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COMPUTING THE INFINITESIMAL INVARIANTS ASSOCIATED TO DEFORMATIONS OF SUBVARIETIES

Randall F. Westhoff

The purpose of this article is to study and describe a method for computing the infinitesimal invariants associated to deformations of subvarieties. An interpretation of the infinitesimal invariant of normal functions as a pairing similar to the infinitesimal Abel-Jacobi mapping is given. The computation of both invariants for certain forms is then reduced to a residue computation at a finite number of points of the subvariety. Applications of this technique include a nonvanishing result for the infinitesimal Abel-Jacobi mapping leading to finiteness results for low degree rational curves on complete intersection threefolds with trivial canonical bundle and a generalization of a formula of Voisin for the infinitesimal invariant of certain normal functions.

1. Introduction.

The problem of understanding the subvarieties of a given projective variety is one of the central problems in algebraic geometry. If Y is a subvariety of a projective variety $X \subseteq \mathbb{P}^m$ and Y is nontrivial in the sense that it is the not the complete intersection of X with another subvariety of $\mathbb{P}^m$, then in many situations one expects the deformations of Y in X to "generate" some of the cohomology of X. The cohomology generated by a subvariety as it deforms is measured by Abel-Jacobi mappings, normal functions and their infinitesimal variants. Understanding the degree to which deformations of a subvariety generate the cohomology of a variety can yield information about the structure of the family of all such subvarieties. (i.e. dimension, smoothness, etc.)

We begin with a brief discussion of Abel-Jacobi mappings. Let X be a smooth projective variety of dimension n and let F be a smooth projective variety parametrizing a family of subvarieties of dimension d on X. Let $E = \{(Y, x) \in F \times X : x \in Y\}$ and let $p : E \longrightarrow F$ and $q : E \longrightarrow X$ be the natural projections. Then the "cohomological" Abel-Jacobi mapping is the morphism of Hodge structures of type $(-d, -d)$ defined by the composition

$$H^*(X, \mathbb{C}) \xrightarrow{q^*} H^*(E, \mathbb{C}) \xrightarrow{p_*} H^{*-2d}(F, \mathbb{C})$$

where p_* the Poincaré dual of p_* on homology when E is smooth and is defined via a desingularization of E when E is not smooth.

With F as above assume further that the generic subvariety parametrized by F is smooth. The tangent space $\mathcal{T}_{F,Y}$ to F at a subvariety $Y \in F$ maps naturally into $H^0(\mathcal{N}_{Y|X})$, the tangent space to the deformation space of Y in X. Then for generic Y in F there is a commutative diagram

$$\begin{array}{ccc} H^d(\Omega_X^{*-d}) & \xrightarrow{p_* q^*} & H^0(\Omega_F^{*-2d}) \\ \Phi\big\downarrow & & \big\downarrow \\ \wedge^{*-2d} H^0(\mathcal{N}_{Y|X})^* & \longrightarrow & \wedge^{*-2d}\mathcal{T}^*_{F,Y}. \end{array}$$

where the map in the top row is identified with the $(* - d, d)$ piece of the Abel-Jacobi mapping and Φ is the infinitesimal Abel-Jacobi mapping which is given by the contraction mapping

$$\Phi : \wedge^{*-2d} H^0(\mathcal{N}_{Y|X}) \otimes H^d(\Omega_X^{*-d}) \longrightarrow H^d(\Omega_Y^d).$$

When $* = 2d + 1$, Φ also computes the differential at Y of the Abel-Jacobi mapping of Griffiths from F into the $(n - d)th$ intermediate Jacobian of X. For details see [4].

As Abel-Jacobi mappings measure the cohomology generated by a subvariety as it deforms in a fixed variety X, normal functions measure the cohomology generated by a subvariety as it deforms with X. An infinitesimal invariant of normal functions was first introduced by Griffiths in [7] and later refined by Green in [6]. Let $S = \mathrm{Spec}(\mathbb{C}[s]/s^2)$. If Y is an algebraic cycle of dimension d on X and Y_S is an infinitesimal deformation of Y in an infinitesimal deformation X_S of X, then there is a natural pairing

$$\Phi : H^0(\mathcal{N}_{Y|X_S}) \otimes H^d(\Omega^{d+1}_{X_S}|_X) \longrightarrow H^d(\Omega_Y^d)$$

given by contraction. If $\eta \in H^0(\mathcal{N}_{Y|X_S})$ is the vector field determined by $Y_S \subseteq X_S$, then it is shown in Section 2 that $\Phi(\eta, \cdot)$ computes the infinitesimal invariant of the normal function associated to $Y_S \subseteq X_S$. For a related interpretation of this invariant see [15].

This pairing is similar in form to the infinitesimal Abel-Jacobi mapping and this observation is exploited to give a general technique for computing both invariants in Section 3. This construction is a generalization of a construction of Clemens in [4] and reduces the computation of Φ for certain forms on X to a residue computation at a finite number of points of Y. In the sections that follow several applications of this technique are given. By appealing to a regularity result for space curves in [8], this technique is used to prove some nonvanishing results for the infinitesimal Abel-Jacobi mapping

$$\Phi : \wedge^{n-2} H^0(\mathcal{N}_{C|X}) \otimes H^1(\Omega_X^{n-1}) \longrightarrow H^1(\Omega_C^1)$$

associated to certain low degree curves C on varieties X of dimension n. In particular, it is shown that the infinitesimal Abel-Jacobi mapping is of maximal rank for low degree smooth rational curves C on complete intersection threefolds X with trivial canonical bundle. Combining this with the fact that the infinitesimal Abel-Jacobi mapping is trivial for such C which deform generically with X forces $H^0(\mathcal{N}_{C|X})$ to vanish. This yields finiteness results for low degree rational curves on complete intersection threefolds with trivial canonical bundle. There has been a great deal of interest in this problem stemming from some computations of physicists working in string theory. The mirror symmetry principle allows them to relate the number of rational curves of a given degree on a generic quintic threefold (or more generally a Calabi-Yau threefold) to the coefficients of a certain Fourier series determined by the variation of Hodge structure of some other family of threefolds with trivial canonical bundle. It would therefore be of great interest to determine whether there was a finite number of rational curves of each degree on a generic complete intersection threefold with trivial canonical bundle. For a nice survey mirror symmetry and its implications in algebraic geometry see [13].

In the last two sections the formulas for the infinitesimal invariant of a normal function developed in Section 3 are refined in the case of subvarieties of hypersurfaces. This leads to a generalization of Voisin's formula [16] for the infinitesimal invariant of normal functions associated to algebraic one-cycles on hypersurface threefolds which are contained in a hyperplane section.

This work is an expanded version of some results in the authors dissertation completed at the University of Utah. The author would like to thank Herb Clemens for suggesting these problems and for many helpful conversations.

2. Normal Functions.

Let X be a smooth projective variety of dimension n and let X_S be an infinitesimal deformation of X. Let Y be an algebraic cycle on X of dimension d. When Y is smooth there is a natural pairing

$$\Phi : H^0(\mathcal{N}_{Y|X_S}) \otimes H^d(\Omega_{X_S}^{d+1}|_X) \longrightarrow H^d(\Omega_Y^d) \tag{2.1}$$

given by contraction. When Y is not smooth this pairing is still defined if we take a desingularization $f : \tilde{Y} \longrightarrow Y$ of the irreducible components of Y

and define the normal bundle to Y in X_S by $\mathcal{N}_{Y|X_S} = f^*\mathcal{T}_{X_S}/\mathcal{T}_{\tilde{Y}}$ and replace Ω^d_Y with $\Omega^d_{\tilde{Y}}$ in (2.1). This agrees with the usual definition of $\mathcal{N}_{Y|X_S}$ when Y is smooth. Let $Y_S \subseteq X_S$ be an infinitesimal deformation of Y in X_S given by $\eta \in H^0(\mathcal{N}_{Y|X_S})$. The goal of this section is to prove that:

Theorem 2.1. $\Phi(\eta,\cdot)$ *computes the infinitesimal invariant of the normal function associated to* $Y_S \subseteq X_S$.

The key step in this process will be the computation of $H^d(\Omega^{d+1}_{X_S}|_X)$. For simplicity, assume that S is one-dimensional. Then there is an exact sequence

$$0 \longrightarrow (\mathcal{N}_{X|X_S})^* \longrightarrow \Omega_{X_S}\,|_X \longrightarrow \Omega_X \longrightarrow 0$$

where $(\mathcal{N}_{X|X_S})^* = \mathcal{I}_{X|X_S}/\mathcal{I}^2_{X|X_S} = \mathcal{I}_{X|X_S} = \mathcal{O}_X \cdot ds \subseteq \Omega_{X_S}|_X$ and $\mathcal{I}_{X|X_S}$ is the ideal sheaf of X in X_S. This induces an exact sequence

$$0 \longrightarrow \Omega^d_X \xrightarrow{\wedge ds} \Omega^{d+1}_{X_S}|_X \longrightarrow \Omega^{d+1}_X \longrightarrow 0. \tag{2.2}$$

To obtain an explicit description of $H^d(\Omega^{d+1}_{X_S}|_X)$ it will be necessary to find a nice Dolbeault type resolution of $\Omega^{d+1}_{X_S}|_X$.

