DILATIONS OF LIMIT ALGEBRAS AND INTERPOLATING SPECTRUM

Allan P. Donsig

We show that limit algebras having interpolating spectrum are characterized by the property that all locally contractive representations have $*$-dilations. This extends a result for digraph algebras by Davidson. It is an open question if such a limit algebra is the limit of a direct system of digraph algebras with interpolating digraphs, although a positive answer would allow one to obtain one direction of our result directly from Davidson's. Instead, we give a 'local' construction of digraph algebras with interpolating digraphs and use this to extend representations.

Tree algebras (in the sense of Davidson, Paulsen, and Power) have been characterized by a commutant lifting property among digraph algebras with interpolating digraphs. We show that the analogous result holds for limit algebras, i.e., limit algebras with the analogous spectral condition are characterized by the same commutant lifting property among the limit algebras with interpolating spectrum.

Dilation theory for operator algebras has its origin in the Sz.-Nagy dilation theorem, that every contractive operator on Hilbert space has a unitary dilation [**SN**]. That is, given a contractive operator X on a Hilbert space $\mathcal{H}$, there a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a unitary operator U on $\mathcal{K}$ so that $X^n = P_{\mathcal{H}} U^n |_{\mathcal{H}}$ for $n \geq 1$. Ando's theorem [**An**] extends this result to two operators, i.e., any pair of commuting contractions has a unitary dilation. This fails for three or more operators [**Pa**]. Closely related to Ando's theorem is the Sz.-Nagy-Foiaş commutant lifting theorem [**SNF1**], that given a contraction T with unitary dilation U and a contraction X commuting with T, there is a contractive dilation of X commuting with U. For a unified treatment of these topics, see the monograph [**SNF2**].

From one point of view, these results are about dilating representations of the disk algebra. By von Neumann's inequality, each contraction T induces a contractive representation of disk algebra, given by $f \mapsto f(T)$. Conversely, a contractive representation of disk algebra gives a contractive operator, the image of the function $f(z) = z$. Thus, Sz.-Nagy's theorem is equivalent to

saying that every representation ρ of the disk algebra that is locally contractive, in the sense that $\|\rho(f)\| \leq 1$ where f is the function $f(z) = z$, can be dilated to a $*$-representation of the C*-algebra $C(\mathbb{T})$.

The seminal papers of Arveson [**A1**, **A2**] consider the general problem: given a (nonselfadjoint) operator algebra, a subalgebra of a C*-algebra, and a representation of the operator algebra, when does the representation extend to a $*$-representation of the C*-algebra? The necessary and sufficient condition is that the original representation be completely contractive. To define this, suppose ρ is the representation of the operator algebra A contained in the C*-algebra C. Then there is a unique C*-norm on $C \otimes M_n$, as M_n is finite dimensional, and this induces a norm on $A \otimes M_n$. Then ρ is defined to be *completely contractive* if for all n, the maps $\rho \otimes 1_{M_n}$ are contractive on $A \otimes M_n$, with respect to this norm.

In [**MM**], McAsey and Muhly proved an analogue of Sz.-Nagy's theorem for analytic crossed products, which implies the theorem holds for finite dimensional nest algebras, i.e., contractive representations of the upper triangular matrix algebras admit dilations to the full matrix algebras. The paper [**PPW**] shows that locally contractive representations of finite dimensional nest algebras admit dilations and extends the result to σ-weakly continuous, contractive representations on infinite dimensional nest algebras, by using the semi-discreteness property. Ando's theorem and the Sz.-Nagy-Foiaş commutant lifting theorem are proved for nest algebras in [**PP1**] using similar methods. These results extend to CSL algebras under various conditions on the CSL algebra: [**PPS**] shows that for finite dimensional CSL algebras, a.k.a. digraph algebras, if the associated digraph is chordal, then all locally contractive representations are completely contractive and so admit dilations, and [**DPP**] shows that if the digraph is generated by a tree, then the digraph algebra satisfies an appropriate analogue of Sz. Nagy-Foiaş commutant lifting theorem. This last result extends to infinite dimensional completely distributive CSL algebras, provided the CSL algebra has properties analogous to those of digraph algebras generated by a tree. This extension is proved in much the same way as the extension argument of [**PPW**].

Recently, [**D**] has given necessary and sufficient conditions on the digraph for a digraph algebra to have every locally contractive representation completely contractive. Digraphs satisfying these conditions, called interpolating digraphs, include all chordal digraphs of course. Moreover, among all interpolating digraphs, those generated by trees are precisely the ones whose associated algebras satisfy the analogue of the commutant lifting theorem. Again, these results extend to suitable classes of completely distributive CSL algebras.

A different analogue of the commutant lifting theorem is considered

in [**MS4**] and characterized, for digraph algebras, by a 'rooted' tree property. The papers [**MS1-MS4**] develop dilation theorems for various subalgebras of groupoid C*-algebras or of 'coordinatized' von Neumann algebras. This framework is quite general and one usually needs some condition on the subalgebra, such as triangularity, i.e., $A \cap A^*$ is maximal abelian in the generated $*$-algebra, or that $A + A^*$ is dense, in the appropriate topology, in the generated $*$-algebra.

Our focus is on limit algebras, algebras which are norm limits of digraph algebras. These algebras are contained in AF C*-algebras and fit naturally into the framework of groupoid C*-algebras. The support of the operator algebra in the groupoid can be realized as a topological binary relation, a natural generalization of the digraph associated to digraph algebras. This topological relation, called the spectrum, is a complete invariant for triangular algebras, up to isometric isomorphism [**P1**]. The monograph [**P3**] is a general reference for limit algebras.

In [**T**], Thelwall showed that a limit algebra with chordal spectrum can be written as a limit of digraph algebras with chordal digraphs. By applying the results of [**PPW**], it follows that every locally contractive representation of a limit algebra with chordal spectrum is completely contractive.

In this paper, we show that a limit algebra has interpolating spectrum if and only if every locally contractive representation is completely contractive. Unlike Thelwall's result, we do not show that a limit algebra with interpolating spectrum is the limit of a direct system of digraph algebras with interpolating digraph. Thus, we cannot directly apply the conclusions of [**D**]. Instead, we extend representations 'locally', adapting Thelwall's work to obtain digraph algebras with interpolating digraphs, and then using the structure of Davidson's 'sloppy unitary dilation'.

