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A GELFAND-NAIMARK THEOREM FOR C*-ALGEBRAS

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We generalize the Gelfand-Naimark theorem for non-commutative C*-algebras in the context of CP-convexity theory. We prove that any C*-algebra A is *-isomorphic to the set of all $B(H)$-valued uniformly continuous quivariant functions on the irreducible representations $\mathrm{Irr}(A:H)$ of A on H vanishing at the limit 0 where H is a Hilbert space with a sufficiently large dimension. As applications, we consider the abstract Dirichlet problem for the CP-extreme boundary, and generalize the notion of semi-perfectness to non-separable C*-algebras and prove its Stone-Weierstrass property. We shall also discuss a generalized spectral theory for non-normal operators.

Introduction.

The classical Gelfand-Naimark theorem states that every commutative C*-algebra A is *-isomorphic to the algebra $C_0(\Omega)$ of all continuous functions on the characters Ω of A which vanish at infinity (cf. [**28**]). This theorem eventually opened the gate to the subject of C*-algebras, and is legitimately called "the fundamental theorem of C*-algebras" (cf. [**16**, **17**] for the history of this theorem). Since then, there have been various attempts to generalize this beautiful representation theorem for non-commutative C*-algebras, i.e., to reconstruct the original algebra as a certain function space on some "generalized spectrum", or the duality theory in the broad sense, for which we can refer to e.g., [**2**], [**3**], [**4**], [**11**], [**14**], [**19**], [**29**], [**32**], [**37**], [**39**]. Among them, we cite the following notable directions which motivated our present work.

One is the approach from convexity theory, where the characters in the commutative case are replaced by the pure states (i.e., the extreme points of the state space), on which one could try to seek a functional representation of the algebra. This approach was initiated by R.V. Kadison ([**29**], [**30**]) representing the self-adjoint part of a unital C*-algebra on the weak*-closure of the pure states as an order isomorphism in the context of Kadison's function representation theorem. This motivated the abstract Dirichlet problem for the extreme boundary of compact convex sets (cf. [**4**], [**5**, Th. II.4.5]), and the duality arguments on the pure states for C*-algebras by F.W. Shultz [37] in the context of the Alfsen-Shultz theory, which was further developed by C.A. Akemann and F.W. Shultz [3] using equivariant functions on irreducible representations (or the closure of irreducible representations) based on Takesaki's duality theorem (cf. [11], [39]). Our method in this paper is essentially based on these results in this approach.

Another important direction was the approach from the method of sectional representation in C*-bundles over the primitive ideal space, which was initiated by J.M.G. Fell [19], and culminated in the Dauns-Hofmann theorem [14] where they showed that every C*-algebra is *-isomorphic to the C*-algebra of all continuous sections of a C*-bundle over the spectrum of the center of its multiplier algebra (cf. also [18] for a simplified proof due to J. Varela [43]). In this approach, one can obtain a generalized Gelfand-Naimark theorem recovering the algebraic structure by the *-isomorphism, however the affine geometric aspect of the duality, which was naturally endowed in the commutative case, could not be restored.

We should also note that there have been some attempts to generalize the Gelfand-Naimark theorem for C*-algebras generated by a single or finitely many elements in search of a non-commutative spectral theory. P. Kruszyński and S.L. Woronowicz [32] constructed a non-commutative analogue of the spectral theorem of finitely generated C*-algebras, which can be extended to separable C*-algebras, where they introduced the notions of "compact domain" and "continuous operator function", which seemingly correspond to "representation space" and "admissible operator field" in Takesaki's duality [39]. On the other hand, W.B. Arveson [7, 8] initiated the method of using completely positive maps in operator theory, where, among others, he introduced "matrix range" as a non-commutative analogue of numerical range, and proved a classification theorem of irreducible operators, which was also intended to study "generalized spectrum" for non-normal operators.

In this situation, it would be natural to try to seek a generalized Gelfand-Naimark theorem for non-commutative C*-algebras which recovers both the algebraic and affine geometric aspects of the duality, and when it is applied to C*-algebras generated by a single element, it should provide a non-commutative spectral theory for non-normal operators, recovering the extremal property of the spectrum. In our previous works [20-24], we introduced "CP-convexity" for completely positive maps and developed a "quantized" convexity theory for C*-algebras. In the present paper, we intend to show that the ideas of the above different approaches in duality theory for C*-algebras can be combined together in our setting of CP-convexity theory to give rise to the "CP-Gelfand-Naimark theorem" which satisfies the above mentioned desired requirements. It therefore should serve as a unification of these different methods, bridging the gaps between them by presenting the common basic principle underlying these subjects. We shall also show some examples of its applications in the Stone-Weierstrass problem and the non-commutative spectral theory. We hope that the readers will find their own useful applications in each field of mathematics and mathematical physics wherever C*-algebras can be applied.

We are now going to summarize the content of this paper. In [**20-22**], we proved the CP-duality theorem for C*-algebras and W*-algebras which generalizes Kadison's function representation theorem recovering the full C*-structure. Furthermore, in [**23**, **25**] we characterized the "CP-extreme" elements of the CP-state space $Q_H(A)$, and showed that they are precisely the set of all irreducible representations $\mathrm{Irr}(A : H)$ of the C*-algebra A on the Hilbert space H if H has a sufficiently large dimension (cf. §1). In the present paper, we proceed forward with this project, and show that, if we take the "CP-extreme" states as our dual object, then we can realize the desired generalization of the Gelfand-Naimark theorem, i.e., A is *-isomorphic to the set of all $B(H)$-valued BW-w uniformly continuous functions on the CP-extreme boundary $\mathrm{Irr}(A : H)$ which preserve unitary equivalence and vanish at the limit 0 (Theorem 2.3). Therefore, the irreducible representations $\mathrm{Irr}(A : H) \cup \{0\}$ equipped with the BW-uniform topology and unitary equivalence relation provides a complete dual object for C*-algebras (Theorem 2.6).

In this setting, the CP-Gelfand-Naimark theorem and the CP-duality theorem are connected by the abstract Dirichlet problem (Theorem 3.1), so we can realize the same situation of the commutative case. Then we can consider the CP-affine extension of continuous equivariant functions on the CP-extreme boundary, and as applications of this argument, we shall give a simple characterization of perfect C*-algebras (Proposition 3.2), the notion of which was initially introduced by F.W. Shultz [**37**] together with its Stone-Weierstrass property. Moreover, we can generalize the notion of semi-perfect C*-algebras to non-separable C*-algebras and prove its Stone-Weierstrass property (Theorem 3.5), which was initially defined and proved by C.J.K. Batty [**10**] for separable C*-algebras in scalar convexity theory.

In the final section, we apply our CP-Gelfand-Naimark theorem to the C*-algebra $C^*(a)$ generated by a single element a, and define the "operator spectrum" of a by the operator range of the CP-Gelfand-Naimark representation of the operator a on the CP-extreme boundary of $C^*(a)$. We can then generalize the spectral theorem for non-normal operators (Theorem 4.4), and the spectral decomposition theorem using CP-measure and integral developed in [**24**] (Theorem 4.5). As an application, we shall prove that an irreducible operator a such that $C^*(a)$ is not NGCR (i.e., $C^*(a)$ contains a non-zero compact operator) is completely determined by its operator spectrum up to unitary equivalence (Theorem 4.7), which improves Arveson's classification theorem for *first order* irreducible operators in [**8**]. (The readers who are mainly interested in the results in this section could take a short course skipping directly from Theorem 2.3 to §4.)

A brief summary of the preliminary version of this paper has appeared in [**23**], and since then the paper has been developed and improved as presented in the present paper.

1. Preliminaries and Notation.

We shall first prepare some notations and basic results on CP-maps and CP-convexity. Let A be a C*-algebra and H be a Hilbert space. We shall denote by $CP(A, B(H))$ the set of all completely positive maps from A to the C*-algebra $B(H)$ of all bounded linear operators on H (cf. [**7**], [**33**], [**38**] for CP-maps). Recall in particular the Stinespring representation theorem ([**38**]), which states that every $\psi \in CP(A, B(H))$ can be written as

$$\psi(a) = V^*\pi(a)V \qquad \text{for all} \ \ a \in A,$$

where π is a representation of A on a Hilbert space K and $V \in B(H, K)$ is a bounded linear operator from H to K, and this representation is unique up to unitary intertwining operators under the minimality condition $K = [\pi(A)VH]$. A CP-map $\psi \in CP(A, B(H))$ is called a *CP-state* if ψ is a contraction, and we define the *CP-state space* of A for H by $Q_H(A) := \{\psi \in CP(A, B(H)); \|\psi\| \leq 1\}$, which generalizes the quasi-state space $Q(A)$ in the scalar convexity theory. If A is a W*-algebra, then $\psi = V^*\pi V \in CP(A, B(H))$ is called *normal* if π is normal, and we denote by $Q_H(A)_n$ the set of all normal CP-states of A for H. In addition to the norm topology in $Q_H(A)$, where we note $\|\psi\| = \|V\|^2$, we consider the BW- [resp. BS-] topology in $Q_H(A)$, which is defined as the pointwise weak [resp. strong] convergence topology.

