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EXPLICIT CAYLEY TRIPLES IN REAL FORMS OF G_2, F_4, AND E_6

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We re-examine the problem of classifying the nilpotent adjoint orbits in semisimple real Lie algebras. We present the list of representatives of these orbits, as well as those of related Cayley triples, in the case when the complexification of the algebra is one of the exceptional complex Lie algebras mentioned in the title.

1. Introduction.

The nilpotent adjoint orbits in non-compact real forms $\mathfrak{g}$ of exceptional complex Lie algebras $\mathfrak{g}^c$ have been classified in our papers [**6**, **7**]. That classification is indirect because it makes use of the Sekiguchi bijection (see the next section). Hence our classification is not as explicit as one would like it to be. For instance the list of representatives of the nilpotent adjoint orbits of $\mathfrak{g}$ is missing. The main objective of this paper is the fill this gap when $\mathfrak{g}^c$ is of the type G_2, F_4, or E_6.

Let $\mathfrak{g}$ be an arbitrary semisimple real Lie algebra (of finite dimension). Let $\mathfrak{g}^c$ be the complexification of $\mathfrak{g}$, and G (resp. G^c) the adjoint group of $\mathfrak{g}$ (resp. $\mathfrak{g}^c$). We fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of $\mathfrak{g}$, with Cartan involution θ (extended also to $\mathfrak{g}^c$). Let K^c be the connected Lie subgroup of G^c corresponding to $\mathfrak{k}^c$.

In Section 2 we describe the Sekiguchi bijection which gives the one-to-one correspondence between the nonzero nilpotent G-orbits in $\mathfrak{g}$ and the nonzero nilpotent K^c-orbits in $\mathfrak{p}^c$.

In Section 3 we sketch a new approach to the problem of classification of nilpotent adjoint orbits in $\mathfrak{g}$. By using a theorem from our note [**9**], we are able to select from the list of the nilpotent G^c-orbits $\mathcal{O} \subset \mathfrak{g}^c$ those orbits that possess real points, i.e., such that $\mathcal{O} \cap \mathfrak{g} \neq \emptyset$. Theorem 2 provides a method of identifying the G-orbits into which $\mathcal{O} \cap \mathfrak{g}$ splits.

In Section 4 we construct Chevalley systems for $\mathfrak{g}^c$ when the latter is of type G_2, F_4, or E_6. We also describe the action of the conjugation σ corresponding to the real form $\mathfrak{g} \subset \mathfrak{g}^c$. These data are necessary if one wants to list the representatives of the nilpotent adjoint orbits in $\mathfrak{g}$.

Finally, in Section 5 we list the representatives (E, H, F) of the G-orbits of the real Cayley triples (defined in Section 2) in $\mathfrak{g}$ for each of the noncompact real forms of G_2, F_4, and E_6. The elements E listed there are the representatives of the nonzero nilpotent G-orbits in $\mathfrak{g}$.

The following misprints have been detected in our paper [**6**]:

p. 511, Table VIII, orbit 1: Replace $\mathfrak{su}(3)$ with $\mathfrak{su}(4)$.
p. 514, Table XI, orbit 85: Replace 1 in column 6 with 7.
p. 516, Table XIII, orbit 16: Replace $\mathfrak{su}(7)$ with $\mathfrak{so}(7)$.
p. 520, Table XV, orbit 11: Replace 62 in column 4 with 63.

We also mention that in [**4**] (where some of the results of [**6**, **7**] are quoted) the Dynkin diagram of F_4, in the table of nilpotent orbits in $E_{6(-26)}$ on p. 152, should have the arrow pointing in the other direction.

2. Cayley transformation and Sekiguchi bijection.

Let $\mathfrak{g}$ be a real semisimple Lie algebra and $\mathfrak{g}^c$ its complexification. We fix a Cartan involution θ of $\mathfrak{g}$ and set $\mathfrak{k} = (1+\theta)\mathfrak{g}$ and $\mathfrak{p} = (1-\theta)\mathfrak{g}$. Hence $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition of $\mathfrak{g}$. We denote by G the adjoint group of $\mathfrak{g}$ and by K the connected Lie subgroup of G corresponding to $\mathfrak{k}$. Similarly, G^c will denote the adjoint group of $\mathfrak{g}^c$ and K^c its connected Lie subgroup corresponding to $\mathfrak{k}^c$.