The second notable result is that, just as finite dimensional tree algebras are characterized by a commutant lifting property among digraph algebras with interpolating digraphs, limit algebras with the analogous spectral condition are characterized by the same property among limit algebras with interpolating spectrum. This result follows from combining the interpolating spectrum result, a theorem of [**PP2**] relating tensor products and the commutant lifting theorem, and a connection between interpolating digraphs and tensor products of tree algebras established in [**D**].

1. Limit Algebras and the Spectrum.

In this section, we relate the properties of finite-dimensional subalgebras to properties of the limit algebra and of the spectrum. A digraph algebra, or finite dimensional CSL algebra, is a subalgebra of the $n \times n$ matrices over the complex numbers that contains the diagonal matrices. Given a digraph algebra B, we use $G(B)$ to denote associated digraph, i.e., the graph with vertices the diagonal matrix units e_{ii} and an edge (e_{ii}, e_{jj}) in the digraph if and only if the matrix unit e_{ji} is in B.

Recall that a *canonical masa* $\mathcal{D}$ is a maximal abelian selfadjoint subalgebra (masa) of $\mathcal{C}$ so that there is a nested sequence of finite dimensional C*-algebras (C_i) so that $\mathcal{C} = \overline{\cup_i C_i}$ and for all i, $D_i = C_i \cap \mathcal{D}$, is a masa in C_i satisfying $\mathcal{D} = \overline{\cup_i D_i}$ and

$$N_{D_i}(C_i) \subseteq N_{D_{i+1}}(C_{i+1}),$$

where $N_Y(X)$ is

$$\{x \in X : x \text{ is a partial isometry and } xyx^*, x^*yx \in Y \text{ for all } y \in Y\}.$$

In general, we call $\mathcal{A} \subset \mathcal{C}$ a *regular* subalgebra if $D = \mathcal{A} \cap \mathcal{D}$ is a masa in $\mathcal{A}$ and $N_D(\mathcal{A}) \subseteq N_{\mathcal{D}}(\mathcal{C})$.

We assume that $\mathcal{A}$ is a subalgebra of an AF C*-algebra $\mathcal{C}$ and that $\mathcal{A}$ contains a canonical masa $\mathcal{D}$. This implies that $\mathcal{A}$ is a regular subalgebra of $\mathcal{C}$ and also that the subalgebras $C_i \cap \mathcal{A}$ are digraph algebras and regular subalgebras of $\mathcal{A}$.

Henceforth, let X be the maximal ideal space of $\mathcal{D}$ and for p a projection in $\mathcal{D}$, let $\hat{p}$ denote $\{x \in X \,|\, x(p) = 1\}$. Each $c \in N_{\mathcal{D}}(\mathcal{C})$ induces a partial homeomorphism h_c, from $\widehat{cc^*}$ to $\widehat{c^*c}$, where $h_c(x)$ is $d \mapsto x(cdc^*)$. If $\hat{c}$ denotes the graph of h_c in $X \times X$, then we can define

$$R(\mathcal{A}) = \bigcup \{\hat{c} : c \in N_{\mathcal{D}}(\mathcal{A})\}.$$

We topologize $R(\mathcal{A})$ by using as a basis of open sets $\hat{c}$ for $c \in N_{\mathcal{D}}(\mathcal{A})$. There are partially defined maps $\tilde{\alpha}_i \colon G(A_{i+1}) \to G(A_i)$, given by sending an edge (a, b) to an edge (c, d) if the matrix unit associated to (a, b) appears in the sum of matrix units that is the image of the matrix unit associated to (c, d). With these maps we can write $R(\mathcal{A})$ as an inverse limit, namely

$$R(\mathcal{A}) = \varprojlim(G(A_i), \tilde{\alpha}_i).$$

We call $R(\mathcal{A})$ the *spectrum* of $\mathcal{A}$. It is a complete invariant for the pair $(\mathcal{A}, \mathcal{D})$, up to isometric isomorphism; see [**P1**] or [**P3**, Chapter 7].

Suppose we write $\mathcal{A}$ as $\varinjlim(A_j, \alpha_j)$ and choose matrix units systems for each A_j so that matrix units in A_j are sums of matrix units in A_{j+1}. Then we can describe the spectrum of $\mathcal{A}$ in terms of the matrix unit systems, which is convenient for computations with concrete examples; see [**HP**] or [**P3**, Chapter 7]. Specifically, $R(\mathcal{A})$ is the union of $\hat{e}$ as e runs over the matrix units in all the A_j and the topology has as a basis the set of $\hat{e}$ as e runs over the matrix units

Definition 1.1. Suppose Y is a finite subset of X. We call a finite-dimensional regular subalgebra of $\mathcal{A}$, say B, a *covering algebra* for $R(\mathcal{A})|_{Y \times Y}$ if

1. there is an isomorphism of digraphs $\phi \colon G(B) \to R(\mathcal{A})|_{Y \times Y}$, and

2. we can identify $G(B)$ with matrix units of B so that $\phi(e) \in \hat{e}$ for $e \in G(B)$.

Covering algebras are far from being unique. Indeed, suppose $B \subset \mathcal{A}$ is a covering algebra for $R(\mathcal{A})|_{Y \times Y}$, $p \in B$ is a minimal diagonal projection and $y_0 \in Y$ satisfies $\phi(p) = (y_0, y_0)$. For each clopen set S with $y_0 \in S \subset \hat{p}$, we can replace p with the characteristic function of S and similarly compress the other matrix units of B to obtain a new covering algebra. In particular, if $Y \subset Z \subset X$, then it does not follow that a covering algebra for $R(\mathcal{A})|_{Y \times Y}$ is a subalgebra of a covering algebra for $R(\mathcal{A})|_{Z \times Z}$.

Lemma 1.2. *If Y is a finite subset of X, then we can find a covering algebra for $R(\mathcal{A})|_{Y \times Y}$.*

Conversely, if B is a regular digraph subalgebra of $\mathcal{A}$, then there is a finite subset of X, call it Y, so that there is an injection from $G(B)$ to $R(\mathcal{A})|_{Y \times Y}$.