Assume that $X_S \longrightarrow S$ comes from a family of smooth projective varieties $\mathcal{X} \longrightarrow T$ where T is a disc in $\mathbb{C}$ centered at the origin $0 \in \mathbb{C}$ and $X = X_0$. After shrinking T, if necessary, there is a diffeomorphism $\Psi : X \times T \longrightarrow \mathcal{X}$. The fibers X_t of $\mathcal{X} \longrightarrow T$ can then viewed as a family of complex structures on the C^∞ manifold X. From this point of view, Kodaira in [12, Chapter 5], describes this variation of the complex structure in terms of the Kodaira-Spencer class of the deformation. Let $\{U_k\}$ be an open cover of X and let $z^1_k, \dots, z^n_k$ be local holomorphic coordinates on U_k for X. Then on each $U_k \times T$

$$\Psi(z^1_k, \dots, z^n_k, t) = (\zeta^1_k(z,t), \dots, \zeta^n_k(z,t), t)$$

where $\zeta^1_k(z,t), \dots, \zeta^n_k(z,t)$ are local C^∞ coordinates on X which are holomorphic with respect to the complex structure on X_t. On each $(U_j \cap U_k) \times T$, write $\zeta^l_k(z,t) = f^l_{j,k}(\zeta_k(z,t),t)$. Note that $f^l_{j,k}$ is holomorphic in $\zeta^1_k, \dots, \zeta^n_k, t$ and $\zeta^l_k(z,t)$ is holomorphic in t for all j,k,l. Thus we can write $\zeta^l_k(z,t)$ as a power series in t, $\zeta^l_k(z,t) = z^l_k + t\alpha^l_k(z) + \cdots$. The Kodaira-Spencer class of the deformation is the image of $\partial/\partial t$ under the Kodaira-Spencer map and is represented by the $\bar\partial$-closed C^∞ vector valued $(0,1)$ form θ where

$$\theta|_{U_k} = \sum \theta^i_{j,k} \frac{\partial}{\partial z^i_k} \otimes d\bar z^j_k \quad \text{and} \quad \theta^i_{j,k} = -\partial\alpha^i_k/\partial\bar z^j_k.$$

Replacing T by S gives the same set up with all equations taken modulo t^2. We wish to determine the $\bar\partial$ operator on X_S in terms of z and s. First notice that

$$\bar\partial\zeta_k^l = \bar\partial(z_k^l + s\alpha_k^l) = s\bar\partial\alpha_k^l = -s\sum\theta_{j,k}^l d\bar z_k^j = -s\theta|_{U_k}(z_k^l).$$

Thus $(\bar\partial + s\theta|_{U_k})(\zeta_k^l) = 0$ for each l and a straight forward computation shows that a C^∞ function $f(z,s)$ on $U_k \times S$ is holomorphic when viewed as a function on $\Psi(U_k \times S)$ if and only if $(\bar\partial + s\theta|_{U_k})(f) = 0$. Thus $\bar\partial + s\theta$ detects the holomorphic functions on X_S. Since $\zeta_k^1, \dots, \zeta_k^n, s$ are local holomorphic coordinates on $U_k \times S \cong \Psi(U_k \times S)$, $(\Omega_{X_S}|_X)|_{U_k}$ is generated by ds and $d\zeta_k^i = dz_k^i + \alpha_k^i ds$, $i = 1, \dots, n$ over $\mathcal{O}_{U_k}$.

Let $\mathcal{A}^{p,q} = \mathcal{A}^{p,q}(X, \mathbb{C})$ denote the sheaf of C^∞ (p,q)-forms on X. The following lemma will enable us to effectively compute $H^d(\Omega_{X_S}^{d+1}|_X)$.

Lemma 2.2. *For each $p \geq 1$ there is a commutative diagram of fine resolutions*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega_X^{p-1} & \longrightarrow & \mathcal{A}^{p-1,0} & \xrightarrow{\bar\partial} & \mathcal{A}^{p-1,1} & \xrightarrow{\bar\partial} & \cdots \\
 & & \downarrow{\scriptstyle \wedge ds} & & \downarrow{\scriptstyle \wedge ds} & & \downarrow{\scriptstyle \wedge ds} & & \\
0 & \longrightarrow & \Omega_{X_S}^p|_X & \longrightarrow & \mathcal{A}^{p-1,0} \oplus \mathcal{A}^{p,0} & \xrightarrow{\bar\partial'} & \mathcal{A}^{p-1,1} \oplus \mathcal{A}^{p,1} & \xrightarrow{\bar\partial'} & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega_X^p & \longrightarrow & \mathcal{A}^{p,0} & \xrightarrow{\bar\partial} & \mathcal{A}^{p,1} & \xrightarrow{\bar\partial} & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $\bar\partial'(\zeta \wedge ds + \omega) = (\bar\partial\zeta + \theta \cdot \omega) \wedge ds + \bar\partial\omega.$

Proof. The commutativity of the diagram is clear and the exactness of the top and bottom rows is Dolbeault's lemma. To complete the proof of the lemma it only remains to be shown that the middle row is exact. Using the fact that $\bar\partial^2 = 0$ and $\bar\partial\theta = 0$ it is easy to check that $\bar\partial' \circ \bar\partial' = 0$. By definition, $\bar\partial\alpha_k^i = -\sum\theta_{j,k}^i d\bar z_k^j$ and $\theta|_{U_k} = \sum\theta_{j,k}^i \frac{\partial}{\partial z_k^i} \otimes d\bar z_k^j$ so that $\bar\partial'(d\zeta_k^i) = \bar\partial'(dz_k^i + \alpha_k^i ds) = 0$. Considering $\Omega_{X_S}^p|_X$ as a subsheaf of $\mathcal{A}^{p-1,0} \wedge ds \oplus \mathcal{A}^{p,0}$, it then follows that

$$\Omega_{X_S}^p|_X \subseteq \ker(\mathcal{A}^{p-1,0} \wedge ds \oplus \mathcal{A}^{p,0} \xrightarrow{\bar\partial'} \mathcal{A}^{p-1,1} \wedge ds \oplus \mathcal{A}^{p,1}).$$

For the opposite inclusion, let $\zeta \wedge ds + \omega \in \mathcal{A}^{p-1,0}(U) \wedge ds \oplus \mathcal{A}^{p,0}(U)$ where U is a small polydisc in X and suppose $\bar\partial'(\zeta \wedge ds + \omega) = 0$. Then $\bar\partial\omega = 0$ and $\bar\partial\zeta = -\theta\cdot\omega$ so that $\omega \in \Omega^p_X(U)$. Let $\tilde\omega$ be any form in $\Omega^p_{X_S}|_X(U)$ which restricts to ω. Then the image of $\tilde\omega$ in $\mathcal{A}^{p-1,0}(U) \wedge ds \oplus \mathcal{A}^{p,0}(U)$ is $\bar\partial'$-closed and of the form $\zeta' \wedge ds + \omega$ where $\bar\partial\zeta' = -\theta\cdot\omega$. It then follows that $\zeta - \zeta' \in \Omega^{p-1}_X(U)$ and $(\zeta-\zeta')\wedge ds + \tilde\omega$ defines an element of $\Omega^p_{X_S}|_X(U)$ whose image $\mathcal{A}^{p-1,0}(U) \wedge ds \oplus \mathcal{A}^{p,0}(U)$ is $\zeta \wedge ds + \omega$.

Now suppose that $q \geq 1$ and $\zeta \wedge ds + \omega \in \mathcal{A}^{p-1,q}(U) \wedge ds \oplus \mathcal{A}^{p,q}(U)$ is $\bar\partial'$-closed. Then $\bar\partial\omega = 0$ and $\bar\partial\zeta = -\theta\cdot\omega$. By Dolbeault's lemma, there is a $\gamma \in \mathcal{A}^{p,q-1}(U)$ such that $\bar\partial\gamma = \omega$. Since $\bar\partial\theta = 0$, $\bar\partial\zeta = -\theta\cdot\bar\partial\gamma = \bar\partial(\theta\cdot\gamma)$ so by Dolbeault's lemma there is a $\beta \in \mathcal{A}^{p-1,q-1}(U)$ such that $\bar\partial\beta = \zeta - \theta\cdot\gamma$. Putting this together one obtains

$$\bar\partial'(\beta \wedge ds + \gamma) = (\bar\partial\beta + \theta\cdot\gamma)\wedge ds + \bar\partial\gamma = \zeta \wedge ds + \omega.$$

Thus the middle row is also exact. □

Proof of Theorem 2.1. The resolutions of Lemma 2.2 compute the cohomology groups of Ω^d_X, $\Omega^{d+1}_{X_S}|_X$ and Ω^{d+1}_X. Furthermore, in the long exact sequence of cohomology groups for the sequence (2.2) the boundary map $H^d(\Omega^{d+1}_X) \longrightarrow H^{d+1}(\Omega^d_X)$ is given by contraction with the Kodaira-Spencer class θ. Thus a cohomology class $\omega \in H^d(\Omega^{d+1}_X)$ extends to a cohomology class $\tilde\omega \in H^d(\Omega^{d+1}_{X_S}|_X)$ if and only if $\theta\cdot\omega = 0$. Lemma 2.2 also shows that in this case $\tilde\omega$ is represented by a form $-\zeta \wedge ds + \omega$ for some $\zeta \in A^{d,d}(X,\mathbb{C})$ with $\bar\partial\zeta = \theta\cdot\omega$.

Consider the exact sequence of normal bundles

$$0 \longrightarrow \mathcal{N}_{Y|X} \longrightarrow \mathcal{N}_{Y|X_S} \longrightarrow \mathcal{N}_{X|X_S}|_Y \longrightarrow 0.$$

Since $s = 0$ is the defining equation for X in X_S and $s^2 = 0$, then $\mathcal{N}_{X|X_S}|_Y \cong \mathcal{O}_Y\cdot(\partial/\partial s)$. If Y_S is an infinitesimal deformation of Y in X_S given by $\eta \in H^0(\mathcal{N}_{Y|X_S})$, then the image of η in $\mathcal{N}_{X|X_S}|_Y$ corresponds to $\partial/\partial s$. In this situation, $\mathcal{N}_{Y|X_S} \cong \mathcal{N}_{Y|X} \oplus \mathcal{N}_{X|X_S}|_Y$ since the obstruction to splitting the sequence is the image of $\partial/\partial s$ in $H^1(\mathcal{N}_{Y|X}) \cong \mathrm{Ext}^1(\mathcal{O}_Y\cdot(\partial/\partial s), \mathcal{N}_{Y|X})$. Then locally $\eta = \eta' + \partial/\partial s$ for some $\eta' \in \mathcal{N}_{Y|X}$ and the pairing

$$\Phi : H^0(\mathcal{N}_{Y|X_S}) \otimes H^d(\Omega^{d+1}_{X_S}|_X) \longrightarrow H^d(\Omega^d_Y) \xrightarrow{\int_Y} \mathbb{C}$$

takes

$$\eta \otimes \tilde\omega = (\eta' + \partial/\partial s)\otimes(-\zeta \wedge ds + \omega) \longrightarrow -\int_Y \zeta + \int_Y \eta\cdot\omega.$$

This is up to a sign the same as the formula given by Griffiths in [7, pp. 302-307] and completes that proof of Theorem 2.1. □

3. Computation of the pairings.

In this section a general method for computing both the infinitesimal Abel-Jacobi mapping and the infinitesimal invariant associated to a normal function will be given. As we have seen these invariants are both given by contracting normal vector fields on a subvariety $Y \subseteq X$ against forms on X. The computation of these invariants will be reduced to a residue computation at a finite number of points on the subvariety. This generalizes a construction of Clemens in [4] for computing the infinitesimal Abel-Jacobi mapping for curves.