The involution θ of $\mathfrak{g}$ extends uniquely to an involutive automorphism of $\mathfrak{g}^c$ which we also denote by θ. Let σ be the conjugation of $\mathfrak{g}^c$ defined by $\mathfrak{g}$. Thus we have $\sigma(X) = X$ and $\sigma(iX) = -iX$ for $X \in \mathfrak{g}$. Then $\sigma\theta = \theta\sigma$ and $\sigma_u := \theta\sigma$ is the conjugation of $\mathfrak{g}^c$ corresponding to the compact real form $\mathfrak{g}_u = \mathfrak{k} \oplus i\mathfrak{p}$ of $\mathfrak{g}^c$.

We say that (E, H, F) is a *standard triple* if $\{E, H, F\} \subset \mathfrak{g}^c$, $E \neq 0$, and they satisfy the relations

$$[H, E] = 2E,\ [H, F] = -2F,\ [F, E] = H.$$

By adopting the terminology of [**4**], we say that H is the *neutral*, E *nilpositive*, and F *nilnegative* element of this triple. (Our definition of standard triples is different from the one in [**4**] where the last relation above is replaced with $[E, F] = H$.)

We say that a standard triple (E, H, F) is a *real Cayley triple* if E, H, F belong to $\mathfrak{g}$ and $\theta(E) = F$. In that case we have $\theta(H) = -H$, i.e., $H \in \mathfrak{p}$.

A standard triple (E, H, F) is called *normal* if $H \in \mathfrak{k}^c$ and $E, F \in \mathfrak{p}^c$. We say that a normal triple (E, H, F) is a *complex Cayley triple* if $\sigma(E) = -F$.

Let (E, H, F) be a real Cayley triple and write $E = U + V$ with $U \in \mathfrak{k}$

and $V \in \mathfrak{p}$. Since $F = \theta(E)$, we have $F = U - V$. It follows easily that

$$[H, U] = 2V, \ [H, V] = 2U, \ H = 2[U, V].$$

If we define the elements:

$$E' = \frac{1}{2}(H + iF - iE) = \frac{1}{2}H - iV,$$

$$H' = i(E + F) = 2iU,$$

$$F' = \frac{1}{2}(-H + iF - iE) = -\frac{1}{2}H - iV,$$

then (E', H', F') is a complex Cayley triple and we refer to the map

$$(E, H, F) \mapsto (E', H', F')$$

as the *Cayley transformation*. We also say that (E', H', F') is the *Cayley transform* of (E, H, F).

The inverse $(E', H', F') \mapsto (E, H, F)$ of the Cayley transformation is given by the formulae

$$U = -\frac{i}{2}H', \ H = E' - F', \ V = \frac{i}{2}(E' + F')$$

i.e.,

$$E = \frac{i}{2}(-H' + E' + F'), \quad H = E' - F', \quad F = -\frac{i}{2}(H' + E' + F').$$

For an element $X \in \mathfrak{g}$ we say that it is *nilpotent* if the linear operator $\mathrm{ad}\,(X)$ is nilpotent. The group G (resp. G^c) acts on $\mathfrak{g}$ (resp. $\mathfrak{g}^c$) via the adjoint representation.