Proof. Let $\mathcal{A} = \varinjlim(A_i, \alpha_i)$. As usual, we identify the A_i with subalgebras of $\mathcal{A}$ when convenient.

Since the topology in $R(\mathcal{A})$ separates points, we can find a k so that for each diagonal matrix unit e in A_k, $\hat{e}$ contains at most one point of Y. By increasing k, we can arrange that each point of $R(\mathcal{A})|_{Y \times Y}$ is in the graph of some matrix unit in A_k. If we let $B \subset A_k$ be the span of the matrix units in A_k that contain a point in $R(\mathcal{A})|_{Y \times Y}$ then it is easy to check that $G(B)$ is isomorphic to $R(\mathcal{A})|_{Y \times Y}$ and that the edge associated to a matrix unit in B is sent to a point in the graph of that matrix unit. Being a span of normalizing matrix units, B is a regular digraph subalgebra of $\mathcal{A}$.

Conversely, if B is a regular digraph subalgebra of $\mathcal{A}$, then there is a system of normalizing matrix units $\{e_{ij}\}$ for $C^*(B)$ so that B is the span of the matrix units that it contains. Let $y \in R(\mathcal{A})$ be an element of $\hat{e}_{11}$ and let

$$Y = \{h_c(y) : c = e_{1j} \text{ for some } j \text{ with } e_{jj} \in B\}.$$

Clearly, there is a bijection between Y and the set of minimal diagonal projections of B. It follows that there is an injection from $G(B)$ into $R(\mathcal{A})|_{Y \times Y}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

2. Interpolating Digraphs and Algebras.

After recalling the definition of interpolating digraphs and Davidson's characterization of digraph algebras with interpolating digraphs, we prove the main result, an analogous characterization for limit algebras with interpolating spectrum.

The $2k$-cycle algebras and their digraphs, D_{2k}, are fundamental examples, both for CSL algebras [**GHL**] and for limit algebras [**P2**, **DoP**]. The digraph D_{2k} has $2k$ vertices, $\{u_i, v_i \mid 1 \leq i \leq k\}$, with edges from u_i to v_i and to $v_{i+1 \mod k}$, i.e.,

$$
A(D_4) = \begin{bmatrix} * & * & * \\ & * & * & * \\ \hline & & * & \\ & & & * \end{bmatrix} \qquad
A(D_{2k}) = \begin{bmatrix} * & & & & * & & & * \\ & * & & & * & * & & \\ & & \ddots & & & * & \ddots & \\ & & & * & & & \ddots & * \\ \hline & & & * & & & * & * \\ & & & & * & & & \\ & & & & & * & & \\ & & & & & & \ddots & \\ & & & & & & & * \\ & & & & & & & * \end{bmatrix}.
$$

Let $\varphi_k \colon A(D_{2k}) \to A(D_{2k})$ be the endomorphism which sends each matrix unit to itself, except the $(1, k+1)$ matrix unit, which is sent to its negative; these maps are locally contractive but not contractive, and so have no $*$-dilation [**D**, **DP**].

By a *subgraph* of a digraph G, we mean a subset of the vertices of G, together with all edges of G between these vertices. Recall that a digraph is *chordal* if it does not contain a subgraph isomorphic to D_{2k} for $k \geq 2$, i.e., every cycle in the digraph has a chord. Also, if G has a subgraph isomorphic to D_4, then call a vertex e an *interpolating vertex* if every edge in the subgraph is a composition of edges through e, i.e., adding e gives the

digraph for the following algebra:

$$\begin{bmatrix} * & * & * & * \\ & * & * & * & * \\ & & e & * & * \\ & & & * & \\ & & & & * \end{bmatrix}.$$

Definition 2.1 [D].   A digraph G is called *interpolating* if:
1. Every subgraph of G isomorphic to D_4 has an interpolating vertex.
2. G does not contain a subgraph isomorphic to D_{2k} for $k \geq 3$.

Clearly, all chordal digraphs are interpolating. The importance of interpolating digraphs is in the following theorem. For a digraph algebra A, a representation $\rho\colon A \to B(\mathcal{H})$ on a separable Hilbert space $\mathcal{H}$ is *locally contractive* if $\|\rho(e)\| \leq 1$ for all matrix units $e \in A$.

Theorem 2.2 [D, Corollary 3.5].   *Let A be a digraph algebra. The following are equivalent:*
1. *A has an interpolating digraph,*
2. *every locally contractive endomorphism of A is contractive, and*
3. *every locally contractive representation has a unitary $*$-dilation.*

If we think of the spectrum as the generalization of digraph for limit algebras, then our result is an extension of Davidson's.

Definition 2.3.   Let $\mathcal{A}$ be a subalgebra of an AF C*-algebra that contains a canonical masa. We say $R(\mathcal{A})$ is *interpolating* if for any finite subset $Y \subset X$, $R(\mathcal{A})|_{Y \times Y}$ is never isomorphic to D_{2n}, $n \geq 3$ and if it is isomorphic to D_4, then there is an interpolating vertex in X.

Together with Lemma 1.2, the next lemma shows that a limit algebra with interpolating spectrum contains many digraph algebras with interpolating digraphs.

Lemma 2.4.   *Let $\mathcal{A}$ be a subalgebra of an AF C*-algebra that contains a canonical masa. If $R(\mathcal{A})$ is interpolating and $Y \subset X$ is finite, then there is a finite set Z with $Y \subset Z \subset X$ so that $R(\mathcal{A})|_{Z \times Z}$ is interpolating.*

Proof. Given that $R(\mathcal{A})$ is interpolating, the only way that $R(\mathcal{A})|_{Y \times Y}$ can fail to be interpolating is if it contains some subgraph isomorphic to D_4 without containing an interpolating vertex. The following argument is adapted

from [**D**, pp. 215-216]; the point is that we will add the needed interpolating vertices so that any new 4-cycles also have interpolating vertices.

Claim 1. If $B, C \subset Y$ with $(b, c) \in R(\mathcal{A})$ for all $b \in B$ and all $c \in C$, then there is $g \in Y$ with $(b, g), (g, c) \in R(\mathcal{A})$ for all $b \in B$ and all $c \in C$.

