in \mathcal{F}$ such that*

$$N_g(1\!\!1_W) = \inf_{V \in \mathcal{F}} N_g(1\!\!1_V). \tag{1.11}$$

Every $W \in \mathcal{F}$ which satisfies (1.11) is a quasiminimizer for $N(\cdot)$ (as in Definition 1.3), with the constant $\alpha = 1 - \dfrac{m}{M}$.

See Section 2 for a proof and more details.

The following converse to Theorem 1.8 will be proved in Section 7.

Theorem 1.12. *Let Q_0, Q_1 be as above, and let $E \subset Q_1$ be an Ahlfors-regular set of dimension $n-1$. Suppose that $\mathbb{R}^n \backslash E$ has exactly two connected components W_0 and W_1, with $\mathrm{int}(Q_0) \subset W_0 \subset Q_1$, that $E = \partial W_0 = \partial W_1$, and that W_0, W_1 are both John domains. Define $g : \mathbb{R}^n \to \mathbb{R}_+$ by*

$$\begin{cases} g(x) = 1 & \text{when} \quad x \in E \\ g(x) = A & \text{when} \quad x \in \mathbb{R}^n \backslash E. \end{cases} \tag{1.13}$$

If the constant A is large enough (depending only on the regularity constant for E, the John constants for W_0 and W_1, n, Q_0 and Q_1), then W_0 is the unique minimizer for $N_g(\cdot)$ in the sense that

$$N_g\left(1\!\!1_{W_0}\right) = \inf_{V \in \mathcal{F}} N_g\left(1\!\!1_V\right) \tag{1.14}$$

and every $W \in \mathcal{F}$ for which $N_g\left(1\!\!1_W\right) = N_g\left(1\!\!1_{W_0}\right)$ satisfies $1\!\!1_W(x) = 1\!\!1_{W_0}(x)$ almost everywhere.

The original motivation for the results of this paper came from the following consequence of Theorem 1.8 and Proposition 1.10.

Theorem 1.15. *For each constant $C_4 > 0$, there is a constant $M = M(C_4, n)$ such that the following holds. Let K be a compact subset of $\mathbb{R}^n$ such that $K \subset \overline{B(0,2)} \backslash B(0,1)$, K separates 0 from ∞ in $\mathbb{R}^n$, and $H^{n-1}(K) \leq C_4$. Then there is an M-Lipschitz graph Γ such that $H^{n-1}(K \cap \Gamma) \geq M^{-1}$.*

By M-Lipschitz graph we mean a set of the form $\Gamma = \{(x, h(x)) : x \in \mathbb{R}^{n-1}\}$, where $h : \mathbb{R}^{n-1} \to \mathbb{R}$ is such that $|h(x) - h(y)| \leq M\,|x - y|$ for $x, y \in \mathbb{R}^{n-1}$, or the image of such a set by a rotation.

Theorem 1.15 answers a question articulated in [**Se2**] and known to the authors for some time. Although the hypotheses of Theorem 1.15 permit

K to have a substantial fractal piece, the conclusion says that K must have a reasonably smooth part too. In order to make a true wall without much mass, K ought to contain something like a surface.

Results of a similar flavor had been proved before, but each time with separation hypotheses that would hold at all scales. A simple example of such a result is that every regular set of codimension 1 that satisfies condition B contains big pieces of Lipschitz graphs at all scales. Theorem 1.15 was proved independently and by a completely different method by P. Jones, N. Katz and A. Vargas [**JKV**]. Here Theorem 1.15 will be obtained as a consequence of the following more precise result.

Theorem 1.16. *Let K be a compact subset of $\mathbb{R}^n$ such that $K \subset \overline{B(0,2)}\setminus B(0,1)$, K separates 0 from ∞, and $H^{n-1}(K) \leq C_4 < +\infty$. Then for each $\varepsilon > 0$ there is a compact set $E \subset \overline{B(0,2)}\setminus B(0,1)$ which is Ahlfors regular of dimension $n-1$, satisfies Condition B, separates 0 from ∞, and for which*

$$H^{n-1}(E\setminus K) \leq \varepsilon. \tag{1.17}$$

(*Thus E is almost contained in K.*) *The regularity and Condition B constants for E depend only on n, C_4, and ε, but not on K.*

In fact, we could take $E = \partial W_0$, where W_0 is associated to a quasiminimizer for $N(\cdot)$ as in Theorem 1.8 (but with cubes replaced by balls), and so E has the additional properties mentioned in that theorem. [That is, $\mathbb{R}^n\setminus E$ has only two components W_0 and W_1, $E = \partial W_0 = \partial W_1$, and W_0, W_1 are John domains.]

There is a variant of Theorem 1.16 where the hypothesis that K separates 0 from ∞ is replaced with the existence of a function $f \in BV(B(0,3)\setminus K)$ that equals 0 on $B\left(0, \frac{1}{2}\right)$ and has a mean value on $B(0,3)\setminus B(0,2)$ which is much larger than its BV norm. See Section 8 for details.

The rest of this paper is organized as follows. In Section 2, we shall describe the relations between minimizers for N_g and quasiminimizers for $N(\cdot)$, and in particular prove Proposition 1.10.

Theorem 1.8 will be proved in Sections 3, 4, 5 and 6. In Section 3, it will be proved that we can associate to each quasiminimizer W for $N(\cdot)$ an open set W_0 such that $1\!\!1_{W_0} = 1\!\!1_W$ almost everywhere, ∂W_0 is Ahlfors-regular and satisfies Condition B. In Section 4 we check, for the convenience of the reader, that the BV norm of characteristic functions $1\!\!1_V$ is dominated by $H^{n-1}(\partial V)$, and that the converse is true for sets like W_0. In Section 5 we prove that ∂W_0 has only two complementary connected components (W_0 and $\mathbb{R}^n \backslash \overline{W}_0$), each of which is a "domain of isoperimetry", i.e., a domain on which an isoperimetric inequality holds. We shall see in Section 6 that every domain of isoperimetry which is bounded, contains a ball, and has an Ahlfors-regular boundary is also a John domain, and Theorem 1.8 will follow.

Section 7 contains the proof of the converse (Theorem 1.12), and also of the related fact that bounded John domains with Ahlfors-regular boundary are domains of isoperimetry.

Section 8 is devoted to the proof of Theorems 1.15 and 1.16, and the variant to which we alluded. It relies only on Section 2-4 for its proof, and can essentially be read independently of all sections other than 1 and 8.

Section 9 deals with the notion of quasiminimizers based on the H^{n-1} measure of the boundary instead of BV norms.

2. Existence of minimizers.

Recall that $BV = BV(\mathbb{R}^n)$ denotes the set of locally integrable functions on $\mathbb{R}^n$ whose distributional gradient is a finite (vector-valued) measure. We shall be particularly interested in functions in BV which are the characteristic function of some set. If A is a measurable set and $\mathbb{1}_A \in BV$, then A is called a Cacciopoli set or a set of finite perimeter. Basic examples include cubes (and then the distributional first derivatives of $\mathbb{1}_A$ are given by the measures on the faces of A that one expects) and bounded domains with smooth boundary. In these examples the BV norm of $\mathbb{1}_A$ is the same as $H^{n-1}(\partial A)$, but there are well-known pathological examples where this fails to hold.

Let us first review some basic results about compactness and lower semicontinuity that will be used to establish the existence of minimizers.

Lemma 2.1. *Let B be a ball in $\mathbb{R}^n$ and let X_k denote the set of L^1-functions which vanish outside B and satisfy $\|f\|_{BV} =: N(f) \leq k$. Then X_k is a compact subset of $L^1(\mathbb{R}^n)$.*

This is well-known and easy to prove. For instance one can check that X_k is closed and that for each $\varepsilon > 0$ there is a finite subset F of $L^1(\mathbb{R}^n)$ such that every element of X_k is within ε of F. The Poincaré inequality is helpful in establishing the latter fact, while the closedness of X_k follows from a lower semicontinuity property of the BV norm.

Our next task is to define the variants $N_g(\cdot)$ of the BV norm, and for this a few simple properties of lower semicontinuous (l.s.c.) functions will be useful. Recall that a real-valued function g on $\mathbb{R}^n$ is said to be lower semicontinuous if $\{x \in \mathbb{R}^n : g(x) > \lambda\}$ is open for every λ. Typical examples of l.s.c. functions include characteristic functions of open sets.

Lemma 2.2. *Let g be a nonnegative l.s.c. function on $\mathbb{R}^n$. Then there is a nondecreasing sequence $\{g_j\}$ of nonnegative smooth functions on $\mathbb{R}^n$ such that $g(x) = \lim_{j\to\infty} g_j(x)$ for all x.*

To see this, define $\mathcal{U} \subset \mathbb{R}^n \times \mathbb{R}$ by

$$\mathcal{U} = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : g(x) > t\}. \tag{2.3}$$

Then $\mathcal{U}$ is open because g is l.s.c. Let $\{\varphi_j(x,t)\}$ be a sequence of smooth functions such that $0 \le \varphi_j \le \varphi_{j+1} \cdots \le \mathbb{1}_{\mathcal{U}}$ and $\lim_{j\to\infty} \varphi_j(x,t) = \mathbb{1}_{\mathcal{U}}(x,t)$ for all (x,t), with uniform convergence on compact subsets of $\mathcal{U}$. (For instance, one can take partial sums of a partition of unity of $\mathcal{U}$.) Set

$$g_j(x) = \int_0^j \varphi_j(x,t)dt. \tag{2.4}$$

Clearly, each g_j is nonnegative and smooth and $g_j \le g_{j+1}$. Since $g(x) = \int_0^{g(x)} dt = \int_0^\infty \mathbb{1}_{\mathcal{U}}(x,t)dt$ for each $x \in \mathbb{R}^n$, we also have that $g(x) = \lim_{j\to\infty} g_j(x)$ for each x. This proves Lemma 2.2.

We now come to the definition of $N_g(\cdot)$. Given $f \in BV$, let μ_j denote the signed Borel measure $\frac{\partial f}{\partial x_j}$, and set $\mu = \sum_{j=1}^n |\mu_j|$. Thus μ is a positive Borel measure and each μ_j is absolutely continuous with respect to μ. We can write μ_j as $h_j\mu$, with $h_j \in L^1(d\mu)$. Let h denote the vector of h_j's, and let $|\nabla f|$ denote the measure $|h|\, d\mu$.

Now let g be a bounded nonnegative l.s.c. function on $\mathbb{R}^n$. Then

$$N_g(f) = \int_{\mathbb{R}^n} g\, |\nabla f| = \int_{\mathbb{R}^n} g\, |h|\, d\mu \tag{2.5}$$

makes sense (and is even finite) for $f \in BV$. We want to establish lower semicontinuity properties for $N(\cdot)$ (the reason for the lower semicontinuity assumption for g), and for this it will be convenient to rewrite (2.5) in various ways. Let $\{g_j\}$ be the sequence of smooth functions provided by Lemma 2.2, and set $N_j(f) = N_{g_j}(f)$. Then

$$N_g(f) = \lim_{j\to\infty} N_j(f) = \sup_j N_j(f) \tag{2.6}$$

by the monotone convergence theorem.

For each $f \in BV$, define $v : \mathbb{R}^n \to \mathbb{R}^n$ by $v(x) = \frac{h(x)}{|h(x)|}$ when $h(x) \neq 0$ and $v(x) = 0$ otherwise. This is a Borel-measurable function, and $v \cdot h = |h|$. Using for instance Lusin's theorem to approximate v by smooth functions, we can obtain that

$$N_j(f) = \sup \Big\{ \int_{\mathbb{R}^n} g_j \varphi \cdot h \; d\mu : \varphi : \mathbb{R}^n \to \mathbb{R}^n \;\text{ is smooth compactly supported, and }\; \|\varphi\|_\infty \leq 1 \Big\}. \tag{2.7}$$

Combining this with (2.6) and the fact that $h \; d\mu$ is the distributional gradient of f, we get that

$$N_g(f) = \sup_j \Big[\sup \Big\{ \int_{\mathbb{R}^n} \operatorname{div}(g_j \varphi) f \; dx : \varphi : \mathbb{R}^n \to \mathbb{R}^n \;\text{ is smooth, compactly supported, and }\; \|\varphi\|_\infty \leq 1 \Big\} \Big]. \tag{2.8}$$

Lemma 2.9. *Suppose that $\{f_k\}$ is a sequence in $BV(\mathbb{R}^n)$ which converges in the sense of distributions to some $f \in BV$. Then*

$$N_g(f) \leq \liminf_{k \to +\infty} N_g(f_k) \; . \tag{2.10}$$

Indeed, our convergence assumption implies that

$$\begin{aligned} \int_{\mathbb{R}^n} \operatorname{div}(g_j \varphi) f \; dx &= \lim_{k \to \infty} \int_{\mathbb{R}^n} \operatorname{div}(g_j \varphi) f_k \; dx \\ &\leq \liminf_{k \to \infty} N_j(f_k) \leq \liminf_{k \to \infty} N_g(f_k) \end{aligned} \tag{2.11}$$

for all compactly supported smooth functions $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ with $\|\varphi\|_\infty \leq 1$ and for all j. We get (2.10) now by taking the supremum of the left side of (2.11) over φ and j, and using (2.8). This proves Lemma 2.9.

We are now ready to prove the first half of Proposition 1.10.

Proposition 2.12. *Let Q_0, Q_1 be closed cubes in $\mathbb{R}^n$, with $Q_0 \subset \operatorname{int} Q_1$ and let $\mathcal{F}$ be, as before, the collection of measurable sets $V \subset \mathbb{R}^n$ such that $\operatorname{int} Q_0 \subset V \subset Q_1$ and $1\!\!1_V \in BV$. Also let $g : \mathbb{R}^n \to \mathbb{R}$ be a* l.s.c. *function such that $0 < m \leq g(x) \leq M < +\infty$ for all $x \in \mathbb{R}^n$. Then there exist a $W \in \mathcal{F}$ such that*

$$N_g(1\!\!1_W) = \inf_{V \in \mathcal{F}} N_g(1\!\!1_V) \; . \tag{2.13}$$

This is quite straightforward. Let $\{V_j\}$ be a sequence in $\mathcal{F}$ for which $\lim\limits_{j\to+\infty} N_g(\mathbb{1}_{V_j}) = \inf\limits_{V\in\mathcal{F}} N_g(\mathbb{1}_V)$. Notice that $\{\mathbb{1}_{V_j}\}$ is bounded in BV, since $\|f\|_{BV} = N(f) \le m^{-1}N_g(f)$ for all $f \in BV$. The compactness result in Lemma 2.1 allows us to find a subsequence of $\{\mathbb{1}_{V_j}\}$ that converges in $L^1(\mathbb{R}^n)$. The limit is the characteristic function of some set V such that $\mathrm{int}(Q_0) \subset V \subset Q_1$. [This may be seen by extracting a subsequence that converges a.e.] Also, $V \in \mathcal{F}$ because the sets X_k of Lemma 2.1 are closed. Moreover, the subsequence of $\{\mathbb{1}_{V_j}\}$ converges to $\mathbb{1}_V$ in the sense of distribution since it converges in L^1, and so Lemma 2.9 implies that $N_g(\mathbb{1}_V) \le \liminf\limits_{j\to+\infty} N_g(\mathbb{1}_{V_j})$, and Proposition 2.12 follows.

Proposition 2.14. *Let Q_0, Q_1, $\mathcal{F}$, and g be as in Proposition 2.12 (or Proposition 1.10). If $W \in \mathcal{F}$ is a minimizer for $N_g(\cdot)$, i.e., satisfies (2.13), then W is a quasiminimizer for $N(\cdot)$ (as in Definition 1.3), with $\alpha = 1 - \frac{m}{M}$.*

To prove this, let $W \in \mathcal{F}$ be a minimizer for $N_g(\cdot)$, and let $V \in \mathcal{F}$ be some other competitor. Then

$$
\begin{aligned}
M\int_{\mathbb{R}^n} |\nabla \mathbb{1}_W| &= \int_{\mathbb{R}^n} g\,|\nabla \mathbb{1}_W| + \int_{\mathbb{R}^n} (M-g)\,|\nabla \mathbb{1}_W| \\
&\le \int g\,|\nabla \mathbb{1}_V| + \int (M-g)\,|\nabla \mathbb{1}_W| \\
&\le M\int |\nabla \mathbb{1}_V| + \int (M-g)\,(|\nabla \mathbb{1}_W| - |\nabla \mathbb{1}_V|) \\
&\le M\int |\nabla \mathbb{1}_V| + \int (M-g)\,|\nabla \mathbb{1}_W - \nabla \mathbb{1}_V| \\
&\le M\int |\nabla \mathbb{1}_V| + (M-m)\int |\nabla \mathbb{1}_W - \nabla \mathbb{1}_V| \\
&= MN(\mathbb{1}_V) + (M-m)N(\mathbb{1}_W - \mathbb{1}_V),
\end{aligned}
\tag{2.15}
$$

from which (1.4) follows at once, with $\alpha = 1 - \frac{m}{M}$. This completes the proof of Proposition 2.14. Proposition 1.10 follows from this and Proposition 2.12.

Remark 2.16. The same kind of argument implies that any quasiminimizer for $N_g(\cdot)$ is also a quasiminimizer for $N(\cdot)$ (with a worse constant). We shall not need this.

3. Ahlfors-regularity and Condition B.

In this section we want to prove that for every quasiminimizer W for $N(\cdot)$, there is an open set W_0 such that $\mathbb{1}_{W_0}(x) = \mathbb{1}_W(x)$ a.e., and ∂W_0 is Ahlfors regular of codimension 1 and satisfies Condition B.

For this part of the argument, we shall only need to compare W with competitors V of the form $W \cup Q$ or $W \backslash Q$, where Q is a cube. Let us state the slightly more precise result that we shall get.

Theorem 3.1. *Let Q_0, Q_1 be closed cubes in $\mathbb{R}^n$, with $Q_0 \subset \operatorname{int}(Q_1)$, and let $W \in \mathcal{F}$ be given. [See (1.2) for the definition of $\mathcal{F}$.] Suppose that there is an $\alpha \in (0,1)$ such that (1.4) holds for all sets $V \in \mathcal{F}$ of the form $V = W \cup Q$ or $V = W \backslash Q$, where Q is a cube. Then there is an open set $W_0 \in \mathcal{F}$ such that $\mathbb{1}_W(x) = \mathbb{1}_{W_0}(x)$ a.e., ∂W_0 is Ahlfors-regular of dimension $n-1$ and satisfies Condition B. The constants for the regularity of ∂W_0 and Condition B can be taken to depend only on n, Q_0, Q_1 and α.*

See Lemma 3.43 for some additional information concerning Condition B.

Let $W \in \mathcal{F}$ be as in the statement of Theorem 3.1 and fixed from now on. We begin the argument with a fairly rough estimate.

Lemma 3.2. *Let Q be a cube, and set $X_+ = \mathbb{1}_{W \cup Q}$ and $X_- = \mathbb{1}_{W \backslash Q}$. Then X_+ and X_- lie in $BV(\mathbb{R}^n)$, ∇X_+ and ∇X_- are both equal to 0 in the interior of Q and to $\nabla \mathbb{1}_W$ outside $\overline{Q}$, and*

$$\int_{\partial Q} |\nabla X_\pm| \leq H^{n-1}(\partial Q). \tag{3.3}$$

The proof of Lemma 3.2 uses a straightforward approximation argument. Let $\{\theta_j\}$ be a sequence of smooth functions such that $0 \leq \theta_j \leq 1$ for all j, $\theta_j \equiv 1$ on a neighborhood of Q and $\theta_j = 0$ outside $(1+2^{-j})Q$. Let us even take θ_j to be of the form

$$\theta_j(x) = \prod_{i=1}^{n} \widetilde{\theta}_j\left(x_i - c_i\right), \tag{3.4}$$

where $x_1, \cdots, x_n$ are the coordinates of x, $c_1, \cdots, c_n$ the coordinates of the center of Q, and $\widetilde{\theta}_j$ is a smooth even function of one variable such that $\int_0^\infty \left|\widetilde{\theta}_j'\right| = 1$. With this choice of function θ_j, we have that

$$\lim_{j \to +\infty} \int_{\mathbb{R}^n} |\nabla \theta_j| = H^{n-1}(\partial Q)\ , \tag{3.5}$$

which will help us get the constant 1 in (3.3). [Such precision is not really needed for the proof of Theorem 3.1, though.]

For each j we have that $(1-\theta_j)\mathbb{1}_W \in BV$, and

$$\nabla\left(\left(1-\theta_j\right)\mathbb{1}_W\right) = -\left(\nabla\theta_j\right)\mathbb{1}_W + \left(1-\theta_j\right)\left(\nabla\mathbb{1}_W\right) \tag{3.6}$$

in the obvious distributional sense. It follows that $X_- = (1 - \mathbb{1}_Q)\, \mathbb{1}_W \in BV$, since it is the limit in L^1_{loc} of $(1-\theta_j)\, \mathbb{1}_W$, and since the sequence $\{(1-\theta_j)\, \mathbb{1}_W\}$ is bounded in BV by (3.6) and (3.5). [See Lemma 2.2.]

It is clear that ∇X_- equals 0 in $\mathrm{int}(Q)$ and $\nabla \mathbb{1}_W$ outside $\overline{Q}$. Also notice that the sequence $\{(1-\theta_j)\, \nabla \mathbb{1}_W\}$ of vector-valued measures converges weakly to $\mathbb{1}_{\mathbb{R}^n \setminus \overline{Q}}\, \nabla \mathbb{1}_W$, and in particular gives zero mass to ∂Q. Applying Lemma 2.9 to the sequence $\{(1-\theta_j)\, \mathbb{1}_W\}$, we get that

$$
\begin{aligned}
\int_{\partial Q} |\nabla X_-| &= N(X_-) - \int_{\mathbb{R}^n \setminus \overline{Q}} |\nabla \mathbb{1}_W| \\
&\leq \liminf_{j \to +\infty} N\left((1-\theta_j)\mathbb{1}_W\right) - \int_{\mathbb{R}^n \setminus \overline{Q}} |\nabla \mathbb{1}_W| \\
&\leq \liminf_{j \to +\infty} \left\{ \int \mathbb{1}_W\, |\nabla \theta_j| \right\} \leq H^{n-1}(\partial Q)
\end{aligned}
\tag{3.7}
$$

by (3.6) and (3.5). This proves our claims concerning X_-.