Let X be a smooth projective variety of dimension n embedded in a smooth projective variety W of dimension m. Let Y be a smooth projective variety immersed as a subvariety of dimension d in X by a morphism $f : Y \longrightarrow X \subseteq W$. Let X_S be an infinitesimal deformation of X in an infinitesimal deformation W_S of W. Throughout this section S will denote either $\mathrm{Spec}(\mathbb{C}[s]/s^2)$ or $\mathrm{Spec}(\mathbb{C})$. In the later case X_S and W_S will be identified with X and W respectively. In any case, for $k \geq 2d+1$ there is a natural pairing

$$\Phi : \wedge^{k-2d} H^0(\mathcal{N}_{Y|X_S}) \otimes H^d(\Omega^{k-d}_{X_S}|_X) \longrightarrow H^d(\Omega^d_Y) \cong \mathbb{C} \tag{3.1}$$

given by contraction. When $S = \mathrm{Spec}(\mathbb{C})$ this reduces to the infinitesimal Abel-Jacobi mapping for Y. When $S = \mathrm{Spec}(\mathbb{C}[s]/s^2)$, $k = 2d+1$ and Y deforms with X in X_S this is the pairing that gives the infinitesimal invariant of a normal function.

Consider the commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{T}_{X_S}|_X & \longrightarrow & \mathcal{T}_{W_S}|_X & \longrightarrow & \mathcal{N}_{X_S|W_S}|_X & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{N}_{Y|X_S} & \longrightarrow & \mathcal{N}_{Y|W_S} & \longrightarrow & f^*\mathcal{N}_{X_S|W_S} & \longrightarrow & 0. \end{array} \tag{3.2}$$

The obstruction $\mu \in \mathrm{Ext}^1(\mathcal{N}_{X_S|W_S}|_X, \mathcal{T}_{X_S}|_X)$ to splitting the top row of (3.2) determines the obstruction $\sigma \in \mathrm{Ext}^1(f^*\mathcal{N}_{X_S|W_S}, \mathcal{N}_{Y|X_S})$ to splitting the bottom row of (3.2) in the sense that applying the functor $\mathrm{Hom}(\mathcal{N}_{X_S|W_S}|_X, \,\cdot\,)$ to the top row and $\mathrm{Hom}(f^*\mathcal{N}_{X_S|W_S}, \,\cdot\,)$ to the bottom row gives a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(\mathcal{N}_{X_S|W_S}|_X, \mathcal{N}_{X_S|W_S}|_X) & \longrightarrow & \mathrm{Ext}^1(\mathcal{N}_{X_S|W_S}|_X, \mathcal{T}_{X_S}|_X) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(f^*\mathcal{N}_{X_S|W_S}, f^*\mathcal{N}_{X_S|W_S}) & \longrightarrow & \mathrm{Ext}^1(f^*\mathcal{N}_{X_S|W_S}, \mathcal{N}_{Y|X_S}) \end{array}$$

such that μ is mapped to σ.

We wish to compute Φ for forms in the image of the composition

(3.3)
$$H^0\left(\Omega^k_{X_S}\otimes\mathcal{N}^{\otimes d}_{X_S|W_S}|_X\right)\longrightarrow H^1\left(\Omega^k_{X_S}\otimes\mathcal{T}_{X_S}\otimes\mathcal{N}^{\otimes(d-1)}_{X_S|W_S}|_X\right)\longrightarrow\cdots$$
$$\longrightarrow H^d\left(\Omega^k_{X_S}\otimes\mathcal{T}^{\otimes d}_{X_S}|_X\right)\longrightarrow H^d\left(\Omega^{k-d}_{X_S}|_X\right)$$

where each map

$$H^i\left(\Omega^k_{X_S}\otimes\mathcal{T}^{\otimes i}_{X_S}\otimes\mathcal{N}^{\otimes(d-i)}_{X_S|W_S}|_X\right)\longrightarrow H^{i+1}\left(\Omega^k_{X_S}\otimes\mathcal{T}^{\otimes(i+1)}_{X_S}\otimes\mathcal{N}^{\otimes(d-i-1)}_{X_S|W_S}|_X\right)$$

is the boundary map given by cup product with μ and the final map is given by contraction.

Similarly, there is a composition of maps

$$H^0\left(f^*\left(\Omega^k_{X_S}\otimes\mathcal{N}^{\otimes d}_{X_S|W_S}\right)\right)\longrightarrow H^1\left(f^*\Omega^k_{X_S}\otimes\mathcal{N}_{Y|X_S}\otimes f^*\mathcal{N}^{\otimes d-1}_{X_S|W_S}\right)\longrightarrow$$
$$\cdots\longrightarrow H^d\left(f^*\Omega^k_{X_S}\otimes\mathcal{N}^{\otimes d}_{Y|X_S}\right)$$

each given by cup product with σ. The commutativity of (3.2) then gives a commutative diagram

(3.4)
$$\begin{array}{ccc}
H^0\left(\Omega^k_{X_S}\otimes\mathcal{N}^{\otimes d}_{X_S|W_S}|_X\right) & \longrightarrow & H^0\left(f^*\left(\Omega^k_{X_S}\otimes\mathcal{N}^{\otimes d}_{X_S|W_S}\right)\right)\\
\downarrow\mu & & \downarrow\sigma\\
H^1\left(\Omega^k_{X_S}\otimes\mathcal{T}_{X_S}\otimes\mathcal{N}^{\otimes(d-1)}_{X_S|W_S}|_X\right) & \longrightarrow & H^1\left(f^*\Omega^k_{X_S}\otimes\mathcal{N}_{Y|X_S}\otimes f^*\mathcal{N}^{\otimes(d-1)}_{X_S|W_S}\right)\\
\downarrow\mu & & \downarrow\sigma\\
\vdots & & \vdots\\
\downarrow\mu & & \downarrow\sigma\\
H^d\left(\Omega^k_{X_S}\otimes\mathcal{T}^{\otimes d}_{X_S}|_X\right) & \longrightarrow & H^d\left(f^*\Omega^k_{X_S}\otimes\mathcal{N}^{\otimes d}_{Y|X_S}\right)\\
\downarrow & & \downarrow\otimes\wedge^{k-2d}H^0(\mathcal{N}_{Y|X_S})\\
H^d\left(\Omega^{k-d}_{X_S}|_X\right) & \xrightarrow{\otimes\wedge^{k-2d}H^0(\mathcal{N}_{Y|X_S})} & H^d(\Omega^d_Y)
\end{array}$$

where the final horizontal and vertical maps are given by contraction. This reduces the computation of Φ for forms in the image of (3.3) to a computation involving objects defined on Y. In particular, this proves the following generalization of [3, Lemma 1.3].

Proposition 3.1 *If $\sigma=0$ or equivalently the sequence*

$$0\longrightarrow\mathcal{N}_{Y|X_S}\longrightarrow\mathcal{N}_{Y|W_S}\longrightarrow f^*\mathcal{N}_{X_S|W_S}\longrightarrow 0$$

splits, then Φ *vanishes on the image* $H^0(\Omega^k_{X_S} \otimes \mathcal{N}^{\otimes d}_{X_S|W_S}|_X)$ *in* $H^d(\Omega^{k-d}_{X_S}|_X)$.

Let $\mathcal{L}_1, \dots, \mathcal{L}_d$ be subline bundles of $\mathcal{N}_{X_S|W_S}|_X$ and for each i, let $\mathcal{T}_i \subseteq \mathcal{N}_{Y|W_S}$ denote the inverse image of $f^*\mathcal{L}_i \subseteq f^*\mathcal{N}_{X_S|W_S}$. Then there is an exact sequence

$$0 \longrightarrow \mathcal{N}_{Y|X_S} \longrightarrow \mathcal{T}_i \longrightarrow f^*\mathcal{L}_i \longrightarrow 0. \tag{3.5}$$

The obstruction σ to splitting the bottom row of (3.2) then restricts to the obstruction $\sigma_i \in \operatorname{Ext}^1(f^*\mathcal{L}_i, \mathcal{N}_{Y|X_S})$ to splitting (3.5). This gives a commutative diagram similar to (3.4) with $\mathcal{N}^{\otimes(d-i+1)}_{X_S|W_S}$ replaced by $\mathcal{L}_i \otimes \cdots \otimes \mathcal{L}_d$ and the vertical column on the right replaced by the composition

$$H^0(f^*(\Omega^k_{X_S} \otimes \mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_d)) \xrightarrow{\sigma_1} H^1(f^*\Omega^k_{X_S} \otimes \mathcal{N}_{Y|X_S} \otimes f^*(\mathcal{L}_2 \otimes \cdots \otimes \mathcal{L}_d))$$
$$\xrightarrow{\sigma_2} \cdots \xrightarrow{\sigma_d} H^d(f^*\Omega^k_{X_S} \otimes \mathcal{N}^{\otimes d}_{Y|X_S}).$$

Let $\mathcal{S}_i$ be a subline bundle of $\mathcal{T}_i$ whose image in $f^*\mathcal{N}_{X_S|W_S}$ generically generates $\mathcal{L}_i$ and let D_i be the effective divisor on Y given by the scheme theoretic degeneracy locus of the morphism of line bundles $\mathcal{S}_i \longrightarrow f^*\mathcal{L}_i$. For notational convenience, set $\mathcal{N} = \mathcal{N}_{Y|X_S}$. Let $\tilde{\mathcal{S}}_i = \mathcal{S}_i(D_i)$ be the sheaf of sections of $\mathcal{S}_i$ which are holomorphic except for poles along the components of D_i of order not exceeding the multiplicity of the component. Let $\tilde{\mathcal{T}}_i$ be the subsheaf of $\mathcal{T}_i(D_i)$ generated by $\mathcal{T}_i$ and $\tilde{\mathcal{S}}_i$ and let $\tilde{\mathcal{N}}_i$ be the kernel of the map $\tilde{\mathcal{T}}_i \longrightarrow f^*\mathcal{L}_i$. Then there is a natural map

$$f^*\mathcal{L}_i \longrightarrow \tilde{\mathcal{S}}_i \subseteq \tilde{\mathcal{T}}_i$$

which gives the meromorphic inverse of $\mathcal{S}_i \longrightarrow f^*\mathcal{L}_i$ and splits the exact sequence

$$0 \longrightarrow \tilde{\mathcal{N}}_i \longrightarrow \tilde{\mathcal{T}}_i \longrightarrow f^*\mathcal{L}_i \longrightarrow 0. \tag{3.6}$$

Let $\tau_i \in \operatorname{Ext}^1(\tilde{\mathcal{N}}_i/\mathcal{N}, \mathcal{N})$ be the obstruction to splitting the exact sequence

$$0 \longrightarrow \mathcal{N} \longrightarrow \tilde{\mathcal{N}}_i \longrightarrow \tilde{\mathcal{N}}_i/\mathcal{N} \longrightarrow 0. \tag{3.7}$$