We prove the claim using induction on the sum of the cardinalities of B and C. If B or C is a singleton, then we can set g equal to that single element; if B and C both have cardinality two, then by the assumption on $R(\mathcal{A})$, a suitable g exists. Suppose now $B = \{b_1, \dots, b_n\}$ and $C = \{c_1, \dots, c_m\}$. By induction, we can find an interpolating point g_1 for the sets $\{b_1, \dots, b_{n-1}\}$ and C. Again by induction, there is an interpolating point g for the sets $\{g_1, b_n\}$ and C. It is easy to check that g has the required properties, proving the claim.

Let $\mathfrak{S}$ be the collection of maximal pairs of subsets of Y, (B, C), with $(b, c) \in R(\mathcal{A})$ for all $b \in B$, $c \in C$. Clearly, $\mathfrak{S}$ is a finite collection, $\{S_1, \dots, S_m\}$. Letting $S_1 = (B_1, C_1)$, Claim 1 implies that there is a $g_1 \in X$ so that $(b, g_1), (g_1, c) \in R(\mathcal{A})$ for all $b \in B_1$, $c \in C_1$. Next, we enlarge each $S_i = (B_i, C_i)$, $i > 1$, as follows. If $(g_1, c) \in R(\mathcal{A})$ for all $c \in C_i$, then we add g_1 to B_i. If $(b, g_1) \in R(\mathcal{A})$ for all $b \in B_i$, then we add g_1 to C_i. Otherwise, S_i is left unchanged. Note that we may add g_1 to both B_i and C_i and that the new sets are maximal pairs of subsets of $Y \cup \{g_1\}$.

Continuing in this way for the (possibly modified) $S_2, \dots, S_m$, we obtain $g_2, \dots, g_m$, and we set $Z = Y \cup \{g_1, \dots g_m\}$. Note that we do not try to add g_j to S_i, for $i \leq j$.

Claim 2. $R(\mathcal{A})|_{Z \times Z}$ is interpolating.

First, we show that $(g_i, g_j) \in R(\mathcal{A})$ if and only if the (original) sets satisfy $B_i \subset B_j$. By the maximality of S_i and the choice of g_i, it follows that $B_i = \{b \in Y : (b, g_i) \in R(\mathcal{A})\}$. If $(g_i, g_j) \in R(\mathcal{A})$, then by transitivity

$$B_i = \{b \in Y : (b, g_i) \in R(\mathcal{A})\} \subset \{b \in Y : (b, g_j) \in R(\mathcal{A})\} = B_j.$$

Conversely, if $B_i \subset B_j$ then by maximality $C_i \supset C_j$ and so either g_j has been added to B_i or g_i has been added to C_j. In either case, $(g_i, g_j) \in R(\mathcal{A})$.

To prove the claim, suppose that $R(\mathcal{A})|_{T \times T}$ is a 4-cycle, i.e., $T = \{x_1, x_2, y_1, y_2\} \subseteq Z$ with $(x_i, y_j) \in R(\mathcal{A})$, $i, j \in \{1, 2\}$. If $x_i \in Y$, let $E_i = \{x_i\}$ and if $x_i = g_j$ for some j, let $E_i = B_j$. If $y_i \in Y$, let $F_i = \{y_i\}$ and if $y_i = g_j$, let $F_i = C_j$. Then $(E_1 \cup E_2, F_1 \cup F_2)$ is contained in some maximal pair (B_k, C_k) in $\mathfrak{S}$. It is straightforward to verify that g_k is interpolating for $R(\mathcal{A})|_{T \times T}$. $\square$

This lemma does not imply that if a limit algebra has interpolating spectrum, then a finite-dimensional regular subalgebra is contained in such a subalgebra with interpolating spectrum. Indeed, this statement is equivalent to showing that every limit algebra with interpolating spectrum that has a presentation using digraph algebras with interpolating digraphs. However, if we work with finitely many algebras instead of only one, then the corresponding statement is true and this is enough to build the required representation.

Lemma 2.5. *Suppose that $\mathcal{A}$ is a subalgebra of an AF C^*-algebra containing a canonical masa $\mathcal{D}$, that $R(\mathcal{A})$ is interpolating, and that $S \subset \mathcal{A}$ is a finite-dimensional regular subalgebra. Then there are finite-dimensional regular subalgebras $B_1, \dots, B_m$ so that each $G(B_i)$ is interpolating and the $\mathcal{D}$-bimodule generated by $B_1, \dots, B_m$ contains S. Moreover, there are orthogonal projections $P_1, \dots, P_m$ in the commutant of S so that $P_i S \subseteq B_i$.*

Proof. We suppose that $G(S)$ is connected; the general case then follows by considering the connected components of $G(S)$.

Fix a diagonal matrix unit in S, call it e. For each point $x \in \hat{e} \subseteq X$, let

$$Y(x) = \{y \in X : (x, y) \text{ or } (y, x) \in \hat{f} \text{ for some matrix unit } f \in S\}.$$

As $R(\mathcal{A})$ is interpolating, by Lemma 2.4, there is a finite set $Z(x)$ containing $Y(x)$ so that $R(\mathcal{A})|_{Z(x) \times Z(x)}$ is interpolating. By Lemma 1.2, there is a covering algebra, call it $A(e)$, for each $R(\mathcal{A})|_{Z(x) \times Z(x)}$. Let $O(x)$ be the graph of the matrix unit in $A(x)$ that covers the point $(x, x) \in \hat{e}$. Then the set $\{O(x) : x \in \hat{e}\}$ is an open cover of $\hat{e}$ and by the compactness of $\hat{e}$, it admits a finite subcover, given by $x_1, \dots, x_m$, say. The bimodule generated by $A(x_1), \dots, A(x_m)$ includes e and all the other matrix units of S and by construction each $A(x_i)$ has an interpolating digraph. Thus the $A(x_i)$ are almost the required subalgebras.

If the $O(x_i)$ overlap, then we can split the algebras $A(x_i)$ so the $O(x_i)$ are either disjoint or identical. Eliminating all but one algebra among those with identical sets $O(x_i)$ and changing m if necessary, we have the required algebras $A(x_1), \dots, A(x_m)$.