We denote by $\mathrm{Rep}(A)$ [resp. $\mathrm{Rep}_c(A)$, $\mathrm{Irr}(A)$] the set of all [resp. cyclic, irreducible] representations of A, and by $\mathrm{Rep}(A : H)$ [resp. $\mathrm{Rep}_c(A : H)$, $\mathrm{Irr}(A : H)$] to specify the Hilbert space H on which the representations are confined. We note that the BW- and BS-topologies coincide on $\mathrm{Rep}(A : H)$ (cf. [**11**] or [**39**]). Following [**3**], we shall denote by $\overline{\mathrm{Irr}(A : H)}^c$ the BW-closure of $\mathrm{Irr}(A : H)$ in $\mathrm{Rep}_c(A : H)$. For $\pi \in \mathrm{Rep}(A)$, H_π denotes the essential subspace of π, and for $\pi \in \mathrm{Rep}(A : H)$, p_π denotes the projection of H onto H_π. More generally, for $\psi = V^*\pi V \in CP(A, B(H))$, we define the *support* (or *essential subspace*) H_ψ of ψ by the support of $\psi(A)$ in H, i.e.,

$H_\psi := [|V|H]$ (which is the essential support of V), and denote by p_ψ the projection of H onto H_ψ. We also define the *cyclic dimension* $\alpha_c(A)$ [resp. *irreducible dimension* $\alpha_i(A)$] by

$$\alpha_c(A)[\,\text{resp.}\,\alpha_i(A)] := \sup\{\dim\, H_\pi;\, \pi \in\, \text{Rep}_c(A)[\,\text{resp. Irr}(A)]\}.$$

In [20-22], we introduced *CP-convexity* in $Q_H(A)$ as follows: $\psi \in Q_H(A)$ is said to be a *CP-convex combination* of $(\psi_\alpha)_{\alpha\in\Lambda} \subset Q_H(A)$ if

$$\psi = \sum_{\alpha\in\Lambda} S_\alpha^* \psi_\alpha S_\alpha \quad \text{with} \;\; S_\alpha \in B(H) \;\; \text{such that} \;\; \sum_{\alpha\in\Lambda} S_\alpha^* S_\alpha \le I_H,$$

where the sum converges in the BS-topology (cf. [22, Prop. 1.2]). In the following, we shall abbreviate the above CP-convex combination by $\psi = CP\text{-}\sum_{\alpha\in\Lambda} S_\alpha^* \psi_\alpha S_\alpha$. A function $\gamma : Q_H(A) \to B(H)$ is defined to be *CP-affine*, if

$$\psi = CP\text{-}\sum_\alpha S_\alpha^* \psi_\alpha S_\alpha \;\; \text{with} \;\; \psi_\alpha \in Q_H(A) \quad \text{implies} \quad \gamma(\psi) = \sum_\alpha S_\alpha^* \gamma(\psi_\alpha) S_\alpha.$$

A CP-affine function γ is called *bounded* if $\|\gamma\| = \sup\{\|\gamma(\psi)\|; \psi \in Q_H(A)\} < \infty$. We denote by $AB(Q_H(A), B(H))$ the set of all bounded CP-affine functions from $Q_H(A)$ to $B(H)$, and by $AC(Q_H(A), B(H))$ the set of all BW-w (or BS-s) continuous elements. Then the CP-duality theorem ([**22**, Th. 2.2.A]) states that, if $\dim H \ge \alpha_c(A)$, then

$$A \cong AC(Q_H(A), B(H)) \qquad (\text{*-isomorphism}),$$

where the product in $AC(Q_H(A), B(H))$ is defined on $\text{Rep}_c(A : H)$; or more precisely, as is shown in [22, Prop. 1.4.A], every $\psi \in Q_H(A)$ can be expressed as

$$\psi = CP\text{-}\sum_\alpha V_\alpha^* \pi_\alpha V_\alpha \quad \text{with} \quad \pi_\alpha \in \text{Rep}_c(A : H),$$

then the product $\gamma_1 \cdot \gamma_2$ for $\gamma_1, \gamma_2 \in AC(Q_H(A), B(H))$ on ψ is defined by

$$(\gamma_1 \cdot \gamma_2)(\psi) := \sum_\alpha V_\alpha^*\, \gamma_1(\pi_\alpha) \cdot \gamma_2(\pi_\alpha)\, V_\alpha,$$

which can be shown to be well-defined (cf. [22] for details). This CP-duality theorem generalizes Kadison's function representation theorem recovering the full C*-structure.

Moreover, if A and B are C*-algebras and $\dim H \ge \sup\{\alpha_c(A), \alpha_c(B)\}$, then $Q_H(A)$ and $Q_H(B)$ are CP-affine BW- (or BS-) homeomorphic if and only if A and B are *-isomorphic ([22, Th. 3.2.A]). This, compared with the Kadison's result on state spaces with scalar convexity (cf. [30]), implies the correspondence that the CP-convexity in the CP-state space is to the C*-structure what the scalar convexity in the state space is to the Jordan structure of the algebra. We can also formulate the above duality theorems for W*-algebras using the normal part of the CP-state spaces (cf. [22]).

We shall summarize our results on the extreme elements of the CP-state space $Q_H(A)$ in the sense of CP-convexity, which would justify our generalization of the Gelfand-Naimark theorem in the context of CP-convexity theory. A nonzero CP-state $\psi \in Q_H(A)$ is defined to be *CP-extreme* if $\psi = \sum_{\alpha\in\Lambda} S_\alpha^* \psi_\alpha S_\alpha$ with $\psi_\alpha \in Q_H(A)$ and $\sum_{\alpha\in\Lambda} S_\alpha^* S_\alpha \leq p_\psi$ implies that ψ_α is unitarily equivalent to ψ for each $\alpha \in \Lambda$, i.e., $\psi_\alpha = U_\alpha \psi U_\alpha^*$ and $S_\alpha = c_\alpha U_\alpha$ ($c_\alpha \in \mathbb{C}$) with a partial isometry U_α such that $U_\alpha^* U_\alpha = p_\psi$ and $U_\alpha U_\alpha^* = p_{\psi_\alpha}$. We denote by $D_H(A)$ the set of all CP-extreme states of A for H. We can then show that, if $\dim H \geq \inf\{\alpha_i(A), \aleph_0\}$, then $D_H(A) = \mathrm{Irr}(A : H)$, and it reduces to the set of all pure states $P(A)$ of A if $H = \mathbb{C}$, i.e., $D_{\mathbb{C}}(A) = P(A)$ (cf. [23], [25]). Throughout this paper, H is assumed to be infinite dimensional with $\dim H \geq \alpha_c(A)$, hence $D_H(A) = \mathrm{Irr}(A : H)$, so that the irreducible representations $\mathrm{Irr}(A : H)$ of A on H can be characterized as CP-extreme elements of the CP-state space $Q_H(A)$. (This material was initially planned to be included in this paper, however we had to move it to another paper [25] because of the length of this paper.)

Finally, we recall that Choquet's boundary measure representation theorem can be generalized in the setting of CP-convexity using *CP-measure and integration* developed in [24], i.e., if A and H are separable, then for any CP-state $\psi \in Q_H(A)$, there exists a *CP-measure*, i.e., $Q_H(B(H))_n$ valued measure, supported by the CP-extreme elements $D_H(A)$ of $Q_H(A)$, such that

$$\psi(a) = \int_{D_H(A)} \hat{a}\, d\lambda_\psi \qquad \text{for all } \; a \in A,$$

where $\hat{a}$ denotes the CP-affine functional representation of $a \in A$ on $Q_H(A)$ (cf. [24, Th. 4.2]).

2. CP-Gelfand-Naimark theorem.

We begin with the following definition.

Definition 2.1. Let A be a C*-algebra and H be a Hilbert space, and $X \subset \mathrm{Rep}(A : H)$ be an invariant subset under the operation of unitary equivalence. Then a function $\gamma : X \to B(H)$ is called *equivariant* if $\gamma(u^*\pi u) = u^*\gamma(\pi)u$ for $\pi \in X$ and for all partial isometry u such that $uu^* \geq p_\pi$. We denote by $A^E(X, B(H))$ the set of all bounded equivariant functions from X to $B(H)$, and we write $A_c^E(X, B(H))$ for the set of all c-continuous elements, where c denotes the weak ($c = w$), strong ($c = s$)

or strong* ($c = s^*$) topology in $B(H)$. We also denote by $A_u^E(X, B(H))$ the set of all uniformly c-continuous elements, and in particular we write $A_{u,0}^E(X, B(H))$ for those elements which vanish at the limit 0.

The following theorem was first proved by C.A. Akemann and F.W. Shultz [**3**] based on the results in F.W. Shultz [**37**], where the Hilbert space H was assumed to have a larger dimension than ours, such that every unitary equivalence of cyclic representations in H can be implemented by a unitary operator on H, and also the continuity of the equivariant functions was restricted to the strong* topology. We shall show that the theorem can be released from these constraints as follows. In consequence, we can assume the Hilbert space H to be separable if A is separable, and also we can discuss the connection between the generalized Gelfand-Naimark theorem and the CP-duality theorem in §3.

Theorem 2.2. *Let A be a C*-algebra and H be an infinite dimensional Hilbert space with $\dim H \geq \alpha_c(A)$. Then,*

$$A \;\cong\; A_c^E(\overline{\mathrm{Irr}(A:H)}^c, B(H)) \qquad (\text{*-isomorphism}).$$

Proof. Consider the following canonical embedding

$$i : A \longmapsto A_c^E(\overline{\mathrm{Irr}(A:H)}^c, B(H)),$$

where $a \in A$ is assigned to the evaluation map $\hat{a}$ at a, i.e.,

$$i(a)(\pi) := \hat{a}(\pi) = \pi(a) \qquad \text{for} \quad \pi \in \overline{\mathrm{Irr}(A:H)}^c.$$

Obviously i is linear, and an isometry, because

$$\begin{aligned}\|a\| &= \sup\{\|\pi(a)\|; \pi \in \mathrm{Irr}(A:H)\} \leq \sup\{\|\pi(a)\|; \pi \in \overline{\mathrm{Irr}(A:H)}^c\} \\ &\leq \sup\{\|\psi(a)\|; \psi \in Q_H(A)\} = \|a\|,\end{aligned}$$

where note that, since $\dim H \geq \alpha_c(A) \geq \alpha_i(A)$, every irreducible representation of A can be realized in H (cf. [**15**, Prop. 2.7.1]). We shall show that i is surjective, hence a *-isomorphism, in the following.

