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A CARLESON ESTIMATE FOR THE COMPLEX MONGE–AMPERE OPERATOR

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Let D be a pseudoconvex domain in $\mathbb{C}^n$ that admits a plurisubharmonic defining function ρ of class C^2. We prove that if $u_1, \dots, u_r$ are bounded plurisubharmonic functions in D and $\omega = dd^c \log 1/(-\rho)$, then $(-\rho)^n dd^c u_1 \wedge \dots \wedge dd^c u_r \wedge \omega^{n-r}/(n-r)!$ is a Carleson measure. This is a global variant of the Chern-Levine-Nirenberg inequality.

1. Introduction.

Let $u_n, \dots, u_n$ be bounded plurisubharmonic functions in some domain Ω in $\mathbb{C}^n$. It was proved in [2] and [3] that the Monge-Ampere expression $dd^c u_1 \wedge \dots \wedge dd^c u_n$ defines a positive measure (defined via an appropriate regularization), and the Chern-Levine-Nirenberg inequality, see [6], states that for each $K \subset\subset \Omega$ there is a $C = C(K, \Omega)$ such that

$$\int_K dd^c u_1 \wedge \dots \wedge dd^c u_n \leq C \sup_\Omega |u_1| \dots \sup_\Omega |u_n|.$$

It follows from the proof that one can replace $\sup_\Omega |u_j|$ by $\int_\Omega |u_j|$ in this estimate for one of the functions u_i.

The purpose of this note is to prove a global version of this inequality. Let D be a pseudoconvex domain in $\mathbb{C}^n$ that admits a plurisubharmonic defining function ρ of class C^2. Let $d^c = (i/2)(\bar\partial - \partial)$, $\omega = dd^c \log(1/-\rho)$, $\beta = dd^c \rho$, $\omega_k = \omega^k/k!$ and $\beta_k = \beta^k/k!$. Moreover, let H^1 be the space of holomorphic functions in Ω such that

$$\|f\|_{H^1} = \sup_{\epsilon_0 > \epsilon > 0} \int_{\rho = -\epsilon} |f| d\sigma$$

is finite, where $d\sigma$ is the surface measure $d\sigma = d^c\rho \wedge \beta_{n-1}$. We say that a positive measure τ in D is a Carleson measure if the estimate

$$\int_D |\phi| d\tau \leq C \|\phi\|_{H^1}, \quad \phi \in H^1, \tag{1.1}$$

holds.

Theorem 1.1. *Let $u_1, \dots, u_r$ be positive plurisubharmonic functions in D (and $r \geq 1$). Then*

$$d\tau = (-\rho)^n dd^c u_1 \wedge \ldots \wedge dd^c u_r \wedge \omega_{n-r} \tag{1.2}$$

is a Carleson measure with norm

$$\leq c_{r,n} e \sup u_1 \ldots \sup u_r,$$

where $c_{n,r} = (r-1)! n!/(n-r)!$.

A few comments are in order.

If u_j are bounded and plurisubharmonic it follows that $d\tau$ is a Carleson measure with norm $\leq c_{n,r} e 2^r \|u_1\|_{L^\infty} \ldots \|u_k\|_{L^\infty}$.

Clearly Theorem 1.1 implies the Chern-Levine-Nirenberg inequality. The variant with L^1 norm of one of the u_j follows from Proposition 2.1 below and its proof.

Theorem 1.1 is sharp. In fact, restricted to one variable it says that $-\rho\Delta u$ is a Carleson measure in D if u is a bounded subharmonic function. This fact was noted in [4] and it is also shown in [4] that any Carleson measure in the unit disk that satisfies a small regularity assumption is given by such an expression, simply by taking ϕ as the Green potential of $\mu/(1-|z|)$. If $k = 1$ and D is the ball, then ω is the Bergman metric form and Theorem 1.1 then says that $(1-|z|^2)^{-1}\Delta u$ is a Carleson measure, where Δ is the Bergman Laplacian (the invariant Laplacian) $\Delta = (1-|z|^2)(\sum \partial^2/\partial z_j \partial \bar{z}_j - \sum z_j \bar{z}_k \partial/\partial z_j \partial \bar{z}_k)$. This was also stated in [4]. Even in this case the converse holds for reasonable μ, see [5]. (A similar statement holds for the Euclidean Laplacian Δ_E; that $(1-|z|^2)\Delta_E u$ is a Carleson measure. Just let $u_2, \dots, u_n$ be $|z|^2$ or ρ or use (2.2) and then proceed as in the proof of Theorem 1.1. However, by the same argument as in [4] it follows that $(1-|z|^2)\Delta_E u$ even is a $\mathbb{R}^{2n}$-Carleson measure.)