Let Q_i be the projection in B_i with graph $O(x_i)$. Let P_i be the sum of $f^* Q_i f$ over all f matrix units of $C^*(S)$ with $f^* f = e$. Clearly, P_i commutes with each matrix unit of S and so is in the commutant of S. By construction, $P_i f \in A(x_i)$ for each matrix unit f of S, so $P_i S \subseteq A(x_i)$, as required. Since the $O(x_i)$ are disjoint, it follows that the $P(x_i)$ are orthogonal. $\square$

If ρ is a representation of a digraph algebra A, then ρ being locally contractive is equivalent to having $\|\rho(e)\| \leq 1$ for all $e \in N_D(A)$ for some masa

$D \subseteq A$. This motivates the following definition.

Definition 2.6. Let $\mathcal{A}$ be a subalgebra of an AF C*-algebra so that $\mathcal{A}$ contains a canonical masa $\mathcal{D}$. A representation $\rho \colon \mathcal{A} \to B(\mathcal{H})$ on a separable Hilbert space $\mathcal{H}$ is *locally contractive* if, for all $e \in N_{\mathcal{D}}(\mathcal{A})$,

$$\|\rho(e)\| \leq 1.$$

A major tool in the proof of Theorem 2.2 and an essential ingredient here is the vertex elimination scheme for anti-symmetric interpolating digraphs [**D**, Corollary 2.5]. That is, for an anti-symmetric interpolating digraph, there is a vertex so that deleting this vertex, and all edges involving this vertex, gives an interpolating sub-digraph. Moreover, this vertex has at most one immediate successor and at most one immediate predecessor. The result is that one can successively delete such vertices until one has the empty graph and so readily prove theorems by induction on the number of vertices.

We will also need the notion of a sloppy unitary dilation from [**D**], which is used there in proving Theorem 2.2. The sloppy unitary dilation of a contraction $T \in B(\mathcal{H})$ is a unitary $U \in B(\mathcal{K})$ where $\mathcal{K} = \mathcal{H}_+ \oplus \mathcal{H} \oplus \mathcal{H}_-$ and $\mathcal{H}_-, \mathcal{H}_+$ are both countable direct sums of copies of $\mathcal{H}$ and where U has the form

$$U = \begin{bmatrix} X & A & B \\ 0 & T & C \\ 0 & 0 & Y^* \end{bmatrix}$$

with

1. X, Y are isomorphic to $S \otimes I_{\mathcal{H}}$ where S is the unilateral shift,

2.   the range of A has infinite codimension in $(I - XX^*)\mathcal{H}_+$, and

3.   the domain of C has infinite codimension in $(I - YY^*)\mathcal{H}_-$.

If V is another sloppy unitary dilation of T, then there is a unitary $W = W_+ \oplus I \oplus W_-$ with $WVW^* = U$ [**D**, Lemma 3.1].

Theorem 2.7.   *Let $\mathcal{A}$ be a unital subalgebra of an AF C*-algebra that contains a canonical masa $\mathcal{D}$. Then the spectrum, $R(\mathcal{A})$, is interpolating, if and only if, every locally contractive representation of $\mathcal{A}$ has a unitary $*$-dilation.*

Proof. Let X be the maximal ideal space of $\mathcal{D}$; since $\mathcal{A}$ is unital, X is compact.

($\Longrightarrow$)   Fix a presentation for $\mathcal{A}$, i.e., $\mathcal{A} = \varinjlim(A_i, \alpha_i)$.

Suppose $\rho\colon \mathcal{A} \to B(\mathcal{H})$ is a locally contractive representation. If ρ is not completely contractive, then a straightforward argument using the density of $\cup A_i$ in $\mathcal{A}$ shows that there is some n so that $\rho|_{A_n}$ is not completely contractive. Thus, it suffices to show that for each n, $\rho|_{A_n}$ has a unitary $*$-dilation.

Fix n. By Lemma 2.5, we have regular subalgebras $B_1, \ldots, B_m$, with each $G(B_i)$ interpolating and A_n contained in the bimodule generated by $B_1, \ldots, B_m$. There is some k so that $B_i \subseteq A_k$ for all i. By possibly splitting each B_i into several isomorphic algebras, we may assume that the matrix units of each B_i are matrix units of A_k.

Next, we put $\rho|_{A_k}$ into a standard form. Let $\{e_{ij}\}$ be the matrix units of $\mathrm{C}^*(A_k)$. We may assume that each $\rho(e_{ii})$ is a projection of infinite rank by adding, if necessary, infinitely many copies of the identity representation. Thus we can identify $\rho(e_{ii})\mathcal{H}$ for all i, call it $\mathcal{H}'$, and so may regard $\rho|_{A_k}$ as a representation on $M_l(B(\mathcal{H}'))$ for some l.

As ρ is locally contractive, each $\rho(e_{ij})$ is a contraction from the j-th copy of $\mathcal{H}'$ to i-th copy. By Theorem 2.2, each $\rho|_{B_i}$ can be dilated to a $*$-representation of $\mathrm{C}^*(B_i)$, call it σ_i. Moreover, these dilations have a common form. Let $\mathcal{K}'$ be the ℓ^2 direct sum of countably many copies of $\mathcal{H}'$ and identify $\mathcal{H}'$ with one of these copies, so that $\mathcal{H}' \subset \mathcal{K}'$. Then we may assume that the all $*$-dilations have a common range, namely $M_l(B(\mathcal{K}'))$. Each matrix unit of $\mathrm{C}^*(A_k)$ is in at most one of $\mathrm{C}^*(B_1), \ldots, \mathrm{C}^*(B_m)$, and if e_{ij} is in one of these algebras, then its image is a unitary sending the j-th copy of $\mathcal{K}'$ to the i-th copy.

Recall, by Lemma 2.5 that we have orthogonal projections $P_1, \ldots, P_m$ in the commutant of A_n with $P_i A_n \subseteq B_i$ for each i. Since P_i are pairwise orthogonal, so are their images, the $\rho(P_i)$. Then we can define $\sigma\colon A_n \to M_l(B(\mathcal{K}'))$ by

$$\sigma(a) = \sum_{i=1}^m \rho(P_i)\sigma_i(P_i a)\rho(P_i).$$

The map σ is contractive, as an element a is the direct sum of the $P_i a$ as i runs from 1 to m and the $\rho(P_i)$ are pairwise orthogonal.