Let $\mathcal{O}$ be a nonzero nilpotent G-orbit in $\mathfrak{g}$. Choose $E_0 \in \mathcal{O}$. By Jacobson-Morozov Theorem, there exist $H_0, F_0 \in \mathfrak{g}$ such that (E_0, H_0, F_0) is a standard triple. By [4, Theorem 9.4.1] there exists $g \in G$ such that $g \cdot (E_0, H_0, F_0) = (E, H, F)$ is a real Cayley triple. Let (E', H', F') be the Cayley transform of (E, H, F). Finally let $\mathcal{O}' := K^c \cdot E'$ be the nilpotent K^c-orbit in $\mathfrak{p}^c$ through E'. The map that assigns $\mathcal{O}'$ to $\mathcal{O}$ is well defined and it was shown by J. Sekiguchi [16] and the author [5] that it establishes a bijection between the nonzero nilpotent G-orbits in $\mathfrak{g}$ and the nonzero nilpotent K^c-orbits in $\mathfrak{p}^c$. Following [4], we shall refer to this map as the *Sekiguchi bijection*.

3. An approach to the classification of nilpotent G-orbits in $\mathfrak{g}$.

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$, and $\mathfrak{m}$ the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ such that $\mathfrak{h} \supset \mathfrak{a}$. Then

$$\mathfrak{h} = \mathfrak{a} \oplus (\mathfrak{h} \cap \mathfrak{m})$$

and $\mathfrak{h} \cap \mathfrak{m}$ is a Cartan subalgebra of $\mathfrak{m}$. We recall that $\mathfrak{m}$ is reductive in $\mathfrak{k}$. The complexification $\mathfrak{h}^c$ is a Cartan subalgebra of $\mathfrak{g}^c$.

Let Φ be the root system of $(\mathfrak{g}^c, \mathfrak{h}^c)$. Each root $\alpha \in \Phi$ is a complex linear functional on $\mathfrak{h}^c$ which takes real values on the real form

$$\mathfrak{h}_0 := \mathfrak{a} \oplus i(\mathfrak{h} \cap \mathfrak{m})$$

of $\mathfrak{h}^c$. Let $\rho(\alpha)$ denote the restriction of $\alpha \in \Phi$ to the subspace $\mathfrak{a}$. We set

$$\Phi_0 := \{\alpha \in \Phi \ : \ \rho(\alpha) = 0\}$$

and $\Phi_1 := \Phi \setminus \Phi_0$. The set

$$\Sigma := \{\rho(\alpha) : \alpha \in \Phi_1\}$$

is a root system (not necessarily reduced) in the dual space of $\mathfrak{a}$.

We choose a base Π of Φ containing a base Π_0 of Φ_0, and we set $\Pi_1 := \Pi \setminus \Pi_0$. Then the set

$$\Theta := \{\rho(\alpha) \ : \ \alpha \in \Pi_1\}$$

is a base of Σ.

Let $C_\Pi \subset \mathfrak{h}_0$ (resp. $C_\Theta \subset \mathfrak{a}$) be the closed fundamental Weyl chamber of Φ (resp. Σ) corresponding to the base Π (resp. Θ). Note that $C_\Theta = \mathfrak{a} \cap C_\Pi$.

Let $\mathcal{O} \subset \mathfrak{g}^c$ be a nonzero nilpotent G^c-orbit. Then we can choose a standard triple (E, H, F) such that $E \in \mathcal{O}$ and $H \in C_\Pi$. The neutral element H of this triple is uniquely determined by $\mathcal{O}$ and is called the *characteristic* of $\mathcal{O}$. One usually identifies H by means of the labels $\alpha(H)$, $\alpha \in \Pi$. Hence the Dynkin diagram of (Φ, Π) together with the labels $\alpha(H)$, $\alpha \in \Pi$, determines uniquely the orbit $\mathcal{O}$. All the labels $\alpha(H)$, $\alpha \in \Pi$, belong to $\{0, 1, 2\}$.

In the case where $\mathfrak{g}^c$ is one of the five simple exceptional Lie algebras G_2, F_4, E_6, E_7, or E_8, one can find the list of nonzero nilpotent orbits, i.e, the corresponding labelled Dynkin diagrams in many places, eg. [**3**, **4**, **10**, **11**]. In the case when $\mathfrak{g}^c$ is a simple classical Lie algebra see [**4**].

Theorem 2 of [**9**], when specialized to the real field $\mathbf{R}$, gives the following result.