Let $\gamma \in A_c^E(\overline{\mathrm{Irr}(A:H)}^c, B(H))$. It is shown in [**3**, Lemma 1.1] that $zA^{**} \cong A^E(\mathrm{Irr}(A:H), B(H))$ (*-isomorphism), where z denotes the central projection in A^{**} such that zA^{**} is the atomic part of A^{**}, so that, since $\gamma \in A^E(\mathrm{Irr}(A:H), B(H))$, there exists a unique element $b_\gamma \in zA^{**}$ such that

$$\gamma(\pi) = \tilde{\pi}(b_\gamma) \qquad \text{for all} \quad \pi \in \mathrm{Irr}(A:H),$$

where $\tilde{\pi}$ denotes the unique normal extension of π to A^{**} (cf. [**36**, 1.21.13]).

Assuming that A has an identity, it is proved in the proof of [3, Th.1.4] that the above b_γ is w^*-continuous on $\overline{P(A) \cup \{0\}}$, hence b_γ is uniformly w^*-continuous on $P(A) \cup \{0\}$, the argument of which is essentially based on [11, II 4) Prop.], and is true for any c $(= w, s$ or $s^*)$ topology in $B(H)$ for the continuity of γ. It can also be checked that, in this part of the proof, the condition on the dimension of H in [3] can be replaced by our condition without any problem, noting that our definition of equivariant functions in Definition 2.1 (which is based on [11]) is more general than that of [3] and so defined that the above argument is true.

Now, L.G. Brown's complement to Shultz's theorem in [12] shows that an element $x \in zA^{**}$ belongs to zA if x is uniformly w^*-continuous on $P(A) \cup \{0\}$. (We note here that the original Shultz's theorem in [37] required the uniform w^*-continuity of x^*x and xx^*, in addition to that of x, on $P(A) \cup \{0\}$.) It follows that, since b_γ is uniformly w^*-continuous on $P(A) \cup \{0\}$, we conclude that $b_\gamma \in zA$, so $b_\gamma = za_\gamma$ for some $a_\gamma \in A$. Then, noting that $\tilde{\pi}(z) = p_\pi$ for all $\pi \in \mathrm{Irr}(A : H)$, we have

$$\gamma(\pi) = \tilde{\pi}(za_\gamma) = \pi(a_\gamma) \qquad \text{for all} \quad \pi \in \mathrm{Irr}(A : H),$$

so that, by the continuity of γ, we have

$$\gamma(\pi) = \pi(a_\gamma) = i(a_\gamma)(\pi) \qquad \text{for all} \quad \pi \in \overline{\mathrm{Irr}(A : H)}^c,$$

which shows that the embedding i is surjective.

It is straightforward to extend the above isomorphism to the case where A is not unital, using the same argument given in the last part of the proof of [3, Th.1.4]. Now that the bijective correspondence of the canonical embedding $i : A \to A_c^E(\overline{\mathrm{Irr}(A : H)}^c, B(H))$ is established, it is clear that i is a *-isomorphism with the natural pointwise C*-operations in $A_c^E(\overline{\mathrm{Irr}(A : H)}^c, B(H))$. □

We can view the above duality theorem as the restriction of the CP-duality theorem on $\overline{\mathrm{Irr}(A : H)}^c$. We shall now proceed forward to establish a duality of A on $\mathrm{Irr}(A : H)$, namely the generalized Gelfand-Naimark theorem for C*-algebras on the CP-extreme boundary $\mathrm{Irr}(A : H)$ (the unital case) or $\mathrm{Irr}(A : H) \cup \{0\}$ (the non-unital case).

Theorem 2.3. *Let A be a C*-algebra and H be an infinite dimensional Hilbert space with $\dim H \geq \alpha_c(A)$.*

(i) *If A is unital, then*

$$A \;\cong\; A_u^E(\mathrm{Irr}(A : H), B(H)) \qquad (\textit{*-isomorphism}).$$

(ii) *If A is any C*-algebra (unital or non-unital), then*

$$\begin{aligned} A &\cong A^E_{u,0}(\mathrm{Irr}(A:H), B(H)) \qquad (\text{*-isomorphism}) \\ &\cong A^E_u(\mathrm{Irr}(A:H)\cup\{0\}, B(H)) \qquad (\text{*-isomorphism}). \end{aligned}$$

Proof. (i) We shall prove the theorem when A is unital. Let $\gamma \in A^E_u(\mathrm{Irr}(A:H), B(H))$, and we first extend γ to $\overline{\mathrm{Irr}(A:H)}^c$ as follows. Let $\pi \in \overline{\mathrm{Irr}(A:H)}^c$, then there exists a net (π_α) in $\mathrm{Irr}(A:H)$ such that $\pi_\alpha \to \pi$. Since γ is uniformly c-continuous, $(\gamma(\pi_\alpha))$ is a bounded Cauchy net in $B(H)$ in the c-topology, hence noting that a bounded ball of $B(H)$ is c-complete, we conclude that there exists a limit in the c-topology in $B(H)$, which we denote by $c\text{-}\lim_\alpha \gamma(\pi_\alpha)$. Let us define the extension of γ by

$$\tilde{\gamma}(\pi) := c\text{-}\lim_\alpha \gamma(\pi_\alpha).$$

We can easily see that this definition of $\tilde{\gamma}$ does not depend on the choice of the net $(\pi_\alpha) \subset \mathrm{Irr}(A:H)$ which converges to π, and that, by the standard argument, $\tilde{\gamma}$ is uniformly c-continuous on $\overline{\mathrm{Irr}(A:H)}^c$.

We shall show that $\tilde{\gamma}$ is equivariant. Let $\pi \in \overline{\mathrm{Irr}(A:H)}^c$. We first note that $\tilde{\gamma}(\pi) = p_\pi\tilde{\gamma}(\pi)p_\pi$. In fact, let $(\pi_\alpha) \subset \mathrm{Irr}(A:H)$ be a net such that $\pi_\alpha \to \pi$. Then, $p_\alpha := p_{\pi_\alpha} = \pi_\alpha(e) \underset{s^*}{\longrightarrow} \pi(e) = p_\pi$, where e is the identity of A, so that

$$\tilde{\gamma}(\pi) = c\text{-}\lim_\alpha \gamma(\pi_\alpha) = c\text{-}\lim_\alpha \gamma(p_\alpha\pi_\alpha p_\alpha) = c\text{-}\lim_\alpha p_\alpha\gamma(\pi_\alpha)p_\alpha = p_\pi\tilde{\gamma}(\pi)p_\pi.$$

Next we let $\kappa = u^*\pi u$ with some partial isometry u in H such that $uu^* \geq p_\pi$, and we shall show that $\tilde{\gamma}(\kappa) = u^*\tilde{\gamma}(\pi)u$. It is enough to show this for u such that $uu^* = p_\pi$; indeed, if it is proved in this case, noting that $\kappa = u^*\pi u = u^*p_\pi\pi p_\pi u = (p_\pi u)^*\pi(p_\pi u)$ with $(p_\pi u)(p_\pi u)^* = p_\pi uu^*p_\pi = p_\pi$, we can conclude that $\tilde{\gamma}(\kappa) = (p_\pi u)^*\tilde{\gamma}(\pi)(p_\pi u) = u^*p_\pi\tilde{\gamma}(\pi)p_\pi u = u^*\tilde{\gamma}(\pi)u$, which will prove our assertion.

We take a net $(\pi_\alpha) \subset \mathrm{Irr}(A:H)$ such that $\pi_\alpha \to \pi$, and let $\xi \in H_\pi$ with $\|\xi\| = 1$ be a cyclic vector of π, and set $\eta = u^*\xi$, and we shall define

$$\omega_\alpha := (\pi_\alpha(\cdot)\xi, \xi) = (u^*\pi_\alpha(\cdot)u\eta, \eta) \quad \text{and} \quad \omega := (\pi(\cdot)\xi, \xi) = (\kappa(\cdot)\eta, \eta),$$

where note that, since $\pi_\alpha \to \pi$, we have $\omega_\alpha \underset{w^*}{\longrightarrow} \omega$ in $Q(A)$. We shall use the following lemma which directly follows from [**11**, II 4) Prop. (i)].

Lemma 2.4. *Let A be a C*-algebra and H be an infinite dimensional Hilbert space with* $\dim H \geq \alpha_c(A)$. *If $\omega_\alpha \underset{w^*}{\longrightarrow} \omega$ in $Q(A)$, then there exist* *representative couples* $(\pi_{\omega_\alpha}, \xi_{\omega_\alpha})$ *and* (π_ω, ξ_ω) *for* ω_α *and* ω *respectively in* $\mathrm{Rep}_c(A : H) \times H$ *such that* $\pi_{\omega_\alpha} \to \pi_\omega$.

By the above Lemma, we can take a representative couple $(\kappa_\alpha, \eta_\alpha)$ of ω_α in $\mathrm{Irr}(A : H) \times H$, such that $\kappa_\alpha \to \kappa$. (It follows from this argument that $\overline{\mathrm{Irr}(A : H)}^c$ is invariant under the operation of unitary equivalence.)

Now, if $\dim H \ominus H_\kappa \le \dim H \ominus H_\pi$, then there exists an isometry $v : H \ominus H_\kappa \to H \ominus H_\pi$. Let $U := u \oplus v : H = H_\kappa \oplus (H \ominus H_\kappa) \to H = H_\pi \oplus (H \ominus H_\pi)$, then U is an isometry in H, i.e., $U^*U = I_H$. Define $\tilde{\pi}_\alpha := U\kappa_\alpha U^*$ and $\tilde{\kappa}_\alpha := \kappa_\alpha$, then $\tilde{\pi}_\alpha, \tilde{\kappa}_\alpha \in \mathrm{Irr}(A : H)$, and $\tilde{\pi}_\alpha \to \pi$ in $UH \subset H$ and $\tilde{\kappa}_\alpha \to \kappa$ in $U^*H = H$.