Finally, σ extends to a $*$-representation on $\mathrm{C}^*(A)$, if and only if, the partial isometries $\sigma(e)$, e a matrix unit, satisfy the same relations as the corresponding matrix units. As each σ_i extends to a $*$-representation, the $\sigma_i(P_i e)$ do satisfy the required relations. As the $\rho(P_i)$ are pairwise orthogonal, each $\sigma(e)$ is the direct sum of the the $\sigma_i(P_i e)$ and so also satisfies these relations.

($\Longleftarrow$) We suppose that $R(\mathcal{A})$ is not interpolating and construct a locally contractive representation of $\mathcal{A}$ that is not contractive and hence has no $*$-dilation; this argument is an elaboration on one direction of [**D**, Theorem 2.2].

By our supposition, there is a finite subset of X, say Y, so that $R(\mathcal{A})|_{Y \times Y}$ is a 4-cycle without an interpolating vertex in $R(\mathcal{A})$, or is a $2k$-cycle, $k \geq 3$. Suppose Y is $\{u_1, \ldots, u_k, v_1, \ldots v_k\}$, $k \geq 2$, where (u_i, v_i) and (u_i, v_{i+1}) are in $R(\mathcal{A})$ (and $v_{k+1} = v_1$). For each l, define U_i for $i = 1, \ldots, k$ by

$$U_i = \{y \in X \mid (u_i, y), (y, u_i) \in R(\mathcal{A})\},$$

and V_j similarly. For $i, j = 1, \ldots, k$, let

$$I_{ij} = \{y \in X \mid (u_i, y), (y, v_j) \in R(\mathcal{A}) \text{ and } y \notin U_i, V_i\}.$$

Finally, we set $H = \{y \in X \mid (u_i, y), (y, v_j) \text{ for some } i, j\}$, i.e., the union of the U_i, the V_j and the I_{ij}.

Note that H is countable, so we can introduce a separable Hilbert space $\mathcal{H}$ with orthonormal basis $\{b_h : h \in H\}$. We define a map $\phi \colon R(\mathcal{A})|_{H \times H} \to B(\mathcal{H})$ given by

$$\phi(x, y) = \alpha(x, y) b_x b_y^*,$$

where $b_x b_y^*$ is the rank one operator $z \mapsto (z, b_y) b_x$ and α is given by

$$\alpha(a, b) = \begin{cases} -1 & \text{if } (a, b) \in (U_1 \cup (I_{11} \cap I_{1k})) \times (V_1 \cup (I_{11} \setminus I_{1k})), \\ 1 & \text{otherwise.} \end{cases}$$

It is straightforward to check that $\alpha(a, b) \alpha(b, c) = \alpha(a, c)$ and hence $\phi(a, b) \phi(b, c) = \phi(a, c)$. Remark 5.6 below motivates this definition. If $H = Y$, then ϕ is φ_k, the locally contractive but not completely contractive representation defined at the start of this section. We can extend ϕ to $R(\mathcal{A})$ by sending elements outside $R(\mathcal{A})|_{H \times H}$ to zero.

To define a representation $\Phi \colon \mathcal{A} \to B(\mathcal{H})$, we first define Φ on $N_{\mathcal{D}}(\mathcal{A})$ by sending such an element e to

$$\Phi(e) = \text{WOT-} \sum_{(x, y) \in \hat{e}} \phi(x, y).$$

As each $\phi(x, y)$ has orthogonal initial and final projections, this sum does indeed converge in the weak operator topology. Being a partial isometry, $\Phi(e)$ has norm at most 1.

Extend Φ by linearity to linear combinations of elements of $N_{\mathcal{D}}(\mathcal{A})$. It is not hard to check that this map is bounded and so we may extend by continuity to obtain a locally contractive representation.

It remains only to show that Φ is not contractive. By Lemma 1.2, we can find a covering algebra for $R(\mathcal{A})|_{Y \times Y}$, say B, with $G(B) = D_{2k}$. As the graph of each matrix unit contains a point of $R(\mathcal{A})|_{Y \times Y}$, it follows that

compressing $\Phi|_B$ by $P = \sum_{y \in Y} b_y b_y^*$ gives a representation isomorphic to φ_{2k}. Hence $\Phi|_B$ is not contractive. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 2.8. An example is the best motivation for the definition of the maps α and ϕ in the previous proof. Suppose that $k = 2$ and A_1 is the following matrix algebra

$$
\begin{bmatrix}
u_1 & & * & * & * & * \\
& u_2 & * & & * & * \\
& & * & & * & \\
& & & * & * & * \\
& & & & v_1 & \\
& & & & & v_2
\end{bmatrix}
$$

where we suppose that $Y = \{u_1, u_2, v_1, v_2\}$. To obtain an extension of φ_2, the (1,5) matrix unit must go to its negative. As the map is multiplicative, for each of the pairs of matrix units, (1,3), (3,5) and (1,4), (4,5), one of the pair must go to its negative. We send the $(1,3)$ and $(4,5)$ matrix units to their negatives, corresponding to the sets $U_1 \times (I_{11} \setminus I_{1k})$ and $(I_{11} \cap I_{1k}) \times V_1$ in the definition of α.

Example 2.9.   Another way one might prove the first direction of Theorem 2.7 is to first show that an interpolating spectrum implies the algebra has a presentation using digraph algebras with interpolating digraphs. It would then be straightforward to apply the characterization for digraph algebras to obtain the result. Indeed, replacing interpolating with chordal, this is the method that Thelwall uses in showing chordal spectrum implies contractive representations are completely contractive.

However, it is unclear how to construct the required presentation. The essential step is, given a limit algebra $\mathcal{A}$ with interpolating spectrum and a finite dimensional regular subalgebra A_1 to construct another finite dimensional regular subalgebra A_2 with $A_1 \subset A_2$ and $G(A_2)$ interpolating. Lemma 2.5 is insufficient, as the algebra generated by the algebras $B_1, \ldots,$ B_m of the lemma need not be interpolating.