Theorem 1. *Let $\mathcal{O} \subset \mathfrak{g}^c$ be a nonzero nilpotent G^c-orbit and $H \in C_\Pi$ its characteristic. Then $\mathcal{O} \cap \mathfrak{g} \neq \emptyset$ if and only if $H \in C_\Theta$.*

The condition $H \in C_\Theta$ can be verified easily by using the Satake diagram of G. Recall that the Satake diagram is obtained from the Dynkin diagram of (Φ, Π) by colouring in black all vertices in Π_0 and by joining by a curved arrow two white vertices $\alpha, \beta \in \Pi_1$ whenever $\rho(\alpha) = \rho(\beta)$. The condition $H \in C_\Theta$ is satisfied if and only if all the weights $\alpha(H) = 0$ for $\alpha \in \Pi_0$ and $\alpha(H) = \beta(H)$ whenever $\alpha, \beta \in \Pi_1$ are joined by a curved arrow.

Let $\mathcal{O} \subset \mathfrak{g}^c$ be a nonzero nilpotent G^c-orbit, and let $H \in C_\Theta$ be its characteristic. Then H defines the $\mathbf{Z}$-gradation

$$\mathfrak{g} = \oplus_{k \in \mathbf{Z}} \mathfrak{g}(k)$$

with

$$\mathfrak{g}(k) := \{x \in \mathfrak{g} \ : \ [H, X] = kX\}.$$

Let $G(0)^c$ (resp. $G(0)$) be the connected Lie subgroup of G^c (resp. G) corresponding to the subalgebra $\mathfrak{g}(0)^c \subset \mathfrak{g}^c$ (resp. $\mathfrak{g}(0) \subset \mathfrak{g}$). The pair $(G(0)^c\ ,\ \mathfrak{g}(2)^c)$ is a prehomogeneous vector space, i.e., there exists a unique $G(0)^c$-orbit, say $\mathcal{O}(2)^c$, in $\mathfrak{g}(2)^c$ which is open (and dense) in $\mathfrak{g}(2)^c$. Moreover we have $\mathcal{O}(2)^c = \mathcal{O} \cap \mathfrak{g}(2)^c$. It is known [**12**, Lemma 5] that the centralizer $Z_{G^c}(H)$ is connected, and so it coincides with $G(0)^c$. Consequently, we have $Z_G(H) = G \cap G(0)^c$ and $G(0)$ is the identity component of $Z_G(H)$.

Let $\mathcal{O}(2) := \mathcal{O}(2)^c \cap \mathfrak{g}(2)$. This is a nonempty Zariski open subset of $\mathfrak{g}(2)$, and so it has only finitely many connected components in the Euclidean topology. Each of these components is a $G(0)$-orbit.

Theorem 2. *The map which assigns to a $Z_G(H)$-orbit $\mathcal{O}_1 \subset \mathcal{O}(2)$ the G-orbit containing $\mathcal{O}_1$ is a bijection from the set of $Z_G(H)$-orbits in $\mathcal{O}(2)$ to the set of G-orbits in $\mathcal{O} \cap \mathfrak{g}$.*

Proof. Let $E \in \mathcal{O} \cap \mathfrak{g}$. By Jacobson-Morozov theorem there exist $H', F \in \mathfrak{g}$ such that (E, H', F) is a standard triple. Since H' is a real semisimple element there exists $g \in G$ such that $g \cdot H' \in \mathfrak{a}$. Hence, without any loss of generality, we may assume that $H' \in \mathfrak{a}$. We can choose an element w in the Weyl group of Σ such that $w \cdot H' \in C_\Theta$. This Weyl group can be identified with $N_G(\mathfrak{a})/Z_G(\mathfrak{a})$. Hence we can replace H' with $w \cdot H'$, i.e., we may assume that $H' \in C_\Theta$ and consequently $H' = H$. Then $E \in \mathcal{O}(2)$, and so the map mentioned in the theorem is surjective.