The corresponding statements for X_+ are proved similarly. This time we write

$$
1 - X_+ = (1 - \mathbb{1}_Q)\,(1 - \mathbb{1}_W) = \lim_{j \to +\infty} (1-\theta_j)\,(1-\mathbb{1}_W) \tag{3.8}
$$

and use the formula

$$
\nabla\left((1-\theta_j)\left((1-\mathbb{1}_W)\right)\right) = -\,(1-\mathbb{1}_W)\,\nabla\theta_j - (1-\theta_j)\,\nabla \mathbb{1}_W \tag{3.9}
$$

instead of (3.6). The fact that $X_+ \in BV$, $\nabla X_+ = 0$ inside $\mathrm{int}(Q)$ and $\nabla X_+ = \nabla \mathbb{1}_W$ outside $\overline{Q}$ follows from the same fact for X_-, and (3.3) now follows because

$$
\begin{aligned}
\int_{\partial Q} |\nabla X_+| &= N\left(X_+\right) - \int_{\mathbb{R}^n \setminus \overline{Q}} |\nabla \mathbb{1}_W| \\
&\leq \liminf_{j \to +\inf} N\left((1-\theta_j)\,(1-\mathbb{1}_W)\right) - \int_{\mathbb{R}^n \setminus \overline{Q}} |\nabla \mathbb{1}_W| \\
&\leq \liminf_{j \to +\inf} \left\{ \int_{\mathbb{R}^n} (1-\mathbb{1}_W)\,|\nabla\theta_j| \right\} \leq H^{n-1}(\partial Q).
\end{aligned}
\tag{3.10}
$$

This completes the proof of Lemma 3.2. We shall need also the following slightly more precise version of (3.3).

Lemma 3.11. *Let Q, X_-, and X_+ be as in Lemma 3.2. Then*

$$
\int_{\partial Q} |\nabla X_-| \leq C \liminf_{\varepsilon \searrow 0} \left\{ (\varepsilon \operatorname{diam} Q)^{-1}\, |((1+\varepsilon)Q \setminus Q) \cap W| \right\} \tag{3.12}
$$

and

$$\int_{\partial Q} |\nabla X_+| \leq C \liminf_{\varepsilon \searrow 0} \left\{ (\varepsilon \operatorname{diam} Q)^{-1} |((1+\varepsilon)Q \backslash Q) \backslash W| \right\}. \tag{3.13}$$

Here C depends only on n and $|A|$ denotes the Lebesgue measure of the set A in $\mathbb{R}^n$.

This lemma follows from the second-to-last inequalities in (3.7) and (3.10), at least if we choose the θ_j's more carefully. (The $\widetilde{\theta}_j$'s should be almost piecewise linear.)

Lemma 3.14. *Let Q be a cube such that $Q \subset Q_1$ or* $\operatorname{int}(Q_0) \cap Q = \emptyset$. *Then*

$$\int_{\overline{Q}} |\nabla 1\!\!1_W| \leq \frac{1+\alpha}{1-\alpha} H^{n-1}(\partial Q) \ . \tag{3.15}$$

To prove this lemma, we want to compare $N(X_+)$ or $N(X_-)$ to $N(1\!\!1_W)$. If $Q \subset Q_1$, then $\operatorname{int}(Q_0) \subset W \cup Q \subset Q_1$ because $W \in \mathcal{F}$, and so $W \cup Q \in \mathcal{F}$, since $X_+ \in BV$ by Lemma 3.2. Similarly, if $\operatorname{int}(Q_0) \cap Q = \emptyset$, then $\operatorname{int}(Q_0) \subset W \backslash Q \subset Q_1$ and $W \backslash Q \in \mathcal{F}$. Thus we may apply (1.4) with $1\!\!1_V = X_+$ or X_-. We get that

$$N(1\!\!1_W) \leq N(X_\pm) + \alpha N(1\!\!1_W - X_\pm), \tag{3.16}$$

and then Lemma 3.2 yields

$$\begin{aligned} \int_{\overline{Q}} |\nabla 1\!\!1_W| &= N(1\!\!1_W) - \int_{\mathbb{R}^n \backslash \overline{Q}} |\nabla 1\!\!1_W| \\ &\leq N(X_\pm) + \alpha N\left(1\!\!1_W - X_\pm\right) - \int_{\mathbb{R}^n \backslash \overline{Q}} |\nabla 1\!\!1_W| \\ &\leq \int_{\overline{Q}} |\nabla X_\pm| + \alpha N\left(1\!\!1_W - X_\pm\right) \\ &\leq (1+\alpha) \int_{\overline{Q}} |\nabla X_\pm| + \alpha \int_{\overline{Q}} |\nabla 1\!\!1_W| \end{aligned} \tag{3.17}$$

and then

$$\begin{aligned} \int_{\overline{Q}} |\nabla 1\!\!1_W| &\leq \frac{1+\alpha}{1-\alpha} \int_{\overline{Q}} |\nabla X_\pm| = \frac{1+\alpha}{1-\alpha} \int_{\partial Q} |\nabla X_\pm| \\ &\leq \frac{1+\alpha}{1-\alpha} H^{n-1}(\partial Q). \end{aligned} \tag{3.18}$$

This proves the lemma.

Now we want to control quantities like

$$h(Q) = r(Q)^{-n}\text{Min}\left(|Q \cap W|, |Q \backslash W|\right), \tag{3.19}$$

where $r(Q)$ denotes the sidelength of the cube Q, and $|E|$ is the Lebesgue measure of $E \subset \mathbb{R}^n$.

Lemma 3.20. *Suppose that*

$$Q \subset Q_1 \quad \text{or} \quad Q \cap \text{int}(Q_0) = \emptyset. \tag{3.21}$$

Then

$$r(Q)^{-n+1} \int_{\overline{Q}} |\nabla 1\!\!1_W| \leq Ch(2Q), \tag{3.22}$$

where C depends on n and α, but not on Q.

To prove this we may as well assume $h(2Q) \leq \delta$ for some small $\delta > 0$ that we get to choose, since otherwise (3.22) follows from (3.15).

Let us first suppose that $h(2Q) = (2r(Q))^{-n}|W \cap 2Q|$ so that $|W \cap 2Q| \leq 2^n \delta r(Q)^n$. If δ is small enough, this implies that $\frac{3}{2}Q \cap \text{int}(Q_0) = \emptyset$, and so $V_\lambda = W \backslash \lambda Q \in \mathcal{F}$ for $1 \leq \lambda \leq \frac{3}{2}$. Thus we may apply (3.18) to the function $X_-^\lambda = 1\!\!1_{V_\lambda}$ that corresponds to λQ and get that

$$\begin{aligned}\int_{\overline{Q}} |\nabla 1\!\!1_W| &\leq \int_{\lambda\overline{Q}} |\nabla 1\!\!1_W| \leq \frac{1+\alpha}{1-\alpha}\int_{\partial(\lambda Q)} |\nabla X_-^\lambda| \\ &\leq \frac{C}{1-\alpha} \liminf_{\varepsilon \searrow 0} \left\{(\varepsilon r(Q))^{-1}|[(1+\varepsilon)\lambda Q \backslash \lambda Q] \cap W|\right\}\end{aligned} \tag{3.23}$$

by (3.12). Thus (3.22) will follow if we can find $\lambda \in [1, \frac{3}{2})$ such that the right-hand side of (3.23) is $\leq \frac{C'}{1-\alpha} r(Q)^{-1}|W \cap 2Q| = \frac{C'}{1-\alpha}2^n r(Q)^{n-1}h(2Q)$. The existence of such a λ comes from the general fact that if μ is a finite nonnegative measure on $[1,2)$, there is a $\lambda \in [1, \frac{3}{2})$ such that

$$\liminf_{\varepsilon \searrow 0} \varepsilon^{-1}\mu\left([s_0, s_0 + \varepsilon]\right) \leq C\mu([1,2)). \tag{3.24}$$

This last fact is itself an easy consequence of the Hardy-Littlewood maximal theorem. (One can make a more elementary argument using Fatou's lemma.)

This proves (3.22) when $h(2Q) = (2r(Q))^{-n}|W \cap 2Q|$.

If $h(2Q) = (2r(Q))^{-n}\,|2Q\backslash W|$, then $|2Q\backslash W| \le 2^n r(Q)^n\delta$. If δ is small enough, this implies that $\frac{3}{2}Q \subset Q_1$, and so $V'_\lambda = W \cup \lambda Q \in \mathcal{F}$ for $1 \le \lambda \le \frac{3}{2}$. Thus we may apply (3.18) to the function $X_+^\lambda = 1\!\!1_{W\cup\lambda Q}$. This gives that

$$\int_{\overline{Q}} |\nabla 1\!\!1_W| \le \frac{1+\alpha}{1-\alpha}\int_{\partial(\lambda Q)} |\nabla X_+^\lambda|\,. \tag{3.25}$$

We then apply (3.13) and choose an appropriate value of λ as before, to conclude that

$$\begin{aligned}\int_{\overline{Q}} |\nabla 1\!\!1_W| &\le \frac{C}{1-\alpha}\liminf_{\varepsilon\searrow 0}\left\{(\varepsilon r(Q))^{-1}\,|[(1+\varepsilon)\lambda Q\backslash\lambda Q]\backslash W|\right\}\\ &\le \frac{C'}{1-\alpha} r(Q)^{n-1}h(2Q).\end{aligned} \tag{3.26}$$

This completes the proof of Lemma 3.20.

Next we want to show that $\int_{\overline{Q}} |\nabla 1\!\!1_W|$ controls $h(Q)$. This will be a consequence of the following Sobolev-Poincaré inequality: for $f \in BV(\mathbb{R}^n)$ and any cube Q,

$$\left\{\int_Q \left|f(x) - \frac{1}{|Q|}\int_Q f\right|^{\frac{n}{n-1}} dx\right\}^{\frac{n-1}{n}} \le C\int_Q |\nabla f|\,. \tag{3.27}$$

This inequality is well-known. If one wanted to prove it from scratch, one could use the argument on p. 128-130 of [St]. $\Big[$A first step would be to notice that $g = \left[f - \frac{1}{|Q|}\int_Q f\right] 1\!\!1_Q$ lies in BV, with a norm $N(g) \le C\displaystyle\int_Q |\nabla f|\,.\Big]$ For the purpose of this paper, the power $\frac{n}{n-1}$ is not crucial, in the sense that anything larger than 1 would work as well. Thus we could manage with more vulgar Sobolev-Poincaré inequalities than (3.27).

Applying (3.27) to $f = 1\!\!1_W$ and then using (3.22) we get that

$$\begin{aligned}&\left\{r(Q)^{-n}\int_Q \left|1\!\!1_W(x) - \frac{|Q\cap W|}{|Q|}\right|^{\frac{n}{n-1}}\right\}^{\frac{n-1}{n}}\\ &\le C\; r(Q)^{-n+1}\int_Q |\nabla 1\!\!1_W| \le Ch(2Q)\end{aligned} \tag{3.28}$$

when Q satisfies (3.21). On the other hand,

$$h(Q) \le 4r(Q)^{-n}\int_Q \left|1\!\!1_W(x) - \frac{|Q\cap W|}{|Q|}\right|^{\frac{n}{n-1}} \tag{3.28$\frac{1}{2}$}$$

because $\left|1\!\!1_W(x) - \frac{|Q\cap W|}{|Q|}\right| \geq \frac{1}{2}$ on $Q\backslash W$ if $\frac{|Q\cap W|}{|Q|} \geq \frac{1}{2}$ and $\left|1\!\!1_W - \frac{|Q\cap W|}{|Q|}\right| \geq \frac{1}{2}$ on $Q\cap W$ when $\frac{|Q\cap W|}{|Q|} \leq \frac{1}{2}$. Therefore

$$h(Q) \leq Ch(2Q)^{\frac{n}{n-1}} \tag{3.29}$$

for all cubes Q that satisfy (3.21). The constant C depends on n α, Q_0 and Q_1, but not on Q. The next lemma will be a rather mechanical consequence of (3.29).

Lemma 3.30. *There is a small number $\varepsilon_0 > 0$, depending only on n, α, Q_0 and Q_1, such that if Q satisifies (3.21) and $h(Q) \leq \varepsilon_0$, then either $\left|\frac{1}{2}Q\cap W\right| = 0$ or $\left|\frac{1}{2}Q\backslash W\right| = 0$.*

Let Q be as in the lemma. For each $x \in \frac{1}{2}Q$ and each integer $j \geq 1$, let $Q_j(x)$ denote the cube with center x, sidelength $2^{-j}r(Q)$, and sides parallel to those of Q. Thus $Q_j(x) \subset Q$ for all $j \geq 1$, and in particular $Q_j(x)$ satisfies the condition (3.21). Our assumption gives that $h(Q_1(x)) \leq 2^n h(Q) \leq 2^n\varepsilon_0$. Also, (3.29) tells us that $h\left(Q_{j+1}(x)\right) \leq C\, h\left(Q_j(x)\right)^{\frac{n}{n-1}}$. If ε_0 is chosen small enough, this implies that $h(Q_{j+1}) \leq \frac{1}{2}h\left(Q_j(x)\right)$ as soon as $h\left(Q_j(x)\right) \leq 2^n\varepsilon_0$. A trivial induction then shows that $h\left(Q_j(x)\right) \leq 2^{-j+n+1}\varepsilon_0$ for all $j \geq 1$ and $x \in \frac{1}{2}Q$.

Now suppose first that $h(Q) = r(Q)^{-n}\left|Q\cap W\right|$. Then $|Q_1(x)\cap W| \leq |Q\cap W| \leq 2^n\varepsilon_0\left|Q_1(x)\right|$, and so $h\left(Q_1(x)\right) = \frac{|Q_1(x)\cap W|}{|Q_1(x)|}$. Since $h\left(Q_1(x)\right)$ is also very small, this implies that $\frac{|Q_2(x)\cap W|}{|Q_2(x)|}$ is so small that $h\left(Q_2(x)\right) = \frac{|Q_2(x)\cap W|}{|Q_2(x)|}$. The argument can be continued like this, and the estimate above gives that

$$|Q_j(x)\cap W| \leq 2^{-j+n+1}\varepsilon_0\left|Q_j(x)\right| \tag{3.31}$$

for all $j \geq 1$ and $x \in \frac{1}{2}Q$. The Lebesgue density theorem implies that $\left|\frac{1}{2}Q\cap W\right| = 0$ in this case.

If $h(Q) = r(Q)^{-n}\left|Q\backslash W\right|$, the same argument as above shows that

$$|Q_j(x)\backslash W| \leq 2^{-j+n+1}\varepsilon_0\left|Q_j(x)\right| \tag{3.32}$$

for all $x \in \frac{1}{2}Q$ and $j \geq 1$, so that $\left|\frac{1}{2}Q\backslash W\right| = 0$. This completes the proof of Lemma 3.30.

Set $W_0 = \left\{x \in \mathbb{R}^n : |B(x,r)\backslash W| = 0 \text{ for some } r > 0\right\}$, $W_1 = \left\{x \in \mathbb{R}^n : |B(x,r)\cap W| = 0 \text{ for some } r > 0\right\}$, and $E = \mathbb{R}^n\backslash\left(W_1 \cup W_2\right)$. Clearly, W_0 and W_1 are open. Let us check that

$$\begin{cases} W_0 \ \text{ is the set of points of density of } \ W, \ \text{ and} \\ W_1 \ \text{ the set of points of density of } \ \mathbb{R}^n\backslash W. \end{cases} \tag{3.33}$$

It is clear that W_0 and W_1 are contained in the sets of points of density of W and $\mathbb{R}^n \backslash W$ respectively. To prove the converse, let $x \in \mathbb{R}^n$ be given. Note that all the small enough cubes centered at x satisfy (3.21). If there is such a cube Q such that $h(Q) < \varepsilon_0$, then Lemma 3.30 says that x lies in W_0 or W_1. This proves the claim (3.33), but it also tells us that

$$\begin{cases} \text{if } \; x \in E, \; \text{ then } \; h(Q) \geq \varepsilon_0 \; \text{ for all the} \\ \text{cubes } \; Q \; \text{ centered on } \; x \text{ and for which (3.21) holds.} \end{cases} \tag{3.34}$$

An immediate consequence of the first part of (3.33) is that

$$|W_0 \backslash W| = |W \backslash W_0| = 0. \tag{3.35}$$

Observe that if $x \in E$, then (3.34) says that $|Q \cap W| > 0$ and $|Q \backslash W| > 0$ for every cube centered at x, so that (3.33) now implies that $x \in \partial W_0 \cap \partial W_1$. Conversely, if $x \in \partial W_0$, then x cannot be in W_0 or W_1 (because they are open), and so $x \in E$. Altogether,

$$E = \partial W_0 = \partial W_1. \tag{3.36}$$

Our next task is to show that E is an Ahlfors-regular set that satisfies Condition B. We shall see in later sections that if W is a quasiminimizer, then W_0 and W_1 are connected and are John Domains. To prove that E is regular, we shall use (3.34) to compare $H^{n-1}(E)$ with $\int |\nabla 1\!\!1_W|$.

Let μ denote the positive measure $|\nabla 1\!\!1_W|$. Observe that supp$\mu \subset E$, since $\nabla 1\!\!1_W = \nabla 1\!\!1_{W_0}$.

Lemma 3.37. *There is a constant $\varepsilon_1 > 0$ such that if Q is a cube centered on E which satisfies (3.21), then*

$$\mu(Q) \geq \varepsilon_1 \, |Q|^{\frac{n-1}{n}} \, . \tag{3.38}$$

Let Q be as in the statement. We may apply the Sobolev-Poincaré estimate (3.27) to $f = 1\!\!1_W$ and get the first inequality in (3.28). Because of ($3.28\frac{1}{2}$), this yields

$$\begin{aligned} h(Q)^{\frac{n-1}{n}} &\leq C r(Q)^{-n+1} \int_Q |\nabla 1\!\!1_W| \\ &= C \, |Q|^{-\frac{n-1}{n}} \, \mu(Q). \end{aligned} \tag{3.39}$$

The lemma follows, because $h(Q) \geq \varepsilon_0$ by (3.34).

Lemma 3.37 implies that

$$E = \operatorname{supp}\mu. \tag{3.40}$$

Moreover, it follows from Lemma 3.14 and Lemma 3.37 that E is Ahlfors-regular of dimension $n-1$. To be precise, these lemmas imply that

$$C^{-1}r^{n-1} \leq \mu(B(x,r)) \leq Cr^{n-1} \tag{3.41}$$

for all $x \in E$ and $0 < r < r_0$, with $r_0 = (2\sqrt{n})^{-1}\operatorname{dist}(\partial Q_0, \partial Q_1)$, say. This last condition on r comes from the fact that we may only apply Lemmas 3.14 and 3.37 to cubes that satisfy (3.21). Of course, this is not a serious restriction, and (3.41) remains true with a different constant C for $x \in E$ and $0 < r < \operatorname{diam} E$. [Recall that $E \subset Q_1$ by definitions.] In Definition 1.5, we required that the estimate (3.41) be satisfied with $H^d_{|E}$ rather than μ, but it is fairly easy to see that if μ is a measure that satisfies (3.41), then there is a constant $\widetilde{C}$ such that

$$\widetilde{C}^{-1}H^d_{|E} \leq \mu \leq \widetilde{C}H^d_{|E}. \tag{3.42}$$

This is fairly easy to check using standard arguments. (A proof was written down for Lemma C.3 in [**Se3**].)

Thus E is regular, and our next task is to prove Condition B. We shall actually prove the following slightly more precise result.

Lemma 3.43. *There is a constant C_1, that depends only on n, Q_0, Q_1 and α, such that for each $x \in E$ and each $0 < r < \operatorname{diam} E$, we can find $z_0 \in W_0 \cap B(x,r)$ and $z_1 \in W_1 \cap B(x,r)$ such that $\operatorname{dist}(z_i, E) \geq C_1^{-1}r$ for $i = 0, 1$.*

The fact that E satisfies Condition B will follow from this, since W_0, W_1 are disjoint open sets whose union is $\mathbb{R}^n \backslash E$. [Compare with Definition 1.6.]

To prove the lemma, let $x \in E$ and $0 < r < \operatorname{diam} E$ be given. Without true loss of generality, we may assume that $r \leq (2\sqrt{n})^{-1}\operatorname{dist}(\partial Q_0, \partial Q_1)$ so that the largest cube Q centered at x and contained in $B(x,r)$ satisfies (3.21). Because of (3.34), $|W \cup Q| \geq \varepsilon_0 |Q|$ and $|Q \backslash W| \geq \varepsilon_0 |Q|$. Since W and W_0 coincide almost everywhere (by (3.35)) and similarly $\mathbb{R}^n \backslash W$ and W_1 are almost the same (by a similar argument or by (3.35) and the fact that $|E| = 0$), we have that

$$|W_0 \cap Q| \geq \varepsilon_0 |Q| \quad \text{and} \quad |W_1 \cap Q| \geq \varepsilon_0 |Q|. \tag{3.44}$$

On the other hand, Ahlfors-regularity of E implies that

$$|\{x \in Q : \operatorname{dist}(x,E) \leq t\, r(Q)\}| \leq Ct\,|Q| \tag{3.45}$$

for all $t \in (0,1)$, and a constant C that does not depend on t or Q. This is easy to prove. For instance one can select a maximal subset $\mathcal{A}$ of $E \cap (2Q)$ whose points are at mutual distances $\geq t\,r(Q)$. Because the balls $B\left(z, \frac{t\,r(Q)}{2}\right)$, $z \in \mathcal{A}$, are pairwise disjoint, (3.41) implies that $\mathcal{A}$ has at most Ct^{n-1} elements; (3.45) follows because the set in (3.45) can be covered by the balls $B(z, 2t\,r(Q))$, $z \in \mathcal{A}$.

If we choose t small enough, depending on ε_0, (3.44) and (3.45) imply the existence of points $z_0 \in W_0 \cap Q$ and $z_1 \in W_1 \cap Q$ such that $\operatorname{dist}(z_i, E) \geq t\,r(Q)$ for $i = 1, 2$. This completes the proof of Lemma 3.43, and of Theorem 3.1 as well.

Remark 3.46 (about uniqueness). In the statement of Theorem 3.1, the set W_0 is not yet unique. It is easy to see that any open set W_2 with the properties described in Theorem 3.1 must be contained in W_0. [Each point of W_2 must be a point of density of W.] On the other hand, one may artificially reduce W_0 by adding a piece of E, in a way that preserves Ahlfors-regularity and Condition B. [See Figure 1.]

This problem does not arise if we also require that W_2 satisfy the conclusion of Lemma 3.43 [i.e., that one can find $z_0 \in W_2 \cap B(x,r)$ and $z_1 \in B(x,r) \backslash W_2$ with $\operatorname{dist}(z_i, \partial W_2) \geq C_1^{-1} r$]. Indeed, if W_2 satisfies this condition and if $x \in W_0 \backslash W_2$, then $x \in \partial W_2$ because x is a point of density of W, and hence of W_2. This contradicts the conclusion of Lemma 3.43, since $x \in W_0$.