Concretely, consider

$$
A_1 = \begin{bmatrix}
* & & & * & * & * \\
& * & & * & * & \\
& & * & * & & * \\
\hline
& & & * & & \\
& & & & * & \\
& & & & & *
\end{bmatrix},
\qquad \text{and} \qquad
A_2 = \begin{bmatrix}
D & S_{12} & S_{13} \\
0 & S_{22} & S_{23} \\
0 & 0 & D
\end{bmatrix}
$$

where each D is the 12×12 diagonal matrices, S_{22} is a 4×4 matrix and the two-by-two block of S_{ij}'s is given by the $*$-diagram in the following matrix.

The inclusion from A_1 to A_2 is given by sending the (1,2) block of A_1 to the (1,3) block of A_2, precisely, to the numbered images in the matrix. Then the (1,1) and (2,2) blocks of A_1 are sent to the (1,1) and (3,3) blocks of A_2, so that the resulting map from A_1 to A_2 is a homomorphism, albeit non-unital.

$$
\left[
\begin{array}{cccc|ccc ccc cccc}
* & * & & & 1\,1\,1 & & & ⊠* & & ⊠ & & * \\
* & & & & 1\,1 & & & * & * & & & \\
& & * & & 1 & \ 1 & & & & * & & * \\
& * & & * & & 2\,2\,2\,⊠ & & * & ⊠ & * & & \\
& * & & & & 2\,2 & & & & * & * & \\
& & * & & & 2 & \ 2 & * & & * & & \\
* & & & *\ ⊠\ * & & ⊠ & & * & 3\,3\,3 & & & \\
* & & & * & * & & & & 3\,3 & & & \\
& & * & & & * & & * & 3 & \ 3 & & \\
& *\ * & & ⊠ & * & ⊠\ * & & & & 4\,4\,4 & & \\
& * & & & & * & * & & & 4\,4 & & \\
& & * & *\ & * & & & & & 4 & \ 4 & \\
\hline
* & & & * & * & & & * & * & & & \\
& * & & & & * & * & & & * & * & \\
& & * & *\ & * & & & & & * & & * \\
& & & * & & * & * & * & * & & * & \\
\end{array}
\right]
$$

Observe that each of the 4-cycle of partial isometries in A_1,

$$e_{1,4}, e_{1,5}, e_{2,5}, e_{2,4} \quad \text{and} \quad e_{1,4}, e_{1,6}, e_{3,6}, e_{3,4},$$

is a sum of four 4-cycles in A_2 and each of these 4-cycles in A_2 has an interpolating point. However, the algebra generated by these 4-cycles and their interpolating points includes new 4-cycles that do not have interpolating points, namely the two boxed 4-cycles in the diagram. What one needs is some new construction that either does not introduce these new 4-cycles, or more likely, also constructs interpolating points for them.

3. Tree Algebras.

We characterize, among the limit algebras with interpolating spectrum, those which have a commutant lifting property or satisfy Ando's theorem. We say an algebra has the *commutant lifting property* if given a contractive representation ρ and a contraction X commuting with the range of ρ, there is a $*$-dilation π and a unitary U commuting with the range of π so that $X = P_{\mathcal{H}} U|_{\mathcal{H}}$. An algebra satisfies Ando's theorem if given a contractive

representation of the algebra and a commuting representation of the disk algebra, then there are commuting $*$-dilations; this is equivalent to a stronger commutant lifting property, where U is required to be a power dilation of X.

Recall that if A is a digraph algebra, then the *reduced digraph* associated to a digraph algebra A, denoted $G_r(A)$, is $G(A)/\approx$ where $p_i \approx p_j$ if (p_i, p_j) and (p_j, p_i) are edges of $G(A)$. This digraph, $G_r(A)$, is a partial order on $\{p_1, \ldots, p_n\}/\approx$. The advantage of the reduced digraph is that its vertices correspond to the summands of $A \cap A^*$. Thus, $G_r(A)$ gives the non-self-adjoint structure of A whereas $G(A)$ is also encumbered with information about $A \cap A^*$.

Definition 3.1. A digraph algebra A is a *tree algebra* if each connected component of $G_r(A)$ is generated by a bilateral tree, a bilateral tree being an undirected finite tree in the usual sense with arbitrary directions specified for each edge.

In [**DPP**], tree algebras are shown to possess the commutant lifting property and to satisfy Ando's theorem. Further, in [**D**] it is shown that, for A a digraph algebra with interpolating digraph, A is a tree algebra if and only if it has the commutant lifting property.

The next theorem gives a local characterization of tree algebras, in terms of the associated digraph. In particular, tree algebras have interpolating digraphs.

Theorem 3.2. [**DPP**] *For A a digraph algebra, A is a tree algebra if and only if $G(A)$ satisfies the following conditions:*

1. *If a subgraph of $G(A)$ contains the following digraph, then it also contains an edge between f_1 and f_2.*
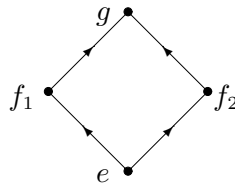
2. *Every subgraph of $G(A)$ isomorphic to D_4 has an interpolating vertex, and*

3. *$G(A)$ does not contain a subgraph isomorphic to D_{2k} for $k \geq 3$.*

Since the three conditions above are all local, we can extend them to the spectrum of a limit algebra in the same way that we defined interpolating

spectrum.

Definition 3.3.   Let $\mathcal{A}$ be a subalgebra of an AF C*-algebra that contains a canonical masa. We say $R(\mathcal{A})$ is *locally generated by trees* if for any finite subset $Y \subset X$, the directed graph $R(\mathcal{A})|_{Y \times Y}$ satisfies the three conditions of Theorem 3.2.

Since $R(\mathcal{A})$ is infinite, the conditions of Theorem 3.2 do not imply that each connected component of $R(\mathcal{A})$ is generated by a single tree; hence our somewhat awkward term for a spectrum satisfying these conditions.