Next let $\mathcal{O}_1, \mathcal{O}_2 \subset \mathcal{O}(2)$ be $Z_G(H)$-orbits and assume that $\mathcal{O}_1$ and $\mathcal{O}_2$ are contained in the same G-orbit. Let $E_1 \in \mathcal{O}_1$ and $E_2 \in \mathcal{O}_2$. By our assumption, $E_2 \in G \cdot E_1$. We can choose $F_1, F_2 \in \mathfrak{g}(-2)$ such that (E_1, H, F_1) and (E_2, H, F_2) are standard triples. By [2, Chapter 8, §11, Proposition 1], there exists $g \in G$ such that

$$g \cdot (E_1, H, F_1) = (E_2, H, F_2).$$

Hence $g \in Z_G(H)$ and $E_2 = g \cdot E_1$. Consequently $\mathcal{O}_1 = \mathcal{O}_2$, and so our map is also injective. □

This theorem reduces the classification problem for nilpotent G-orbits in $\mathfrak{g}$ to the problem of classifying the $Z_G(H)$-orbits in $\mathcal{O}(2)$ for each of the characteristics H satisfying $\sigma(H) = H$. In spite of its attractiveness, this method is hard to apply in practise. We used it in order to check the (known) classification in the case where $\mathfrak{g}$ is the split real form of G_2.

The general problem of classifying the nilpotent G-orbits in $\mathfrak{g}$, reduces easily to the case where $\mathfrak{g}$ is absolutely simple, i.e., $\mathfrak{g}^c$ is simple. If $\mathfrak{g}$ is of classical type, the classification is well known [17]. When $\mathfrak{g}$ is a noncompact real form of an exceptional algebra $\mathfrak{g}^c$, the classification was obtained in our papers [6, 7]. In fact we classified the nonzero nilpotent K^c-orbits in $\mathfrak{p}^c$. In view of Sekiguchi bijection, this provides indirectly also a classification of the nonzero nilpotent G-orbits in $\mathfrak{g}$.

Let us recall the following basic theorem of Kostant and Rallis [13].

Theorem 3. *Every nonzero nilpotent element $E \in \mathfrak{p}^c$ is the nilpositive element of a normal triple. If (E, H, F) and (E', H', F') are normal triples such that $E = E'$ or $H = H'$, then there exists $g \in K^c$ such that $g \cdot (E, H, F) = (E', H', F')$.*

Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{k}$, Δ the root system of $(\mathfrak{k}^c, \mathfrak{t}^c)$ and $C_\Delta \subset i\mathfrak{t}$ the closed fundamental Weyl chamber corresponding to some base Γ of Δ. Let $\mathcal{O} \subset \mathfrak{p}^c$ be a nilpotent K^c-orbit. The above theorem implies that there exists a normal triple (E, H, F) with $E \in \mathcal{O}$ and $H \in C_\Delta$. Furthermore the neutral element H of this triple is uniquely determined by $\mathcal{O}$ and we refer to it as the *characteristic* of $\mathcal{O}$.

From now on we assume that $\mathfrak{g}^c$ is an exceptional simple complex Lie algebra and that $\mathfrak{g}$ is noncompact.

Assume first that $\mathfrak{k}$ is semisimple. Then the Dynkin diagram of (Δ, Γ) together with the labels $\alpha(H)$, $\alpha \in \Gamma$, determines the characteristic H uniquely. (The labels $\alpha(H)$ are nonnegative integers.)

Next assume that $\mathfrak{k}$ is not semisimple. Then $\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{z}$ where $\mathfrak{k}_0$ is the derived subalgebra of $\mathfrak{k}$ and $\mathfrak{z}$ is the 1-dimensional center of $\mathfrak{k}$. In this case the labels $\alpha(H)$, $\alpha \in \Gamma$, determine only the component of H in $\mathfrak{t}^c \cap \mathfrak{k}_0^c$, and say nothing about the component of H in $\mathfrak{z}^c$. Since $\mathfrak{t}^c$ is also a Cartan subalgebra of $\mathfrak{g}^c$, the root system $\tilde{\Delta}$ of $(\mathfrak{g}^c, \mathfrak{t}^c)$ contains Δ as a closed root subsystem. Furthermore