Applying the functor $\operatorname{Hom}(f^*\mathcal{L}_i, \ \cdot\ )$ to the exact sequence (3.7) we obtain an exact sequence

$$\cdots \longrightarrow \operatorname{Hom}(f^*\mathcal{L}_i, \tilde{\mathcal{N}}_i) \longrightarrow \operatorname{Hom}(f^*\mathcal{L}_i, \tilde{\mathcal{N}}_i/\mathcal{N}) \xrightarrow{\tau_i}$$
$$\operatorname{Ext}^1(f^*\mathcal{L}_i, \mathcal{N}) \longrightarrow \operatorname{Ext}^1(f^*\mathcal{L}_i, \tilde{\mathcal{N}}_i) \longrightarrow \cdots .$$

Since the sequence (3.6) splits the the obstruction $\sigma_i \in \operatorname{Ext}^1(f^*\mathcal{L}_i, \mathcal{N})$ to splitting (3.5) goes to zero in $\operatorname{Ext}^1(f^*\mathcal{L}_i, \tilde{\mathcal{N}}_i)$. Then there is a $\sigma'_i \in$

$\operatorname{Hom}(f^*\mathcal{L}_i, \tilde{\mathcal{N}}_i/\mathcal{N})$ such that $\tau_i \circ \sigma_i' = \sigma_i$. In fact, σ_i' is represented by the composition

$$f^*\mathcal{L}_i \longrightarrow \tilde{\mathcal{T}}_i \longrightarrow \tilde{\mathcal{T}}_i/\mathcal{T}_i \cong \tilde{\mathcal{N}}_i/\mathcal{N}.$$

Assume that the D_i intersect properly so that $P = D_1 \cap D_2 \cap \cdots \cap D_d$ is a possibly nonreduced set of points and set

$$\begin{aligned} D^{[i]} &= D_i + \cdots + D_d \\ \mathcal{L}^{[i]} &= \mathcal{L}_i \otimes \cdots \otimes \mathcal{L}_d \\ \tilde{\mathcal{N}}/\mathcal{N}^{[i]} &= \tilde{\mathcal{N}}_i/\mathcal{N} \otimes \cdots \otimes \tilde{\mathcal{N}}_d/\mathcal{N} \end{aligned}$$

for $i = 1, \ldots, d$. Then we have a commutative diagram

(3.8)

$$\begin{array}{ccccccc}
H^0(f^*(\Omega^k_{X_S} \otimes \mathcal{L}^{[1]})) & \xrightarrow{\sigma_1} & H^1(f^*\Omega^k_{X_S} \otimes \mathcal{N} \otimes f^*\mathcal{L}^{[2]}) & & & & \\
\downarrow{\scriptstyle \sigma_1'} & & \downarrow{\scriptstyle \|} & & & & \\
H^0(f^*\Omega^k_{X_S} \otimes \tilde{\mathcal{N}}_1/\mathcal{N} \otimes f^*\mathcal{L}^{[2]}) & \xrightarrow{\tau_1} & H^1(f^*\Omega^k_{X_S} \otimes \mathcal{N} \otimes f^*\mathcal{L}^{[2]}) & \xrightarrow{\sigma_2} & & & \\
\downarrow{\scriptstyle \sigma_2'} & & \downarrow{\scriptstyle \sigma_2'} & & & & \\
\vdots & & \vdots & & & & \\
\downarrow{\scriptstyle \sigma_d'} & & \downarrow{\scriptstyle \sigma_d'} & & & & \\
H^0(f^*\Omega^k_{X_S} \otimes \tilde{\mathcal{N}}/\mathcal{N}^{[1]}) & \xrightarrow{\tau_1} & H^1(f^*\Omega^k_{X_S} \otimes \mathcal{N} \otimes \tilde{\mathcal{N}}/\mathcal{N}^{[2]}) & \xrightarrow{\tau_2} & \cdots & \xrightarrow{\tau_d} & H^d(f^*\Omega^k_{X_S} \otimes \mathcal{N}^{\otimes d}) \\
\downarrow{\scriptstyle \wedge^{k-2d} \otimes H^0(\mathcal{N})} & & \downarrow{\scriptstyle \wedge^{k-2d} \otimes H^0(\mathcal{N})} & & & & \downarrow{\scriptstyle \wedge^{k-2d} \otimes H^0(\mathcal{N})} \\
H^0\left(\frac{\Omega^d_Y(D^{[1]})}{\Omega^d_Y(D^{[1]})^*}\right) & \longrightarrow & H^1\left(\frac{\Omega^d_Y(D^{[2]})}{\Omega^d_Y(D^{[2]})^*}\right) & \longrightarrow & \cdots & \longrightarrow & H^d(\Omega^d_Y)
\end{array}$$

where $\Omega^d_Y(D^{[i]})$ is the sheaf of meromorphic d-forms on Y which are allowed to have poles along the divisors $D_i, \ldots, D_d$ but are otherwise holomorphic and $\Omega^d_Y(D^{[i]})^*$ is the subsheaf of $\Omega^d_Y(D^{[i]})$ consisting of those forms which are holomorphic along at least one of the divisors $D_i, \ldots, D_d$. The bottom row of (3.8) can then be identified with the usual residue mapping for meromorphic d-forms on Y at the points of P

$$H^0\left(\frac{\Omega^d_Y(\sum_{i=1}^d D_i)}{\sum_{j=1}^d \Omega^d_Y(\sum_{i \neq j} D_i)}\right) \xrightarrow{\operatorname{Res}_P} \mathbb{C}.$$

For $i = 1, \dots, d$, let s_i be a nontrivial meromorphic section of $\mathcal{S}_i$ and let $\bar{s}_i$ be the image of s_i in $f^*\mathcal{L}_i$. Then it follows that $\sigma'_i \in \operatorname{Hom}(f^*\mathcal{L}_i, \tilde{\mathcal{N}}_i/\mathcal{N})$ is represented by

$$s_i/\bar{s}_i \in \operatorname{Hom}(f^*\mathcal{L}_i, \tilde{\mathcal{S}}_i) \subseteq \operatorname{Hom}(f^*\mathcal{L}_i, \tilde{\mathcal{T}}_i).$$

If $\omega \in H^d(\Omega^{k-d}_{X_S}|_X)$ is the image of $\tilde{\omega} \in H^0(\Omega^k_{X_S}|_X \otimes \mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_d)$ under the map given by (3.3) and $\eta \in \wedge^{k-2d} H^0(\mathcal{N}_{Y|X_S})$ then combining the above results we find that

$$\Phi(\eta, \omega) = \sum \operatorname{Res}_p \left(\frac{\eta \wedge s_d \wedge \cdots \wedge s_1}{\bar{s}_1 \cdots \bar{s}_d} \rightharpoonup f^*\tilde{\omega} \right) \tag{3.9}$$

where the sum is taken over all points p in the support of $P = D_1 \cap D_2 \cap \cdots \cap D_d$.

A geometric method of obtaining such a setup in the case when X_S has codimension one in W_S is to take d subvarieties $Z_i \subseteq W_S$ each containing Y in its smooth locus such that $\dim Z_i = \dim Y + 1$ and the intersections $Z_i \cdot X_S = Y + Y_i$ have the property that Y_i intersects Y properly along a divisor D_i whose support is contained in the smooth locus of Y. If $\mathcal{N}_{Y|Z_i}$ is invertible for each i, then in the above construction we can take $\mathcal{L}_i = \mathcal{N}_{X_S|W_S}|_X$, $\mathcal{T}_i = \mathcal{N}_{Y|W_S}$ and $\mathcal{S}_i = \mathcal{N}_{Y|Z_i}$ for each i. If $\mathcal{N}_{Y|Z_i}$ is not invertible then $\mathcal{S}_i$ can be taken to be any invertible subsheaf of $\mathcal{N}_{Y|Z_i}$. This construction also works if the Z_i are only defined in a first order neighborhood of Y in W_S which amounts to choosing d global sections s_i of $\mathcal{N}_{Y|W_S}$ and taking $\mathcal{S}_i$ to be the subline bundle of $\mathcal{N}_{Y|W_S}$ generated by s_i. In fact, it was these examples that initially motivated the above construction.

In the case when $S = \operatorname{Spec}(\mathbb{C})$, $k = n$ and the pairing Φ given by (3.1) is the usual infinitesimal Abel-Jacobi mapping we can produce sections of Ω^n_X via the isomorphism

$$\Omega^m_W \otimes \wedge^{m-n}\mathcal{N}_{X|W} \longrightarrow \Omega^n_X. \tag{3.10}$$

This isomorphism is given locally as follows. Let ρ be a meromorphic section of $\wedge^{m-n}\mathcal{T}_W$ whose image $\bar{\rho}$ in $\wedge^{m-n}\mathcal{N}_{X|W}$ is not zero. Then the isomorphism given by (3.10) can be viewed as contraction with $\rho/\bar{\rho}$. If we choose a different ρ, then the results of the two contraction will agree where they are both defined. This gives a convenient way to produce local representatives of forms in $H^0(\Omega^n_X)$. Similarly, local representatives of forms in $H^0(f^*(\Omega^n_X))$ can be obtained by contracting sections of $H^0(f^*(\Omega^m_W \otimes \wedge^{m-n}\mathcal{N}_{X|W}))$ against $\rho/\bar{\rho}$ where ρ is now any meromorphic section of $\wedge^{m-n} f^*\mathcal{T}_W$ whose image $\bar{\rho}$

in $\wedge^{m-n} f^*\mathcal{N}_{X|W}$ is not zero. Consider the commutative diagram

$$\begin{array}{ccc} H^0(\Omega^m_W \otimes \wedge^{m-n}\mathcal{N}_{X|W} \otimes \mathcal{L}^{[1]}) & \longrightarrow & H^0(f^*(\Omega^m_W \otimes \wedge^{m-n}\mathcal{N}_{X|W} \otimes \mathcal{L}^{[1]})) \\ \downarrow & & \downarrow \\ H^0(\Omega^n_X \otimes \mathcal{L}^{[1]}) & \longrightarrow & H^0(f^*(\Omega^n_X \otimes \mathcal{L}^{[1]})). \end{array}$$

If $\omega \in H^d(\Omega^{n-d}_X)$ is the image of $\tilde{\omega} \in H^0(\Omega^m_W \otimes \wedge^{m-n}\mathcal{N}_{X|W} \otimes \mathcal{L}^{[1]})$ under the composition

$$(3.11)\ H^0(\Omega^m_W \otimes \wedge^{m-n}\mathcal{N}_{X|W} \otimes \mathcal{L}^{[1]}) \longrightarrow H^0(\Omega^n_X \otimes \mathcal{L}^{[1]}) \longrightarrow H^d(\Omega^{n-d}_X)$$

and $\eta \in \wedge^{n-2d} H^0(\mathcal{N}_{Y|X})$ then (3.9) can be rewritten

$$\Phi(\eta, \omega) = \sum Res_p \left(\frac{\eta \wedge s_d \wedge \cdots \wedge s_1 \wedge \rho}{\bar{s}_1 \cdots \bar{s}_d \bar{\rho}} \rightharpoonup f^*\tilde{\omega} \right) \tag{3.12}$$

where the s_i and $\bar{s}_i$ are defined as before and ρ is any meromorphic section of $\wedge^{m-n}\mathcal{N}_{Y|W} = \wedge^{m-n}(f^*\mathcal{T}_W/\mathcal{T}_Y)$ whose image $\bar{\rho}$ in $\wedge^{m-n} f^*\mathcal{N}_{X|W}$ does not vanish at any of the points $p \in P$.