If $\dim H \ominus H_\kappa > \dim H \ominus H_\pi$, then there exists a co-isometry $w : H \ominus H_\kappa \to H \ominus H_\pi$. Let $U := u \oplus w : H = H_\kappa \oplus (H \ominus H_\kappa) \to H = H_\pi \oplus (H \ominus H_\pi)$, then U is a co-isometry in H, i.e., $UU^* = I_H$. Define $\tilde{\pi}_\alpha := \pi_\alpha$ and $\tilde{\kappa}_\alpha := U^*\pi_\alpha U$, then $\tilde{\pi}_\alpha, \tilde{\kappa}_\alpha \in \mathrm{Irr}(A : H)$, and $\tilde{\pi}_\alpha \to \pi$ in $UH = H$ and $\tilde{\kappa}_\alpha \to \kappa$ in $U^*H \subset H$.

In either case, we have defined nets $(\tilde{\pi}_\alpha)$ and $(\tilde{\kappa}_\alpha)$ in $\mathrm{Irr}(A : H)$ such that $\tilde{\pi}_\alpha \to \pi$ in UH and $\tilde{\kappa}_\alpha \to \kappa$ in U^*H where $\tilde{\kappa}_\alpha = U^*\tilde{\pi}_\alpha U$. Noting that $UU^* \ge p_{\tilde{\pi}_\alpha}$ and $p_\pi U = u$, we have

$$\begin{aligned}\tilde{\gamma}(\kappa) &= c\text{-}\lim_\alpha \gamma(\tilde{\kappa}_\alpha) = c\text{-}\lim_\alpha \gamma(U^*\tilde{\pi}_\alpha U) = c\text{-}\lim_\alpha U^*\gamma(\tilde{\pi}_\alpha)U \\ &= U^*\tilde{\gamma}(\pi)U = U^*p_\pi\tilde{\gamma}(\pi)p_\pi U = u^*\tilde{\gamma}(\pi)u,\end{aligned}$$

which shows that $\tilde{\gamma}$ is equivariant.

Obviously, the extension map

$$\gamma \in A_u^E(\mathrm{Irr}(A : H), B(H)) \quad \longmapsto \quad \tilde{\gamma} \in A_c^E(\overline{\mathrm{Irr}(A : H)}^c, B(H))$$

is one to one and onto, and is a *-isomorphism, hence our assertion follows from Theorem 2.2.

(ii) We shall prove the CP-Gelfand-Naimark theorem for any C*-algebras including the non-unital case in general. It is clear that, if A is unital, then $A \cong A_{u,0}^E(\mathrm{Irr}(A : H), B(H))$. Indeed, we already know from (i) that $A \cong A_u^E(\mathrm{Irr}(A : H), B(H))$, and note that, as we have shown in the above proof of (i), for any $\gamma \in A_u^E(\mathrm{Irr}(A : H), B(H))$ and $\pi \in \overline{\mathrm{Irr}(A : H)}^c$, $\tilde{\gamma}(\pi) = p_\pi\tilde{\gamma}(\pi)p_\pi$. Note here that $0 \in \overline{\mathrm{Irr}(A : H)}^c$; in fact, let (π_α) be a net in $\mathrm{Irr}(A : H)$ with mutually orthogonal essential subspaces, then $\pi_\alpha \to 0$. Hence, $\tilde{\gamma}(0) = p_0\tilde{\gamma}(0)p_0 = 0\tilde{\gamma}(0)0 = 0$, so that γ vanishes at the limit 0, i.e., $\gamma \in A_{u,0}^E(\mathrm{Irr}(A : H), B(H))$. Therefore, if A is unital, we have the *-isomorphism $A \cong A_{u,0}^E(\mathrm{Irr}(A : H), B(H))$.

We shall next assume that A is not unital, and show that $A \cong A_u^E(\mathrm{Irr}(A : H) \cup \{0\}, B(H))$ using the similar argument as in [**3**, Th. 1.4]. Since it is obvious that A is injectively embedded into $A_u^E(\mathrm{Irr}(A : H) \cup \{0\}, B(H))$

by the natural evaluation map, it suffices to show that this embedding is surjective.

Let $\tilde{A}$ be the C*-algebra obtained from A by adjoining the identity e. Then, from (i), we have $\tilde{A} \cong A_u^E(\mathrm{Irr}(\tilde{A} : H), B(H))$. Let $\gamma \in A_u^E(\mathrm{Irr}(A : H) \cup \{0\}, B(H))$, and we shall define

$$\tilde{\gamma}(\rho) := \gamma(\rho|_A) \quad \text{for} \;\; \rho \in \mathrm{Irr}(\tilde{A} : H),$$

where note that, since A is an ideal of $\tilde{A}$, if $\rho \in \mathrm{Irr}(\tilde{A} : H)$ then $\rho|_A \in \mathrm{Irr}(A : H) \cup \{0\}$, and it is clear that $\tilde{\gamma} \in A_u^E(\mathrm{Irr}(\tilde{A} : H), B(H))$. Hence, by (i), there exists $a_{\tilde{\gamma}} \in A$ and $\lambda_{\tilde{\gamma}} \in \mathbb{C}$ such that

$$\rho(a_{\tilde{\gamma}} + \lambda_{\tilde{\gamma}}\, e) = \tilde{\gamma}(\rho) \quad \text{for all} \;\; \rho \in \mathrm{Irr}(\tilde{A} : H).$$

If we choose $\rho \in \mathrm{Irr}(\tilde{A} : H)$ such that $\rho|_A = 0$ and $p_\rho \neq 0$, then

$$\lambda_{\tilde{\gamma}}\, p_\rho = \rho(a_{\tilde{\gamma}}) + \lambda_{\tilde{\gamma}}\, \rho(e) = \tilde{\gamma}(\rho) = \gamma(\rho|_A) = \gamma(0) = 0,$$

where the last equality follows from the equivariance of γ, so that $\lambda_{\tilde{\gamma}} = 0$. Hence,

$$\rho(a_{\tilde{\gamma}}) = \tilde{\gamma}(\rho) \qquad \text{for all} \;\; \rho \in \mathrm{Irr}(\tilde{A} : H).$$

Now, for each $\pi \in \mathrm{Irr}(A : H)$, we shall define $\tilde{\pi} \in \mathrm{Irr}(\tilde{A} : H)$ by

$$\tilde{\pi}(a + \lambda e) := \pi(a) + \lambda p_\pi \quad \text{for all} \;\; a \in A \;\; \text{and} \;\; \lambda \in \mathbb{C},$$

then substituting $a = a_{\tilde{\gamma}}$, $\lambda = \lambda_{\tilde{\gamma}} = 0$ in the above, we have

$$\pi(a_{\tilde{\gamma}}) = \tilde{\pi}(a_{\tilde{\gamma}}) = \tilde{\gamma}(\tilde{\pi}) = \gamma(\tilde{\pi}|_A) = \gamma(\pi) \quad \text{for all} \;\; \pi \in \mathrm{Irr}(A : H).$$

As the above equality obviously holds for $\pi = 0$, we have

$$\pi(a_{\tilde{\gamma}}) = \gamma(\pi) \qquad \text{for all} \;\; \pi \in \mathrm{Irr}(A : H) \cup \{0\},$$

which shows that the natural embedding $A \to A_u^E(\mathrm{Irr}(A : H) \cup \{0\}, B(H))$ is surjective, hence a *-isomorphism.

To complete our proof, we have to show the *-isomorphism

$$A_{u,0}^E(\mathrm{Irr}(A : H), B(H)) \cong A_u^E(\mathrm{Irr}(A : H) \cup \{0\}, B(H)).$$

It is obvious that every $\gamma \in A_u^E(\mathrm{Irr}(A\!:\!H) \cup \{0\}, B(H))$ restricts to $\gamma|_{\mathrm{Irr}(A:H)} \in A_{u,0}^E(\mathrm{Irr}(A : H), B(H))$, since $\gamma(0) = 0$ because of the equivariance of γ. Conversely, by the same argument in the first part of the proof in (i), each $\gamma \in A_{u,0}^E(\mathrm{Irr}(A : H), B(H))$ can be uniquely extended to a $B(H)$-valued uniformly c-continuous function $\tilde{\gamma}$ on $\mathrm{Irr}(A : H) \cup \{0\}$. Since γ vanishes at the limit 0, we have $\tilde{\gamma}(0) = 0$, so that $\tilde{\gamma}$ is equivariant at 0, i.e., $\tilde{\gamma} \in A_u^E(\mathrm{Irr}(A : H) \cup \{0\}, B(H))$. Hence, we have proved the *-isomorphism claimed above, and the proof is complete. ☐

Remark. We note that the passage from the CP-Gelfand-Naimark theorem to the classical Gelfand-Naimark theorem for commutative C*-algebras can be realized by the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{\ i\ } & A_{u,0}^E(\mathrm{Irr}(A:H), B(H)) \\ & \searrow^{g} & \downarrow j \\ & & C_0(P(A)) \end{array}$$

where i is the CP-Gelfand representation; j is defined by

$$j(\gamma)(\omega) := (\gamma(\omega P_h)h, h) \quad \text{for } \gamma \in A_{u,0}^E(\mathrm{Irr}(A : H), B(H)) \text{ and } \omega \in P(A),$$

where one should note that every $\varphi \in \mathrm{Irr}(A : H)$ is a one dimensional representation, since A is commutative, so it is of the form $\varphi = \omega(\cdot)P_h$ with $\omega \in P(A)$, and P_h denotes the projection of H onto the one-dimensional subspace $[h]$ for $h \in H$ with $\|h\| = 1$; and g represents the Gelfand representation. Note that the infinity ∞ for $P(A)$ corresponds to the limit 0 for $\mathrm{Irr}(A : H)$. (It should be remarked that the above passage cannot be carried through by simply letting $H = \mathbb{C}$, since H is assumed to be infinite dimensional in Theorem 2.3.)