$$\text{rank }(\tilde{\Delta}) = \text{ rank }(\Delta) + 1.$$

In this case we can choose a root $\beta \in \tilde{\Delta}$ such that $\tilde{\Gamma} = \Gamma \cup \{\beta\}$ is a base of $\tilde{\Delta}$ ([6]). This β is not unique: There are exactly two such choices. Once the choice of β is made then the component of H in $\mathfrak{z}^c$ is uniquely determined by the integer $\beta(H)$. Hence the labels $\alpha(H)$, $\alpha \in \tilde{\Gamma}$, determine H uniquely.

4. Structure constants and the action of σ.

In the next section we shall tabulate the representatives (E, H, F) of K-orbits of the real Cayley triples in $\mathfrak{g}$ when $\mathfrak{g}^c$ is of type G_2, F_4, or E_6. The nilpositive elements E of these triples are the representatives of the nonzero nilpotent G-orbits in $\mathfrak{g}$. In order to do this we have to choose a suitable basis of $\mathfrak{g}^c$ and describe the action of σ in terms of that basis. This section is devoted to that task.

Let n be the rank of $\mathfrak{g}^c$ and N the number of positive roots of (Φ, Π). We start by enumerating the simple roots :

$$\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$$

in the same way as in [1]. A positive root

$$\alpha = k_1\alpha_1 + k_2\alpha_2 + \cdots + k_n\alpha_n$$

will be also represented by the symbol $k_1k_2 \cdots k_n$. (Note that each k_i is a single digit.) The height of α is defined by

$$\text{ht}(\alpha) = k_1 + k_2 + \cdots + k_n.$$

We extend the enumeration of simple roots to obtain an enumeration

$$\alpha_1, \alpha_2, \dots, \alpha_n, \dots, \alpha_N$$

of all positive roots such that $\text{ht}(\alpha_i) \le \text{ht}(\alpha_j)$ for $i < j$. In particular α_N is the highest root of (Φ, Π). The negative root $-\alpha_i$ will also be written as α_{-i}, $1 \le i \le N$.

The extended Dynkin diagrams of G_2, F_4, and E_6 are given in Fig. 1.

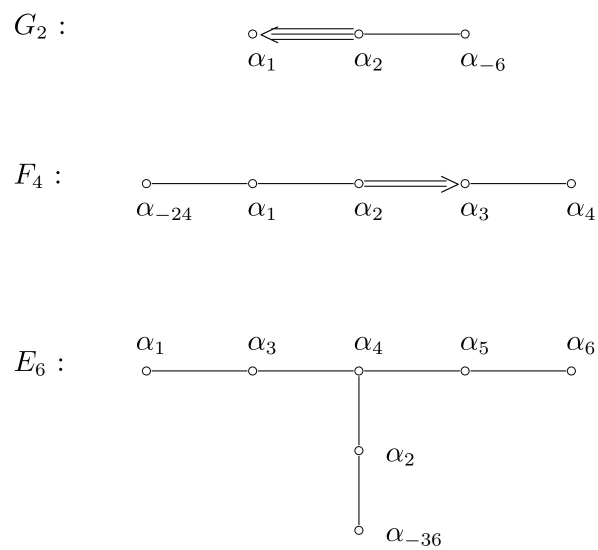

Figure 1.

The co-root H_{α_i} will be now denoted by H_i. Hence

$$\Pi^\vee := \{H_1, \dots , H_n\}$$

is a base of the dual root system $\Phi^\vee$ of Φ. A positive co-root H_i, $i > 0$, can be written as

$$H_i = k'_1 H_1 + \cdots + k'_n H_n$$

where k'_i are nonnegative integers. Thus we can represent H_i by the symbol $k'_1 k'_2 \cdots k'_n$.

If $\mathfrak{g}^c$ is simply laced, say of type E_6, then a positive root α_i and its co-root H_i will be represented by the same symbol. This is not the case if $\mathfrak{g}^c$ is triply or doubly laced (say of type G_2 or F_4).