4. Nonvanishing results for K_X nef.

In this section we will use the results of Section 3 to prove some nonvanishing results for the infinitesimal Abel-Jacobi mapping for low degree curves on projective varieties X with nef canonical bundle K_X. These results depend on the regularity theorem for space curves of Gruson, Lazarsfeld and Peskine in [8]. Let ω denote the natural map $\wedge^{n-2} H^0(\mathcal{N}_{C|X}) \longrightarrow H^0(\wedge^{n-2}\mathcal{N}_{C|X})$.

Theorem 4.1. *Let $X \subseteq \mathbb{P}^m$ be a smooth projective variety of dimension n with K_X nef and let $f : C \longrightarrow X \subseteq \mathbb{P}^m$ be a smooth curve of genus g and degree d on X with $\deg(\mathcal{N}_{C|X}) < 0$. Assume that X is a divisor on a smooth variety $Z \subseteq \mathbb{P}^m$ such that $\mathcal{N}_{C|Z}$ is generated by global sections and $\Omega^n_X \otimes \mathcal{N}_{X|Z} \cong \mathcal{O}_X(k)$ for some positive integer k with $d \leq k+2$. Then the infinitesimal Abel-Jacobi mapping $\Phi : \wedge^{n-2} H^0(\mathcal{N}_{C|X})/\ker\omega \longrightarrow H^1(\Omega^{n-1}_X)^*$ is injective.*

Proof. In the construction of Section 3 take $\mathcal{L} = \mathcal{N}_{X|Z} \subseteq \mathcal{N}_{X|\mathbb{P}^m}$. Then (3.5) is replaced by the exact sequence

$$0 \longrightarrow \mathcal{N}_{C|X} \longrightarrow \mathcal{N}_{C|Z} \longrightarrow f^*\mathcal{N}_{X|Z} \longrightarrow 0.$$

By our hypothesis, $H^0(\Omega^m_{\mathbb{P}^m} \otimes \wedge^{m-n}\mathcal{N}_{X|\mathbb{P}^m} \otimes \mathcal{N}_{X|Z}) \cong H^0(\Omega^n_X \otimes \mathcal{N}_{X|Z}) \cong H^0(\mathcal{O}_X(k))$ and thus can be identified with a quotient of

$$\{G\Omega : G \in S^k\}$$

where S^k is the set of homogeneous polynomials of degree k on $\mathbb{P}^m$ and

$$\Omega = \sum (-1)^j x_j dx_0 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_m$$

is a nonzero global section of $\Omega^m_{\mathbb{P}^m}(m+1) \cong \mathcal{O}_{\mathbb{P}^m}$. Denote by ω_G the image of $G\Omega$ in $H^1(\Omega^{n-1}_X)$ under the mapping defined by (3.11) for each $G \in S^k$.

Fix an $\eta \in \wedge^{n-2} H^0(\mathcal{N}_{C|X})$ whose image in $H^0(\wedge^{n-2}\mathcal{N}_{C|X})$ is nonzero. By our hypothesis, $\deg(\mathcal{N}_{C|X}) < 0$ so that $\mathcal{N}_{C|X}$ cannot be semipositive . If we denote by $\mathcal{N}^+_{C|X}$ the subsheaf of $\mathcal{N}_{C|X}$ generated by its global sections, then $\mathcal{N}_{C|Z}$ is semipositive implies that the exact sequence

$$0 \longrightarrow \mathcal{N}_{C|X}/\mathcal{N}^+_{C|X} \longrightarrow \mathcal{N}_{C|Z}/\mathcal{N}^+_{C|X} \longrightarrow f^*\mathcal{N}_{X|Z} \longrightarrow 0 \tag{4.1}$$

cannot split. In this situation, there is a global section $s \in H^0(\mathcal{N}_{C|Z})$ whose image $\bar{s} \in H^0(f^*\mathcal{N}_{X|Z})$ is not zero and a simple point p_1 in the zero locus of $\bar{s}$ such that $\eta \wedge s$ does not vanish at p_1. To see this pick s so that $\bar{s}$ has distinct zeros and does not vanish at any of the zeros of η. Since (4.1) does not split there is a point p_1 in the zero locus of $\bar{s}$ such that s is not zero in the geometric fiber of $(\mathcal{N}_{C|Z}/\mathcal{N}^+_{C|X})$ at p_1. Since $\eta \in \wedge^{n-2}H^0(\mathcal{N}^+_{C|X})$ does not vanish at p_1, then $\eta \wedge s$ will not vanish at p_1. Let $l = \deg(f^*\mathcal{N}_{X|Z})$ and let $p_2, \ldots, p_l$ be the other zeros of $\bar{s}$. Let ρ be any global section of $\wedge^{m-n}\mathcal{N}_{C|\mathbb{P}^m}$ whose image $\bar{\rho}$ in $\wedge^{m-n} f^*\mathcal{N}_{X|\mathbb{P}^m}$ does not vanish at any of the p_j. Then (3.12) can be rewritten

$$\Phi(\eta, \omega_G) = \sum \mathrm{Res}_{p_j} \left(\frac{\eta \wedge s \wedge \rho}{\bar{s}\bar{\rho}} \rightharpoonup f^*(G\Omega) \right)$$

for any $G \in S^k$. By construction $\eta \wedge s \wedge \rho$ defines a global section of $\wedge^{m-1}\mathcal{N}_{C|\mathbb{P}^m}$ which does not vanish at p_1.

Thus to complete the proof of the theorem we need to show that it is possible to pick $G \in S^k$ such that G vanishes at every p_j, except p_1 and does not vanish at p_1. Since C is smooth of degree $d \leq k+2$, then by the main result of [8] the natural map

$$H^0(\mathcal{O}_{\mathbb{P}^m}(k)) \longrightarrow H^0(\mathcal{O}_C(k))$$

is surjective. Since $\deg(\mathcal{N}_{C|X}) = -K_X \cdot C + 2g - 2 < 0$, then $K_X \cdot C \geq 2g - 1$. Also note that by the definitions of k and l, $K_X \cdot C = dk - l$. Thus $dk - (l-1) \geq 2g$. Since S^k cuts out a complete linear system on C of degree dk and $dk - (l-1) \geq 2g$, then it follows from the Riemann-Roch theorem that we can choose G to vanish at any $l-1$ points of C and not vanish at p_1. □

Let $X \subseteq \mathbb{P}^m$ be a complete intersection of $m-n$ hypersurfaces $Y_1, \ldots, Y_{m-n}$ in $\mathbb{P}^m$. For each j, let k_j denote the degree of Y_j and let $Z_j = \cap_{i \neq j} Y_i$.

Then X has dimension n and when X is smooth the canonical bundle of X is isomorphic to $\mathcal{O}(\sum k_j - m - 1)$. We call such an X a complete intersection of type $(k_1, \dots, k_{m-n})$ in $\mathbb{P}^m$. For complete intersections, Theorem 4.1 has the following form:

Corollary 4.2. *Let X be a generic complete intersection of type $(k_1, \dots, k_{m-n})$ in $\mathbb{P}^m$ such that K_X is nef or equivalently $\sum k_j \geq m+1$. Let $f : C \longrightarrow X \subseteq \mathbb{P}^m$ be a smooth curve of genus g and degree d on X and assume there is an i such that $d \leq \sum k_j - m + 1 + k_i$, $d(m+1-\sum k_j)+2g-2 < 0$ and $\mathcal{N}_{C|Z_i}$ is generated by global sections. Then $\Phi : \wedge^{n-2} H^0(\mathcal{N}_{C|X})/\ker\omega \longrightarrow H^1(\Omega_X^{n-1})^*$ is injective.*

Proof. Since X is a generic complete intersection we may assume that Z_i is smooth. Then the $\deg(\mathcal{N}_{X|Z_i}) = k_i$ and $\Omega_X^n \otimes \mathcal{N}_{X|Z_i} \cong \mathcal{O}_X(\sum k_j - m - 1 + k_i)$. Also notice that $\deg(\mathcal{N}_{C|X}) = -K_X \cdot C + 2g - 2 = d(m+1-\sum k_j)+2g-2 < 0$ so that all the hypotheses of Theorem 4.1 are satisfied. □

Remark. When the canonical bundle of X is not necessarily nef the basic argument of Theorem 4.1 will still go through with some modification. For example, it can be shown that if X is a smooth hypersurface of degree m in $\mathbb{P}^m$, $m \geq 4$ and $f : C \longrightarrow X \subseteq \mathbb{P}^m$ is a smooth rational curve of degree $d \leq m+1$ such that the global sections of $\mathcal{N}_{C|X}$ generate a subsheaf of rank $\geq m-3$ then the infinitesimal Abel-Jacobi mapping is nonzero. For this and related results see [17].

5. Rational curves on K-trivial complete intersection threefolds.

Let C be a smooth curve on a smooth threefold X. Then there is an exact sequence

$$H^0(\mathcal{T}_X) \longrightarrow H^0(\mathcal{N}_{C|X}) \longrightarrow \mathcal{H}^1 \longrightarrow H^1(\mathcal{T}_X) \xrightarrow{\phi} H^1(\mathcal{N}_{C|X}) \tag{5.1}$$

where $\mathcal{H}^1$ is the first hypercohomology group of the map $\mathcal{T}_X \longrightarrow \mathcal{N}_{C|X}$ and classifies the first order infinitesimal deformations of the pair (C, X). When C is rational and X has trivial canonical bundle, the map ϕ can be identified with the infinitesimal Abel-Jacobi mapping via the commutative diagram

$$\begin{array}{ccc} H^1(\mathcal{T}_X) & \xrightarrow{\phi} & H^1(\mathcal{N}_{C|X}) \\ \downarrow \wr & & \downarrow \wr \text{ Serre duality} \\ H^1(\Omega_X^2) & \xrightarrow{\Phi} & H^0(\mathcal{N}_{C|X})^*. \end{array}$$

When C deforms generically with X to first order the mapping $\mathcal{H}^1 \longrightarrow H^1(\mathcal{T}_X)$ of (5.1) is surjective and the infinitesimal Abel-Jacobi mapping must vanish. This gives the following vanishing result.