Thus W_0 is unique if we require that it satisfy the condition stated in Lemma 3.43, in addition to the conclusion of Theorem 3.1.

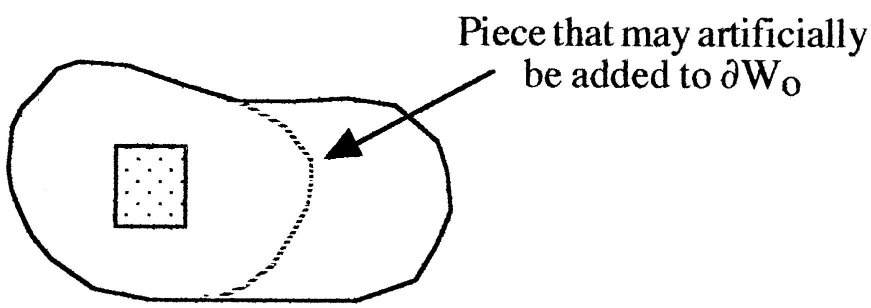

Figure 1.

4. $|\nabla 1\!\!1_W|$ and Hausdorff measure on ∂W.

In this section we want to compare the sizes of the measures $|\nabla 1\!\!1_\Omega|$ and $H^{n-1}|_{\partial\Omega}$ when Ω is an open set such that $H^{n-1}(\partial\Omega) < +\infty$. This will be used in the next section to prove that the open set W_0 associated to a quasiminimizer for $N(\cdot)$ as in Section 3 is connected and even a "domain of isoperimetry". It will also be used later to derive geometric consequences like Theorems 1.15 and 1.16 from properties of quasiminimizers.

Proposition 4.1. *Let W be a quasiminimizer for $N(\cdot)$ and $E = \partial W_0$ be as in the last section. Then $|\nabla 1\!\!1_W|$ is the restriction of H^{n-1} to E.*

The plan of this section is to first derive Proposition 4.1 from standard results on Caccioppoli sets, and then give a direct proof of the weaker fact that $|\nabla 1\!\!1_W|$ is equivalent to $H^{n-1}|_{\partial\Omega}$ with constants that depend only on the dimension n. The point of doing so is that the proof is slightly simpler (both because we are in a simpler situation and prove less), and we shall typically need only the weaker estimate.

To each set W such that $1\!\!1_W \in BV$, one associates a reduced boundary $\partial^* W$, which is the set of points $x \in \mathbb{R}^n$ with the following properties:

$$\int_{B(x,r)} |\nabla 1\!\!1_W| > 0 \quad \text{for all} \quad r > 0 \tag{4.2}$$

and, if we define vectors $\nu_r(x)$ by

$$\nu_r(x) = \frac{\int_{B(x,r)} \nabla 1\!\!1_W}{\int_{B(x,r)} |\nabla 1\!\!1_W|} \quad \text{for } r > 0, \tag{4.3}$$

then

$$\nu(x) = \lim_{r\to 0} \nu_r(x) \text{ exists, and its length is } |\nu(x)| = 1. \tag{4.4}$$

This is Definition 3.3 on p. 43 of the book of E. Giusti [**Gi**], which we also refer to for more information on Caccioppoli sets. Observe that $\partial^* W$ does not change when we replace W with an equivalent set, i.e., a set $\widetilde{W}$ such that $\left|\widetilde{W}\backslash W\right| = \left|W\backslash \widetilde{W}\right| = 0$. It is clear from (4.2) that

$$\partial^* W \subset \partial W, \tag{4.5}$$

and of course this information may become more interesting if we replace W by a correctly chosen equivalent set, like the set W_0 of the previous section.

The theorem of Besicovitch on differentiation of measures implies that $|\nabla 1\!\!1_W|$-almost every point of the support of $|\nabla 1\!\!1_W|$ lies in $\partial^* W$. [See 3.3 on pp. 43-44 in [**Gi**].] Moreover,

$$\begin{cases} \text{the measure } \ |\nabla 1\!\!1_W| \ \text{ coincides with} \\ \text{the restriction to } \ \partial^* W \ \text{ of } \ H^{n-1}. \end{cases} \tag{4.6}$$

This is (4.1) on p. 52 of [**Gi**].

In the special case when W is a quasiminimizer for $N(\cdot)$, (4.5) applied to the open set W_0 instead of W tells us that $\partial^* W \subset E = \partial W_0$. Also, we know from (3.40) and (3.41) that $E = \operatorname{supp}(|\nabla 1\!\!1_W|)$ and that $|\nabla 1\!\!1_W|$ and $H^{n-1}|_E$ are equivalent in size. [See also (3.42).] Because of this, H^{n-1}-almost every point of E lies in $\partial^* W$, and Proposition 4.1 follows from (4.6).

The next lemma is an easy consequence of (4.5) and (4.6), but we shall prove it for the convenience of the reader.

Lemma 4.7. *If Ω is a Borel set such that $H^{n-1}(\partial\Omega) < +\infty$, then $1\!\!1_\Omega \in BV$ and we have the following inequality between measures:*

$$|\nabla 1\!\!1_\Omega| \le C_n \, H^{n-1}|_{\partial\Omega} . \tag{4.8}$$

We know from (4.5) and (4.6) that we can take $C_n = 1$, but we shall only prove (4.8) with a constant C_n that depends only on n. The point is to give a very direct proof.

To prove the lemma, we shall establish that for every choice of an open set ω, a test-function φ with compact support in ω, and an index $j \in \{1, 2, \dots, n\}$, we have that

$$\left| \left\langle \frac{\partial}{\partial x_j} 1\!\!1_\Omega, \varphi \right\rangle \right| \le C_n' \, \|\varphi\|_\infty \, H^{n-1}(\omega \cap \partial\Omega) \tag{4.9}$$

where $\left\langle \frac{\partial}{\partial x_j} 1\!\!1_\Omega, \varphi \right\rangle = -\int_{\mathbb{R}^n} 1\!\!1_\Omega \frac{\partial \varphi}{\partial x_j}$ is the result of the action of the distribution $\frac{\partial}{\partial x_j} 1\!\!1_\Omega$ on the test function φ.

Let us check that the lemma will follow from (4.9). If (4.9) holds with $\omega = \mathbb{R}^n$, the Riesz representation theorem tells us that each $\frac{\partial}{\partial x_j} 1\!\!1_\Omega$ is a finite measure, and so $1\!\!1_\Omega \in BV$. Then (4.9) and the regularity of the finite Borel measures $\left|\frac{\partial}{\partial x_j} 1\!\!1_\Omega\right|$ and $H^{n-1}|_{\partial\Omega}$ imply that $\left|\frac{\partial}{\partial x_j} 1\!\!1_\Omega\right| \le C_n' \, H^{n-1}|_{\partial\Omega}$, from which (4.8) follows.

We shall only prove (4.9) with $j = 1$. Let H denote the hyperplane $\{x \in \mathbb{R}^n : x_1 = 0\}$ and let π be the orthogonal projection into H. We want to use the fact that

$$\int_H N(y) dy \le C H^{n-1}(\omega \cap \partial\Omega), \tag{4.10}$$

where $N(y)$ denotes the number of elements in $\omega\cap\partial\Omega\cap\pi^{-1}(y)$. This formula will be proved later, but let us first see why it implies (4.9).

Let φ be a test-function with compact support in ω. By Fubini,

$$
\begin{aligned}
\left\langle \frac{\partial}{\partial x_1}1\!\!1_\Omega, \varphi \right\rangle &= -\int_{\mathbb{R}^n} 1\!\!1_\Omega \frac{\partial \varphi}{\partial x_1} \\
&= -\int_H \left\{ \int_{\mathbb{R}} 1\!\!1_\Omega\,(x_1,y)\,\frac{\partial\varphi}{\partial x_1}\,(x_1,y)\,dx_1 \right\} dy,
\end{aligned}
\tag{4.11}
$$

where we wrote points of $\mathbb{R}^n$ as $x=(x_1,y)$ with $x_1\in\mathbb{R}$ and $y\in H$. Let $y\in H$ be given, and suppose that $N(y)<+\infty$. (By (4.10), this is the case for almost every $y\in H$.) Denote by $\{I_k\}$, $k\in K$, the connected components of $\{x_1\in\mathbb{R} : (x_1,y)\in\Omega\}$. For each $k\in K$,

$$
\int_{I_k} 1\!\!1_\Omega\,(x_1,y)\,\frac{\partial\varphi}{\partial x_1}\,(x_1,y)\,dx_1 = \varphi(a_k^+,y)-\varphi(a_k^-,y),
$$

where a_k^- and a_k^+ are the extremities of I_k. Notice that $\varphi(a_k^\pm,y)=0$ unless $(a_k^\pm,y)\in\omega$, in which case $(a_k^\pm,y)\in\omega\cap\partial\Omega$. Thus

$$
\begin{aligned}
&\left| \int_{\mathbb{R}} 1\!\!1_\Omega\,(x_1,y)\,\frac{\partial\varphi}{\partial x_1}\,(x_1,y)\,dx_1 \right| \\
&\le \sum_{k\in K} \left(|\varphi(a_k^+,y)| + |\varphi(a_k^-,y)| \right) \le 2N(y)\,\|\varphi\|_\infty\,,
\end{aligned}
\tag{4.12}
$$

where the factor 2 comes from the fact that a given $(x_1,y)\in\omega\cap\partial\Omega$ may correspond to two intervals I_k. The desired estimate (4.9) follows from (4.11), (4.12) and (4.10).

Let us now verify (4.10). Given $\varepsilon>0$, choose a sequence of closed subsets E_k in $\mathbb{R}^n$ such that $\partial\Omega\cap\omega\subset\bigcup_k E_k$, $\operatorname{diam} E_k\le\varepsilon$ for all k, and $\sum_k(\operatorname{diam} E_k)^{n-1}\le H^{n-1}(\partial\Omega\cap\omega)+\varepsilon$. Set $F_k=\pi(E_k)$ and, for $y\in H$, denote by $N_\varepsilon(y)$ the supremum of the integers $\ell\ge 0$ such that there exist ℓ points $x_1,\cdots,x_\ell\in\partial\Omega\cap\omega\cap\pi^{-1}(y)$ which satisfy $|x_i-x_j|>2\varepsilon$ for $i\ne j$. Then $N_\varepsilon(y)\le\sum_k 1\!\!1_{F_k}(y)$, and so

$$
\begin{aligned}
\int_H N_\varepsilon(y)dy &\le C\sum_k (\operatorname{diam} E_k)^{n-1} \\
&\le CH^{n-1}(\partial\Omega\cup\omega)+C\varepsilon.
\end{aligned}
\tag{4.13}
$$

By sending ε to 0 we get (4.10) (from Fatou's lemma). We can finesse the issue of measurability of $N(y)$ by observing that the proof shows that $N(y)$

is less than or equal to a measurable function whose integral is bounded by $CH^{n-1}(\partial\Omega \cap \omega)$, which suffices for the proof of (4.9). This completes our proof of Lemma 4.7.

Next we want to prove that when W is a quasiminimizer for $N(\cdot)$, then

$$H^{n-1}\big|_E \le C_n \, |\nabla 1\!\!1_W| \tag{4.14}$$

(an inequality between measures), where $E = \partial W_0$ is the Ahlfors-regular set introduced in the last section. This estimate is less precise than Proposition 4.1, but it is better than (3.41) or (3.42) because in (4.14) we do not allow C_n to depend on anything other than the dimension. It will also be sufficient for the purposes of this paper.

The proof of (4.14) will only use the fact that W_0 is an open set that satisfies Condition B, as in the following variant of Definition 1.6.

Definition 4.15. Let W be a bounded open set in $\mathbb{R}^n$. We say that W satisfies Condition B if ∂W is an Ahlfors regular set of dimension $n-1$ (as in Definition 1.5) and if there is a constant C_1 such that, for each $x \in \partial W$ and each radius $0 < r \le \operatorname{diam} W$, we can find two balls B_1 and B_2 of radius $C_1^{-1} r$, contained in $B(x,r)$, and such that $B_1 \subset W$ and $B_2 \subset \mathbb{R}^n \backslash \overline{W}$.

Notice that we do not require that W or $\mathbb{R}^n \backslash \overline{W}$ be connected. The condition in this definition is slightly more restrictive than requiring that ∂W be a Condition B set; we also demand that one of the balls B_i be contained in one of the connected components of W and the other one in one of the components of $\mathbb{R}^n \backslash \overline{W}$. Of course a good example of Condition B open set is our set W_0 of the previous section (by Lemma 3.43).

Lemma 4.16. *There is a constant $C_n > 0$ such that, if W is an open set that satisfies Condition B, then*

$$H^{n-1}\big|_{\partial W} \le C_n \, |\nabla 1\!\!1_W| \, . \tag{4.17}$$

It is important that C_n does not depend on the Condition B constants for W. In fact, as in Proposition 4.1, one can even take $C_n = 1$, but we shall typically not need this fact.

Let W be an open set that satisfies Condition B, and set $E = \partial W$. The first property of E that we want to use is its rectifiability, and more precisely the existence of tangent planes to E almost everywhere. For each $x \in E$ and $t > 0$, set

$$\beta(x,t) = \inf_P \left\{ \sup_{y \in E \cap B(x,t)} t^{-1} \operatorname{dist}(y,P) \right\}, \tag{4.18}$$

where the infimum is taken over all affine hyperplanes P. We claim that

$$\lim_{t\to 0} \beta(x,t) = 0 \quad \text{for } H^{n-1}\text{-almost every point } x \in E. \tag{4.19}$$

One way to see this is to observe that, since E satisfies Condition B, E "has big pieces of Lipschitz graphs" (see [**D**], [**DJ**], or [**DS1**]), and in particular E is rectifiable. Then E has approximate tangent planes almost everywhere, and it is easy to check that for Ahlfors regular sets, approximate tangent planes are the same as tangent planes. In terms of total amount of work, though, the best proof of (4.19) is probably to use Theorem 1.20 and Proposition 1.18 in [**DS1**] to get that E satisfies the "weak geometric lemma". (See Definition 1.16 in [**DS1**].) The fact that the weak geometric lemma implies (4.19) is only a matter of definitions.

Denote by μ the restriction to E of H^{n-1}. Because of Lemma 4.7, the Radon-Nikodym theorem says that there is $f \in L^\infty(\mu)$ such that

$$|\nabla \mathbb{1}_W| = f d\mu. \tag{4.20}$$

To prove Lemma 4.16, it will be enough to show that $f(x) \geq C_n^{-1}$ μ-almost everywhere. Let us first check that there is a constant $C(n)$, that depends only on the dimension n, such that

$$\int_{B(x,t)} |\nabla \mathbb{1}_W| \geq C(n)^{-1} t^{n-1} \tag{4.21}$$

for each $x \in E$ and each $t \leq \operatorname{diam} W$ such that $\beta(x,t) \leq (10C_1)^{-1}$, say, where C_1 is the constant in the definition of Condition B.

Indeed, suppose that $0 < t \leq \operatorname{diam} W$ and $\beta(x,t) \leq (10C_1)^{-1}$, let P be a plane that realizes the infimum in the definition (4.18) of $\beta(x,t)$, and denote by $\mathcal{U}^\pm$ the two connected components of $\{y \in B(x,t) : \operatorname{dist}(y,P) > t\beta(x,t)\}$. By definition of P, $\mathcal{U}^+$ and $\mathcal{U}^-$ do not meet E.

Because W satisfies Condition B, we can find two balls B_1 and B_2, of radius $C_1^{-1}t$, such that $B_1 \subset B(x,t) \cap W$ and $B_2 \subset B(x,t) \setminus \overline{W}$. The set $B(x,t) \setminus (\mathcal{U}^+ \cup \mathcal{U}^-)$ is too thin to contain B_1 or B_2, and so each of them meets $\mathcal{U}^+$ or $\mathcal{U}^-$. Suppose for definiteness that B_1 meets $\mathcal{U}^+$. Then $\mathcal{U}^+ \subset W$ because it is connected and does not meet E. Then B_2 cannot meet $\mathcal{U}^+$ (because $B_2 \subset B(x,t)\setminus\overline{W}$), and so B_2 meets $\mathcal{U}^-$. Then $\mathcal{U}^- \subset B(x,t)\setminus\overline{W}$ for the same reason as above. Thus $\mathbb{1}_W$ equals 0 on $\mathcal{U}^-$ and 1 on $\mathcal{U}^+$, and (4.21) follows fairly easily. Let us sketch the argument. Without loss of generality, we may assume that $x = 0$ and P is parallel to the hyperplane $\{x_1 = 0\}$. Choose a bump function of the form $\varphi(x) = \varphi_1\left(\frac{x_1}{t}\right)\psi\left(\frac{x_2}{t}, \cdots, \frac{x_n}{t}\right)$, where

ψ is a nonnegative bump function with integral 1 and small support around 0, and φ_1 is even, supported in $[-\frac{1}{2}, \frac{1}{2}]$ equal to 1 on $[-\frac{1}{4}, \frac{1}{4}]$ and decreasing on $]\frac{1}{4}, \frac{1}{2}[$. Then

$$\begin{aligned} t^{n-1} &= \left| \int \frac{\partial \varphi}{\partial x_1} \mathbb{1}_W \right| = \left| \int \varphi \frac{\partial}{\partial x_1} \mathbb{1}_W \right| \\ &\leq \|\varphi\|_\infty \int_{B(x,t)} |\nabla \mathbb{1}_W| \\ &= \|\varphi_1\|_\infty \, \|\psi\|_\infty \int_{B(x,t)} |\nabla \mathbb{1}_W| \, . \end{aligned} \tag{4.22}$$

This proves (4.21).

Because of (4.21) and (4.19),

$$\liminf_{t \to 0} t^{1-n} \int_{B(x,t)} |\nabla \mathbb{1}_W| \geq C(n)^{-1} \tag{4.23}$$

for almost every point $x \in E$.

On the other hand, it is well known that there is a constant $C'(n)$ such that

$$\limsup_{t \to 0} \frac{H^{n-1}(E \cap B(x,t))}{t^{n-1}} \leq C'(n) \tag{4.24}$$

for every compact set E such that $H^{n-1}(E) < +\infty$ and H^{n-1}-almost every point $x \in E$, and so

$$\liminf_{t \to 0} \frac{t^{n-1}}{\mu(B(x,t))} \geq C'(n)^{-1}$$

for μ-almost every $x \in E$. Thus

$$\liminf_{t \to 0} \mu(B(x,t))^{-1} \int_{B(x,t)} |\nabla \mathbb{1}_W| \geq C(n)^{-1} C'(n)^{-1}. \tag{4.25}$$

By a standard differentiation theorem (see [**Fe**] or [**Ma**]),

$$f(x) = \lim_{t \to 0} \mu(B(x,t))^{-1} \int_{B(x,t)} |\nabla \mathbb{1}_W| \tag{4.26}$$

for μ-almost every $x \in E$. Note that in the present case, (4.26) can be obtained more easily than for general measures, because E is Ahlfors-regular and so we may use the standard proof of the Lebesgue differentiation theorem in $\mathbb{R}^n$.

From (4.26) and (4.25), we deduce that $f(x) \geq C(n)^{-1} C'(n)^{-1}$ μ-almost everywhere, and Lemma 4.16 follows.

5. Domains of isoperimetry.

Let Q_0, Q_1 be closed cubes in $\mathbb{R}^n$, with $Q_0 \subset \mathrm{int}(Q_1)$. We have seen in Section 3 that to each quasiminimizer W for $N(\cdot)$, we can associate a unique open set W_0 such that $|W_0 \backslash W| = |W \backslash W_0| = 0$ and which satisfies Condition B. [See Theorem 3.1 and Lemma 3.43 for the existence, the end of Remark 3.46 for uniqueness, and Definition 4.15 for Condition B open sets.] We shall call such a set W_0 a *normalized quasiminimizer* for $N(\cdot)$.

The aim of this section is to prove that if W_0 is a normalized quasiminimizer for $N(\cdot)$, then W_0 and $\mathbb{R}^n \backslash W_0$ are connected, and are even "domains of isoperimetry". We shall see in Section 6 that this, together with Condition B, implies that W_0 and $\mathbb{R}^n \backslash W_0$ are John domains.

Definition 5.1. An open set $W \subset \mathbb{R}^n$ is a domain of isoperimetry if there is a constant $C_2 > 0$ so that if Ω is any open set in W, then

$$\min\left(|\Omega|, |W \backslash \Omega|\right) \leq C_2 H^{n-1}\left(W \cap \partial\Omega\right)^{\frac{n}{n-1}}. \tag{5.2}$$

Remark 5.3. It is clear from the definition that domains of isoperimetry are connected if they have finite measure or if they contain the complement of a ball.

Note that (5.2) is just an isoperimetric inequality for Ω as a space in its own right, without regard to the ambient space. Cubes, balls, or $\mathbb{R}^n$ itself are domains of isoperimetry (with uniform constants).

Theorem 5.4. *Let Q_0, Q_1 be closed cubes in $\mathbb{R}^n$, with $Q_0 \subset \mathrm{int}(Q_1)$, and let W be a normalized quasiminimizer for $N(\cdot)$. Then W and $\mathbb{R}^n \backslash \overline{W}$ are domains of isoperimetry. Moreover, the constant C_2 in (5.2) can be taken to depend only on n, Q_0, Q_1, and the constant α in (1.4).*

To prove this it will be easier to work with a minor reformulation of the conditions that W and $\mathbb{R}^n \backslash W$ be domains of isoperimetry. We start with W.

Lemma 5.5. *Let Q_0, Q_1 be closed cubes in $\mathbb{R}^n$, with $Q_0 \subset \mathrm{int}(Q_1)$, and let W be an open set such that $\mathrm{int}(Q_0) \subset W \subset Q_1$. Then W is a domain of isoperimetry if and only if there is a constant $C > 0$ so that*

$$|W \backslash \Omega| \leq C H^{n-1}\left(\partial\Omega \cap W\right)^{\frac{n}{n-1}} \tag{5.6}$$

for all open subsets Ω of W with $\mathrm{int}(Q_0) \subset \Omega$.

If W is a domain of isoperimetry and if Ω is an open subset of W such that $\mathrm{int}(Q_0) \subset \Omega$, then (5.6) holds because $|W \backslash \Omega| \leq C\,|\Omega|$.

To prove the converse, let W be as in the lemma, and let us show that W is a domain of isoperimetry. Let Ω be an open subset of W. If $\partial\Omega \cap W$ has positive Lebesgue measure, then (5.2) holds trivially. Thus it is enough to consider the case when

$$|Q_0 \cap \Omega| \geq \frac{1}{2}|Q_0|, \tag{5.7}$$

and prove that

$$|W\backslash\Omega| \leq C_2 H^{n-1}(W \cap \partial\Omega)^{\frac{n}{n-1}}, \tag{5.8}$$

since otherwise we can work with $W\backslash\overline{\Omega}$ instead of Ω.