As a direct consequence of CP-Gelfand-Naimark theorem, and to prepare for our application in the subsequent section, we shall study the dual maps on the irreducible representations induced by *-isomorphisms.

Definition 2.5. Let A and B be C*-algebras and H be an infinite dimensional Hilbert space with $\dim H \geq \sup\{\alpha_i(A), \alpha_i(B)\}$. Then we call a map $\Phi : \mathrm{Irr}(A : H) \cup \{0\} \to \mathrm{Irr}(B : H) \cup \{0\}$ to be *equivariant* if $\Phi(u^*\pi u) = u^*\Phi(\pi)u$ for $\pi \in \mathrm{Irr}(A : H) \cup \{0\}$ and for all partial isometry u in $B(H)$ such that $uu^* \geq p_\pi$.

The following theorem is an immediate consequence from Theorem 2.3, and refines Theorem 3.2 in [**22**] (cf. §1).

Theorem 2.6. *Let A and B be C*-algebras and H be an infinite dimensional Hilbert space with $\dim H \geq \sup\{\alpha_c(A), \alpha_c(B)\}$. Then, there exists an isomorphism $\Theta : \mathrm{Irr}(A : H) \cup \{0\} \to \mathrm{Irr}(B : H) \cup \{0\}$ such that Θ and Θ^{-1} are equivariant maps which are BW-uniform homeomorphisms, if and only* *if A and B are *-isomorphic. If A and B are unital, then Θ can be reduced to the described dual map between* $\mathrm{Irr}(A:H)$ *and* $\mathrm{Irr}(B:H)$.

Proof. Assume first that $\theta : A \to B$ is a *-isomorphism. Define $\theta^\natural : \mathrm{Irr}(B : H) \cup \{0\} \to \mathrm{Irr}(A : H) \cup \{0\}$ by $\theta^\natural(\pi) = \pi \circ \theta$ for $\pi \in \mathrm{Irr}(B : H) \cup \{0\}$. Then, $\theta^\natural$ and $(\theta^\natural)^{-1}$ are equivariant maps which are BW-uniform homeomorphisms.

Conversely, let $\Theta : \mathrm{Irr}(A : H) \cup \{0\} \to \mathrm{Irr}(B : H) \cup \{0\}$ be an isomorphism such that Θ and Θ^{-1} are equivariant BW-uniform homeomorphisms. Define $\Theta^\natural : A_u^E(\mathrm{Irr}(B : H) \cup \{0\}, B(H)) \to A_u^E(\mathrm{Irr}(A : H) \cup \{0\}, B(H))$ by $\Theta^\natural(\gamma) := \gamma \circ \Theta$ for $\gamma \in A_u^E(\mathrm{Irr}(B : H) \cup \{0\}, B(H))$. Then $\Theta^\natural$ is obviously an *-isomorphism, so by Theorem 2.3 (ii), we have $A \cong B$.

If A and B are unital, then the assertion is proved similarly using Theorem 2.3 (i). ☐

Remark. This result could be compared with F.W. Shultz's result in [**37**] on the duality of C*-algebras on the pure states, where he showed that a C*-algebra A is determined up to *-isomorphism by the pure states $P(A) \cup \{0\}$ equipped with the w^*-uniform topology, transition probability and orientation. In our duality, A is determined by the irreducible representations $\mathrm{Irr}(A : H) \cup \{0\}$ with the BW-uniform topology and unitary equivalence relation, where A can be explicitly recovered as a C*-algebra of $B(H)$-valued uniformly continuous equivariant functions on $\mathrm{Irr}(A : H) \cup \{0\}$.

3. CP-Dirichlet problem and applications.

For a unital commutative C*-algebra A, every continuous function $f \in C(P(A))$ on the pure states (characters) $P(A)$ can be uniquely extended to a continuous affine function $\tilde{f} \in A(S(A))$ over the whole state space $S(A)$ of A. This property is characterized as the fact that the state space $S(A)$ is a Bauer simplex (cf. e.g., [**5**]), and connects the Gelfand-Naimark theorem with Kadison's function representation theorem. In this section, we shall discuss the abstract Dirichlet problem for non-commutative C*-algebras in the CP-convexity context. Although we only consider unital C*-algebras in this section, the generalization of the results to the non-unital case would be straightforward, taking care of the behavior of functions at infinity or zero.

In §2, we have seen that the CP-Gelfand-Naimark theorem is exactly the restriction of the CP-duality theorem on the CP-extreme boundary (irreducible representations), hence we have shown the following.

Theorem 3.1. *Let A be a unital C*-algebra and H be an infinite dimensional Hilbert space with* $\dim H \geq \alpha_c(A)$. *Then, every $B(H)$-valued uniformly continuous equivariant function $\gamma \in A_u^E(\mathrm{Irr}(A : H), B(H))$ on the*

CP-extreme boundary $\mathrm{Irr}(A:H)$ *can be uniquely extended to a continuous CP-affine function* $\tilde{\gamma} \in AC(Q_H(A), B(H))$ *over the whole CP-state space* $Q_H(A)$.

Thus the abstract Dirichlet problem in the CP-convexity theory can generalize the same situation of the commutative case, which was not the case in the scalar convexity theory on state spaces (cf. [**4**], [**5**, Th. II. 4.5], or [**37**]). We note however that the above extension was concerned with the uniformly continuous functions, so that one could naturally ask about the abstract Dirichlet problem for continuous functions, which we shall discuss in the following.

In the convexity approach on state spaces of C*-algebras, some important classes of C*-algebras were studied concerning the general Stone-Weierstrass problem, notably perfect C*-algebras and semi-perfect C*-algebras. We shall show that these algebras can be simply characterized in the abstract Dirichlet problem in our setting of CP-convexity theory.

Following F.W. Shultz [**37**], A_c is defined as the set of all those $a \in zA^{**}$ such that a, a^*a, aa^* are w^*-continuous on $P(A) \cup \{0\}$. Then A is defined to be *perfect* if $A_c = zA$. We recall that C.A. Akemann and F.W. Shultz [**3**, Cor. 2.4] showed that, if A is unital, then A_c is *-isomorphic to the set of all $B(H)$-valued s^*-continuous equivariant functions on $\mathrm{Irr}(A:H)$, i.e.,

$$A_c \cong A^E_{s^*}(\mathrm{Irr}(A:H), B(H)) \quad (\text{*-isomorphism}),$$

while, from our CP-Gelfand-Naimark theorem (Theorem 2.3 (i)), we have

$$A \cong A^E_u(\mathrm{Irr}(A:H), B(H)) \quad (\text{*-isomorphism}),$$

so that A_c can be considered as the "continuous closure" of A on $\mathrm{Irr}(A:H)$. Then we can simply characterize perfect C*-algebras in the setting of the abstract Dirichlet problem as follows.

Proposition 3.2. *Let* A *be a unital C*-algebra and* H *be an infinite dimensional Hilbert space with* $\dim H \geq \alpha_c(A)$. *Then,* A *is perfect if and only if every* $\gamma \in A^E_{s^*}(\mathrm{Irr}(A:H), B(H))$ *can be uniquely extended to* $\tilde{\gamma} \in AC(Q_H(A), B(H))$, *or equivalently* $A^E_{s^*}(\mathrm{Irr}(A:H), B(H)) = A^E_u(\mathrm{Irr}(A:H), B(H))$.

Proof. By Akemann-Shultz's theorem, we have $A_c \cong A^E_{s^*}(\mathrm{Irr}(A:H), B(H))$. On the other hand, by CP-Gelfand-Naimark theorem (Theorem 2.3 (i)), and noting that the map $a \in A \to za \in zA$ is a *-isomorphism, we have $zA \cong A^E_u(\mathrm{Irr}(A:H), B(H))$. Since these isomorphisms are defined by the natural evaluation maps, A is perfect (i.e., $A_c = zA$) if and only if

$A^E_{s^*}(\mathrm{Irr}(A:H), B(H)) = A^E_u(\mathrm{Irr}(A:H), B(H))$. Then, our assertion follows from Theorem 3.1. □

The notion of perfectness was generalized by C.J.K. Batty in [**10**], where he defined a separable C*-algebra A to be *semi-perfect* if every element in A_c is annihilated by each boundary affine dependence on the state space $S(A)$ of A, or equivalently, there exists a weak expectation $E : A_c \to A^{**}$ (i.e., a contractive linear map from A_c into A^{**} extending the natural *-isomorphism $zA \cong A$), and moreover he showed that E is actually an embedding (i.e., an into *-isomorphism) (cf. [**10**, Prop. 4.1]). In the following, we shall generalize the notion of semi-perfectness to non-separable C*-algebras.