Lemma 5.1. *Let X be a smooth threefold with trivial canonical bundle. If C is a smooth rational curve on X, then C deforms generically with X to first order if and only if the infinitesimal Abel-Jacobi mapping is zero.*

If X is a smooth complete intersection threefold with trivial canonical bundle, then it is easy to check that X is a complete intersection of type (5), (2, 4), (3, 3), (3, 2, 2), or (2, 2, 2, 2). For the generic (5) and (2, 4) complete intersection, the Z_i in Corollary 4.2 can be taken to be $\mathbb{P}^4$ and a smooth quadric fourfold respectively. Then $\mathcal{N}_{C|Z}$ is then generated by global sections since Z_i is a homogeneous space. For the remaining cases the arguments of [5, Lecture 21] show that when C deforms generically with X there is an i such that $\mathcal{N}_{C|Z_i}$ is generated by global sections. Combining this with Lemma 5.1 and Corollary 4.2 gives the following finiteness result for smooth rational curves on such X.

Theorem 5.2. *Let X be a generic complete intersection threefold with trivial canonical bundle. Then X has only a finite number of smooth rational curves of degree d if*

(i) $d \leq 7$ *and X is of type* (5)

(ii) $d \leq 6$ *and X is of type* (2, 4)

(iii) $d \leq 5$ *and X is of type* (3, 3)

(iv) $d \leq 4$ *and X is of type* (2, 2, 3) *or* (2, 2, 2, 2).

This result in the case of quintic threefolds was originally proven by Katz in [11] and recently strengthened by Johnsen and Kleiman in [10], and Nijsse in [14] to $d \leq 9$. A similar result for complete intersection threefolds was also obtained independently by Huybrechts in [9]. Huybrechts method only yields the result for $d \leq 4$ in the (2, 4) complete intersection case.

Let S be the moduli space of all smooth complete intersection threefolds of one of the types (5), (2, 4), (3, 3), (2, 2, 3), or (2, 2, 2, 2) and let $S_{d,a}$ denote the set of smooth threefolds X in S which admit a smooth rational curve of degree d with normal bundle $\mathcal{N}_{C|X} \cong \mathcal{O}(a) \oplus \mathcal{O}(-a-2)$. Then $S_{d,a}$ is a locally closed, possibly empty, subvariety of S for each d and $a \geq -1$.

Theorem 5.3. *If d is as in the statement of Theorem 5.2, then the codimension of $S_{d,a}$ in S is $\geq a+1$.*

Proof. Let X be a smooth threefold in $S_{d,a}$ and let C be a smooth rational curve on X with $\mathcal{N}_{C|X} \cong \mathcal{O}(a) \oplus \mathcal{O}(-a-2)$. By Theorem 4.1, for d in the above range the infinitesimal Abel-Jacobi mapping $\Phi : H^0(\mathcal{N}_{C|X}) \longrightarrow H^1(\Omega_X^2)^*$ is injective. Then its dual ϕ is surjective and since $H^0(\mathcal{T}_X) = 0$ the sequence

$$0 \longrightarrow H^0(\mathcal{N}_{C|X}) \longrightarrow \mathcal{H}^1 \longrightarrow H^1(\mathcal{T}_X) \xrightarrow{\phi} H^1(\mathcal{N}_{C|X}) \longrightarrow 0$$

is exact. Since $H^1(\mathcal{N}_{C|X}) \cong H^0(\mathcal{N}_{C|X})$ and so has dimension $a+1$ the image of $\mathcal{H}^1$ in $H^1(\mathcal{T}_X)$ has codimension $a+1$. Thus C deforms to first order over a codimension $a+1$ subset of the tangent space to X in S. Since the rank of $H^0(\mathcal{N}_{C|X})$ can only drop as C moves generically with X, this shows that $S_{d,a}$ has codimension $\geq a+1$ in S. □

These results will also be valid for low degree rational curves on any smooth threefold X with trivial canonical bundle if one can show that X is a divisor in some smooth variety Z satisfying the hypothesis of Theorem 4.1.

6. Some computations for hypersurfaces

In this section we will apply the results of Section 3 to obtain some formulas for the infinitesimal invariant of normal functions associated to subvarieties of hypersurfaces. There is an added difficulty in applying the results of Section 3 in the case when $S = \operatorname{Spec}(\mathbb{C}[s]/s^2)$ in that the cohomology groups $H^i(\Omega_{X_S}^{n-i}|_X)$ are more difficult to compute. Let X be a smooth hypersurface of degree m in $\mathbb{P}^{n+1}, n \geq 3$ defined by a homogeneous polynomial F. Then $F + sG$ defines an infinitesimal deformation $X_G \subseteq \mathbb{P}_S^{n+1}$ of X for any homogeneous polynomial G of degree m. Let $x_0, \dots, x_{n+1}$ be homogeneous coordinates for $\mathbb{P}^{n+1}$ and let F_i denote the partial derivative $\partial F/\partial x_i$ for $i = 0, \dots, n+1$. Set:

$S =$ the graded ring $\mathbb{C}[x_0, \dots, x_{n+1}]/(F)$
$J =$ the homogeneous ideal generated by $F_0, \dots, F_{n+1}$
$R = S/J$, the Jacobian ring of F
$S^a, J^a, R^a =$ the *ath* graded piece of S, J, R respectively.

Let $\Omega = \Omega_{\mathbb{P}^{n+1}} = \sum (-1)^i x_i dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{n+1}$ and for each $J = (i_0, \dots, i_a), i_j \in \{0, \dots, n+1\}$, let $\Omega_J = \partial/\partial x_{i_a} \rightharpoonup \cdots \rightharpoonup \partial/\partial x_{i_0} \rightharpoonup \Omega$ and $F_J = F_{i_0} \cdots F_{i_a}$. Then Ω is a nonzero global section of $\Omega_{\mathbb{P}^{n+1}}^{n+1}(n+2) \cong \mathcal{O}_{\mathbb{P}^{n+1}}$ and since X is smooth $\mathcal{U} = \{U_i\}$ where $U_i = \{F_i \neq 0\}$ is an open cover X. In [1], it is shown that $R^{(a+1)m-n-2}$ can be identified with the primitive Čech cohomology group $H^a(\mathcal{U}, \Omega_X^{n-a})^\circ = \check{H}^a(\Omega_X^{n-a})^\circ$ via the map

$$H \in R^{(a+1)m-n-2} \longrightarrow \omega_H = \{H\Omega_J/F_J\}_{|J|=a+1}.$$

Lemma 6.1. *The sections of* $\check{H}^0(\Omega^n_{X_G} \otimes \mathcal{O}_X(dm))/\check{H}^0(\Omega^{n-1}_X \otimes \mathcal{O}_X(dm))$ *are in one to one correspondence with*

$$\left\{H \in R^{(d+1)m-n-2} : GH = 0 \text{ in } R^{(d+2)m-n-2}\right\}.$$

Furthermore, if $GH = \sum K_j F_j$ *for some* $K_j \in S^{(d+1)m-n-1}$, *then*

$$\left\{\frac{H\Omega_i \ - \ \sum K_j \Omega_{j,i} \wedge ds}{F_i}\right\}$$

defines a Čech 0*-cycle in* $\check{H}^0(\Omega^n_{X_G} \otimes \mathcal{O}_X(dm))$ *corresponding to* H.

Proof. In the long exact sequence in cohomology for the exact sequence

$$0 \longrightarrow \Omega^{n-1}_X \otimes \mathcal{O}(dm) \xrightarrow{\wedge ds} \Omega^n_{X_G}|_X \otimes \mathcal{O}(dm) \longrightarrow \Omega^n_X \otimes \mathcal{O}(dm) \longrightarrow 0,$$

the boundary map $\check{H}^0(\Omega^n_X \otimes \mathcal{O}(dm)) \longrightarrow \check{H}^1(\Omega^{n-1}_X \otimes \mathcal{O}(dm))$ takes $\{H\Omega_i/F_i\}$ to the 1-cocycle $\{GH\Omega_{i,j}/F_iF_j\}$. As in [1], $\{GH\Omega_{i,j}/F_iF_j\} = 0$ in $\check{H}^1(\Omega^{n-1}_X \otimes \mathcal{O}(dm))$ if and only if $GH \in J^{(d+2)m-n-2}$. Now suppose $GH = \sum K_j F_j \in J^{(d+2)m-n-2}$. It follows from the identity

$$0 \ = \ dF \wedge (\Omega_{l,i,j} \wedge ds) \ = \ F_j \Omega_{l,i} \wedge ds \ - \ F_i \Omega_{l,j} \wedge ds \ + \ F_l \Omega_{i,j} \wedge ds$$

that

$$\delta\left\{\frac{\sum K_l \Omega_{l,i} \wedge ds}{F_i}\right\} = \left\{\frac{K_l(F_i\Omega_{l,j} - F_j\Omega_{l,i}) \wedge ds}{F_iF_j}\right\} = \left\{\frac{\sum K_l F_l \Omega_{i,j} \wedge ds}{F_iF_j}\right\}$$

$$= \left\{\frac{GH\Omega_{i,j} \wedge ds}{F_iF_j}\right\} = \delta\left\{\frac{H\Omega_i}{F_i}\right\}.$$

Thus $\{(H\Omega_i \ - \ \sum K_j \Omega_{j,i} \wedge ds)/F_i\}$ defines a Čech cocycle in $\check{H}^0(\Omega^n_{X_G} \otimes \mathcal{O}_X(dm))$. □