Let us apply (5.6) to the open set $\Omega' = \Omega \cup \text{int}(Q_0)$. We have that $\partial\Omega' \cap W \subset (\partial\Omega \cap W) \cup [(\partial Q_0\backslash\Omega) \cap W]$, and so (5.6) yields

$$|W\backslash\Omega'| \leq CH^{n-1}(\partial\Omega \cap W)^{\frac{n}{n-1}} + CH^{n-1}((\partial Q_0\backslash\Omega) \cap W)^{\frac{n}{n-1}}. \tag{5.9}$$

Also,

$$\begin{aligned} |W\backslash\Omega| &\leq |W\backslash\Omega'| + |\text{int}(Q_0)\backslash\Omega| \\ &\leq |W\backslash\Omega'| + CH^{n-1}(\text{int}(Q_0) \cap \partial\Omega)^{\frac{n}{n-1}}, \end{aligned} \tag{5.10}$$

because $\text{int}(Q_0)$ is a domain of isoperimetry and $|\text{int}(Q_0)\backslash\Omega| \leq |\text{int}(Q_0) \cap \Omega|$ by (5.7). Thus (5.8) will follow as soon as we prove that

$$H^{n-1}((\partial Q_0\backslash\Omega) \cap W) \leq CH^{n-1}(\partial\Omega \cap W). \tag{5.11}$$

[This looks rather reasonable, $\partial Q_0\backslash\Omega$ and $\partial\Omega \cap Q_0$ are the two pieces of the boundary of $\text{int}(Q_0)\backslash\Omega$, and one expects the first one not to be much larger than the second one.] Note that $H^{n-1}(\partial Q_0 \cap \partial\Omega \cap W)$ is under control, and so it is enough to show that

$$H^{n-1}\left(\partial Q_0\backslash\overline{\Omega}\right) \leq CH^{n-1}(\partial\Omega \cap W). \tag{5.12}$$

Set $Q = \lambda Q_0$, where $\lambda \in (0,1)$ is chosen so that $|Q_0\backslash Q| = \frac{1}{4}|Q_0|$. Thus $|\Omega \cap Q| \geq \frac{1}{4}|Q_0|$, by (5.7). Given $p \in \Omega \cap Q$, define $r_p : \mathbb{R}^n\backslash\{p\} \to \partial Q_0$ to be the obvious "radial projection". [In other words, $r_p(x) - p$ is a positive multiple of $x - p$.] For each $p \in \Omega \cap Q$ we have that

$$\partial Q_0\backslash\overline{\Omega} \subset r_p(\text{int}(Q_0) \cap \partial\Omega), \tag{5.13}$$

because for each $z \in \partial Q_0\backslash\overline{\Omega}$, the line segment (p, z) must meet $\partial\Omega$.

Of course the mapping $x \to r_p(x)$ is not Lipschitz with uniform estimates, even if we choose p as far as possible from $\partial\Omega$, but we still have that

(5.14)
$$H^{n-1}\left(\partial Q_0 \backslash \overline{\Omega}\right) \leq \int_{\mathrm{Int}(Q_0)\cap\partial\Omega} |\nabla r_p(x)|^{n-1}\, dH^{n-1}(x) \\ \leq C \int_{\mathrm{Int}(Q_0)\cap\partial\Omega} \left(\frac{\operatorname{diam} Q_0}{|x-p|}\right)^{n-1} dH^{n-1}(x).$$

We now get (5.12) by averaging (5.14) over $p \in \Omega \cap Q$, using also the fact that $|\Omega \cap Q| \geq \frac{1}{4}|Q_0|$. This completes our proof of Lemma 5.5.

Here is the analogue of Lemma 5.5 for unbounded domains.

Lemma 5.15. *Let W_1 be an open set such that $\mathbb{R}^n \backslash Q_1 \subset W_1$. Then W_1 is a domain of isoperimetry if and only if there is a constant $C > 0$ so that*

$$|W_1 \backslash \Omega| \leq C H^{n-1}\left(\partial\Omega \cap W_1\right)^{\frac{n}{n-1}} \tag{5.16}$$

for all open subsets Ω of W_1 such that $\Omega \supset \mathbb{R}^n \backslash Q_1$.

If W_1 is a domain of isoperimetry that contains $\mathbb{R}^n \backslash Q_1$, and if Ω is as in the lemma, (5.16) holds because it is exactly the same as (5.2) in this case. To prove the converse, we give ourselves an open set W_1 with the property of Lemma 5.15 and an open set $\Omega \subset W_1$, and we want to check (5.2). As before, the case $|\partial\Omega \cap W_1| > 0$ is trivial, and since we may always replace Ω by $W_1 \backslash \overline{\Omega}$, it is enough to prove (5.2) when

$$|(\mathbb{R}^n \backslash Q_1) \backslash \Omega| \leq |(\mathbb{R}^n \backslash Q_1) \cap \Omega|. \tag{5.17}$$

Observe that in this case $|\Omega| = |(\mathbb{R}^n \backslash Q_1) \cap \Omega| = +\infty$, and (5.2) is the same as

$$|W_1 \backslash \Omega| \leq C_2 H^{n-1}\left(W_1 \cap \partial\Omega\right)^{\frac{n}{n-1}}. \tag{5.18}$$

It is not too hard to check that $\mathbb{R}^n \backslash Q_1$ is a domain of isoperimetry. Thus

$$|(\mathbb{R}^n \backslash Q_1) \backslash \Omega)| \leq C H^{n-1}\left((\mathbb{R}^n \backslash Q_1) \cap \partial\Omega\right)^{\frac{n}{n-1}} \\ \leq C H^{n-1}\left(W_1 \cap \partial\Omega\right)^{\frac{n}{n-1}}. \tag{5.19}$$

Now apply (5.16) to the open set $\Omega' = \Omega \cup (\mathbb{R}^n \backslash Q_1)$. Since $W_1 \backslash \Omega' = (W_1 \cap Q_1) \backslash \Omega$ and $\partial\Omega' \subset \partial\Omega \cup (\partial Q_1 \backslash \Omega)$, we get that

$$|(W_1 \cap Q_1) \backslash \Omega| \leq C H^{n-1}\left(W_1 \cap \partial\Omega\right)^{\frac{n}{n-1}} + C H^{n-1}\left(W_1 \cap \partial Q_1 \backslash \Omega\right)^{\frac{n}{n-1}}. \tag{5.20}$$

Putting (5.19) and (5.20) together, we get that

$$|W_1 \backslash \Omega| \leq C H^{n-1} (W_1 \cap \partial\Omega)^{\frac{n}{n-1}} + C H^{n-1} (W_1 \cap \partial Q_1 \backslash \Omega)^{\frac{n}{n-1}}. \tag{5.21}$$

Moreover, if $|(W_1 \cap Q_1) \backslash \Omega| \leq 2\, |(\mathbb{R}^n \backslash Q_1) \backslash \Omega|$, then we do not need (5.20), and (5.18) follows directly from (5.19). Thus it is enough to prove (5.18) when

$$|(\mathbb{R}^n \backslash Q_1) \backslash \Omega| \leq \frac{1}{2} |Q_1|. \tag{5.22}$$

Let us check that

$$H^{n-1} ((W_1 \cap \partial Q_1) \backslash \Omega) \leq C H^{n-1} (W_1 \cap \partial\Omega) \tag{5.23}$$

when (5.22) holds; the lemma will then follow at once from this and (5.21).

The proof of (5.23) is practically the same as for (5.12). It suffices to show that

$$H^{n-1} \left(\partial Q_1 \backslash \overline{\Omega}\right) \leq C H^{n-1} (W_1 \cap \partial\Omega). \tag{5.24}$$

Let $Q_{1,j}$, $1 \leq j \leq 3^n - 1$ denote the various translates of Q_1, adjacent to Q_1, which fill up $3Q_1 \backslash \mathrm{Int}(Q_1)$ in the obvious way. In order to get (5.24) it is clearly sufficient to show that

$$H^{n-1} \left(\partial Q_{1,j} \backslash \overline{\Omega}\right) \leq C H^{n-1} (\mathrm{Int}\,(Q_{1,j}) \cap \partial\Omega) \tag{5.25}$$

for each j. These estimates in turn follow from exactly the same argument as used to establish (5.12), because our assumption (5.22) implies that $|Q_{1,j} \cap \Omega| \geq \frac{1}{2} |Q_{1,j}|$ for each j. (Also $\mathrm{Int}(Q_{1,j}) \subseteq W_1$ for each j.) This completes the proof of Lemma 5.15.

Let us now return to the proof of Theorem 5.4. Let W be a normalized quasiminimizer; we want to prove that W satisfies the condition of Lemma 5.5, and so we give ourselves an open set Ω such that

$$\mathrm{int}(Q_0) \subset \Omega \subset W, \tag{5.26}$$

and we want to prove that (5.6) holds. Of course we may assume that $H^{n-1} (\partial\Omega \cap W) < +\infty$ because otherwise there is nothing to prove. Notice that $H^{n-1}(\partial\Omega) \leq H^{n-1} (W \cap \partial\Omega) + H^{n-1}(\partial W) < +\infty$, and so $1\!\!1_\Omega \in BV$ by Lemma 4.7. Thus $\Omega \in \mathcal{F}$, and (1.4) says that

$$N (1\!\!1_W) \leq N (1\!\!1_\Omega) + \alpha N (1\!\!1_W - 1\!\!1_\Omega). \tag{5.27}$$

If we want to use (5.27), it will be useful to understand better the measures $\nu = \nabla 1\!\!1_W$, $\nu_+ = \nabla 1\!\!1_\Omega$ and $\nu_- = \nabla 1\!\!1_{W\setminus\Omega}$. The more interesting part of the discussion will concern the parts of ν_+ and ν_- that live on the boundary of W, i.e. $\nu_\pm^b = 1\!\!1_{\partial W}\, \nu_\pm$. Let us first enumerate the obvious relations

$$\nu = \nu_+ + \nu_- = \nu_+^b + \nu_-^b, \tag{5.28}$$

which follow from the identity $1\!\!1_W = 1\!\!1_\Omega + 1\!\!1_{W\setminus\Omega}$ and the fact that ν is supported on $E = \partial W$.

Denote by μ the restriction of H^{n-1} to E. Because of Lemma 4.7 (or (4.6)) and the Radon-Nikodym theorem, there are bounded, measureable, vector-valued functions f, f_+ and f_- such that

$$\nu = f d\mu, \quad \nu_+^b = f_+ d\mu \quad \text{and} \quad \nu_-^b = f_- d\mu. \tag{5.29}$$

Obviously, $f = f_+ + f_-$. We shall get more information about these functions by looking more closely at the geometry of the situation.

Lemma 5.30. *For μ-almost every $x \in E$, we have either*

$$\lim_{r\to 0} r^{-n} \left|\Omega \cap B(x,r)\right| = 0 \tag{5.31}$$

or

$$\lim_{r\to 0} r^{-n} \left|(W\setminus\Omega) \cap B(x,r)\right| = 0. \tag{5.32}$$

In other words, at almost every point $x \in E$, Ω is asymptotically either as thick as it can be or as thin as it can be in W. To prove the lemma, consider the finite Borel measure $\eta = H^{n-1}_{|W\cap\partial\Omega}$. We claim that

$$\lim_{r\to 0} r^{1-n} \eta(B(x,r)) = 0 \ \text{ for } \mu\text{-almost all } x \in E. \tag{5.33}$$

This will only use the facts that $\eta(E) = 0$ and that μ is Ahlfors-regular of dimension $n-1$, i.e. that

$$C^{-1} r^{n-1} \le \mu(B(x,r)) \le C r^{n-1} \tag{5.34}$$

for all $x \in E$ and $0 < r \le \operatorname{diam} W$. [See (3.41) and (3.42), and remember that then μ denoted $|\nabla 1\!\!1_W|$ rather than $H^{n-1}_{|E}$.] To prove (5.33), notice first that for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\eta\left(\{y \in \mathbb{R}^n : \operatorname{dist}(y,E) \le \delta\}\right) < \varepsilon. \tag{5.35}$$

We then define a maximal function by

$$\eta^*(z) = \sup\left\{r^{1-n}\eta(B(z,r)) : 0 < r < \delta\right\}, \tag{5.36}$$

and we conclude that

$$\mu\left(\{z \in E : \eta^*(z) > \lambda\}\right) \leq \frac{C}{\lambda}\eta\left(\{y \in \mathbb{R}^n : \operatorname{dist}(y,E) \leq \delta\}\right) \leq \frac{C\varepsilon}{\lambda}, \tag{5.37}$$

by the standard proof of the Hardy-Littlewood maximal theorem that uses a covering argument of Vitali type, for instance. [The point is that all the balls that we consider are centered on E, and (5.34) gives a very good control on their mass for μ.] Applying (5.37) with $\lambda = \varepsilon^{\frac{1}{2}}$ we conclude that the lim sup of the quantity in (5.33) is $> \varepsilon^{\frac{1}{2}}$ only on a set of measure $\leq C\varepsilon^{\frac{1}{2}}$. Since $\varepsilon > 0$ is arbitrary it is easy to conclude that (5.33) holds.

We now return to the proof of Lemma 5.30. Let $x \in E$ be such that

$$\lim_{r\to 0} r^{1-n} H^{n-1}\left(W \cap \partial\Omega \cap B(x,r)\right) = 0, \tag{5.38}$$

and also $\lim\limits_{r\to 0} \beta(x,r) = 0$, where $\beta(x,r)$ is as in (4.18). We know from (5.33) and (4.19) that these equalities hold for μ-almost every $x \in E$.

Suppose that r is so small that $\beta(x,r) \leq \varepsilon$, where $\varepsilon > 0$ is given in advance. Let P denote a hyperplane that realizes the infimum in the definition (4.18) of $\beta(x,r)$, and denote by $\mathcal{U}^+$ and $\mathcal{U}^-$ the two connected components of $B(x,r)\backslash\{y \in \mathbb{R}^n : \operatorname{dist}(y,P) \leq \varepsilon r\}$. Since $\beta(x,r) \leq \varepsilon$, we know that E does not meet $\mathcal{U}^+$ or $\mathcal{U}^-$. If ε is small enough, depending on the condition B constant for the open set W, we conclude that $\mathcal{U}^+ \subset W \cap B(x,r) \subset B(x,r)\backslash\mathcal{U}^-$, or else $\mathcal{U}^- \subset W \cap B(x,r) \subset B(x,r)\backslash\mathcal{U}^+$. Suppose for definiteness that we are in the first case. If r is small enough, (5.38) and the isoperimetric inequality tell us that either almost all of $\mathcal{U}^+$ is contained in Ω, or else almost all of $\mathcal{U}^+$ is contained in $W\backslash\Omega$. (This is not hard to check. One can make it look like more standard statements using the fact that $\mathcal{U}^+$ is bilipschitz equivalent to a ball and a cube, with uniform bounds.) This argument shows that

$$\lim_{r\to 0} \operatorname{Min}\left(d^+(r), d^-(r)\right) = 0, \tag{5.39}$$

where we set $d^+(r) = r^{-n}\left|\Omega \cap B(x,r)\right|$ and $d^-(r) = r^{-n}\left|(W\backslash\Omega) \cap B(x,r)\right|$. This is not exactly the same as the alternative (5.31) or (5.32), which requires that either $d^+(r)$ or $d^-(r)$ tend to 0, because one could imagine a situation where $d^+(r)$ is small for some values of r, and $d^-(r)$ is small for the other ones. It is easy to see that, since $d^+(r) + d^-(r) \geq C^{-1}$ for r small enough, this situation cannot occur for continuity reasons. This completes our proof of Lemma 5.30.

Remark 5.40. Lemma 5.30 is also true, with the same proof, if we replace W by $W_1 = \mathbb{R}^n \setminus \overline{W}$, and Ω by any open set $\Omega' \subset \mathbb{R}^n \setminus \overline{W}$ such that $H^{n-1}(\partial\Omega') < +\infty$.

Let us now come back to our measures ν, $\nu_\pm^b$ and our functions f and $f_\pm$.

Lemma 5.41. *For μ-almost every point $x \in E$, we either have that*

$$f_+(x) = 0 \qquad \text{and} \qquad f(x) = f_-(x) \tag{5.42}$$

or

$$f_-(x) = 0 \qquad \text{and} \qquad f(x) = f_+(x). \tag{5.43}$$

Let $x \in E$ be a Lebesgue point for f_+ and f_- (with respect to μ), and suppose that (5.38) holds and that we have (5.31) or (5.32). We know from Lemma 5.30 and (5.33) that this is the case for μ-almost every point of E. We want to show that $f_+(x) = 0$ or $f_-(x) = 0$.

For each small $r > 0$, choose a bump function φ_r on $\mathbb{R}^n$ supported in $B(x,r)$ and such that $0 \le \varphi_r \le Cr^{-n+1}$, $|\nabla\varphi_r| \le Cr^{-n}$, and $\int_E \varphi_r d\mu = 1$. Since x is a Lebesgue point for $f_\pm$ we have that

$$f_\pm(x) = \lim_{r\to 0} \int_E \varphi_r f_\pm d\mu. \tag{5.44}$$

On the other hand,

$$\begin{aligned} \int_E \varphi_r f_\pm d\mu &= \int \varphi_r d\nu_\pm^b \\ &= \int \varphi_r d\nu_\pm - \int_W \varphi_r d\nu_\pm. \end{aligned} \tag{5.45}$$

Because of Lemma 4.7 (or (4.6)),

$$\begin{aligned} \left|\int_W \varphi_r d\nu_\pm\right| &= \left|\int_W \varphi_r \nabla \mathbb{1}_\Omega\right| \\ &\le C \int_{\partial\Omega\cap W} |\varphi_r|\, dH^{n-1} \\ &\le C \left\|\varphi_r\right\|_\infty H^{n-1}\left(\partial\Omega \cap W \cap B(x,r)\right) \end{aligned} \tag{5.46}$$

which tends to 0 by (5.38).

Suppose that (5.31) holds. Then

$$\left|\int \varphi_r d\nu_+\right| = \left|\int \nabla\varphi_r 1\!\!1_\Omega\right| \le \|\nabla\varphi_r\|_\infty |\Omega \cap B(x,r)|, \tag{5.47}$$

which tends to 0 when $r \to 0$ because of (5.31). In this case $f_+(x) = 0$.

If (5.32) holds, then

$$\left|\int \varphi_r d\nu_-\right| = \left|\int \nabla\varphi_r 1\!\!1_{W\setminus\Omega}\right| \le \|\nabla\varphi_r\|_\infty |(W\setminus\Omega) \cap B(x,r)|, \tag{5.48}$$

which tends to 0 when $r \to 0$. This time, $f_-(x) = 0$. This proves that $f_+(x) = 0$ or $f_-(x) = 0$ for μ-almost every $x \in E$. Lemma 5.41 follows, because we already know that $f = f_+ + f_-$.

We are now ready to use the information given by (5.27). Let E_+ denote the piece of E where $f_- = 0$, and E_- denote the rest of E.

Lemma 5.41 tells us that

$$N(1\!\!1_\Omega) = \int_W |\nabla 1\!\!1_\Omega| + \int_{E_+} |\nabla 1\!\!1_W| \tag{5.49}$$

and

$$N(1\!\!1_W - 1\!\!1_\Omega) = \int_W |\nabla 1\!\!1_\Omega| + \int_{E_-} |\nabla 1\!\!1_W| \tag{5.50}$$

(because $\nabla 1\!\!1_\Omega = -\nabla 1\!\!1_{W\setminus\Omega}$ on W), while

$$N(1\!\!1_W) = \int_{E_+} |\nabla 1\!\!1_W| + \int_{E_-} |\nabla 1\!\!1_W|. \tag{5.51}$$

Thus (5.27) is the same as

$$(1-\alpha)\int_{E_-} |\nabla 1\!\!1_W| \le (1+\alpha)\int_W |\nabla 1\!\!1_\Omega|. \tag{5.52}$$

Now use (4.6) (or Lemma 4.7 if you do not want sharp constants) to estimate the right-hand side, and then (5.50) again to get

$$N(1\!\!1_{W\setminus\Omega}) \le \frac{2}{1-\alpha} H^{n-1}(W \cap \partial\Omega). \tag{5.53}$$

By the (standard) isoperimetric inequality in $\mathbb{R}^n$,

$$|W\backslash\Omega| \leq CN\left(1\!\!1_{W\backslash\Omega}\right)^{\frac{n}{n-1}} \leq CH^{n-1}\left(W \cap \partial\Omega\right)^{\frac{n}{n-1}}, \tag{5.54}$$

which is (5.6). Thus we proved that W satisfies the condition of Lemma 5.5, and that it is a domain of isoperimetry.

To complete the proof of Theorem 5.4, we still need to prove that $W_1 = \mathbb{R}^n\backslash\overline{W}$ is also a domain of isoperimetry. Fortunately, the proof is very similar to what we did for W.

This time we want to show that W_1 satisfies the condition of Lemma 5.15, and so we give ourselves an open subset Ω of W_1 such that $\Omega \supset \mathbb{R}^n\backslash Q_1$, and we want to prove (5.16).

We may as well assume that $H^{n-1}(\partial\Omega) < +\infty$, because otherwise there is nothing to prove. Thus $1\!\!1_{\mathbb{R}^n\backslash\Omega}$ lies in BV. Moreover, $W_1 = \mathbb{R}^n\backslash\overline{W} \supset \Omega \supset \mathbb{R}^n\backslash Q_1$, and so $\operatorname{int}(Q_0) \subset W \subset \overline{W} \subset \mathbb{R}^n\backslash\Omega \subset Q_1$, so that $V = \mathbb{R}^n\backslash\Omega$ lies in $\mathcal{F}$ and the quasiminimality of W implies that

$$N\left(1\!\!1_W\right) \leq N\left(1\!\!1_V\right) + \alpha N\left(1\!\!1_{V\backslash W}\right). \tag{5.55}$$

This time we want to study the measures

$$\nu = \nabla 1\!\!1_{W_1}, \quad \nu_+ = \nabla 1\!\!1_{\Omega} \quad \text{and} \quad \nu_- = 1\!\!1_{W_1\backslash\Omega},$$

as well as their boundary parts $\nu_\pm^b = 1\!\!1_E\ \nu_\pm$. We still have the identity (5.28) for the same reason as before, and we still can use Lemma 4.7 and the Radon-Nikodym theorem to write ν, ν_+^b and ν_-^b as density measures relative to $\mu = H^{n-1}_{|E}$ as in (5.29).