In Tables 1-3 we list the positive roots α_i and co-roots H_i for G_2, F_4, and E_6, respectively.

Table 1.
Positive roots of G_2.

i	α_i	H_i	i	α_i	H_i	i	α_i	H_i
1	10	10	3	11	13	5	31	11
2	01	01	4	21	23	6	32	12

Table 2.
Positive roots of F_4.

i	α_i	H_i	i	α_i	H_i	i	α_i	H_i
1	1000	1000	9	0120	0110	17	1221	2421
2	0100	0100	10	0111	0211	18	1122	1111
3	0010	0010	11	1120	1110	19	1231	2431
4	0001	0001	12	1111	2211	20	1222	1211
5	1100	1100	13	0121	0221	21	1232	2432
6	0110	0210	14	1220	1210	22	1242	1221
7	0011	0011	15	1121	2221	23	1342	1321
8	1110	2210	16	0122	0111	24	2342	2321

Table 3.
Positive roots of E_6.

i	α_i, H_i	i	α_i, H_i	i	α_i, H_i	i	α_i, H_i
1	100000	10	000110	19	011110	28	011211
2	010000	11	000011	20	010111	29	112210
3	001000	12	101100	21	001111	30	111211
4	000100	13	011100	22	111110	31	011221
5	000010	14	010110	23	101111	32	112211
6	000001	15	001110	24	011210	33	111221
7	101000	16	000111	25	011111	34	112221
8	010100	17	111100	26	111210	35	112321
9	001100	18	101110	27	111111	36	122321

If $\sigma(\alpha_i) = \alpha_j$, we shall also write $\sigma(i) = j$. In this way σ defines an involutory permutation of the set

$$\{\pm 1, \pm 2, \dots, \pm N\}. \tag{4.1}$$

In this notation, we have $\sigma(H_i) = H_{\sigma(i)}$ for all i. We can choose nonzero elements $X_i \in \mathfrak{g}_i^c$, where $\mathfrak{g}_i^c := \mathfrak{g}_{\alpha_i}^c$ is the root space of α_i, such that (see [2]):

(i) $[X_{-i}, X_i] = H_i$ for all i;

(ii) the linear map $\omega : \mathfrak{g}^c \to \mathfrak{g}^c$ which is -1 on $\mathfrak{h}^c$ and satisfies $\omega(X_i) = X_{-i}$ for all i is an automorphism of $\mathfrak{g}^c$.

If $i + j = 0$, then the bracket $[X_i, X_j]$ is determined by (i). Assume now that $i+j \neq 0$. If $\alpha_i+\alpha_j$ is not a root then $[X_i, X_j] = 0$ and we set $N(i, j) = 0$. If $\alpha_i + \alpha_j = \alpha_k \in \Phi$, then we define $N(i, j)$ by

$$[X_i, X_j] = N(i, j)X_k.$$

The conditions (i) and (ii) imply that $N(i, j)$ is an integer. More precisely, let $p, q \geq 0$ be the largest integers such that

$$\alpha_j + p\alpha_i,\ \alpha_j - q\alpha_i \in \Phi.$$

Assuming that $\alpha_i + \alpha_j = \alpha_k \in \Phi$, i.e., $p \geq 1$, we have

$$N(i, j) = \varepsilon(i, j)(q + 1) \tag{4.2}$$

where $\varepsilon(i, j) = \pm 1$.

The X_i's together with the $H_1, \dots, H_n$ form a basis of $\mathfrak{g}^c$. All the brackets of these basic elements are uniquely determined except for the signs $\varepsilon(i, j)$.

The conditions (i) and (ii) do not determine the X_i's uniquely. Different choices of the X_i's may produce different signs $\varepsilon(i, j)$. As shown by J. Kurtzke [14], the X_i's can be chosen, not only to satisfy (i) and (ii), but also

(iii) $\varepsilon(i, j) = 1$ for $1 \leq i \leq n < j \leq N$;

(iv) $\varepsilon(i, j) = -\varepsilon(j, k)$ whenever $\alpha_i, \alpha_j, \alpha_k$ are consecutively linked simple roots in the Dynkin diagram.