Let Y be a smooth variety of dimension d immersed as a subvariety of X by a morphism $f : Y \longrightarrow X$. If $H \in S^{(d+1)m-n-2}$ and $GH = \sum K_j F_j \in J^{(d+2)m-n-2}$, then by Lemma 6.1, $\{(H\Omega_i \ - \ \sum K_j \Omega_{j,i} \wedge ds)/F_i\}$ defines a Čech 0-cycle in $\check{H}^0(\Omega^n_{X_G} \otimes \mathcal{O}_X(dm)) \cong H^0(\Omega^n_{X_G} \otimes \mathcal{N}^{\otimes d}_{X_G|\mathbb{P}^{n+1}_S}|_X)$. Denote by $\omega_{H,K.}$ the image of this Čech 0-cycle in $H^d(\Omega^{n-d}_{X_G}|_X)$. We wish to compute

$$\Phi : \wedge^{n-2d} H^0(\mathcal{N}_{Y|X_G}) \otimes H^d(\Omega^{n-d}_{X_G}|_X) \longrightarrow H^d(\Omega^d_Y)$$

for such forms $\omega_{H,K.}$. Let $s_1, \dots, s_d$ be global sections of $\mathcal{N}_{Y|\mathbb{P}^{n+1}_S}$ whose images $\bar{s}_1, \dots, \bar{s}_d \in H^0(f^*\mathcal{N}_{X_G|\mathbb{P}^{n+1}_S})$ have the property that the zero loci D_i

of $\bar{s}_i$ intersect transversly at a reduced set of points P. Then by (3.9), if $\eta \in \wedge^{n-2d} H^0(\mathcal{N}_{Y|X_G})$

(6.1)
$$\Phi(\eta, \omega_{H,K.}) = \sum \mathrm{Res}_p \left(\frac{\eta \wedge s_d \wedge \cdots \wedge s_1}{\bar{s}_1 \cdots \bar{s}_d} \rightharpoonup f^* \left(\frac{H\Omega_i \; - \; \sum K_j \Omega_{j,i} \wedge ds}{F_i} \right) \right)$$

where the sum is taken over all points $p \in P$.

Consider the exact sequences

$$0 \longrightarrow \mathcal{T}_Y \longrightarrow f^* \mathcal{T}_{\mathbb{P}^{n+1}_S} \longrightarrow \mathcal{N}_{Y|\mathbb{P}^{n+1}_S} \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow f^* \mathcal{O}_{\mathbb{P}^{n+1}}(1)^{n+2} \oplus \mathcal{O}_Y \longrightarrow f^* \mathcal{T}_{\mathbb{P}^{n+1}_S} \longrightarrow 0.$$

If $\beta_0, \ldots, \beta_{n+1} \in H^0(f^* \mathcal{O}_{\mathbb{P}^{n+1}}(1))$ and $c \in H^0(\mathcal{O}_Y) = \mathbb{C}$, then the above exact sequences show that $s = \sum \beta_j \partial/\partial x_j + c\partial/\partial s$ defines a section of $H^0(\mathcal{N}_{Y|\mathbb{P}^{n+1}_S})$. Let V denote the subspace of $H^0(\mathcal{N}_{Y|\mathbb{P}^{n+1}_S})$ given by all such s. It follows that $s = \sum \beta_j \partial/\partial x_j + c\partial/\partial s \in H^0(\mathcal{N}_{Y|X_G})$ if and only if the image of s in $H^0(f^* \mathcal{N}_{X_G|\mathbb{P}^{n+1}_S})$ is zero or equivalently $\sum \beta_j f^* F_j + c f^* G = 0$ when considered as an element of the graded ring $A = \oplus_k H^0(f^* \mathcal{O}_{\mathbb{P}^{n+1}}(k))$. For notational convenience we will set $g = f^* G$, $f_i = f^* F_i$, etc.

Notice that if we take $\rho = \partial/\partial x_i$, then

$$\left(\frac{\rho}{\bar{\rho}} \right) \rightharpoonup f^* \left(H\Omega \; - \; \sum K_j \Omega_j \wedge ds \right) \; = \; f^* \left(\frac{H\Omega_i \; - \; \sum K_j \Omega_{j,i} \wedge ds}{F_i} \right).$$

If $\rho = \sum \beta_j \partial/\partial x_j + c\partial/\partial s \in V \cap H^0(\mathcal{N}_{Y|X_G})$ and $g \neq 0$, then it can be verified that $\rho \rightharpoonup f^*(H\Omega \; - \; \sum K_j \Omega_j \wedge ds) = 0$. As in Section 3 it then follows that if $\rho \in V \cap H^0(\mathcal{N}_{Y|\mathbb{P}^{n+1}_S})$ and the image $\bar{\rho}$ of ρ in $H^0(f^* \mathcal{N}_{X_G|\mathbb{P}^{n+1}_S})$ is nonzero, then $\rho/\bar{\rho} \rightharpoonup f^*(H\Omega \; - \; \sum K_j \Omega_j \wedge ds)$ is a local representative of

$$\left\{ \frac{H\Omega_i \; - \; \sum K_j \Omega_{j,i} \wedge ds}{F_i} \right\} \in \check{H}^0(\Omega^n_{X_G} \otimes \mathcal{O}_X(dm)).$$

Thus we may take $\rho = \partial/\partial s$ and rewrite (6.1) as

$$\text{(6.2)} \quad \Phi(\eta, \omega_{H,K.}) = \sum \mathrm{Res}_p \left(\frac{(\eta \wedge s_d \wedge \cdots \wedge s_1 \wedge \sum k_j \partial/\partial x_j) \rightharpoonup f^* \Omega}{g \bar{s}_1 \cdots \bar{s}_d} \right)$$

whenever $\bar{\rho} = g$ does not vanish at any of the points $p \in P$.

The formula (6.2) has a rather nice interpretation when $Y = C$ is a rational curve on a hypersurface threefold X. In this case C is defined by a morphism

$$f([a,b]) \; = \; [\alpha_0(a,b), \ldots, \alpha_4(a,b)]$$

where each α_i is a homogeneous polynomial of degree d in a, b, the homogeneous coordinates for $\mathbb{P}^1$. A variant of an argument in [2] for the reduced case shows that in this case $H^0(\mathcal{N}_{C|\mathbb{P}^4_S})$ can be identified with

$$\frac{\{\sum \beta_i \partial/\partial x_i + c\partial/\partial s : \beta_i \in H^0(\mathcal{O}_{\mathbb{P}^1}(d)), c \in \mathbb{C}\}}{\{\sum (l(\alpha_i)_a + l'(\alpha_i)_b)\partial/\partial x_i : l, l' \in H^0(\mathcal{O}_{\mathbb{P}^1}(1))\}}$$

where $(\alpha_i)_a$ and $(\alpha_i)_b$ are the partial derivatives of α_i with respect to a and b respectively. If $\eta \in H^0(\mathcal{N}_{C|X_G})$, then $\eta = \sum \gamma_i \partial/\partial x_i + \partial/\partial s$ for some $\gamma_0, \ldots, \gamma_4 \in H^0(\mathcal{O}_{\mathbb{P}^1}(d))$ satisfying $\sum \gamma_i f_i = -g$. If $s = \sum \beta_i \partial/\partial x_i$, then

$$\begin{aligned}
&\Big(\eta \wedge \sum \beta_j \partial/\partial x_j \wedge \sum k_l \partial/\partial x_l\Big) \rightharpoonup f^*\Omega \\
&= \sum \gamma_i \beta_j k_l \partial/\partial x_i \rightharpoonup \partial/\partial x_j \rightharpoonup \partial/\partial x_l \rightharpoonup f^*\Omega \\
&= \sum_{i,j,l} \sum_{m<n} \operatorname{sign}(ijlmn)\ \gamma_i \beta_j k_l\ (\alpha_m d\alpha_n - \alpha_n d\alpha_m) \\
&= (1/d) \sum_{i,j,l} \sum_{m<n} \operatorname{sign}(ijlmn)\ \gamma_i \beta_j k_l\ ((\alpha_m)_b(\alpha_n)_a - (\alpha_n)_b(\alpha_m)_a)(bda - adb) \\
&= (1/d) \det(M)(bda - adb)
\end{aligned}$$

where M is the 5×5 matrix

$$M = \begin{pmatrix} \gamma_0 & \cdots & \gamma_4 \\ \beta_0 & \cdots & \beta_4 \\ k_0 & \cdots & k_4 \\ (\alpha_0)_b & \cdots & (\alpha_4)_b \\ (\alpha_0)_a & \cdots & (\alpha_4)_a \end{pmatrix}.$$

Then (6.2) can be rewritten as

$$\Phi(\eta, \omega_{H,K.}) = (1/d) \sum \operatorname{Res}_p \left(\frac{\det(M)(bda - adb)}{g(\sum \beta_j f_j + cg)} \right)$$

where the p are the distinct zeros of g. Similar formulas can be derived for the infinitesimal Abel-Jacobi mapping for rational curves on hypersurfaces.

7. Some reduction results after Voisin

Let X be a smooth hypersurface of degree m in $\mathbb{P}^{n+1}$ defined by a homogeneous polynomial F and for each homogeneous polynomial G of degree m let $X_G \subseteq \mathbb{P}^{n+1}_S$ be the infinitesimal deformation of X defined by $F + sG$. Let Y be a smooth variety of dimension d immersed as a subvariety of X by a morphism $f : Y \longrightarrow X$. Assume further that the image of Y is contained in a smooth hyperplane section $Z = X \cap \mathbb{P}^n$ of X. By a change of coordinates for $\mathbb{P}^{n+1}$, we may assume that $\mathbb{P}^n = \{x_{n+1} = 0\}$. Denote by F', G' the restrictions of F, G to $\mathbb{P}^n$. If Y deforms to first order with X in X_G and remains in a hyperplane section, then after a change of coordinates for $\mathbb{P}^{n+1}_S$ we may assume that the deformation of Y lies in $Z_{G'} \subseteq \mathbb{P}^n_S$, where $Z_{G'}$ is defined by $F' + sG'$.

In [16], Voisin shows that when X is a threefold and Y is an algebraic one-cycle which deforms with X in a fixed hyperplane, the infinitesimal invariant of the resulting normal function can be related to the infinitesimal Abel-Jacobi mapping for $Y \subseteq Z = X \cap \mathbb{P}^3$. The following theorem is a generalization of this result for arbitrary n and d with $n \geq 2d+1$.