The analogue of Lemma 5.30 (but with W_1 instead of W) still holds, as we observed in Remark 5.40, and Lemma 5.41 also works with the same proof. [We have used the fact that W is a Condition B domain, but not the fact that it is bounded.]

Let us still decompose E into E_+, where $f = f_+$ and $f_- = 0$, and $E_- = E\backslash E_+$, where $f = f_-$ and $f_+ = 0$ μ-almost everywhere. Then

$$\begin{aligned} N\left(1\!\!1_V\right) &= N\left(1\!\!1_\Omega\right) \\ &= \int_{W_1} \left|\nabla 1\!\!1_\Omega\right| + \int_{E_+} \left|\nabla 1\!\!1_{W_1}\right|, \end{aligned} \tag{5.56}$$

$$\begin{aligned} N\left(1\!\!1_{V\backslash W}\right) &= N\left(1\!\!1_{V\backslash\overline{W}}\right) = N\left(1\!\!1_{W_1\backslash\Omega}\right) \\ &= \int_{W_1} \left|\nabla 1\!\!1_\Omega\right| + \int_{E_-} \left|\nabla 1\!\!1_{W_1}\right|, \end{aligned} \tag{5.57}$$

and

$$N\left(1\!\!1_W\right) = N\left(1\!\!1_{\overline{W}}\right) = N\left(1\!\!1_{W_1}\right) = \int_{E_+} \left|\nabla 1\!\!1_{W_1}\right| + \int_{E_-} \left|\nabla 1\!\!1_{W_1}\right|. \tag{5.58}$$

Now (5.55) implies that

$$(1-\alpha)\int_{E_-} \left|\nabla 1\!\!1_{W_1}\right| \leq (1+\alpha)\int_{W_1} \left|\nabla 1\!\!1_{\Omega}\right|. \tag{5.59}$$

We may use (4.6) and (5.57) again to get

$$N\left(1\!\!1_{W_1\setminus\Omega}\right) \leq \frac{2}{1-\alpha}\int_{W_1} \left|\nabla 1\!\!1_{\Omega}\right| \leq \frac{2}{1-\alpha} H^{n-1}\left(W_1 \cap \partial\Omega\right), \tag{5.60}$$

and then the isoperimetric inequality to deduce (5.16) from (5.60). Hence W_1 satisfies the condition of Lemma 5.15, and is a domain of isoperimetry as well.

This completes our proof of Theorem 5.4.

6. Isoperimetry and John domains.

The main goal of this section is to prove the following result.

Theorem 6.1. *Let W be a bounded domain that contains $B(0,1)$. Suppose that W satisfies Condition B and is a domain of isoperimetry. Then W is a John domain with center 0. The constants for the John condition can be bounded in terms of diam W and the Condition B and domain-of-isoperimetry constants for W.*

See Definition 4.15 for Condition B, Definition 5.1 for domains of isoperimetry, and Definition 1.7 for John domains.

Note that John domains are domains of isoperimetry, as in Theorem 5.1 of [**B**] (with $p = 1$, $p^* = \frac{n}{n-1}$).

There is an obvious analogue of Theorem 6.1 for unbounded domains. In this version we ask that the given domain W_1 contain the complement of a ball, that it be a domain of isoperimetry, and that it satisfy the natural variant of Condition B in this situation, in which the radius r in Definition 4.15 is constrained by $r \leq \operatorname{diam}(\mathbb{R}^n \setminus W_1)$. We can derive the analogue of Theorem 6.1 for this case from the statement given above in the following manner. Let x_0 be a point in the complement of W_1 such that dist $(x_0, W_1) \geq C^{-1} \operatorname{diam}(\mathbb{R}^n \backslash W_1)$. The existence of such a point is provided by our Condition B assumption. Let ζ be the inversion of $\mathbb{R}^n$ about the sphere $|x - x_0| = \operatorname{diam}(\mathbb{R}^n \backslash W_1)$, say, and set $W = \zeta(W_1) \cup \{x_0\}$. Then W is a bounded domain which contains the ball centrered at x_0 with radius $\operatorname{diam}(\mathbb{R}^n \backslash W_1)$, and it is contained in a ball which is larger by only a bounded factor. It is easy to check that W satisfies Condition B and is a domain of isoperimetry, the latter because of the characterization provided by Lemmas 5.5 and 5.15. Theorem 6.1 implies that W is a John domain with center x_0, and it follows that W_1 itself is a John domain. Furthermore, the John constant for W_1 depends only on the diameter of $\mathbb{R}^n \backslash W_1$, and the Condition B and domain-of-isoperimetry constants for W_1.

Theorem 1.8 will follow as soon as we prove Theorem 6.1. Indeed, let $W \in \mathcal{F}$ be a quasiminimizer for $N(\cdot)$. Then the proof of Theorem 3.1 provides us with two open sets W_0, W_1 such that $E = \partial W_0 = \partial W_1$ is Ahlfors-regular, and $W_1 = \mathbb{R}^n \backslash \overline{W}_0$. [See (3.33), (3.36) and the remark before Lemma 3.43.] Moreover, W_0 and W_1 satisfy Condition B by Lemma 3.43, and the functions $1\!\!1_W$ and $1\!\!1_{W_0}$ are the same almost everywhere by (3.35). The uniqueness of W_0 (and hence $W_1 = \mathbb{R}^n \backslash \overline{W}_0$) with these properties was discussed in Remark 3.46. Then W_0 and W_1 are domains of isoperimetry by Theorem 5.4 and John domains by Theorem 6.1 and its analogue for unbounded domains.

Thus we can forget about quasiminimizers for the moment and concentrate on Theorem 6.1.

Remark 6.2. The John condition and the property of being a domain of isoperimetry are both quantitative strengthenings of connectedness. However, it is not true that a domain of isoperimetry is necessarily John, i.e., we cannot drop the Condition B assumption from Theorem 6.1. The point is that we can cut out a fine grid from a nice domain and get a domain of isoperimetry which is not John. Let us give an example of this phenomenon in the plane. [See Figure 2.] Given a positive integer n, set $E_n = \{t \in (0,1) : \frac{2j}{2n} \leq t \leq \frac{2j+1}{2n}$ for some integer j, $0 \leq j \leq n-1\}$. Set $W_n = ((0,1) \times (0,1)) \backslash (\{1/2\} \times E_n)$. These are domains of isoperimetry in $\mathbb{R}^2$ with uniformly bounded constant, their boundaries are Ahlfors regular and uniformly rectifiable with bounded constants, but the John constants tend to $+\infty$ as $n \to +\infty$. We could also construct a domain of isoperimetry which is not John although ∂W is Ahlfors-regular and uniformly rectifiable, as in Figure 3.

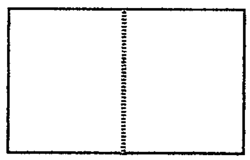

Figure 2.

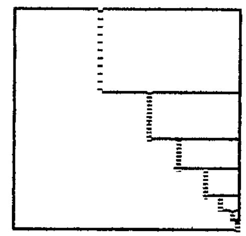

Figure 3.

One could prevent the preceding type of example by requiring that $\mathbb{R}^n \setminus \overline{W}$ be also a domain of isoperimetry, and that it have the same boundary as W. If one also demands that ∂W be Ahlfors-regular, than one can perhaps derive Condition B from arguments like the ones used in Section 3. Without the assumption of Ahlfors-regularity of the boundary, it seems that one can construct counterexamples in $\mathbb{R}^3$. The generalized Eiffel tower of Figure 4 should be a domain W such that W and $\mathbb{R}^n \setminus \overline{W}$ are domains of isoperimetry with a common boundary E, W and $\mathbb{R}^n \setminus W$ satisfy the same conditions for the existence of balls as in the definition of Condition B, but E is not regular and W is not John.

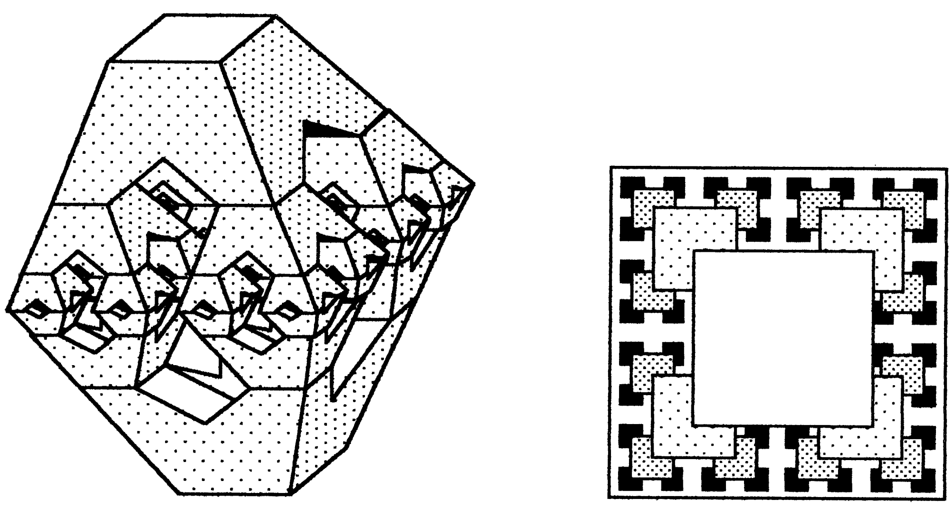

Figure 4. On the left, a picture of a finite approximation of the generalized Eiffel tower. On the right, its sections by various horizontal planes.

Let us now prove Theorem 6.1. From now on we let W be as in Theorem 6.1. We begin with a technical reduction.

Lemma 6.3. *In order to prove that W is John, it suffices to show that there is a constant $k > 0$ so that for each ball $B_0 \subset W$ with radius r we can find a ball $B_1 \subset W$ and a path γ in W with the following properties:*

$$\gamma \text{ connects } B_0 \text{ to } B_1; \tag{6.4}$$

$$\text{either } B_1 = B(0,1) \text{ or } B_1 \text{ has radius } 2r; \tag{6.5}$$

$$\operatorname{diam}\gamma \le kr; \tag{6.6}$$

$$\operatorname{dist}(\gamma, \mathbb{R}^n \backslash W) \ge k^{-1} r. \tag{6.7}$$

This lemma is a straightforward consequence of the definition of the John condition, and we leave the proof as an exercise.

Now we want to prove that W has the property stated in the lemma. Let $B_0 = B(x_0, r) \subset W$ be given. We want to construct various open subsets of W, and then apply our hypothesis that W is a domain of isoperimetry.

For each choice of $0 < t < r$ and $R > r$, set

$$W_{R,t} = \{x \in W \cap B(x_0, R) : \operatorname{dist}(x, \mathbb{R}^n \backslash W) > t\} \tag{6.8}$$

and then denote by $G = G_{R,t}$ the connected component of $W_{R,t}$ that contains x_0. This makes sense because $x_0 \in W_{R,t}$. Also note that $G_{R,t} \subset G_{R',t'}$ when $t \ge t'$ and $R \le R'$. Because of Lemma 6.3, it will be enough to prove that there is a $t \ge k^{-1} r$ and an $R \le kr$ such that

$$\begin{cases} \text{either } \ G_{R,t} \ \text{ contains a point } \ x_1 \text{ such that} \\ \operatorname{dist}(x_1, \mathbb{R}^n \backslash W) \ge 2r, \quad \text{or} \quad G_{R,t} \cap B(0,1) \ne \emptyset. \end{cases} \tag{6.9}$$

We want to replace $G_{R,t}$ by a larger domain with as little boundary in W as possible. To do so, it will be convenient to use the collection Δ_t of cubes in $\mathbb{R}^n$ which are cartesian products of intervals of the form $[jt, (j+1)t]$, $j \in \mathbb{Z}$. Thus the cubes in Δ_t cover $\mathbb{R}^n$ and have disjoint interiors. Set

$$\widetilde{\Delta}_t = \{Q \in \Delta_t : \operatorname{dist}(Q, G_{R,t}) \le 10t\}, \tag{6.10}$$

and then

$$V = V_{R,t} = W \cap \left(\bigcup_{Q \in \widetilde{\Delta}_t} Q \right). \tag{6.11}$$

We shall want to evaluate the size of the part of $\partial V \cap W$ which is not too close to $\partial B(x_0, R)$. Let us assume that

$$100nt \leq r \leq \frac{R}{10} \leq \frac{kr}{10} \tag{6.12}$$

(other conditions will show up later). We are mostly interested in the set

$$E_{R,t} = \{x \in \partial V \cap W \cap B(x_0, 8R/10)\}. \tag{6.13}$$

This set will be controlled using the Condition B hypothesis on W. Let $\beta(x,t)$ be as in (4.18), only with E replaced with ∂W.

Lemma 6.14. *There is a constant $\tau > 0$, that depends only on n and the Condition B constant for W, such that for each $x \in E_{R,t}$,*

$$\mathrm{dist}(x, \partial W) \leq 20nt \tag{6.15}$$

and

$$\beta(y, 100nt) \geq \tau \qquad \textit{for all } \ y \in \partial W \cap B(x, 50nt). \tag{6.16}$$

To prove the lemma, let $x \in E_{R,t}$ be given. Then x lies in cubes Q_1, $Q_2 \in \Delta_t$, with $Q_1 \in \widetilde{\Delta}_t$ and $Q_2 \notin \widetilde{\Delta}_t$. Since $\mathrm{dist}\,(Q_1, G_{R,t}) \leq 10t$ and $\mathrm{dist}\,(Q_2, G_{R,t}) > 10t$, we have that $\mathrm{dist}\,(x, \partial G_{R,t}) \leq 10t + 2nt$. Since $G_{R,t}$ is a connected component of $W_{R,t}$, we get that $\mathrm{dist}\,(x, \partial W_{R,t}) \leq 10t + 2nt$. Because x is very far from $\partial B(x_0, R)$ (compared to t), this means that x lies within $10t + 2nt$ of $\{z \in W : \mathrm{dist}(z, \mathbb{R}^n \backslash W) \leq t\}$. This proves (6.15).

Now suppose that in addition there is a point $y \in \partial W \cap B(x, 50nt)$ such that $\beta(y, 100nt) \leq \tau$. We want to find a contradiction. Let P denote a hyperplane such that $\mathrm{dist}(z, P) \leq 100nt\tau$ for all $z \in \partial W \cap B(y, 100nt)$, and call H^+ and H^- the two connected components of $\{z \in B(y, 100ny) : \mathrm{dist}(z, P) > 100nt\tau\}$. By our choice of P, $H^{\pm}$ does not meet ∂W, and so each of H^+ and H^- is contained in W or in $\mathbb{R}^n \backslash \overline{W}$. If τ is small enough, depending on the Condition B constant for W, we even get that H^+ or H^- (say H^+ for definiteness) is contained in W and the other is contained in $\mathbb{R}^n \backslash \overline{W}$. [This is because there is not enough space between H^+ and H^- to fit a ball of reasonable size.]

Next we claim that if τ is small enough, then the set

$$H_0^+ = \{z \in H^+ \cap B(y, 90nt) : \mathrm{dist}(z, P) \geq 2t\} \tag{6.17}$$

is contained in $G_{R,t}$.

Notice that $H_0^+ \subset W_{R,t}$ because $x \in E_{R,t}$ (and hence $B(y, 90nt) \subset B(x_0, R)$ by (6.12) and (6.13)). Thus the claim will follow if we show that some point of H_0^+ lies in $G_{R,t}$. Let $w \in G_{R,t}$ be such that $|x - w| \le 11t + 2nt$. Such a point exists because $x \in Q_1$ and $\mathrm{dist}(Q_1, G_{R,t}) \le 10t$. If $w \in H_0^+$, then we are happy. So let us assume that $w \notin H_0^+$. Note that $|w - y| \le |w - x| + |x - y| \le 63nt$, so w is still well inside $B(y, 90nt)$. What happens in this case is that w is too close to P, but this will be fairly easy to fix.

We know that $w \in G_{R,t} \subset W_{R,t}$, and so $\mathrm{dist}(w, \mathbb{R}^n \backslash W) > t$. If τ is small enough, this implies that $w \in H^+$, and also that $\mathrm{dist}(w, \mathbb{R}^n \backslash W) \le 3t$ (because $w \notin H_0^+$ and $H^- \subseteq \mathbb{R}^n \backslash W$). [See Figure 5.]

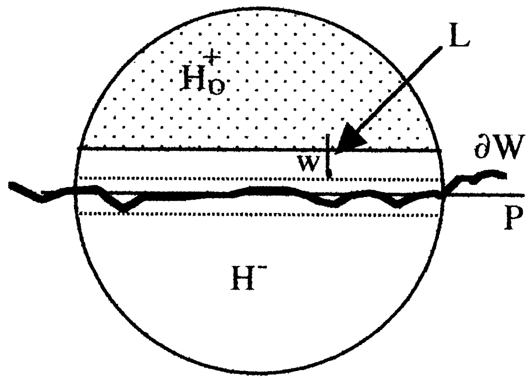

Figure 5.

Let L denote a line segment of length $2t$ which goes from w in the direction perpendicular to P and away from P. Thus L connects w to H_0^+. If $z \in L$, then $\mathrm{dist}(z, \partial W \cap B(y, 100nt)) \le 5t$, which is much smaller than the distance from z to the rest of ∂W or $\partial B(x_0, R)$. Therefore $\mathrm{dist}(z, \partial W) = \mathrm{dist}(z, \partial W \cap B(y, 100nt))$. Notice also that the distance from z to any point of $\partial W \cap B(y, 100nt)$ increases when z runs along L from w to H_0^+. [This is because $w \in H^+$ and $\partial W \cap B(y, 100nt)$ lies under H^+.] Thus L is contained in $W_{R,t}$, and hence in $G_{R,t}$. This proves our claim, that $H_0^+ \subset G_{R,t}$.

The desired contradiction follows from the claim, because $x \in Q_2$ and so $\mathrm{dist}(Q_2, G_{R,t}) \le \mathrm{dist}(x, H_0^+) \le 3t$ (since $x \in W \cap B(y, 50nt)$), which contradicts the definition of Q_2. This completes our proof of Lemma 6.14.

Recall from (6.13) and (6.11) that $E_{R,t}$ is composed of pieces of boundary of cubes $Q \in \Delta_t$. Lemma 6.14 and the Ahlfors-regularity of ∂W imply that

$$H^{n-1}(E_{R,t}) \le C H^{n-1}(\{y \in \partial W \cap B(x_0, R) : \beta(y, 100nt) \ge \tau\}), \tag{6.18}$$

where τ is as in (6.16). That is, to each ∂Q from (6.11) one associates a ball of radius t centered on ∂W, these balls have bounded overlap, and they lead to the estimate (6.18).

Let us now use Condition B to control the right-hand side of (6.18) on average. As was mentioned earlier, every regular set of dimension $n-1$ that satisfies Condition B satisfies a "weak geometric lemma". See [**DS1**], Theorem 1.20 on p. 680, Proposition 1.18 and Definition 1.16. (To be precise, this result is only stated and proved in [**DS1**] for unbounded regular sets, but if E is any bounded regular set that satisfies Condition B, then the union of E with any hyperplane that touches E is an unbounded regular set that satisfies Condition B.) Thus our set ∂W satisfies the weak geometric lemma. This means that for every choice of $\tau > 0$, there is a constant $C = C(\tau)$ that depends only on τ and the regularity and Condition B constants for W and for which

$$\int_0^R \int_{\{x \in \partial W \cap B(x_0,R) : \beta(x,t) > \tau\}} \frac{dH^{n-1}(x)dt}{t} \leq CR^{n-1}. \tag{6.19}$$

Because of (6.18), this implies that

$$\int_0^{(100n)^{-1}r} H^{n-1}\left(E_{R,t}\right) \frac{dt}{t} \leq CR^{n-1}. \tag{6.20}$$

(Do not bother to worry about the measurability of $H^{n-1}\left(E_{R,t}\right)$, it is enough to know that it is bounded by a measurable function which satisfies this estimate.)

Let us now choose t, depending on R, so that

$$k^{-1}r \leq t \leq (100n \operatorname{diam} W)^{-1} r \tag{6.21}$$

and

$$H^{n-1}\left(E_{R,t}\right) \leq C \left[\operatorname{Log} \frac{k}{100n \operatorname{diam} W}\right]^{-1} R^{n-1}. \tag{6.22}$$

The value of $\frac{R}{r}$ will be chosen soon, and then we shall choose k very large, so that the right-hand side of (6.22) will be very small compared to r^{n-1}. For the moment, we want to apply our hypothesis that W be a domain of isoperimetry to the sets

$$V(s) = V \cap B(x_0, s), \quad r < s < \frac{8R}{10}, \tag{6.23}$$

where V continues to be as in (6.11).

Remember that we are trying to find t and R such that (6.9) holds. If $G_{R,t}$ meets $B(0,1)$, then we are happy. So we may assume that $G_{R,t}$ does not meet $B(0,1)$. Let us first check that

$$|V(s)| \leq C\,|W\backslash V(s)|\,. \tag{6.24}$$

By construction, every point of V lies within $10t+2nt$ of $G_{R,t}$ and, since $t \leq (100n \operatorname{diam} W)^{-1} r \leq (100n)^{-1}$, V does not meet $B(0,1/2)$. Then (6.24) follows from the fact that W is bounded.

Because of (6.24), the isoperimetry condition (5.2) yields

$$\begin{aligned} |V(s)| &\leq C\,H^{n-1}\left(\partial V(s)\cap W\right)^{\frac{n}{n-1}} \\ &\leq CH^{n-1}\left(V\cap\partial B\left(x_0,s\right)\right)^{\frac{n}{n-1}} + CH^{n-1}\left(E_{R,t}\right)^{\frac{n}{n-1}} \end{aligned} \tag{6.25}$$

by the definitions of $V(s)$ and $E_{R,t}$.