Proposition 3.3. *Let A be a unital C*-algebra, and H be an infinite dimensional Hilbert space with* $\dim H \geq \alpha_c(A)$. *Then, the following conditions are equivalent.*

(i) *There exists a weak expectation $E : A_c \to A^{**}$.*

(ii) *There exists a CP-affine extension map $\tilde{E} : \gamma \in A^E_{s^*}(\mathrm{Irr}(A:H), B(H)) \mapsto \tilde{\gamma} \in AB(Q_H(A), B(H))$ with $\tilde{\gamma}|_{\mathrm{Irr}(A:H)} = \gamma$ and $\|\tilde{\gamma}\| = \|\gamma\|$, so that $\tilde{E}$ is an isometry which extends the natural extension map from $A^E_u(\mathrm{Irr}(A:H), B(H))$ to $AC(Q_H(A), B(H))$.*

Proof. We shall show (i) ⇒ (ii). Assume that there exists a weak expectation $E : A_c \to A^{**}$. From the *-isomorphisms $i : A_c \to A^E_{s^*}(\mathrm{Irr}(A:H), B(H))$ by Akemann-Shultz's theorem [**3**, Cor. 2.4], and $j : A^{**} \to AB(Q_H(A), B(H))$ by the CP-duality theorem [**22**, Th. 2.2.B], we shall define, for each $\gamma \in A^E_{s^*}(\mathrm{Irr}(A:H), B(H))$,

$$\tilde{\gamma} = \tilde{E}(\gamma) := j \circ E \circ i^{-1}(\gamma) \in AB(Q_H(A), B(H)).$$

We shall show that $\tilde{\gamma}$ is a CP-affine extension of γ, i.e., $\tilde{\gamma}|_{\mathrm{Irr}(A:H)} = \gamma$. Observe that, for any $\pi \in \mathrm{Rep}(A:H)$,

$$\tilde{\gamma}(\pi) = j \circ E \circ i^{-1}(\gamma)(\pi) = \pi(E \circ i^{-1}(\gamma)) = (\pi \circ E)(i^{-1}(\gamma)) := \tilde{\pi}(a_\gamma),$$

where we denoted $\tilde{\pi} := \pi \circ E$ and $a_\gamma := i^{-1}(\gamma) \in A_c$. We note here that every weak expectation is a completely positive map (cf. [**6**, Prop. 2.1]), which is contractive, so $\tilde{\pi} \in Q_H(A_c)$ and $\tilde{\pi}|_A = \pi$ (where we are identifying A with zA as in [**3**]). By [**3**, Prop. 2.9], each pure state of A extends uniquely to $Q(A_c)$ and the extension is in $P(A_c)$, which can be restated in terms of irreducible representations that each $\pi \in \mathrm{Irr}(A:H)$ extends uniquely to $Q_H(A_c)$ and the extension is in $\mathrm{Irr}(A_c:H)$. From this, we conclude that $\tilde{\pi} \in \mathrm{Irr}(A_c:H)$ for each $\pi \in \mathrm{Irr}(A:H)$. Hence, if we denote by s the extension map, i.e., $s : \pi \in \mathrm{Irr}(A:H) \mapsto s(\pi) := \tilde{\pi} \in \mathrm{Irr}(A_c:H)$, then

$\gamma = \hat{a}_\gamma \circ s$, where $\hat{a}_\gamma$ denotes the CP-Gelfand representation of a_γ in the *-isomorphism $A_c \cong A_u^E(\mathrm{Irr}(A_c : H), B(H))$. Then, for each $\pi \in \mathrm{Irr}(A : H)$,

$$\tilde{\gamma}(\pi) = \tilde{\pi}(a_\gamma) = (s(\pi))(a_\gamma) = \hat{a}_\gamma(s(\pi)) = (\hat{a}_\gamma \circ s)(\pi) = \gamma(\pi),$$

so that we have shown $\tilde{\gamma}|_{\mathrm{Irr}(A:H)} = \gamma$.

We note here that $\|\tilde{\gamma}\| = \|\gamma\|$; in fact, we have $\|\tilde{\gamma}\| = \|\tilde{E}(\gamma)\| \leq \|\gamma\|$ since $\|\tilde{E}\| = \|j \circ E \circ i^{-1}\| \leq \|E\| \leq 1$, and $\|\tilde{\gamma}\| \geq \|\gamma\|$ is obvious as $\tilde{\gamma}$ extends γ. Hence, $\tilde{E}$ is an isometry, which extends the natural extension map from $A_u^E(\mathrm{Irr}(A : H), B(H))$ to $AC(Q_H(A), B(H))$, since E extends the natural *-isomorphism $zA \cong A$.

Conversely assume (ii), then $E = j^{-1} \circ (j \circ E \circ i^{-1}) \circ i = j^{-1} \circ \tilde{E} \circ i$ is an isometry from A_c to A^{**} which extends the *-isomorphism $zA \cong A \subset A^{**}$, so that E is a weak expectation, and the above conditions are equivalent. □

As we mentioned above, C.J.K. Batty [**10**] showed that a separable C*-algebra A is semi-perfect if and only if there exists a natural embedding $E : A_c \to A^{**}$, i.e., the weak expectation in Proposition 3.3 is multiplicative. It should be noted here that his proof uses Choquet theory and essentially depends on the separability of the algebra, so it is not straightforward to extend the argument to the non-separable case (cf. Remark at the end of this section). Therefore, in our present paper, we shall define a non-separable generalization of semi-perfect C*-algebras as follows.

Definition 3.4. A C*-algebra A is defined to be *quasi-perfect* if there exists a natural embedding from A_c into A^{**}.

Thus a unital C*-algebra A is quasi-perfect if and only if there exists a CP-affine extension map $\tilde{E} : A_{s^*}^E(\mathrm{Irr}(A : H), B(H)) \to AB(Q_H(A), B(H))$ which is a *-isomorphism, generalizing the situation of perfect C*-algebras (cf. Proposition 3.2). We shall now prove the following Stone-Weierstrass theorem for quasi-perfect C*-algebras, which is a non-separable generalization of Batty's result on separable semi-perfect C*-algebras [**10**, Th. 5.1].

Theorem 3.5. *Let A be a unital C*-algebra, B be a quasi-perfect C*-subalgebra of A which contains the identity of A, and suppose that B separates $P(A)$. Then, $A = B$.*

Proof. From our result in [**27**], B separates $\mathrm{Irr}(A : H)$, and by [**15**, Lemma 11.1.7] the restriction $\pi|_B$ of $\pi \in \mathrm{Irr}(A : H)$ to B is in $\mathrm{Irr}(B : H)$, and every $\pi \in \mathrm{Irr}(B : H)$ has an extension to $\mathrm{Irr}(A : H)$ (cf. [**15**, Lemma 11.1.2]), so that the restriction map $r : \pi \in \mathrm{Irr}(A : H) \mapsto \pi|_B \in \mathrm{Irr}(B : H)$ is bijective. Since $Q_H(A)$ is BW-compact (cf. [**7**]), and r is BW-continuous, every relatively BW-closed (hence compact) subset of $\mathrm{Irr}(A:H)$ is mapped to a relatively BW-closed (compact) subset of $Irr(B:H)$ by r. Therefore, $r:\mathrm{Irr}(A:H)\to\mathrm{Irr}(B:H)$ is a BW-homeomorphism.

In this situation, A is embedded into B_c by the map $s^*: a\in A\mapsto i^{-1}\circ\hat{a}\circ s\in B_c$, where $s:=r^{-1}:\mathrm{Irr}(B:H)\to\mathrm{Irr}(A:H)$, $\hat{a}$ is the CP-Gelfand representation of a on $\mathrm{Irr}(A:H)$, and $i:B_c\to A^E_{s^*}(\mathrm{Irr}(B:H),B(H))$ is the Akemann-Shultz's isomorphism. Furthermore, since B is quasi-perfect, B_c is embedded into B^{**}, which we denote by $I:B_c\to B^{**}$, therefore there exists an embedding $\tilde{I}:=I\circ s^*:A\to B^{**}$ such that $\tilde{I}|_B=id_B$. Let us denote $\tilde{A}:=\tilde{I}(A)\subset B^{**}$. Then, we can see that $\tilde{A}\supset B$, and B separates $\mathrm{Irr}(\tilde{A}:H)$, because by Theorem 2.6 we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Irr}(A:H) & \xleftarrow{\tilde{I}^\natural} & \mathrm{Irr}(\tilde{A}:H) \\ r\big\downarrow & \swarrow_{\tilde{r}} & \\ \mathrm{Irr}(B:H) & & \end{array}$$

where $\tilde{I}^\natural(\pi)=\pi\circ\tilde{I}$ for $\pi\in\mathrm{Irr}(\tilde{A}:H)$ and $\tilde{r}:=r\circ\tilde{I}^\natural$. Thus, we have to solve the Stone-Weierstrass problem in the setting $B\subset\tilde{A}\subset B^{**}$, which is now straightforward as follows.

Indeed, let $\pi\in\mathrm{Rep}_c(\tilde{A})$ be any cyclic representation of $\tilde{A}$ with cyclic vector ξ_π. We shall take its unique normal extension $\tilde{\pi}$ to $\tilde{A}^{**}$, then, since $B^{**}\subset\tilde{A}^{**}$, $\tilde{\pi}|_{B^{**}}$ is normal, and $\tilde{\pi}|_B=\pi|_B$, so $\pi(\tilde{A})\subset\tilde{\pi}(B^{**})=\pi(B)''$, hence $\pi(\tilde{A})''\subset\pi(B)''$. Since $\pi(\tilde{A})''\supset\pi(B)''$ is obvious from $\tilde{\pi}(A)\supset\pi(B)$, we have $\pi(\tilde{A})''=\pi(B)''$. Therefore, $\pi|_B$ is a non-degenerate cyclic representation of B on H_π with the same cyclic vector ξ_π. Since $\pi\in\mathrm{Rep}_c(\tilde{A})$ can be arbitrary, by Akemann's theorem [**1**, Th. III.7], this can be possible only when $\tilde{A}=B$. Therefore, $A=\tilde{I}^{-1}(\tilde{A})=\tilde{I}^{-1}(B)=B$, since $\tilde{I}|_B=id_B$. This completes the proof. □

Remark. As another possible non-separable generalization of semi-perfectness, C.J.K. Batty [**10**, §10.I] suggested to define a C*-algebra B to be semi-perfect if every element in B_c is annihilated by each "induced" boundary affine dependence on $S(B)$ which is defined by "induced" boundary measures on a σ-algebra on $P(B)$ in the sense of Bishop-de Leeuw's theorem (cf. [**5**, Th. I. 4.14]). As he pointed out, this method had a difficulty to prove the Stone-Weierstrass theorem (cf. [**10**, Th.5.1]), since it was not clear that, if B separates $P(A)$, the restriction map $r:S(A)\to S(B)$ takes maximal measures into maximal measures, however this problem has been solved recently by S. Teleman [**41**, **42**]. Although it is not clear at the present that these two generalizations are equivalent, or one includes the other, we hope that it would be possible to improve these results in the future to cover the semi-perfectness in the sense of Proposition 3.3. Our method in this section will be developed further to discuss general Stone-Weierstrass theorems for C*-algebras in [**26**].