Theorem 7.1. *With the above assumptions and notation, let* $\eta \in \wedge^{n-2d-1} H^0(\mathcal{N}_{Y|Z})$ *and assume* $\eta' \in H^0(\mathcal{N}_{Y|Z_{G'}})$ *represents the deformation of* Y *in* $Z_{G'}$*. If* H *is a homogeneous polynomial of degree* $m(d+1)-n-2$ *and* $GH = \sum K_j F_j$ *for some homogeneous polynomials* K_j*, then*

$$\Phi_X(\eta \wedge \eta', \omega_{H,K.}) = (-1)^{n+d+1} \Phi_Z(\eta, \omega_{K'_{n+1}})$$

where K'_{n+1} *is the restriction of* K_{n+1} *to* $\mathbb{P}^n$ *and* $\omega_{H,K.} \in H^d(\Omega^{n-d}_{X_G}|_X)$ *and* $\omega_{K'_{n+1}} \in H^d(\Omega^{n-d-1}_Z)$ *are defined as in Section 6.*

Proof. By (6.2)

$$\begin{aligned} &\Phi_X(\eta \wedge \eta', \omega_{H,K.}) \\ &= \sum \mathrm{Res}_{p \in P} \left(\frac{(\eta \wedge \eta' \wedge s_d \wedge \cdots \wedge s_1 \wedge \sum k_j \partial/\partial x_j) \rightharpoonup f^* \Omega_{\mathbb{P}^{n+1}}}{g \bar{s}_1 \cdots \bar{s}_d} \right) \end{aligned}$$

where $g = f^*G$ and $s_1, \ldots, s_d$ are any global sections of $\mathcal{N}_{Y|\mathbb{P}^{n+1}_S}$ whose images $\bar{s}_1, \ldots, \bar{s}_d \in H^0(f^* \mathcal{N}_{X_G|\mathbb{P}^{n+1}_S})$ have the property that the zero loci of the $\bar{s}_i$ intersect transversly at a reduced set of points P. If we choose $s_1, \ldots, s_d \in H^0(\mathcal{N}_{Y|\mathbb{P}^n_S})$, then $\eta, \eta', s_1, \ldots, s_d$ are all sections of $\mathcal{N}_{Y|\mathbb{P}^n_S}$ and it follows that

$$\begin{aligned} &\left(\eta \wedge \eta' \wedge s_d \wedge \cdots \wedge s_1 \wedge \sum k_j \partial/\partial x_j\right) \rightharpoonup f^* \Omega_{\mathbb{P}^{n+1}} \\ &= (\eta \wedge \eta' \wedge s_d \wedge \cdots \wedge s_1) \rightharpoonup f^*(K_{n+1} \partial/\partial x_{n+1} \rightharpoonup \Omega_{\mathbb{P}^{n+1}}). \end{aligned}$$

Consider the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{N}_{Y|Z} & \longrightarrow & \mathcal{N}_{Y|\mathbb{P}^n} & \longrightarrow & f^*\mathcal{N}_{Z|\mathbb{P}^n} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow\wr & & \\
0 & \longrightarrow & \mathcal{N}_{Y|Z_{G'}} & \longrightarrow & \mathcal{N}_{Y|\mathbb{P}^n_S} & \longrightarrow & f^*\mathcal{N}_{Z_{G'}|\mathbb{P}^n_S} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & f^*\mathcal{N}_{Z|Z_{G'}} & \xrightarrow{\sim} & f^*\mathcal{N}_{\mathbb{P}^n|\mathbb{P}^n_S} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{7.1}$$

Since Y deforms with X in X_G, then as in the proof of Theorem 2.1 the two vertical exact sequences in (7.1) split. Referring to (7.1), $\eta' \in H^0(\mathcal{N}_{Y|Z_{G'}})$ has image $\partial/\partial s \in H^0(f^*\mathcal{N}_{\mathbb{P}^n|\mathbb{P}^n_S})$. Since the image of $\partial/\partial s$ in $H^0(f^*\mathcal{N}_{Z_{G'}|\mathbb{P}^n_S})$ is $g = f^*G$ and the image of η' in $H^0(f^*\mathcal{N}_{Z_{G'}|\mathbb{P}^n_S})$ is zero, then it follows that the image of η' under the composition

$$H^0(\mathcal{N}_{Y|Z_{G'}}) \longrightarrow H^0(\mathcal{N}_{Y|\mathbb{P}^n_S}) \longrightarrow H^0(\mathcal{N}_{Y|\mathbb{P}^n}) \longrightarrow H^0(f^*\mathcal{N}_{Z|\mathbb{P}^n})$$

is $-g$. If we denote by ρ the image of η' in $H^0(\mathcal{N}_{Y|\mathbb{P}^n})$, then the image $\bar{\rho}$ of ρ in $H^0(f^*\mathcal{N}_{Z|\mathbb{P}^n})$ will be $-g$. Also note that $\partial/\partial x_{n+1} \rightharpoonup \Omega_{\mathbb{P}^{n+1}} = (-1)^n\Omega_{\mathbb{P}^n}$. Thus

$$\begin{aligned}
&\Phi_X(\eta\wedge\eta', \omega_{H,K.})\\
&= \sum \operatorname{Res}_{p\in P}\left(\frac{(\eta\wedge\eta'\wedge s_d\wedge\cdots\wedge s_1\wedge\sum k_j\partial/\partial x_j) \rightharpoonup f^*\Omega_{\mathbb{P}^{n+1}}}{g\bar{s}_1\cdots\bar{s}_d}\right)\\
&= \sum \operatorname{Res}_{p\in P}\left(\frac{(\eta\wedge\eta'\wedge s_d\wedge\cdots\wedge s_1) \rightharpoonup f^*(K_{n+1}\partial/\partial x_{n+1} \rightharpoonup \Omega_{\mathbb{P}^{n+1}})}{g\bar{s}_1\cdots\bar{s}_d}\right)\\
&= (-1)^{n+d+1}\sum \operatorname{Res}_{p\in P}\left(\frac{(\eta\wedge s_d\wedge\cdots\wedge s_1\wedge\rho) \rightharpoonup f^*(K_{n+1}\Omega_{\mathbb{P}^n})}{\bar{s}_1\cdots\bar{s}_d\bar{\rho}}\right)\\
&= (-1)^{n+d+1}\Phi_Z(\eta, \omega_{K'_{n+1}})
\end{aligned}$$

by (3.12). $\square$

When $n = 2d+1$ the infinitesimal Abel-Jacobi mapping for $f : Y \longrightarrow Z$ is given by pullback

$$\Phi_Z : H^d(\Omega^d_Z) \xrightarrow{f^*} H^d(\Omega^d_Y) \xrightarrow{\int_Y} \mathbb{C}.$$

In this case Theorem 7.1 can be rewritten as

Corollary 7.2 *If $n = 2d + 1$ then with the notation of Theorem 7.1,*

$$\Phi_X(\eta', \omega_{H,K.}) = (-1)^d \int_Y f^* \omega_{K'_{n+1}}.$$

There are similar results for the infinitesimal Abel-Jacobi mapping associated to subvarieties of hypersurfaces which are contained in a hyperplane section. With the notation introduced earlier, assume that Y is contained in a smooth hyperplane section $Z = X \cap \mathbb{P}^n$ where $\mathbb{P}^n = \{x_{n+1} = 0\}$. Assume that Y deforms to first order with Z in X or equivalently there is an $\eta' \in H^0(\mathcal{N}_{Y|X})$ whose image in $H^0(f^*\mathcal{N}_{Z|X})$ is f^*L for some $L \in H^0(\mathcal{N}_{Z|X}) = H^0(\mathcal{O}_Z(1))$.

Theorem 7.3 *With the above assumptions and notation, let $\eta \in \wedge^{n-2d-1} H^0(\mathcal{N}_{Y|Z})$ and let $\eta' \in H^0(\mathcal{N}_{Y|X})$ be as above. If H is a homogeneous polynomial of degree $m(d+1) - n - 2$, then*

$$\Phi_X(\eta \wedge \eta', \omega_H) = (-1)^{n+d+1} \Phi_Z(\eta, \omega_{LH'})$$

where H' is the restriction of H to $\mathbb{P}^n$ and $\omega_H \in H^d(\Omega_X^{n-d})$ and $\omega_{LH'} \in H^d(\Omega_Z^{n-d-1})$ are defined as in Section 6.

Proof. By (3.12)

$$\Phi_X(\eta \wedge \eta', \omega_H) = \sum \mathrm{Res}_{p \in P} \left(\frac{(\eta \wedge \eta' \wedge s_d \wedge \cdots \wedge s_1 \wedge \rho) \rightharpoonup f^*(H\Omega_{\mathbb{P}^{n+1}})}{\bar{s}_1 \cdots \bar{s}_d \bar{\rho}} \right)$$

where $s_1, \ldots, s_d, \rho$ are any global sections of $\mathcal{N}_{Y|\mathbb{P}^{n+1}}$ whose images $\bar{s}_1, \ldots, \bar{s}_d, \bar{\rho} \in H^0(f^*\mathcal{N}_{X|\mathbb{P}^{n+1}})$ have the property that the zero loci of the $\bar{s}_i$ intersect transversly at a reduced set of points P and $\bar{\rho}$ does not vanish at any $p \in P$. As in the proof of Theorem 7.1 we may choose $s_1, \ldots, s_d, \bar{\rho} \in H^0(\mathcal{N}_{Y|\mathbb{P}^n})$, so that $\eta, s_1, \ldots, s_d, \rho$ are all sections of $\mathcal{N}_{Y|\mathbb{P}^n}$. Then by our assumption on η' and the fact that $\partial/\partial x_{n+1} \rightharpoonup \Omega_{\mathbb{P}^{n+1}} = (-1)^n \Omega_{\mathbb{P}^n}$

$$\begin{aligned}
&(\eta \wedge \eta' \wedge s_d \wedge \cdots \wedge s_1 \wedge \rho) \rightharpoonup f^*(H\Omega_{\mathbb{P}^{n+1}}) \\
&= (-1)^{d+1} (\eta \wedge s_d \wedge \cdots \wedge s_1 \wedge \rho) \rightharpoonup f^*(LH\partial/\partial x_{n+1} \rightharpoonup \Omega_{\mathbb{P}^{n+1}}) \\
&= (-1)^{n+d+1} (\eta \wedge s_d \wedge \cdots \wedge s_1 \wedge \rho) \rightharpoonup f^*(LH\Omega_{\mathbb{P}^n}).
\end{aligned}$$

Thus

$$\begin{aligned}
&\Phi_X(\eta \wedge \eta', \omega_H) \\
&(-1)^{n+d+1} \sum \mathrm{Res}_{p \in P} \left(\frac{(\eta \wedge s_d \wedge \cdots \wedge s_1 \wedge \rho) \rightharpoonup f^*(LH\Omega_{\mathbb{P}^n})}{\bar{s}_1 \cdots \bar{s}_d \bar{\rho}} \right) \\
&= (-1)^{n+d+1} \Phi_Z(\eta, \omega_{LH'})
\end{aligned}$$

by (3.12). □

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Bemidji State University
Bemidji, MN 56601
E-mail address: westhoff@vax1.bemidji.msus.edu