Let us first dispose of the last term. We shall soon choose $R \leq C_1 r$, where the constant C_1 will depend on n, $\operatorname{diam} W$, and the isoperimetry, regularity and Condition B constants for W, but not on k. We choose k, depending on C_1 in particular, so large that if we apply to (6.25) the estimate that comes from (6.22), then the contribution of $H^{n-1}\left(E_{R,t}\right)$ in (6.25) is less than $\frac{1}{2}\left|B\left(x_0,\frac{r}{2}\right)\right|$, say. Observe that $B\left(x_0,\frac{r}{2}\right) \subset W_{R,t}$ by (6.8) and (6.21), so that $|V(s)| \geq \left|B\left(x_0,\frac{r}{2}\right)\right|$. Therefore, with this choice of k, (6.25) yields

$$|V(s)| \leq CH^{n-1}\left(V\cap\partial B\left(x_0,s\right)\right)^{\frac{n}{n-1}}\,. \tag{6.26}$$

For each $s \in [r,\frac{4R}{10})$, choose $\lambda \in (s,2s)$ such that $H^{n-1}(V\cap\partial B(x_0,\lambda)) \leq s^{-1}\left|V\cap B\left(x_0,2s\right)\right|$. [This is possible by Fubini and Tchebytchev.] We may now apply (6.26) to $V(\lambda)$ and get

$$|V(s)| \leq |V(\lambda)| \leq C\left[\frac{\left|V\cap B\left(x_0,2s\right)\right|}{s}\right]^{\frac{n}{n-1}}, \tag{6.27}$$

for $r \leq s \leq \frac{4R}{10}$. We rewrite this as

$$h(2s) \geq C^{-1}h(s)^{\frac{n-1}{n}}, \tag{6.28}$$

where $h(s) = s^{-n}\,|V(s)|$. Observe that

$$h(r) \geq C^{-1}, \tag{6.29}$$

because $V(r) \supset B\left(x_0, \frac{r}{2}\right)$, as we said earlier. From (6.28) and (6.29) we deduce that

$$h(2^m r) \geq C_2^{-1} \tag{6.30}$$

for all integers m such that $2^m r < \frac{4R}{10}$, and with a constant C_2 that depends only on the various constants in the statement of Theorem 6.1, but not on $\frac{R}{r}$ or k. (If $h(2^m r)$ were ever too small, then $h(r)$ would have to be even smaller, by iterating (6.28).)

Lemma 6.31. *If m is sufficiently large (depending only on n, the regularity constant for ∂W, and C_2) but $2^m r < \frac{4R}{10}$, then there is a point $z \in V(2^m r)$ such that* $\operatorname{dist}(z, \partial W) \geq 3r$.

To prove the lemma, choose a maximal subset Z of $V(r)$ with the property that $|z_1 - z_2| \geq 10r$ for z_1, $z_2 \in Z$, $z_1 \neq z_2$. Because of (6.30), Z has at least $C^{-1}C_2^{-1}2^{nm}$ points, where C depends on n and nothing else. Set $Z_1 = \{z \in Z : B(z, 3r) \text{ meets } \partial W\}$. For each $z \in Z_1$, $H^{n-1}(\partial W \cap B(z, 5r)) \geq C^{-1}r^{n-1}$ because ∂W is Ahlfors-regular, and since these sets are disjoint, the number of points in Z_1 is $\leq Cr^{1-n}H^{n-1}(\partial W \cap B(x_0, (2^m + 5)r)) \leq C2^{m(n-1)}$ again by Ahlfors-regularity of ∂W. If m is large enough, then Z has more elements than Z_1, and the lemma follows.

We may now choose m as in Lemma 6.31, and take $R = 2^{m+3}r$. Thus we keep our promise that $\frac{R}{r}$ would not depend on k, and this allows us to choose k and t as was explained earlier.

Let z be as in Lemma 6.31. Because $z \in V(2^m r) \subset V$, there is a point $x_1 \in G_{R,t}$ such that $|x_1 - z| \leq 10t + 2nt$. [See the definitions (6.10) and (6.11) of V.] Then $\operatorname{dist}(x_1, \mathbb{R}^n \backslash W) = \operatorname{dist}(x_1, \partial W) > 3r - 10t - 2nt > 2r$ by Lemma 6.31 and (6.21). (Recall that $\operatorname{diam} W \geq 2$ since $W \supset B(0,1)$.)

Thus we have finally proved that (6.9) holds. Theorem 6.1 follows, as was explained just before (6.9).

7. Regular bi-John domains are quasiminimizers.

So far we have obtained a lot of information about the structure of minimizers and quasiminimzers for $N(\cdot)$. Our next result says that if a domain W satisfies the conclusions of Theorem 1.8, then $1\!\!1_W$ is a minimizer for some functional $N_g(\cdot)$, and in particular it is a quasiminimizer for $N(\cdot)$.

Definition 7.1. Let W be a bounded domain in $\mathbb{R}^n$. We say that W is a regular bi-John domain if ∂W is an Ahlfors-regular set of dimension $n-1$, $\partial W = \partial(\mathbb{R}^n \backslash \overline{W})$, and W and $\mathbb{R}^n \backslash \overline{W}$ are both John domains.

See Definitions 1.5 and 1.7 for regular sets and John domains. Notice that if W is a regular bi-John domain, then W and $\mathbb{R}^n \backslash \overline{W}$ are connected open sets that satisfy Condition B. The following is a restatement of Theorem 1.12.

Theorem 7.2. *Let W be a bounded regular bi-John domain. Let Q_0, Q_1 be two cubes in $\mathbb{R}^n$, with $\overline{Q}_0 \subset \operatorname{int}(Q_1)$ and $\operatorname{int}(Q_0) \subset W \subset Q_1$. For each constant $A > 1$, define a lower-semicontinuous function $g : \mathbb{R}^n \to \mathbb{R}_+$ by*

$$\begin{cases} g(x) = A & \text{when} \quad x \in \mathbb{R}^n \backslash \partial W \\ g(x) = 1 & \text{when} \quad x \in \partial W. \end{cases} \tag{7.3}$$

If A is large enough, depending only on Q_0, Q_1 and the regular bi-John constants for W, then

$$N_g(\mathbb{1}_W) = \inf_{V \in \mathcal{F}} N_g(\mathbb{1}_W) \tag{7.4}$$

and $\mathbb{1}_W$ is the unique minimizer for N_g in the sense that if $V \in \mathcal{F}$ satisfies $N_g(\mathbb{1}_V) = N_g(\mathbb{1}_W)$, then $\mathbb{1}_V(x) = \mathbb{1}_W(x)$ almost everywhere.

See (1.2) for the definition of $\mathcal{F}$, and (2.5) for N_g. Recall from Proposition 2.12 that minimizers for N_g exist anyway, so the content of Theorem 7.2 is that if $\mathbb{1}_V$ is a minimizer for N_g, then $\mathbb{1}_V = \mathbb{1}_W$ a.e. The following is an immediate consequence of Theorem 7.2 and Proposition 2.14.

Corollary 7.5. *If W is a bounded regular bi-John domain and Q_0, Q_1 are as in Theorem 7.2, then W is a quasiminimizer for $N(\cdot)$.*

It might be helpful to think about Theorem 7.2 in the easier case when ∂W is something like a Lipschitz graph. The point is that if we penalize the mass outside ∂W sufficiently, then W should be the only minimizer, in the same way that a flat surface minimizes ordinary area. It is amusing that we can manage to establish quasiminimality under conditions that exactly match the conclusions of Theorem 1.8.

Before we prove Theorem 7.2, we shall establish a related result which is a variation of converses to Theorem 6.1, and which might well be known.

Recall that $BV(W)$ (the set of functions of bounded variation on W) consists of the locally integrable functions whose distributional gradients are finite measures on the open set W. Such functions need not extend to BV functions on all $\mathbb{R}^n$, even when $\mathbb{1}_W \in BV$.

Theorem 7.6. *Suppose that W is a bounded John domain in $\mathbb{R}^n$ such that ∂W is Ahlfors-regular with dimension $n-1$. Then W is a domain of* *isoperimetry.* [*See Definition* 5.1.] *Moreover,* $BV(W) \subset L^1(W)$*, and if for each* $f \in BV(W)$ *we define an extension* $\widehat{f}$ *on* $\mathbb{R}^n$ *by*

$$\begin{cases} \widehat{f}(x) = f(x) & \text{for } x \in W \\ \widehat{f}(x) = \dfrac{1}{|W|} \displaystyle\int_W f & \text{for } x \in \mathbb{R}^n \setminus W, \end{cases} \tag{7.7}$$

then $\widehat{f} \in BV(\mathbb{R}^n)$ *and*

$$\left\| \widehat{f} \right\|_{BV(\mathbb{R}^n)} \le C \left\| f \right\|_{BV(W)}, \tag{7.8}$$

where C *depends only on* n*,* $\operatorname{diam} W$*, and the regularity and John constants.*

The assertion that W be a domain of isoperimetry will follow once we prove the BV statement. Indeed, if $\Omega \subset W$ is open and $H^{n-1}(\partial\Omega \cap W) < +\infty$, then $1\!\!1_\Omega \in BV(W)$ and $\|1\!\!1_\Omega\|_{BV(W)} \le CH^{n-1}(\partial\Omega \cap W)$. This follows from the proof of Lemma 4.7 (which is local). Then we get that the function $\widehat{f}$ which is equal to 1 on Ω, 0 on $W \setminus \Omega$ and $|\Omega|/|W|$ outside W lies in $BV(\mathbb{R}^n)$, whit a norm $\le CH^{n-1}(\partial\Omega \cap W)$. The desired estimate (5.2) follows from the usual Sobolev-Poincaré inequalities for $BV(\mathbb{R}^n)$. Note that it is even true that every John domain is a domain of isoperimetry, even when its boundary is not Ahlfors-regular. See [**B**].

Thus it suffices to prove the part about extending BV functions. Roughly speaking, the point is to control the L^1-norm on ∂W of the boundary values of a BV function on W. The concept of "boundary values" is somewhat ambiguous here, but it turns out that a crude version will suffice for our purposes.

Let W be a bounded John domain with an Ahlfors-regular boundary, and let z_0 be the center of W given by the John condition. [See Definition 1.7.] Thus there is a ball $B_0 = B(z_0, r_0)$ such that $2B_0 \subset W$ and $r_0 \ge C^{-1}$.

Define $\delta(x) = \operatorname{dist}(x, \mathbb{R}^n \setminus W)$. Given $f \in L^1_{\mathrm{loc}}(W)$, define $f_\sharp$ on W by

$$f_\sharp(x) = \delta(x)^{-n} \int_{B\left(x, \frac{\delta(x)}{10}\right)} |f(y) - m_0 f| \, dy, \tag{7.9}$$

where

$$m_0 f = \frac{1}{|B_0|} \int_{B_0} f(x) dx. \tag{7.10}$$

Define a sort of nontangential maximal function on ∂W by

$$Nf(z) = \sup \{ f_\sharp(x) : x \in W \text{ and } |x - z| \le 10\delta(x) \}. \tag{7.11}$$

Lemma 7.12. *If $f \in BV(W)$, then*

$$\int_{\partial W} Nf \; dH^{n-1} \le C \int_W |\nabla f| \,. \tag{7.13}$$

Let $\eta > 0$ be small, to be chosen soon. Given $z \in \partial W$, define the "cone" $\Gamma(z)$ by

$$\Gamma(z) = \{x \in W : \eta\,|x-z| \le \delta(x)\}. \tag{7.14}$$

Sublemma 7.15. *If η is small enough, then*

$$Nf(z) \le C \int_{\Gamma(z)} |z-x|^{-n+1} \, |\nabla f(x)| \, dx. \tag{7.16}$$

Let $z \in \partial W$ and $x \in W$ be given, with $|x-z| \le 10\delta(x)$; we want to estimate $f_\sharp(x)$. Let $\alpha : [0, |x-z_0|] \to W$ be a path as in Definition 1.7, so that α is C-Lipschitz, $\alpha(0) = x$, $\alpha\left(|x-z_0|\right) = z_0$ and $\mathrm{dist}\left(\alpha(t), \mathbb{R}^n \backslash W\right) \ge C^{-1}t$ for $0 \le t \le |x-z_0|$. Because $\alpha(0) = x$ and α is Lipschitz, we even have that

$$\mathrm{dist}(\alpha(t), \mathbb{R}^n \backslash W) \ge C^{-1}(t + \delta(x)), \tag{7.17}$$

perhaps with a larger C. Let $\tau = (3C)^{-1}$, where C is as in (7.17), and set

$$B(t) = B(\alpha(t), \tau t + \tau\delta(x)) \tag{7.18}$$

and

$$h(t) = |B(t)|^{-1} \int_{B(t)} f(y) dy \tag{7.19}$$

for $0 \le t \le |x-z_0|$.

Suppose $|t-u| \le a(t+\delta(x))$, where the constant a is chosen so small that $B(u) \subset 2B(t)$. Then

$$\begin{aligned} |h(t) - h(u)| &= |B(t)|^{-1} \, |B(u)|^{-1} \left| \int_{B(t)\times B(u)} [f(y) - f(w)] \, dy dw \right| \\ &\le C \, (t + \delta(x))^{-n+1} \int_{2B(t)} |\nabla f| \end{aligned} \tag{7.20}$$

by the Poincaré inequality.

Note that if $y \in 2B(t)$, then

$$|y-z| \le |y-\alpha(t)| + |\alpha(t)-x| + |x-z| \le C\,(t+\delta(x))$$

because α is C-Lipschitz, and by our assumption that $|x-z| \le 10\delta(x)$. On the other hand, $\operatorname{dist}(y, \mathbb{R}^n \backslash W) \ge \operatorname{dist}(\alpha(t), \mathbb{R}^n \backslash W) - 2\tau(t+\delta(x)) \ge \tau(t+\delta(x))$ by (7.17) and our choice of τ. Because of this, $2B(t) \subset \Gamma(z)$ if η is small enough, $2B(t) \cap 2B(t') = \emptyset$ when $t' \ge C(t+\delta(x))$ with C large enough, and (7.20) is the same as

$$|h(t)-h(u)| \le C \int_{2B(t)} |z-y|^{-n+1}\, |\nabla f(y)| \tag{7.21}$$

when $|t-u| \le a(t+\delta(x))$.

Define a finite sequence $\{t_m\}$ by $t_0 = 0$ and $t_{m+1} = t_m + a\,(t_m + \delta(x))$ until $t_m + a\,(t_m+\delta(x)) > |x - z_0|$, at which point we stop. Let t_∞ denote the last of the points t_m of the sequence. By repeated applications of (7.21), we get

$$\begin{aligned} |h(t_\infty) - h(0)| &\le C \sum_m \int_{2B(t_m)} |z-y|^{-n+1}\, |\nabla f(y)| \\ &\le C \int_{\Gamma(z)} |z-y|^{-n+1}\, |\nabla f(y)|\,, \end{aligned} \tag{7.22}$$

because we know that the $2B(t_m)$ are contained in $\Gamma(z)$ and have bounded overlap.

To complete the proof of Sublemma 7.15, we still need to control what happens at the endpoints. Let us systematically denote by $m_B f$ the mean value of f on B. We have that

$$\begin{aligned} \left|h(0) - m_{B\left(x, \frac{\delta(x)}{10}\right)} f\right| &= \left| m_{B(x, \tau\delta(x))} f - m_{B\left(x, \frac{\delta(x)}{10}\right)} f\right| \\ &\le C\delta(x)^{-n+1} \int_{B\left(x, \frac{\delta(x)}{10}\right)} |\nabla f| \end{aligned} \tag{7.23}$$

by Poincaré (and assuming, without true loss of generality, that $\tau \le \frac{1}{10}$). Similarly,

$$\begin{aligned} |h(t_\infty) - m_0 f| &= \left| m_{B(t_\infty)} f - m_{B_0} f \right| \\ &\le C \int_{2B(t_\infty) \cup 2B_0} |\nabla f| \end{aligned} \tag{7.24}$$

because the balls $2B_0$ and $2B(t_\infty)$ both have radii $\ge C^{-1}$ and have a non-trivial intersection, by our choices of t_∞ and a. The desired estimate (7.16) follows from (7.22), (7.23), (7.24), the Poincaré inequality, and the fact that $2B_0$ and $B\left(x, \frac{\delta(x)}{10}\right)$, just like the $2B(t_m)$'s, are contained in $\Gamma(z)$ if η is small enough. This proves Sublemma 7.15.

We are now ready to prove Lemma 7.12. By (7.16) and Fubini,

(7.26)
$$\begin{aligned}&\int_{\partial W} Nf\, dH^{n-1}\\&\leq C\int_W \left\{\int_{\partial W} |z-y|^{-n+1}\, 1\!\!1_{\Gamma(z)}(y) dH^{n-1}(z)\right\} |\nabla f(y)|\, dy.\end{aligned}$$

For each $y \in W$, the inside integral is taken over the set $\{z \in \partial W : |z-y| \leq \eta^{-1}\delta(y)\}$. The measure of this set is $\leq C\delta(y)^{n-1}$ because ∂W is Ahlfors-regular. We also have that $|z-y| \geq \delta(y)$ on this set by definition of $\delta(y)$. Lemma 7.12 follows from this.

We have the control we wanted on the "boundary values of f". Next we want to deal with the distribution theory. Roughly speaking, the idea is that the jump of f across ∂W should control the singular part of $|\nabla f|$ on ∂W.

Lemma 7.27. *For each (small) $t > 0$ we can find a smooth function φ_t on W such that $0 \leq \varphi_t \leq 1$, $\varphi_t(x) \equiv 0$ when $\delta(x) \leq t$, $\varphi_t(x) \equiv 1$ when $\delta(x) \geq 2t$, and $|\nabla\varphi_t| \leq Ct^{-1}$.*

This is standard. If we did not require φ_t to be smooth we could simply take it to be a function of $\delta(x)$. It is easy to get a smooth function by regularization. This proves Lemma 7.27.

Let $f \in BV(W)$ be given. For $t > 0$ small define f_t on $\mathbb{R}^n$ by

$$f_t = \varphi_t f + (1-\varphi_t)\, m_0 f. \tag{7.28}$$

These are locally integrable functions on $\mathbb{R}^n$, and we have that

$$\nabla f_t = \varphi_t \nabla f + [f - m_0 f]\, \nabla\varphi_t. \tag{7.29}$$

Thus each f_t lies in $BV(\mathbb{R}^n)$, since $supp\varphi_t$ is a compact subset of W.

Lemma 7.30.
$$\int_{\mathbb{R}^n} |\nabla f_t| \leq C\int_W |\nabla f|$$
for all t, where C depends only on the constants implicit in the statement of Theorem 7.6, but not on t.

This comes down to

$$\int_{W_t} |f(x) - m_0 f|\, dx \leq Ct\int_W |\nabla f|, \tag{7.31}$$

where $W_t = \{x \in W : t \leq \delta(x) \leq 2t\}$. Let $\{Q_i\}$ be a family of closed cubes with diameter $\frac{t}{2}$ such that $W_t \subset \bigcup_i Q_i$, each Q_i intersects W_t, and the Q_i's have disjoint interiors. Choose, for each Q_i, a point $z_i \in \partial W$ such that $\operatorname{dist}(z_i, Q_i) = \operatorname{dist}(\partial W, Q_i) \leq 2t$, and set $B(i) = B(z_i, t) \cap \partial W$. It is easy to see that the $B(i)$'s have bounded overlap, since the Q_i's have diameter $\frac{t}{2}$ and bounded overlap.

Let us check that

$$\sup_{x \in Q_i} f_\sharp(x) \leq \inf_{z \in B_i} Nf(z). \tag{7.32}$$

According to the definition (7.11) of $N(f)$, this comes down to the statement that

$$|x - z| \leq 10\delta(x) \text{ when } x \in Q_i \text{ and } z \in B(i). \tag{7.33}$$

If $x \in Q_i$, then $\delta(x) \geq \frac{t}{2}$, since Q_i intersects W_t and has diameter $\frac{t}{2}$. If also $z \in B(i)$, then

$$\begin{aligned} |x - z| &\leq |x - z_i| + |z_i - z| \\ &\leq \operatorname{dist}(z_i, Q_i) + \operatorname{diam} Q_i + |z_i - z| \\ &\leq 2t + \frac{t}{2} + t \leq 4t. \end{aligned} \tag{7.34}$$

This implies (7.33), since $\delta(x) \geq \frac{t}{2}$; (7.32) follows. Next

$$\begin{aligned} \int_{Q_i} |f - m_0 f| \, dx &\leq C \int_{Q_i} f_\sharp(x) dx \\ &\leq Ct \int_{B(i)} Nf dH^{n-1} \end{aligned} \tag{7.35}$$

by (7.9) and Fubini, (7.32), and the fact that $H^{n-1}(B(i)) \geq C^{-1} t^{n-1}$ because ∂W is regular.

We may now sum over i and use the fact that the $B(i)$'s have bounded overlap to get that

$$\begin{aligned} \int_{W_t} |f(x) - m_0 f| \, dx &\leq \sum_i \int_{Q_i} |f - m_0 f| \\ &\leq Ct \sum_i \int_{B(i)} Nf \; dH^{n-1} \\ &\leq Ct \int_{\partial W} Nf \; dH^{n-1} \\ &\leq Ct \int_{\partial W} |\nabla f| \, , \end{aligned} \tag{7.36}$$

by Lemma 7.12. This is the same as (7.31), and Lemma 7.30 follows.

From Lemma 7.30 and the Poincaré inequality (applied to a ball that contains W) we get that

$$\int_W |f_t - m_{B_0} f_t| \leq C \int_{\mathbb{R}^n} |\nabla f_t| \leq C \int_W |\nabla f| \,. \tag{7.37}$$

When $t > 0$ is small enough we have that $f_t = f$ on B_0 and hence

$$\int_W |f_t - m_0 f| \leq C \int_W |\nabla f| \,. \tag{7.38}$$

Sending t to 0 we get

$$\int_W |f - m_0 f| \leq C \int_W |\nabla f| \,. \tag{7.39}$$

In particular, $f \in L^1(W)$ and so $f_t \to f$ in $L^1(W)$ as $t \to 0$.

Define $\widetilde{f}$ on $\mathbb{R}^n$ by

$$\begin{cases} \widetilde{f}(x) = f(x) & \text{for } \ x \in W \\ \widetilde{f}(x) = m_0 f & \text{for } \ x \in \mathbb{R}^n \backslash W. \end{cases} \tag{7.40}$$

Since f_t and $\widetilde{f}$ coincide on $\mathbb{R}^n \backslash W$, we get that $f_t \to \widetilde{f}$ in $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$. From Lemma 7.30 we conclude that $\widetilde{f} \in BV(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} \left|\nabla \widetilde{f}\right| \leq C \int_W |\nabla f| \,.$

This proves Theorem 7.6, modulo the difference between $\widehat{f}$ and $\widetilde{f}$, which is given by

$$\widehat{f} - \tilde{f} = \left[\frac{1}{|W|} \int_W f - m_{B_0} f\right] 1\!\!1_{\mathbb{R}^n \backslash W}. \tag{7.41}$$

The difference of the mean values is $\leq C \int_W |\nabla f|$, by (7.39). It follows that

$$\int_{\mathbb{R}^n} \left|\nabla(\widehat{f} - \tilde{f})\right| \leq C \left[\int_W |\nabla f|\right] H^{n-1}(\partial W), \tag{7.42}$$

by Lemma 4.7. This completes the proof of Theorem 7.6.