In a forthcoming paper, [1], Theorem 1.1 will be used to obtain new estimates for the so-called H^p-corona theorem in strictly pseudoconvex domains in $\mathbb{C}^n$.

2. Proof.

The proof of Theorem 1.1 is based on the following proposition.

Proposition 2.1. *Let $u_1, \dots, u_r$ be positive plurisubharmonic functions in $C^\infty(\overline{D})$. Then*

(2.1)
$$\int_D (-\rho)^n dd^c u_1 \wedge \dots \wedge dd^c u_r \wedge \omega_{n-r} \le c_{n,r} \sup u_1 \dots \sup u_{r-1} \int_{\partial D} u_r d\sigma,$$

where $c_{n,r} = (r-1)!n!/(n-r)!$.

This proposition can be considered as a global variant of the Chern-Levine-Nirenberg estimate, [**6**]. From Proposition 2.1 we can easily derive our main theorem.

Proof of Theorem 1.1. To begin with we assume that all the functions u_j as well as ϕ are smooth up to the boundary. Notice that

$$e^v dd^c v \le dd^c e^v$$

for any real function v. Therefore we get for any $t > 0$,

$$t dd^c u |\phi| \le e^{tu + \log|\phi|} dd^c tu \le e^{tu+\log|\phi|} dd^c(tu + \log|\phi|) \le dd^c e^{tu+\log|\phi|}.$$

If we just replace u_r in Proposition 2.1 by $\exp(u_r + \log|\phi|)$ we get that

$$\int_D |\phi| d\tau \le c_{n,r} \sup u_1 \dots \sup u_{r-1} \frac{1}{t} \sup e^{tu_r} \int_{\partial D} |\phi| d\sigma.$$

Taking $t = 1/\sup u_r$ we get the desired estimate for smooth functions.

The general case is then obtained by applying this result to smaller domains and a regularization procedure. Let $u_1, \dots, u_r$ be arbitrary positive plurisubharmonic functions in D and let $D_\epsilon = \{\rho < -\epsilon\}$. By a standard regularization there are smooth plurisubharmonic u_j^δ that decrease to u_j in a neighborhood of $\overline{D_\epsilon}$ when $\delta \to 0$. According to Theorem 2.1 in [**3**],

$$\tau_\delta = dd^c u_1^\delta \wedge \dots dd^c u_r^\delta \wedge \omega_{n-r} \to dd^c u_1 \wedge \dots dd^c u_r \wedge \omega_{n-r} = \tau$$

weakly as measures in this neighborhood. From the smooth case (applied to D_ϵ) we now have

$$\int_{D_\epsilon} (-\rho - \epsilon)^n \tau_\delta |\phi| \le c_{n,r} \int_{\partial D_\epsilon} |\psi| d\sigma \le c_{n,r} \|\phi\|_{H^1}.$$

Hence,

$$\int_{D_\epsilon} (-\rho-\epsilon)^n \tau |\psi| \le \liminf_\delta \int_{D_\epsilon} (-\rho - \epsilon)^n \tau_\delta |\psi| \le c_{n,r} \|\psi\|_{H^1},$$

and by monotone convergence it finally follows that

$$\int_D (-\rho)^n \tau |\psi| \leq c_{n,r} \|\psi\|_{H^1}.$$

□

Proof of Proposition 2.1. As expected the proof consists in a number of integrations by parts. The crucial point is to keep check of all signs so that negative terms can be discarded.