4. CP-spectral theorem.

In this section, we shall discuss the generalized Gelfand-Naimark theorem for C*-algebras generated by a single element. In the following, let A be a C*-algebra, $a \in A$ be an arbitrary element, and let $C^*(a)$ be the C*-algebra generated by a.

Assume first that $C^*(a)$ is a commutative C*-algebra, e.g., consider the case that a is normal. We then denote by $\sigma(a)$ the *spectrum* of a, i.e.,

$$\sigma(a) := \{\lambda \in \mathbb{C}; (a - \lambda) \text{ is not invertible in } C^*(a, 1)\},$$

where $C^*(a, 1)$ denotes the C*-algebra generated by a and the identity 1. Note that $0 \in \sigma(a)$ if $C^*(a)$ is not unital, and $\sigma(a) \cup \{0\} = \hat{a}(P(C^*(a))) \cup \{0\}$ where the map $\hat{a} : P(C^*(a)) \cup \{\infty\} \to \sigma(a) \cup \{0\}$ is a homeomorphism which maps the infinity ∞ to 0 in the limit, so that, by the Gelfand-Naimark theorem, $C^*(a)$ is *-isomorphic to the algebra $C_0(\sigma(a))$ of all continuous functions on $\sigma(a)$ which vanish at 0. We shall generalize this spectral theorem to the case where $C^*(a)$ is not commutative.

We note that, since $C^*(a)$ is a separable C*-algebra, we have $\alpha_c(C^*(a)) \leq \aleph_0$, so any infinite dimensional Hilbert space H satisfies the condition $\dim H \geq \alpha_c(C^*(a))$.

Definition 4.1. Let A be a C*-algebra, $C^*(a)$ be the C*-algebra generated by $a \in A$, and let H be an infinite dimensional Hilbert space. Then we define the *operator spectrum* $\sigma_H(a) \subset B(H)$ of a by the operator range of the CP-Gelfand representation $\hat{a}$ of a on $\mathrm{Irr}(C^*(a) : H) \cup \{0\}$, i.e.,

$$\sigma_H(a) := \hat{a}(\mathrm{Irr}(C^*(a) : H) \cup \{0\}) = \hat{a}(\mathrm{Irr}(C^*(a) : H)) \cup \{0\}.$$

Remark. It follows from the above definition that, if $a \in A$ is self-adjoint, [resp. positive, unitary], then the operator spectrum $\sigma_H(a)$ of a consists of self-adjoint [resp. positive, partially unitary] operators in $B(H)$, which can be symbolically expressed as

$$a \in A_{\mathrm{sa}} \text{ [resp. } A^+, U(A)] \quad \Longrightarrow \quad \sigma_H(a) \subset B(H)_{\mathrm{sa}} \text{ [resp. } B(H)^+, U_0(H)],$$

where the notations A_{sa}, A^+, $U(A)$ etc are self-explanatory, and $U_0(H)$ denotes the set of all partial unitaries on H, i.e., $U_0(H) := \{u \in B(H); u^*u = uu^* = \text{a projection on } H\}$.

Proposition 4.2. *Let A be a C*-algebra, H be a Hilbert space, and $C^*(a)$ be the C*-algebra generated by $a \in A$, and let X be a subset of* $\mathrm{Rep}(C^*(a) : H)$ *and set $Y := \hat{a}(X)$. Then, the map $\hat{a} : \pi \in X \mapsto \pi(a) \in Y$ is uniformly homeomorphic with the strong* topology in Y.*

Proof. We first note that $\hat{a} : X \to Y$ is bijective; in fact, it is injective since $C^*(a)$ is generated by a, and it is surjective by definition.

It is obvious that $\hat{a}$ is uniformly continuous, since, if (π_α) is a Cauchy net in X, i.e., for each $x \in C^*(a)$, the net $(\pi_\alpha(x))$ is Cauchy in the strong topology, then, letting $x = a$ and a^*, we conclude that $(\pi_\alpha(a))$ is a Cauchy net in the strong* topology.

Conversely, assume that a net (b_α) in Y is Cauchy in the strong* topology. Since $\hat{a} : X \to Y$ is bijective, for each $b_\alpha \in Y$, there exists $\pi_\alpha \in X$ such that $b_\alpha = \pi_\alpha(a)$. Then, $(\pi_\alpha(p(a, a^*)))$ is a Cauchy net in the strong* topology for any polynomial $p(a, a^*)$ of a and a^*, and, since such polynomials are dense in $C^*(a)$, we conclude that $(\pi_\alpha(x))$ is a Cauchy net in the strong* topology for all $x \in C^*(a)$, i.e., (π_α) is Cauchy in X. Thus, $\hat{a}$ is uniformly homeomorphic. □

It follows from the above that the map $\hat{a} : \mathrm{Irr}(C^*(a) : H) \cup \{0\} \to \sigma_H(a)$ is uniformly homeomorphic with the strong* topology in $\sigma_H(a)$.

Definition 4.3. Let $X \subset \mathrm{Rep}(A : H)$ be invariant under the operation of unitary equivalence, and let $Y := \hat{a}(X)$. A function $f : Y \to B(H)$ is defined to be *equivariant* if, for any $b = \pi(a) \in Y$ with $\pi \in X$, f satisfies

$$f(u^*bu) = u^*f(b)u \quad \text{for} \;\; b = \pi(a) \in Y \quad \text{and for all partial isometry} \;\; u \;\; \text{with} \;\; uu^* \geq p_\pi.$$

We denote by $C(Y)$ [resp. $C_u(Y)$] the set of all bounded $B(H)$-valued continuous [resp. uniformly continuous] equivariant functions on Y with the strong* topology in Y and c-topology in $B(H)$.

We can now prove our CP-spectral theorem in the algebraic form as follows.

Theorem 4.4. *Let A be a C*-algebra, and let $C^*(a)$ be the C*-algebra generated by $a \in A$, and let H be an infinite dimensional Hilbert space. Then*

$$C^*(a) \;\cong\; C_u(\sigma_H(a)) \qquad (\text{*-isomorphism}),$$

where a corresponds to the identity map on $\sigma_H(a)$.

Proof. For each $x \in C^*(a)$, we define a function $f_x : \sigma_H(a) \to B(H)$ by

$$f_x(\pi(a)) := \pi(x) \qquad \text{for} \;\; \pi(a) \in \sigma_H(a) \quad \text{with} \quad \pi \in \mathrm{Irr}(C^*(a) : H) \cup \{0\}.$$

It is immediate to see that $f_x \in C_u(\sigma_H(a))$. Indeed, f_x is equivariant, since

$$f_x(u^*\pi(a)u) = (u^*\pi u)(x) = u^*\pi(x)u = u^* f_x(\pi(a))u,$$

where u is any partial isometry such that $uu^* \geq p_\pi$; and f_x is s^*-c uniformly continuous, because, if a net $(b_\alpha) = (\pi_\alpha(a))$ in $\sigma_H(a)$ is Cauchy in the strong* topology, then the net $(f_x(b_\alpha)) = (\pi_\alpha(x))$ is Cauchy in any c-topology for all $x \in C^*(a)$ since $C^*(a)$ is generated by a, as we have seen in the proof of Proposition 4.2. Hence, we can define a map

$$\mathcal{F} : x \in C^*(a) \to f_x \in C_u(\sigma_H(a)),$$

which maps a to the identity map $f_a : \pi(a) \mapsto \pi(a)$ on $\sigma_H(a)$.

We shall next show that $\mathcal{F}$ is bijective. We first note that, for each $f \in C_u(\sigma_H(a))$, the function

$$f \circ \hat{a} : \pi \in \mathrm{Irr}(C^*(a) : H) \cup \{0\} \mapsto f(\pi(a)) \in B(H)$$

is equivariant and uniformly c-continuous, hence $f \circ \hat{a} \in A_u^E(\mathrm{Irr}(C^*(a) : H) \cup \{0\}, B(H))$. We then define $\mathcal{G} : C_u(\sigma_H(a)) \to C^*(a)$ by

$$\mathcal{G}(f) := i^{-1} \circ f \circ \hat{a} \quad \text{for} \quad f \in C_u(\sigma_H(a)),$$

where i is the CP-Gelfand representation of Theorem 2.3, i.e., $i : x \in C^*(a) \to \hat{x} \in A_u^E(\mathrm{Irr}(C^*(a) : H) \cup \{0\}, B(H))$.

It is now straightforward to to see that $\mathcal{G} \circ \mathcal{F} = id$ and $\mathcal{F} \circ \mathcal{G} = id$. In fact, observe that, for every $x \in C^*(a)$,

$$(\mathcal{G} \circ \mathcal{F})(x) = i^{-1} \circ f_x \circ \hat{a} = i^{-1} \circ \hat{x} = x,$$

and for any $f \in C_u(\sigma_H(a))$ and $\pi \in \mathrm{Irr}(C^*(a) : H) \cup \{0\}$,

$$(\mathcal{F} \circ \mathcal{G})(f)(\pi(a)) = f_{i^{-1} \circ f \circ \hat{a}}(\pi(a)) = \pi(i^{-1} \circ f \circ \hat{a}) = (f \circ \hat{a})(\pi) = f(\pi(a)).$$

Hence, we have shown that $\mathcal{F} : C^*(a) \to C_u(\sigma_H(a))$ is bijective.