Remark 7.43. Theorem 7.6 has an analogue for domains in $\mathbb{R}^n$ with bounded complement. Namely, if W is a domain in $\mathbb{R}^n$ such that $\mathbb{R}^n \backslash W$ is bounded, W is a John domain and ∂W is Ahlfors-regular with dimension $n-1$, then W is a domain of isoperimetry, and every function $f \in BV(W)$ can be extended to a function $\widehat{f} \in BV(\mathbb{R}^n)$, with the same sort of estimates as in Theorem 7.6. This is easily deduced from Theorem 7.6; one can first cut out a small ball from the middle of W (this does not change the problem) and then use an inversion centered on this ball to reduce to the case of a bounded domain.

We now come to the proof of Theorem 7.2. Let W be a bounded regular bi-John domain, and let Q_0, Q_1, A and g be as in the statement of the theorem. Also let $V \in \mathcal{F}$ be such that $N_g(\mathbb{1}_V)$ is minimal. As we observed just after the statement of Theorem 7.2, such a V always exists, and it is enough to prove that $\mathbb{1}_V = \mathbb{1}_W$ almost everywhere when A is sufficiently large.

We know from Proposition 2.14 that V is a quasiminimizer for $N(\cdot)$, and so we may apply to V the results of Section 3. In particular, we may replace V by an open set which is equivalent to V (in the sense that the characteristic functions are equal almost everywhere) and which has a regular boundary and satisfies Condition B. Assuming this substitution, we shall prove that $V = W$.

Note that the Ahlfors-regularity and Condition B constants for V may depend on A. Thus, although we can use these conditions, we cannot do so in a "quantitative way".

Our assumption is that

$$\int g\,|\nabla \mathbb{1}_V| = \int g\,|\nabla \mathbb{1}_W|\,, \tag{7.44}$$

but we can simplify this expression because Proposition 4.1 tells us that $|\nabla \mathbb{1}_V|$ is the same as the restriction of H^{n-1} to ∂V. Similarly, (4.5) and (4.6) tell us that $|\nabla \mathbb{1}_W| \le H^{n-1}_{|\partial W}$. Thus $\int g\,|\nabla \mathbb{1}_V| = H^{n-1}(\partial V \cap \partial W) + AH^{n-1}(\partial V \backslash \partial W)$ and $\int g\,|\nabla \mathbb{1}_W| \le H^{n-1}(\partial W)$, so that (7.44) yields

$$AH^{n-1}(\partial V \backslash \partial W) \le H^{n-1}(\partial W \backslash \partial V)\,. \tag{7.45}$$

Notice that for the first time we have used the full strength of Proposition 4.1, i.e., the fact that $|\nabla \mathbb{1}_V|$ and $H^{n-1}_{|\partial W}$ are actually equal, rather than equivalent in size. It is probable that one can avoid doing so, at the price of a more careful examination of $\nabla(\mathbb{1}_V - \mathbb{1}_W)$ on $\partial V \cap \partial W$. [We need to be able to say that the contributions of $|\nabla \mathbb{1}_V|$ and $|\nabla \mathbb{1}_W|$ to $\partial V \cap \partial W$ are the same.]

Lemma 7.46. *We have that*

$$H^{n-1}\left(\partial W \backslash \overline{V}\right) \le CH^{n-1}(\partial V \cap W) \tag{7.47}$$

and

$$H^{n-1}(\partial W \cap V) \le C H^{n-1}\left(\partial V \backslash \overline{W}\right), \tag{7.48}$$

with a constant C that depends only on n and the various constants for W.

Let us first derive Theorem 7.2 from this lemma. If A is large enough, a comparison of (7.45) with the sum of (7.47) and (7.48) gives that $H^{n-1}(\partial V \backslash \partial W) = H^{n-1}(\partial W \backslash \partial V) = 0$. Since ∂V and ∂W are both (closed) regular sets, we get that $\partial V = \partial W$, and then $V = W$ because $\mathbb{R}^n \backslash \partial W$ has only two connected components. The theorem follows.

To prove (7.47), we want to use the same construction as in the proof of Theorem 7.6. Consider the function $f = 1\!\!1_{W \backslash V}$ on W, and define the functions $f_\sharp$ and Nf as in (7.9) and (7.11), except that we replace 10 in (7.11) with a larger constant that depends only on the John constant for W. Remember that we can choose the center z_0 for the John domain W and the ball $B_0 = B(z_0, r_0)$ so that $B_0 \subset Q_0$. With this choice, $m_0 f = 0$ and

$$f_\sharp(x) = \delta(x)^{-n} \left| B\left(x, \frac{\delta(x)}{10}\right) \backslash V \right|. \tag{7.49}$$

With the present definition of Nf, we have that $Nf(z) \ge C^{-1}$ for every $z \in \partial W \backslash \overline{V}$. (This uses the John condition to find plenty of "good" points $x \in W$ near z.) Then $H^{n-1}(\partial W \backslash \overline{V}) \le C \int_{\partial W} Nf dH^{n-1} \le C \int_W |\nabla f|$ by Lemma 7.12, and (7.47) follows because $\int_W |\nabla f| = \int_W |\nabla 1\!\!1_V| = H^{n-1}(\partial V \cap W)$.

The proof of (7.48) is similar. The simplest approach is probably to observe that the proof of Lemma 7.12 is stable under a suitable inversion (or that the proof extends without difficulty to domains with bounded complement).

This completes the proof of Theorem 7.2.

8. Separation and rectifiability.

Let Q_0 and Q_1 be closed cubes in $\mathbb{R}^n$, with $Q_0 \subseteq \operatorname{Int} Q_1$. Let K be a compact subset of $Q_1 \backslash \operatorname{Int} Q_0$ which separates $\operatorname{Int} Q_0$ from $\mathbb{R}^n \backslash Q_1$ and satisfies $H^{n-1}(K) \le C_5 < \infty$. Also let $\varepsilon \ge 0$ be given. We want to find a regular set E of dimension $n-1$ which is almost entirely contained in K in the sense that

$$H^{n-1}(E \backslash K) \le \varepsilon \tag{8.1}$$

holds. We also want $E \subseteq Q_1 \backslash \operatorname{Int} Q_0$ and for E to separate $\operatorname{Int} Q_0$ from $\mathbb{R}^n \backslash Q_1$, and that E satisfy Condition B. We shall obtain E as the boundary of a normalized quasiminimizer (as defined at the beginning of Section 5). Theorem 1.16 will follow once we have found such an E, because the statement of Theorem 1.16 obviously behaves well under bilipschitz mappings.

Let $\mathcal{F}$ denote the class of measurable sets V such that $Q_0 \subset V \subset Q_1$ and $1\!\!1_V \in BV$. Also let $A > 1$ be a large number (to be chosen soon), and define $g : \mathbb{R}^n \to \mathbb{R}^+$ by

$$\begin{cases} g(x) = A & \text{when} \quad x \in \mathbb{R}^n \setminus K \\ g(x) = 1 & \text{when} \quad x \in K. \end{cases} \tag{8.2}$$

This function is lower semicontinous, because K is compact by assumption.

Proposition 1.10 tells us that there is a set $W \in \mathcal{F}$ such that $N_g(1\!\!1_W) = \int g\,|\nabla 1\!\!1_W|$ is minimal among such sets. It also tells us that W is a quasiminimizer for $N(\cdot)$, and then Theorem 1.8 says that we can modify W on a set of measure zero to get an open set W_0 with the following properties: the boundary $E = \partial W_0$ is Ahlfors-regular of dimension $n-1$ and satisfies Condition B; $\mathbb{R}^n \setminus E$ has exactly two connected components W_0 and $\mathbb{R}^n \setminus \overline{W}_0$, each of which is a John domain; and $E = \partial W_0 = \partial(\mathbb{R}^n \setminus \overline{W}_0)$. [See Definitions 1.5, 1.6 and 1.7.] For the purposes of Theorems 1.16 and 1.15, we shall not need all this information, and Theorem 3.1 (instead of Theorem 1.8) would be enough.

Note that $E \subset Q_1 \setminus \operatorname{Int} Q_0$ and E separates $\operatorname{Int} Q_0$ from $\mathbb{R}^n \setminus Q_1$ because $W_0 \in \mathcal{F}$.

Let us now check that we can choose A, depending on n, ε, and C_4, so that (8.1) holds. We want to compare $N_g(1\!\!1_W)$ with $N_g(1\!\!1_V)$, where V denotes the connected component of $\mathbb{R}^n \setminus K$ that contains $\operatorname{Int} Q_0$. Clearly $\operatorname{Int} Q_0 \subset V \subset Q_1$ because $K \subset Q_1 \setminus \operatorname{Int} Q_0$ and by the separation hypothesis for K.

Because of Lemma 4.7 for instance, $1\!\!1_V \in BV$ and

$$\begin{aligned} \int |\nabla 1\!\!1_V| \le C_n H^{n-1}(\partial V) &\le C_n H^{n-1}(K) \\ &\le C_n C_5, \end{aligned} \tag{8.3}$$

where C_n is a constant that depends only on n (and could even be taken to be 1, by (4.5) and (4.6)). Thus $V \in \mathcal{F}$, and

$$N_g(1\!\!1_V) = \int_K |\nabla 1\!\!1_V| \le C_n C_5, \tag{8.4}$$

because $|\nabla 1\!\!1_V|$ lives on K where g equals 1. On the other hand, Lemma 4.16 (or the stronger Proposition 4.1) yields

$$H^{n-1}(E\backslash K) \le C'_n \int_{E\backslash K} |\nabla 1\!\!1_{W_0}| \le A^{-1}C'_n \int_{E\backslash K} g\,|\nabla 1\!\!1_{W_0}| \le A^{-1}C'_n N_g(1\!\!1_{W_0}) \le A^{-1}C'_n N_g(1\!\!1_V), \tag{8.5}$$

since W (and hence W_0) is a minimizer for N_g. If we choose $A > C_n C'_n C_5 \varepsilon^{-1}$, (8.4) and (8.5) imply the desired estimate (8.1). Observe that the various constants that describe the good properties of E depend only on n and A. This completes the proof of Theorem 1.16.

Let us now explain why Theorem 1.15 follows from Theorem 1.16. Let K be as in these theorems, and apply Theorem 1.16 with $\varepsilon = \frac{\omega_{n-1}}{2}$, where ω_{n-1} is the volume of the unit ball in $\mathbb{R}^{n-1}$. We get a regular set $E \subset \overline{B(0,2)}\backslash B(0,1)$ that satisfies Condition B, separates 0 from ∞, and for which (8.1) holds.

Let P be any hyperplane through 0, and denote by π the orthogonal projection onto P. Every line through $B(0,1)$ and orthogonal to P must meet E somewhere between $B(0,1) \cap P$ and ∞ (which are not in the same component of $\mathbb{R}^n\backslash E$), and so

$$H^{n-1}(\pi(E)) \ge \omega_{n-1}. \tag{8.6}$$

This and (8.1) imply that

$$H^{n-1}(\pi(E \cap K)) \ge \frac{1}{2}\omega_{n-1}. \tag{8.7}$$

Now we want to apply Theorem 2.11 on p. 863 of [**DS1**] to the set E and the projection π. The main hypothesis that E be an $n-1$-dimensional regular set that satisfies the weak geometric lemma follows from the fact that E is regular and satisfies Condition B. [See Theorem 1.20 and Proposition 1.18 of [**DS1**].] There is a minor difference between our definitions and those in [**DS1**], because regular sets there are unbounded. This is not a problem, because we can always apply the theorems of [**DS1**] to the union of E and some hyperplane that touches $\partial B(0,3)$ for instance. The statement of Theorem 2.11 in [**DS1**] talks about a "dyadic cube" Q_0, but we don't need to worry about this; we can either cover E with a finite collection of such cubes with diameters comparable to 1, or observe that we can modify the cubes so that E is a single cube Q_0.

The function $f = \pi$ is well approximated by affine functions because it is affine, and so the quantity (2.10) in [**DS1**] is simply 0.

The result is that we can find subsets $F_1, \cdots, F_L$ of E such that π is bilipschitz on each F_j, i.e.

$$|\pi(x) - \pi(y)| \geq \tau\, |x - y| \quad \text{for} \quad x, y \in F_j, \tag{8.8}$$

and

$$H^{n-1}\left(\pi\left(E \backslash \bigcup_{j=1}^{L} F_j\right)\right) \leq \frac{\omega_{n-1}}{4}. \tag{8.9}$$

The constants $L \in \mathbb{N}$ and $\tau > 0$ depend on our choice of $\frac{\omega_{n-1}}{4}$ in (8.9), and on the regularity and Condition B constants for E (which themselves depend on n and C_5).

From (8.7) and (8.9) we deduce that there is a $j \in \{1, \cdots, L\}$ such that $H^{n-1}\left(\pi(F_j \cap K)\right) \geq \frac{\omega_{n-1}}{4L}$. Because of (8.8), $Fj \cap K$ is contained in some τ^{-1}-Lipschitz graph Γ, and

$$H^{n-1}(\Gamma \cap K) \geq H^{n-1}(Fj \cap K) \geq \frac{\omega_{n-1}}{4L}$$

because π is 1-Lipschitz. This completes the proof of Theorem 1.15.

The separation hypothesis in Theorems 1.15 and 1.16 can be relaxed, as follows.

Theorem 8.10. *Let K be a compact subset of $\mathbb{R}^n$ such that $K \subset \overline{B(0,2)} \backslash B(0,1)$. Suppose that $H^{n-1}(K) \leq C_5$ and that there is a smooth function f on $\overline{B}(0,2) \backslash K$ such that*

$$\int_{\overline{B}(0,2)\backslash K} |\nabla f| \leq 1 \tag{8.11}$$

and

$$f(x) = 0 \text{ on } B(0,1) \text{ and } f(x) \geq L \text{ on } \partial B(0,2) \tag{8.12}$$

for some large L. Then for each $\varepsilon > 0$ there is a compact subset E of $\overline{B(0,2)} \backslash B(0,1)$ which is Ahlfors-regular of dimension $n-1$ and satisfies

Condition B (with constants that depend only on n, C_5 and ε), separates 0 *from* ∞*, and satisfies*

$$H^{n-1}(E \backslash K) \leq \varepsilon + L^{-1}. \tag{8.13}$$

Note that the conclusion of Theorem 8.10 is still valid when L is small, but it is completely useless then because (8.13) does not forbid the trivial case where $E = \partial B(0,1)$ and $E \cap K = \emptyset$.

The hypotheses on the function f can be weakened somewhat; see Remark 8.18 below.

The set E will come from an application of Theorem 1.16, and so it satisfies the usual additional properties that $\mathbb{R}^n \backslash E$ has exactly two connected components, each of which is a John domain with boundary E.

Let us say a few words about our "almost separation" condition on K. Although K may not separate 0 from ∞, it is quite hard for an average curve to go from 0 to $\partial B(0,2)$ if L is large. One can think about the simple case when K is the sphere $\{|x| = \frac{3}{2}\}$ with a small set Z removed. When L is very large, (8.11) and (8.12) force Z to be very small.

It is very pleasant to think of the "almost separation" property in terms of families of curves. If $K = \emptyset$, then there are plenty of curves in $\overline{B(0,2)} \backslash K$ which connect $B(0,1)$ to $\partial B(0,2)$ and which are sufficiently spread out to imply the negation of (8.12) (when (8.11) holds and L is large enough) through the usual trick of averaging the fundamental theorem of calculus over the curves. If K is very scattered - e.g., if it is totally unrectifiable - then one can find such a family of curves that avoids K, and (8.11) and (8.12) cannot hold at the same time. In general, (8.11) and (8.12) prevent the existence of a large family of curves that connect $B(0,1)$ to $\partial B(0,2)$ without meeting K. One can think of this in terms of networks, or highways; (8.11) and (8.12) imply that in order to travel from $B(0,1)$ to $\partial B(0,2)$, one has to go through a small number of bottlenecks. Our proof will consist in closing off these bottlenecks by adding them to K, so that the result separates 0 from $+\infty$. The bottlenecks will be found with the help of the coarea formula.

We are now ready to prove Theorem 8.10. Let K and f be as in the statement, and set

$$\Omega_t = \{x \in B(0,2) \backslash K : f(x) < t\} \tag{8.14}$$

for $0 < t < L$. From the coarea formula we get that

$$\int_0^L H^{n-1}\left(\partial \Omega_t \cap (B(0,2) \backslash K)\right) dt \leq \int_{B(0,2) \backslash K} |\nabla f| \leq 1. \tag{8.15}$$

[In this case the coarea formula is simpler than usual, since f is smooth, so that almost all t are regular values for f.] Hence there is a $t \in (0, L)$ such that

$$H^{n-1}\left(\partial\Omega_t \cap (B(0,2)\backslash K)\right) \leq L^{-1}. \tag{8.16}$$

Since $0 < t < L$, we have that $B(0,1) \subset \Omega_t$ and $\overline{\Omega}_t \subset B(0,2)$. Set $K' = K \cup \partial\Omega_t$. Then K' is a compact subset of $B(0,2)\backslash B(0,1)$ which separates 0 from ∞ and satisfies

$$H^{n-1}(K'\backslash K) \leq L^{-1}. \tag{8.17}$$

We can now apply Theorem 1.16 to K' to get a set $E \subset \overline{B(0,2)}\backslash B(0,1)$ with the usual good properties and $H^{n-1}(E\backslash K') \leq \varepsilon$. This estimate and (8.17) imply (8.13). Note that the regularity, Condition B, and John constants for E depend on n, ε, and $C_5 = C_5' + L^{-1}$, which is a bound for $H^{n-1}(K')$. We can take C_5 independent of L, though, because the theorem is trivial when L is small.

This completes the proof of Theorem 8.10.

Remark 8.18. In the statement of Theorem 8.10, we may replace our hypotheses on f by the weaker assumptions that $f \in BV(B(0,3)\backslash K)$,

$$\int_{B(0,1)} f(x)dx = 0, \tag{8.19}$$

and

$$\int_{B(0,3)\backslash B(0,2)} |f| \geq L \int_{B(0,3)\backslash K} |\nabla f|. \tag{8.20}$$

The conclusions of Theorem 8.10 remain valid with these weaker assumptions, except that we replace (8.13) with

$$H^{n-1}(E\backslash K) \leq \varepsilon + C_n L^{-1}, \tag{8.21}$$

where C_n is a constant that depends only on n.

To prove this observation, let $f \in BV(B(0,3)\backslash K)$ satisfy (8.19) and (8.20). First multiply f by a cut-off function φ_1 such that $\varphi_1 \equiv 0$ in a neighborhood of $B\left(0, \frac{1}{2}\right)$, $\varphi_1 \equiv 1$ on $B(0,3)\backslash B(0,2)$, $0 \leq \varphi \leq 1$ everywhere, and $|\nabla\varphi_1| \leq C$. We get a new function $f_1 \in BV(B(0,3)\backslash K)$ such that

$$\begin{aligned}\int_{B(0,3)\backslash K} |\nabla f_1| &\leq \int_{B(0,3)\backslash K} \varphi\,|\nabla f| + \int_{B(0,1)\backslash B\left(0,\frac{1}{2}\right)} |f|\,|\nabla\varphi| \\ &\leq C\int_{B(0,3)\backslash K} \varphi\,|\nabla f|\end{aligned} \tag{8.22}$$

by (8.19) and the Poincaré inequality.

Next replace f_1 by $f_2 = \varphi_2\,|f_1| + (1-\varphi_2)\,A$, where A is the mean value of $|f|$ on $B(0,3)\backslash B(0,2)$ and φ_2 is a smooth cut-off function such that $\varphi_2 \equiv 1$ on $B(0,2)$, $\varphi_2 \equiv 0$ on a neighbourhood of $B(0,3)\backslash B\left(0,\frac{5}{2}\right)$, $0 \le \varphi_2 \le 1$ everywhere, and $|\nabla\varphi_2| \le C$. Then

$$\begin{aligned}\int_{B(0,3)\backslash K} |\nabla f_2| &\le \int_{B(0,3)\backslash K} |\nabla|f_1|| + \int_{B\left(0,\frac{5}{2}\right)\backslash B(0,2)} ||f_1|-A|\,|\nabla\varphi_2| \\ &\le C\int_{B(0,3)\backslash K} |\nabla|f_1|| \le C\int_{B(0,3)\backslash K} |\nabla f| \end{aligned} \tag{8.23}$$

by definition of A, Poincaré, the fact that $f_1 = f$ on $B(0,3)\backslash B(0,2)$, and (8.22). Note that $f_2 \equiv 0$ on a neighborhood of $B\left(0,\frac{1}{2}\right)$ and $f_2 \equiv A$ on a neighborhood of $B(0,3)\backslash B(0,5/2)$.

We can also replace f_2 by a smooth function f_3 which is equal to 0 on $B\left(0,\frac{1}{2}\right)$ and to A on $B(0,3)\backslash B\left(0,\frac{5}{2}\right)$, at the expense of multiplying again $\int_{B(0,3)\backslash K} |\nabla f_2|$ by at most a constant. Also, $A \ge C^{-1}L\int_{B(0,3)\backslash K} |\nabla f_3|$ by (8.20).

We are now almost in the same situation as in the proof of Theorem 8.10, with the small difference that we have to apply the co-area formula on the larger domain $B\left(0,\frac{5}{2}\right)\setminus\left[B\left(0,\frac{1}{2}\right)\cup K\right]$. We find a compact set $K' \subset B\left(0,\frac{5}{2}\right)\backslash B\left(0,\frac{1}{2}\right)$ such that $H^{n-1}(K'\backslash K) \le CL^{-1}$. If K' is not contained in $\overline{B(0,2)}\backslash B(0,1)$, we take $K'' = \Psi(K')$, where Ψ equals the identity in $B(0,2)\backslash B(0,1)$, Ψ equals the radial projection onto $\partial B(0,1)$ on $B(0,1)\backslash\left(0,\frac{1}{2}\right)$, and Ψ equals the radial projection onto $\partial B(0,2)$ on $B\left(0,\frac{5}{2}\right)\backslash B(0,2)$. Thus $K'' \subseteq \overline{B(0,2)}\backslash B(0,1)$, K'' has approximately the same properties as K' in terms of Hausdorff measure, and one can check that K'' still separates 0 from ∞. We can apply Theorem 1.16 and conclude as in the proof of Theorem 8.10. This proves Remark 8.18.