Recall from [**PP2**] the minimal operator norm on a tensor product of two operator algebras. Given completely isometric isomorphisms $\tau_i \colon \mathcal{A}_i \to B(\mathcal{H}_i)$ for $i = 1, 2$, we have a norm on the algebraic tensor product of $\mathcal{A}_1$ and $\mathcal{A}_2$, defined by

$$\|a_1 \otimes a_2\| = \|\tau_1(a_1) \otimes \tau_2(a_2)\|_{B(\mathcal{H}_1 \otimes \mathcal{H}_2)}.$$

By [**PP2**, Theorem 2.2], this norm is independent of the choice of τ_1 and τ_2 and equals the spatial operator norm. In general this norm is not minimal among the complete operator cross-norms [**ER**]. Since we will have no cause to refer to other tensor product norms, here $\mathcal{A}_1 \otimes \mathcal{A}_2$ will always refer to the completion of the algebraic tensor product with respect to this norm.

Our interest in this operator norm is motivated by the following pair of results. Given algebras $\mathcal{A}_1$ and $\mathcal{A}_2$, and representations $\tau_i \colon \mathcal{A}_i \to B(\mathcal{H})$ so that $\tau_1(a_1)\tau_2(a_2) = \tau_2(a_2)\tau_1(a_1)$ for all $a_1 \in \mathcal{A}_1$ and $a_2 \in \mathcal{A}_2$, we use $\tau_1 \odot \tau_2$ to denote the representation of $\mathcal{A}_1 \otimes \mathcal{A}_2$ on $\mathcal{H}$ given by

$$(\tau_1 \odot \tau_2)(a_1 \otimes a_2) = \tau_1(a_1) \otimes \tau_2(a_2).$$

A unital operator algebra $\mathcal{A}$ has the commutant lifting property if and only if for all pairs of commuting completely contractive representations, τ_1 of $\mathcal{A}$ and τ_2 of T_2, the representation $\tau_1 \odot \tau_2$ of $\mathcal{A} \otimes T_2$ has a $*$-dilation [**PP2**, Propositions 2.5 and 2.6]. An operator algebra $\mathcal{A}$ satisfies Ando's theorem if and only if for all n and for all pairs of commuting completely contractive representations, τ_1 of $\mathcal{A}$ and τ_2 of T_n, the representation $\tau_1 \odot \tau_2$ of $\mathcal{A} \otimes T_n$ has a $*$-dilation [**PP2**, Theorem 3.2].

If $\mathcal{A}_i$, $\mathcal{B}_i$ are operator algebras and $\mathcal{A}_i$ is a subalgebra of $\mathcal{B}_i$ for $i = 1, 2$, then the natural inclusion of $\mathcal{A}_1 \otimes \mathcal{A}_2$ into $\mathcal{B}_1 \otimes \mathcal{B}_2$ is a complete isometry [**PP2**, Corollary 2.3]. In particular, if $\mathcal{A} = \varinjlim(A_i, \alpha_i)$ then as the α_i are $*$-extendible, they are complete isometries and so the injection

$$A_i \otimes T_n \xrightarrow{\alpha_i \otimes \mathrm{Id}} A_{i+1} \otimes T_n$$

is a complete isometry (and hence $*$-extendible). This map is easily seen to be regular and so we have a limit algebra $\varinjlim(A_i \otimes T_n, \alpha_i \otimes \mathrm{Id})$, which is isomorphic to $\mathcal{A} \otimes T_n$.

With these results, we can now prove the main theorem of this section. The arguments are adaptations of those in [**D**, Section 4].

Theorem 3.4.   *Let $\mathcal{A}$ be a unital subalgebra of an AF C^*-algebra that contains a canonical masa $\mathcal{D}$ and suppose that $R(\mathcal{A})$ is interpolating. The following are equivalent:*

1. *The spectrum, $R(\mathcal{A})$, is locally generated by trees,*

2. *$\mathcal{A}$ satisfies Ando's theorem, and*

3. *$\mathcal{A}$ has the commutant lifting property.*

Proof. $(1 \implies 2)$ To show $\mathcal{A}$ satisfies Ando's theorem, it suffices to show that for every n we can lift a representation $\tau_1 \odot \tau_2$ of $\mathcal{A} \otimes T_n$. By the previous discussion,

$$\mathcal{A} \otimes T_n = \varinjlim(A_i \otimes T_n, \alpha_i \otimes \mathrm{Id}),$$

and so $R(\mathcal{A} \otimes T_n) = R(\mathcal{A}) \otimes G(T_n)$. Lemma 4.4 and Theorem 4.6 of [**D**] show that a digraph is a tree digraph if and only if its tensor product with each $G(T_n)$ is interpolating. It follows that $R(\mathcal{A} \otimes T_n)$ is interpolating if and only if $R(\mathcal{A})$ is locally generated by trees. By Theorem 2.7, each representation of $\mathcal{A} \otimes T_n$, and in particular, each representation of the form $\tau_1 \odot \tau_2$, has a $*$-dilation.

$(2 \implies 3)$ This is immediate from the formulation above of both conditions in terms of tensor products.

$(3 \implies 1)$ Suppose the spectrum is not locally generated by trees.

In the proof of [**D**, Theorem 4.6], Davidson shows that for a digraph algebra A with interpolating digraph, if A is not a tree algebra, then the digraph of $A \otimes T_2$ is not interpolating. It follows that if the spectrum of $\mathcal{A}$ is not locally generated by trees, then the spectrum of $\mathcal{A} \otimes T_2$ is not interpolating.

Hence by Theorem 2.7, $\mathcal{A} \otimes T_2$ has a locally contractive representation, say τ, that does not have a $*$-dilation. Let λ_1 be the inclusion of $\mathcal{A}$ in $\mathcal{A} \otimes T_2$ and λ_2 be the inclusion of T_2 in $\mathcal{A} \otimes T_2$. Then letting $\tau_i = \tau \circ \lambda_i$ for $i = 1, 2$, we have $\tau_1 \odot \tau_2 = \tau$.

Since $\mathcal{A}$ and T_2 have interpolating spectrum, the locally contractive representations τ_1 and τ_2 are completely contractive. Thus we have a representation of $\mathcal{A} \otimes T_2$ of the form $\tau_1 \odot \tau_2$ that has no $*$-dilation. By Paulsen and Power's result mentioned above, it follows that $\mathcal{A}$ does not have the commutant lifting property. $\square$

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UNIVERSITY OF WATERLOO
WATERLOO, CANADA, N2L 3G1

Current address:
UNIVERSITY OF NEBRASKA-LINCOLN
LINCOLN, NE 68588-0323
E-mail address: adonsig@math.unl.edu