Moreover, $\mathcal{F}$ is an isometry, since, for any $x \in C^*(a)$,

$$\begin{aligned} \|f_x\| &= \sup\{\|f_x(\pi(a))\|;\, \pi(a) \in \sigma_H(a),\, \pi \in \mathrm{Irr}(C^*(a) : H) \cup \{0\}\} \\ &= \sup\{\|\pi(x)\|;\, \pi \in \mathrm{Irr}(C^*(a) : H) \cup \{0\}\} = \|x\|. \end{aligned}$$

It is obvious from the definition of $\mathcal{F}$ that $\mathcal{F}$ is indeed a *-isomorphism. This completes the proof. □

Remark. If $C^*(a)$ is commutative (e.g., a is normal), then the above CP-spectral theorem reduces to the classical spectral theorem $C^*(a) \cong C_0(\sigma(a))$

by the similar argument as in Remark to Theorem 2.3. If $C^*(a)$ is unital (i.e., a is invertible), we could define the operator spectrum of a by $\sigma_H(a) := \hat{a}(\mathrm{Irr}(C^*(a) : H))$, then we can show as above that $C^*(a) \cong C_u(\sigma_H(a))$ using Theorem 2.3 (i). On the other hand, if we define the *extended operator spectrum* $\rho_H(a)$ of $a \in A$ by

$$\rho_H(a) := \hat{a}(\overline{\mathrm{Irr}(A : H)}^c) = \{\pi(a) : \pi \in \overline{\mathrm{Irr}(A : H)}^c\},$$

then, from Theorem 2.2 and Proposition 4.2, we can show the following generalized spectral theorem:

$$C^*(a) \;\cong\; C(\rho_H(a)) \qquad (\text{*-isomorphism}),$$

the proof of which is parallel to that of Theorem 4.4.

We shall next generalize the spectral decomposition theorem for non-normal operators with respect to operator spectrum, i.e., the CP-spectral theorem in the analytic form as follows.

Theorem 4.5. *Let $a \in B(H)$ be an operator on an infinite dimensional separable Hilbert space H. Then, there exists a CP-measure μ on $\sigma_H(a)$ such that*

$$a = \int_{\sigma \in \sigma_H(a)} \sigma \, d\mu(\sigma).$$

Moreover, the correspondence $\mathcal{F} : x \in C^*(a) \leftrightarrow f_x \in C_u(\sigma_H(a))$ is given by

$$x = \int_{\sigma \in \sigma_H(a)} f_x(\sigma) \, d\mu(\sigma).$$

In particular, for polynomials $f \in P(\sigma_H(a))$, i.e., $f(\sigma) = \sum_{i,j} c_{ij}\sigma^i(\sigma^*)^j$ for $c_{ij} \in \mathbb{C}$ and $\sigma \in \sigma_H(a)$, we have

$$f(a) = \int_{\sigma \in \sigma_H(a)} f(\sigma) \, d\mu(\sigma).$$

Proof. Let π_0 be the identity representation of $C^*(a)$ on H, then by the CP-Choquet theorem [**24**, Th. 4.2], there exists a $Q_H(B(H))_n$ valued CP-measure λ, supported by $\mathrm{Irr}(C^*(a) : H)$, which represents π_0. Hence, for any $x \in C^*(a)$,

$$\begin{aligned} x = \pi_0(x) &= \int_{\pi \in \mathrm{Irr}(C^*(a):H)} \pi(x) \, d\lambda(\pi) = \int_{\pi \in \mathrm{Irr}(C^*(a):H)} f_x(\pi(a)) \, d\lambda(\pi) \\ &= \int_{\sigma \in \sigma_H(a)} f_x(\sigma) \, d\mu(\sigma), \end{aligned}$$

where recall that $f_x(\pi(a)) = \pi(x)$ for $\pi \in \mathrm{Irr}(C^*(a) : H)$, and we set $\sigma := \pi(a) \in \sigma_H(a)$, and $\mu := \lambda \circ \hat{a}^{-1}$ is the induced CP-measure from λ on $\sigma_H(a)$ by the homeomorphism $\hat{a} : \mathrm{Irr}(C^*(a) : H) \cup \{0\} \to \sigma_H(a)$ (cf. Proposition 4.2). In particular, setting $x = a$, we have the spectral decomposition of a. This completes the proof. □

Let A and B be C*-algebras, then we say that $a \in A$ and $b \in B$ are *algebraically equivalent* if there is a *-isomorphism $\Phi : C^*(a) \to C^*(b)$ such that $\Phi(a) = b$. We recall that normal operators a and b are algebraically equivalent if and only if they have the same spectrum, i.e., $\sigma(a) = \sigma(b)$. We shall show that our operator spectrum enables us to generalize this fact for non-normal operators.

Theorem 4.6. *Let A and B be C*-algebras, and $a \in A$ and $b \in B$, and let H be an infinite dimensional Hilbert space. Then, a and b are algebraically equivalent if and only if a and b have the same operator spectrum, i.e., $\sigma_H(a) = \sigma_H(b)$.*

Proof. Assume that $\sigma_H(a) = \sigma_H(b)$, then by the CP-spectral theorem, $C^*(a) \cong C_u(\sigma_H(a)) = C_u(\sigma_H(b)) \cong C^*(b)$. In these isomorphisms, a and b correspond to the identity map on $\sigma_H(a) = \sigma_H(b)$, so that a and b are algebraically equivalent.

Conversely, assume that a and b are algebraically equivalent, i.e., there exists a *-isomorphism $\Phi : C^*(a) \to C^*(b)$ such that $\Phi(a) = b$. Then there exists a conjugate isomorphism $\Phi^\natural : \mathrm{Irr}(C^*(b) : H) \cup \{0\} \to \mathrm{Irr}(C^*(a) : H) \cup \{0\}$ defined by

$$\Phi^\natural(\pi) := \pi \circ \Phi \quad \text{for} \quad \pi \in \mathrm{Irr}(C^*(b) : H) \cup \{0\}.$$

Hence, we can deduce that

$$\begin{aligned}
\sigma_H(b) &= \hat{b}\,(\mathrm{Irr}(C^*(b) : H) \cup \{0\}) = \{\pi(b); \pi \in \mathrm{Irr}(C^*(b) : H) \cup \{0\}\} \\
&= \{\pi \circ \Phi(a); \pi \in \mathrm{Irr}(C^*(b) : H) \cup \{0\}\} \\
&= \{\rho(a) = \Phi^\natural(\pi)(a); \rho := \Phi^\natural(\pi) \in \mathrm{Irr}(C^*(a) : H) \cup \{0\}\} \\
&= \hat{a}\,(\mathrm{Irr}(C^*(a) : H) \cup \{0\}) = \sigma_H(a).
\end{aligned}$$

□

As an application of operator spectrum, we shall give a classification theorem for irreducible operators, which could be compared with Arveson's result [8, Th. 2.4.3] for first order irreducible operators formulated by matrix range. We recall that an operator $a \in B(H)$ on a Hilbert space H is said to be irreducible if $C^*(a)$ acts irreducibly on H.

Theorem 4.7. *Let a and b be irreducible operators on Hilbert spaces H_a and H_b respectively such that neither $C^*(a)$ nor $C^*(b)$ is NGCR. Then a and b are unitarily equivalent if and only if they have the same operator spectrum.*

Proof. The only if part is trivial from Theorem 4.6. We fix an infinite dimensional Hilbert space H, and suppose that $\sigma_H(a) = \sigma_H(b)$. Then, by Theorem 4.6, a and b are algebraically equivalent, i.e., there exists a *-isomorphism $\Phi : C^*(a) \to C^*(b)$ such that $\Phi(a) = b$. Noting that $C^*(a)$ is not NGCR, i.e., $C^*(a) \cap C(H_a) \neq \{0\}$, where $C(H_a)$ denotes the set of all compact operators on H_a, and that $C^*(a)$ is a C*-algebra acting irreducibly on H_a, we must have $C^*(a) \supset C(H_a)$ (cf. [**9**, Cor. 2 to Th. 1.4.2] or [**34**, Lemma 6.1.4]). Note also that $C^*(b)$ is irreducible, so that the *-isomorphism $\Phi : C^*(a) \to C^*(b) \subset B(H_b)$ is an irreducible representation of $C^*(a)$ on H_b, hence by the same argument as in the proof of [**8**, Th. 2.4.3], or more directly by [**34**, Lemma 6.1.4], we conclude that Φ is unitarily equivalent to the identity representation. Thus, Φ is implemented by an unitary operator u from H_a onto H_b, i.e., $b = \Phi(a) = uau^*$. This completes the proof. □

Remark. W.B. Arveson [**8**] introduced the *matrix range* for an operator a, generalizing numerical rage, by

$$W_n(a) := \{\psi(a); \psi \in CP(C^*(a), M_n), \psi(I) = I_n\} \quad (n = 1, 2, \cdots),$$

where M_n denotes the C*-algebra of all complex $n \times n$ matrices, and he showed that they determine the operator a up to unitary equivalence, if a is a first order irreducible operator (i.e., the span $\{I + a\}$ has sufficiently many boundary representations; see [**7**, **8**] for details) and $C^*(a)$ is not NGCR. Our result shows that, if we use "operator spectrum", we can omit the condition on boundary representations, since it can automatically assure the algebraic equivalence as shown in Theorem 4.6, so that the theorem can be stated for the more general setting.

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