9. H^{n-1} quasiminimizers.

Much of this paper is written in the language of BV. Much of it could be reformulated into the language of Hausdorff measure, i.e., by defining minimizers and quasiminimizers directly in terms of Hausdorff measures of the boundaries of sets. This is more direct and pleasant, but the language of BV has the advantage of an easy existence theory for minimizers. This is a well-known and crucial point; the lower semicontinuity theorems needed for the direct method of the calculus of variations do no work when we simply take the Hausdorff measure of the boundary. Actually, for some of the purposes of this paper one could get an adequate existence theory for Hausdorff minimizers by restricting one's attention to finite sets of competitors with controlled local structure, like polyhedral structure. The point is that our methods and results are very stable and give uniform estimates which do not depend on the scale of the discretization.

In this section we look at reformulations of our definitions and results adapted to Hausdorff measure instead of BV. We shall follow closely our earlier work, and we shall see that many of the arguments become simpler.

Fix closed cubes Q_0, Q in $\mathbb{R}^n$, with $Q_0 \subseteq \mathrm{Int}(Q_1)$.

Definition 9.1. Let $\mathcal{F}_0$ denote the class of sets $V \subset \mathbb{R}^n$ such that $\mathrm{int}(Q_0) \subset V \subset Q_1$ and $H^{n-1}(\partial V) < +\infty$. We say that $W \in \mathcal{F}_0$ is a quasiminimizer for $H^{n-1}(\cdot)$ if there is an $M \geq 1$ such that

$$H^{n-1}(\partial W \backslash \partial V) \leq M H^{n-1}(\partial V \backslash \partial W) \tag{9.2}$$

for every competitor $V \in \mathcal{F}_0$.

This is quite similar to Definition 1.3, and the analogy is even more obvious if we observe that (9.2) is equivalent to

$$H^{n-1}(\partial W) \leq H^{n-1}(\partial V) + \alpha H^{n-1}(\partial W \backslash \partial V) + \alpha H^{n-1}(\partial V \backslash \partial W); \tag{9.3}$$

with $M = \frac{\alpha+1}{\alpha-1}$. This is easy to check: if $a = H^{n-1}(\partial W \backslash \partial V)$, $b = H^{n-1}(\partial V \backslash \partial W)$ and $c = H^{n-1}(\partial W \cap \partial V)$, then (9.2) says that $a \leq Mb$ while (9.3) says that $a + c \leq b + c + \alpha a + \alpha b$.

Proposition 9.4. *Theorem 3.1 and Lemma 3.43 are also valid with quasiminimizers for $N(\cdot)$ replaced with quasiminimizers for $H^{n-1}(\cdot)$.*

To prove this we fix $W \in \mathcal{F}_0$ such that (9.2) holds for every $V \in \mathcal{F}_0$ of the form $V = W \cup Q$ or $V = W \backslash Q$, where Q is a cube.

Lemma 9.5. *We have that*

$$H^{n-1}(\partial W \cap \overset{o}{Q}) \leq M H^{n-1}(\partial Q \backslash \overset{o}{W}) \tag{9.6}$$

for all cubes Q such that $Q \subset Q_1$, and

$$H^{n-1}(\partial W \cap \overset{o}{Q}) \leq M H^{n-1}(\partial Q \cap \overline{W}) \tag{9.7}$$

for all cubes Q such that $Q \cap \mathrm{int}(Q_0) = \emptyset$. Here $\overset{o}{W} = \mathrm{Int}\,(W)$, etc.

If $Q \subset Q_1$, then $\mathrm{int}(Q_0) \subset W \cup Q \subset Q_1$ and so $W \cup Q \in \mathcal{F}_0$. Also, $\partial(W \cup Q) \subset (\overline{W} \cup \overline{Q}) \backslash (\overset{o}{W} \cup \overset{o}{Q}) = (\partial W \backslash \overset{o}{Q}) \cup (\partial Q \backslash \overset{o}{W})$, and (9.6) follows from (9.2). Similarly if $Q \cap \mathrm{int}(Q_0) = \emptyset$, then $\mathrm{int}(Q_0) \subset W \backslash Q \subset Q_1$ and $W \backslash Q \in \mathcal{F}_0$. This time $\partial(W \backslash Q) \subset \overline{W} \cap (\overline{\mathbb{R}^n \backslash Q}) \setminus (\overset{o}{W} \cap (\mathbb{R}^n \backslash Q)^0) \subset (\partial W \setminus \overset{o}{Q}) \cup (\partial Q \cap \overline{W})$, so that (9.7) also follows from (9.2). This proves Lemma 9.5.

Let us now prove the analogue of Lemma 3.20.

Lemma 9.8. *Suppose that Q satisfies* (3.21). *Then*

$$H^{n-1}(\partial W \cap \overset{o}{Q}) \leq C\; r(Q)^{n-1} h(2Q), \tag{9.9}$$

where $h(\cdot)$ is still defined by (3.19).

The proof is very similar to the proof of Lemma 3.20, but let us sketch it anyway. Because of Lemma 9.5 we have that

$$H^{n-1}(\partial W \cap \overset{o}{Q}) \leq M H^{n-1}(\partial Q), \tag{9.10}$$

and so we may assume that $h(2Q) < \delta$ for some very small $\delta > 0$. Let us first assume that $h(2Q) = r(2Q)^{-n} |W \cap 2Q|$. Then $|W \cap 2Q| \leq \delta\; r(2Q)^n$, and so $(\frac{3}{2}Q) \cap \mathrm{int}(Q_0) = \emptyset$ (if δ is small enough). Thus we may apply (9.7) to the cube λQ for $\lambda \in (1, \frac{3}{2})$.

Note that $\left|\overline{W} \cap 2Q\right| = |W \cap 2Q| = r(2Q)^n h(2Q)$ because $|\partial W| = 0$ (since $H^{n-1}(\partial W) < +\infty$). By Tchebytchev, we can choose $\lambda \in (1, \frac{3}{2})$ such that $H^{n-1}\left(\overline{W} \cap \partial(\lambda Q)\right) \leq Ch(2Q)\; r(Q)^{n-1}$, and then $H^{n-1}(\partial W \cap \overset{o}{Q}) \leq H^{n-1}(\partial W \cap \lambda \overset{o}{Q}) \leq M H^{n-1}(\overline{W} \cap \partial(\lambda Q)) \leq Ch(2Q)\; r(Q)^{n-1}$ by (9.7) and our choice of λ. This proves (9.9) in this first case.

When $h(2Q) = r(2Q)^{-n} |2Q \backslash W|$, we have that $|2Q \backslash W| \leq \delta r(2Q)^n$ and, if δ is small enough, $\frac{3}{2}Q \subset Q_1$. Then $W \cup \lambda Q \in \mathcal{F}_0$ for $\lambda \in (1, \frac{3}{2})$, and we may apply (9.6). By Tchebytchev we can choose $\lambda \in (1, \frac{3}{2})$ so that $H^{n-1}(\partial(\lambda Q) \setminus \overset{o}{W}) \leq Ch(2Q)\; r(Q)^{n-1}$, and (9.9) follows from (9.6) with this choice of λQ. This proves Lemma 9.8.

Lemma 9.11. *We have that*

$$h(Q)^{\frac{n-1}{n}} \leq C r(Q)^{-n+1} H^{n-1}(\partial W \cap \overline{Q}). \tag{9.12}$$

This is simply the right version of the isoperimetric inequality for a cube. If one insists one can reduce it to the case of a sphere by a doubling argument. One could instead use (3.27).

From (9.12) and (9.9) we deduce that

$$h(Q) \le Ch(3Q)^{\frac{n}{n-1}} \tag{9.13}$$

for all cubes Q such that $\frac{3}{2}Q$ satisfies (3.21). [The factor 3 (instead of 2) is a small precaution which we need because (9.9) does not give us control over $\partial W \cap \partial Q$.] Of course this estimate is just as good as (3.29), and we may continue the proof almost exactly as in the case of minimizers for $N(\cdot)$. In particular, Lemma 3.30 is still valid (when $\frac{3}{2}Q$ satisfies (3.21), and with essentially the same proof), and the sets E, W_0, and W_1 defined just before (3.33) still satisfy the properties (3.33)-(3.36) in the present situation.

Note that $E \subseteq \partial W$, by the definition of E, and that $H^{n-1}(\partial W \backslash E) = 0$, because of the quasiminimality property (9.2) applied to $V = W_0$ (and using (3.36)).

We can continue the argument with $\mu = H^{n-1}_{|\partial W}$ (rather than $\mu = |\nabla 1\!\!1_W|$). Lemma 3.37 still holds, but (3.39) has to be replaced with (9.12). The rest of the proof is the same. This completes our proof of Proposition 9.4.

Thus to each quasiminimizer W we can associate a unique open set W_0 such that $|W \backslash W_0| = |W_0 \backslash W| = 0$, $\partial W_0 \subseteq \partial W$, $H^{n-1}(\partial W \backslash \partial W_0) = 0$, and W_0 satisfies Condition B (Definition 4.15). This open set W_0 is also a quasiminimizer and we call it a normalized quasiminimizer for H^{n-1}.

Theorem 9.14. *If W is a normalized quasiminimizer for H^{n-1}, then W and $\mathbb{R}^n \backslash \overline{W}$ are domains of isoperimetry. The isoperimetry constants C_2 (as in (5.2)) for W and $\mathbb{R}^n \backslash \overline{W}$ depend only on n, Q_0, Q_1 and M.*

This is the analogue of Theorem 5.4, and we follow its proof.

Let us first prove that normalized quasiminimizers for H^{n-1} are domains of isoperimetry. We want to use Lemma 5.5, and so we give ourselves a normalized quasiminimizer W for H^{n-1} and an open subset Ω of W such that $\mathrm{int}(Q_0) \subset \Omega$. We want to prove that (5.6) holds.

Of course Ω is a competitor, but it is probably not good enough. One problem with Ω is that it may have a lot of tiny bubbles away from its main core and close to ∂W, so that $\partial\Omega$ contains the whole ∂W "artificially", and (9.2) holds too easily. We want to remove these bubbles from Ω before we apply (9.2). Similarly, $Z = W \backslash \Omega$ may have lots of tiny bubbles inside Ω and near ∂W. We don't like this because we want to estimate $|Z|$ in terms of ∂Z by the isoperimetric inequality, and we would be happy to say that $\partial Z \backslash W \subset \partial W \backslash \partial\Omega$. So we want to remove these tiny bubbles of Z as well.

Notice that all of these considerations have conterparts in the earlier story for BV.

Set $E = \partial W$ and $\mu = H^{n-1}_{|E}$. We want to decompose ∂W into a piece E_+ that clearly belongs to $\partial\Omega$, a piece E_- that clearly belongs to ∂Z, and a set of μ-measure zero. Let $\beta(x, r)$ be as in (4.18). Since E is an Ahlfors-regular set that satisfies Condition B, the proof of Lemma 5.30 tells us that μ-almost every point of E satisfies

$$\lim_{r\to 0} r^{-n} \, |\Omega \cap B(x, r)| = 0 \tag{9.15}$$

or

$$\lim_{r\to 0} r^{-n} \, |Z \cap B(x, r)| = 0. \tag{9.16}$$

Notice that (9.15) and (9.16) cannot hold at the same time, because $W = \Omega \cup Z$ satisfies Condition B. Thus the sets $E_+ = \{x \in E : x \text{ satisfies } (9.16)\}$ and $E_- = \{x \in E : x \text{ satisfies } (9.15)\}$ are disjoint. Let E_+^* be the set of points $x \in E_+$ where E_- has density zero, i.e., such that $\lim_{r\to 0} r^{1-n}\mu(E_- \cap B(x, r)) = 0$.

Standard density arguments show that $\mu(E_+ \backslash E_+^*) = 0$. Thus

$$\mu(E \backslash (E_+^* \cup E_-)) = 0 \tag{9.17}$$

by the discussion above.

Let $\varepsilon > 0$ be very small, to be chosen later. It will be allowed to depend on W and on Ω. For each $x \in E_+^*$, choose a radius $r(x) \in (0, 1)$ such that

$$|B(x, r(x)) \cap Z| \leq \varepsilon r(x)^n, \tag{9.18}$$

$$H^{n-1}\left(\partial B(x, r(x)) \cap Z\right) \leq \varepsilon r(x)^{n-1}, \tag{9.19}$$

and

$$\mu(B(x, 2r(x)) \cap E_-) \leq \varepsilon r(x)^{n-1}. \tag{9.20}$$

This is possible because of (9.16), Tchebytchev, and the fact that E_- has vanishing density at x. Choose a Vitali covering of E_+^* by balls $B\left(x, \frac{r(x)}{2}\right)$, $x \in I_1 \subset E_+^*$, with the property that the balls $B\left(x, \frac{r(x)}{10}\right)$, $x \in I_1$, are disjoint. See for instance the first pages of [**St**] for the existence of such a covering. Select a finite subset I of I_1 such that

$$\mu\left(E_+^* \backslash \bigcup_{x\in I} B\left(x, \frac{r(x)}{2}\right)\right) \leq \varepsilon. \tag{9.21}$$

For each $y \in E_-^* := E_- \setminus \bigcup_{x \in I} B(x, 2r(x))$, choose a radius $r(y) > 0$ such that

$$r(y) < \frac{1}{10} \inf_{x \in I} r(x), \tag{9.22}$$

$$r(y) < \operatorname{dist}(y, Q_0), \tag{9.23}$$

and

$$H^{n-1}(\partial B(y, r(y)) \cap \Omega) \leq \varepsilon r(y)^{n-1}. \tag{9.24}$$

[Observe that (9.23) is easy to get, because $y \notin \partial Q_0$ when (9.15) holds; (9.24) is also easy to get from (9.15) and Tchebytchev.] Next cover E_-^* by balls $B\left(y, \frac{r(y)}{2}\right)$, $y \in J_1$, with the property that the balls $B\left(y, \frac{r(y)}{10}\right)$ are disjoint. Select a finite subset J of J_1 so that

$$\mu\left(E_-^* \setminus \bigcup_{x \in J} B\left(y, \frac{r(y)}{2}\right)\right) \leq \varepsilon. \tag{9.25}$$

Set $A_1 = \bigcup_{x \in I} B(x, r(x))$ and $A_2 = \bigcup_{x \in J} B(y, r(y))$; notice that $\overline{A}_1 \cap \overline{A}_2 = \emptyset$ because of (9.22) and the definition of E_-^*. Finally set

$$V = [\Omega \cup (W \cap A_1)] \setminus A_2. \tag{9.26}$$

Observe that $V \in \mathcal{F}_0$ because $V \subset W$ and $\operatorname{int}(Q_0) \subset V$ (by 9.23). We are now ready to estimate $|W \setminus \Omega|$. Oberve that $W \setminus \Omega = Z \subset (W \setminus V) \cup (Z \cap A_1)$, and $|Z \cap A_1| \leq \sum_{x \in I} |Z \cap B(x, r(x))| \leq \varepsilon \sum_{x \in I} r(x)^n$, by (9.18). The balls $B\left(x, \frac{r(x)}{10}\right)$, $x \in I$, are disjoint, and so $\sum_{x \in I} r(x)^n \leq C(1 + |Q_1|)$, where C depends only on n. Thus

$$|W \setminus \Omega| \leq |W \setminus V| + C\varepsilon. \tag{9.27}$$

The isoperimetric inequality tells us that

$$|W \setminus V| \leq C\left[H^{n-1}(\partial(W \setminus V))\right]^{\frac{n}{n-1}}; \tag{9.28}$$

our next task is to control $\partial(W \setminus V)$. We start with $\partial_1 = W \cap \partial(W \setminus V)$. Notice that $\partial_1 = W \cap \partial V$. If $z \in \partial_1 \setminus \partial\Omega$, then z must lie on some sphere $\partial B(x, r(x))$, $x \in I$ or $\partial B(y, r(y))$, $y \in J$. In the first case, $z \in Z$ because otherwise a whole neighborhood of z would be contained in Ω and hence in V. [We use the fact that $\overline{A}_1 \cap \overline{A}_2 = \emptyset$, so that A_2 is far from z.] In the second case, z cannot be in the interior of Z for the same reason. Since it does not lie in $\partial\Omega$ either, it must lie in Ω. Altogether

(9.29)
$$\begin{aligned}
H^{n-1}(\partial_1) &\le H^{n-1}(\partial\Omega\cap W) + H^{n-1}(\partial_1\backslash\partial\Omega)\\
&\le H^{n-1}(\partial\Omega\cap W) + \sum_{x\in I} H^{n-1}(\partial B(x,r(x))\cap Z)\\
&\quad + \sum_{y\in J} H^{n-1}(\partial B(y,r(y))\cap\Omega)\\
&\le H^{n-1}(\partial\Omega\cap W) + \varepsilon\sum_{x\in I} r(x)^{n-1} + \varepsilon\sum_{y\in J} r(y)^{n-1}\\
&\le H^{n-1}(\partial\Omega\cap W) + C\varepsilon\mu(E)
\end{aligned}$$

by (9.19), (9.24), the fact that the balls $B\left(x,\frac{r(x)}{10}\right)$ and $B\left(y,\frac{r(y)}{10}\right)$ are pairwise disjoint, and the Ahlfors regularity of E.

The other piece of $\partial(W\backslash V)$ is $\partial_2 = E\cap\partial(W\backslash V)$. Obviously ∂_2 does not meet any of the $B\left(x,\frac{r(x)}{2}\right)$, $x\in I$. (Compare with (9.26).) Hence

$$\begin{aligned}
H^{n-1}(\partial_2) &= \mu\left(\partial_2\cap E_+^*\right) + \mu\left(\partial_2\cap E_-\right)\\
&\le \varepsilon + \mu(E_-)
\end{aligned} \tag{9.30}$$

by (9.17) and (9.21). Next we want to control $E_-\cap\partial V$. If $z\in E_-\cap\partial V$, then z cannot lie in any $B\left(y,\frac{r(y)}{2}\right)$, $y\in J$, and so $z\in E_-^*\setminus\bigcup_{y\in J}B\left(y,\frac{r(y)}{2}\right)$ or else $z\in E_-\setminus E_-^*\subset\bigcup_{x\in I}B(x,2r(x))$. Therefore

$$\begin{aligned}
\mu(E_-) &= \mu(E_-\setminus\partial V) + \mu(E_-\cap\partial V)\\
&\le \mu(E\backslash\partial V) + \varepsilon + \sum_{x\in I}\varepsilon r(x)^{n-1}\\
&\le \mu(E\backslash\partial V) + \varepsilon + C\varepsilon\mu(E)
\end{aligned} \tag{9.31}$$

by (9.25), (9.20), the disjointness of the $B\left(x,\frac{r(x)}{10}\right)$'s, and the Ahlfors regularity of E. Altogether,

(9.32)
$$\begin{aligned}
H^{n-1}(\partial(W\backslash V)) &\le H^{n-1}(\partial_1) + H^{n-1}(\partial_2)\\
&\le H^{n-1}(\partial\Omega\cap W) + \mu(E\backslash\partial V) + C\varepsilon(1+\mu(E))
\end{aligned}$$

by (9.29), (9.30) and (9.31).

Now we may use the fact that $V \in \mathcal{F}_0$. It follows from (9.2) that

$$\mu(E \backslash \partial V) = H^{n-1}(\partial W \backslash \partial V) \leq M H^{n-1}(\partial V \backslash \partial W). \tag{9.33}$$

Since $\partial V \backslash \partial W = W \cap \partial V = \partial_1$, (9.29) and (9.33) yield $\mu(E \backslash \partial V) \leq M H^{n-1}(\partial\Omega \cap W) + C\varepsilon\mu(E)$. The desired conclusion

$$|W \backslash \Omega| \leq C \left[H^{n-1}(\partial\Omega \cap W)\right]^{\frac{n}{n-1}} \tag{9.34}$$

follows from (9.27), (9.28), (9.32) and this last inequality by taking ε small enough. This is the same as (5.6). This completes the proof that W is a domain of isoperimetry.

It remains to show that $\mathbb{R}^n \backslash \overline{W}$ is a domain of isoperimetry as well. The proof is very similar to the one for W, and we leave it to the reader. This finishes the proof of Theorem 9.14.

Once we have Theorem 9.14, we can get the John conditions as in Theorem 6.1.

One can also make a story as in Section 7 using H^{n-1} instead of BV norms, but we shall not bother with that.

References

[Al] F. Almgren, *Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints*, Mem. Amer. Math. Soc., **165**(**4**) (1976), 1-199.

[Bl] B. Bojarski, *Remarks on Sobolev imbedding inequalities*, in 'Complex Analysis', Joensuu 1987 (Laine, Rickman, and Sorvali, Ed.), Lecture Notes in Math., **1351** (1988), 52-68, Springer Verlag.

[D] G. David, *Morceaux de graphes lipschitziens et intégrales singulières sur une surface*, Rev. Math. Iberoamericana, **4** (1988), 73-114.

[DJ] G. David and D. Jerison, *Lipschitz approximations to hypersurfaces, harmonic measure, and singular integrals*, Indiana Math. J., **39** (1990), 831-845.

[DS1] G. David and S. Semmes, *Quantitative rectifiability and Lipschitz mappings*, Transactions Amer. Math. Soc., **337** (1993), 855-889.

[DS2] ———, *Analysis of and on Uniformly Rectifiable Sets*, Mathematical Surveys and Monographs, **38** (1993), American Mathematical Society.

[Fe] H. Federer, *Geometric Measure Theory*, Springer-Verlag, 1969.

[Gi] E. Giusti, *Minimal Surfaces and Functions of Bounded Variation*, Monographs in Mathematics, **80** (1984), Birkhäuser.

[JKV] P. Jones, N. Katz and A. Vargas, *Checkerboards, Lipschitz functions, and uniform rectifiability*, Revista Mat. Iberoamericana, **13** (1997), 189-210.

[Ma] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces*, Cambridge Studies in Advanced Math., **44**, Cambridge Univ. Press, 1995.

[Se1] S. Semmes, *A criterion for the boundedness of singular integrals on hypersurfaces*, Trans. Amer. Math. Soc., **311** (1989), 501-513.

[Se2] ———, *Finding structure in sets with little smoothness*, Proc. I.C.M. (Zürich, 1994), Birkhäuser, 875-885.

[Se3] ———, *Finding Curves on General Spaces through Quantitative Topology with Applications for Sobolev and Poincaré Inequalities*, Selecta Math. (N.S.), **2** (1996), 155-295.

[St] E.M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.

[V] J. Väisälä, *Invariants for quasisymmetric, quasimöbius and bilipschitz maps,* J. d'Analyse Math., **50** (1988), 201-223.

Received October 30, 1996. The second author was supported in part by the U.S. National Science Foundation.

Université Paris XI et Institut Universitaire de France
Bâtiment 425
91405 Orsay, France
E-mail address: Guy.DAVID@math.u-psud.fr
and
IHES
91440 Bures-sur-Yvette, France

Rice University, P.O. Box 1892
Houston, TX 77251
E-mail address: semmes@math.rice.edu