To begin with we claim that if a is a closed (smooth) (r,r)-form, then

$$\int_D (-\rho)^n \omega^{n-r} \wedge a = \frac{n}{r} \int_D (-\rho)^r \beta^{n-r} \wedge a. \tag{2.2}$$

In fact,

$$\int_D (-\rho)^n \omega^{n-r} \wedge a = \int_D (-\rho)^r \wedge \beta^{n-r} \wedge a - \frac{(n-r)}{r} \int_D d(-\rho)^r \wedge d^c\rho \wedge \beta^{n-r-1} \wedge a$$

and an integration by parts in the last integral then yields (2.2).

Lemma 2.2. *If a is a positive closed $(r-1,r-1)$-form, u is positive and plurisubharmonic and $\alpha > 2$, then*

$$\int_D (-\rho)^\alpha dd^c u \wedge a \leq \sup u \alpha \int_D (-\rho)^{\alpha-1} a \wedge \beta. \tag{2.3}$$

Proof of Lemma 2.2. First notice that

$$\begin{aligned} dd^c(-\rho)^\alpha &= \alpha(\alpha-1)(-\rho)^{\alpha-2} d\rho \wedge d^c\rho - \alpha(-\rho)^{\alpha-1} dd^c\rho \\ &\leq \alpha(\alpha-1)(-\rho)^{\alpha-2} d\rho \wedge d^c\rho. \end{aligned} \tag{2.4}$$

Hence, since $d\rho \wedge d^c\rho$ is a positive form,

$$\begin{aligned} \int_D (-\rho)^\alpha dd^c u \wedge a &= \int_D u dd^c(-\rho)^\alpha \wedge a \\ &\leq \sup u \int_D \alpha(\alpha-1)(-\rho)^{\alpha-2} d\rho \wedge d^c\rho \wedge a. \end{aligned} \tag{2.5}$$

However, since a is closed we can replace $\alpha(\alpha-1)(-\rho)^{\alpha-2} d\rho \wedge d^c\rho$ by $\alpha(-\rho)^{\alpha-1} dd^c\rho$ on the right hand side of (2.5), since, in view of the equality in (2.4), the difference is a closed form that vanishes on the boundary. This proves the lemma. □

It is now easy to conclude the proof of Proposition 2.1. By repeated use of the lemma we get, for $\alpha > r$, that

$$\int_D (-\rho)^\alpha dd^c u_1 \wedge \ldots dd^c u_r \wedge \beta^{n-r}$$
$$\leq \alpha \ldots (\alpha - r + 2) \sup u_1 \ldots \sup u_{r-1} \int_D (-\rho)^{\alpha - r + 1} dd^c u_r \wedge \beta^{n-1}.$$

Moving dd^c from u_r to $(-\rho)^{\alpha-r+1}$ in the last integral and applying the inequality in (2.4) we get

$$\int_D (-\rho)^\alpha dd^c u_1 \wedge \ldots dd^c u_r \wedge \beta^{n-r} \leq \alpha \ldots (\alpha - r) \int_D (-\rho)^{\alpha-r-1} u_r d\rho \wedge d^c\rho \wedge \beta^{n-1}.$$

When $\alpha \to r$ this becomes

$$\int_D (-\rho)^r dd^c u_1 \wedge \ldots dd^c u_r \wedge \beta^{n-r} \leq r!(n-1)! \int_{\partial D} u_r d\sigma.$$

Combining with (2.2) we finally get (2.1). □

References

[1] M. Andersson and H. Carlsson, *Estimates of solutions of the H^p and* BMOA *corona problem*, preprint, Göteborg, 1998.

[2] E. Bedford and B.A. Taylor, *The Dirichlet problem for a complex Monge-Ampere equation*, Invent. Math., **37** (1976), 1-37.

[3] ———, *A new capacity for plurisubharmonic functions*, Acta Math., **149** (1982), 1-40.

[4] B. Berndtsson, *$\bar\partial_b$ and Carleson type inequalities*, Complex Analysis II, Univ. of Maryland, 1985-86, Springer Lecture Notes, **1276** (1987), 42-54.

[5] M. Borke, *Carleson measures in the unit ball of $\mathbb{C}^n$*, preprint, Göteborg, 1991.

[6] S.S. Chern, H.I. Levine and L. Nirenberg, *Intrinsic norms on a complex manifold*, Global Analysis (Papers in honor of K. Kodaira), Univ. of Tokyo Press, Tokyo, (1969), 119-139.

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