For G_2 and F_4 we specify that $\varepsilon(1, 2) = 1$, and for E_6 we specify that $\varepsilon(1, 3) = 1$. All other $\varepsilon(i, j)$ are then uniquely determined by (iii) and (iv) (see [14]). In the Appendix we give the tables of the signs $\varepsilon(i, j)$, $1 \leq i, j \leq N$, for the complex simple Lie algebras G_2, F_4, and E_6.

If $Y \in \mathfrak{h}^c$ and we replace the X_i's with $\exp(\operatorname{ad} Y)(X_i)$, then (i-iv) remain valid. In fact the structure constants $N(i, j)$ do not change at all. One can choose such Y so that the new X_i's satisfy also the condition

(v) $\sigma(X_i) = \xi_i X_{\sigma(i)},\ \xi_i = \pm 1$, for all i.

For details, see [15]. Furthermore one knows that $\xi_i = 1$ for all $\alpha_i \in \Phi_0$, and $\xi_{-i} = \xi_i$ for all i. If $\alpha_i + \alpha_j = \alpha_k$, then

$$\xi_k N(i,j) = \xi_i \xi_j N(\sigma(i), \sigma(j)) \tag{4.3}$$

holds. It follows that all the ξ_i's can be computed provided that $\xi_1, \ldots, \xi_n$ are known.

For all non-compact real forms of G_2, F_4, and E_6 we may choose $\xi_i = 1$ for $1 \le i \le n$, with one exception: For the real form $E_{6(-14)}$ of E_6 we can set $\xi_i = 1$ for $1 \le i \le 5$, while $\xi_6 = -1$.

In order to pin down the action of σ on the X_i's, we still need to determine the corresponding permutation of the set (4.1). This permutation is determined by the Satake diagram of $\mathfrak{g}$ (see [15]). For readers convenience we shall describe this permutation below.

If $\mathfrak{g}$ is split, then σ is the identity permutation. Assume now that $\mathfrak{g}$ is not split. Since σ induces an automorphism of Φ, it suffices to specify $\sigma(i)$ for $1 \le i \le n$. If the vertex α_i in the Satake diagram of $\mathfrak{g}$ is black, then we have $\sigma(i) = -i$. For the white vertices of the Satake diagram, the action of σ is given as follows:

$F_{4(-20)}:$ $\sigma(4) = 19;$
$E_{6(2)}:$ $\sigma(1) = 6,\ \sigma(2) = 2,\ \sigma(3) = 5,\ \sigma(4) = 4,\ \sigma(5) = 3,\ \sigma(6) = 1;$
$E_{6(-14)}:$ $\sigma(1) = 21,\ \sigma(2) = 24,\ \sigma(6) = 18;$
$E_{6(-26)}:$ $\sigma(1) = 29,\ \sigma(6) = 31.$

5. Explicit Cayley triples.

In Tables 4-10 we list the representatives (E, H, F) of the G-orbits of real Cayley triples in $\mathfrak{g}$. Here $\mathfrak{g}$ is one of the non-compact real forms of G_2, F_4, or E_6. Thus $\mathfrak{g}$ is one of the algebras:

$$G_{2(2)},\ F_{4(4)},\ F_{4(-20)},\ E_{6(6)},\ E_{6(2)},\ E_{6(-14)},\ E_{6(-26)},$$

where the number in parentheses is the Cartan index of $\mathfrak{g}$, i.e, the difference $\dim \mathfrak{p} - \dim \mathfrak{k}$.

The neutral element H is always chosen to be the characteristic of the nonzero nilpotent orbit $G^c \cdot E$. By Theorem 1 we know that $H \in \mathfrak{a}$. We list both: The labels $\alpha_i(H)$, $i = 1, \ldots, n$, and the coefficients $k_1, \ldots, k_n$ in

$$H = k_1 H_1 + \cdots + k_n H_n. \tag{5.1}$$

We tabulate only the neutral elements H and the nilpositive elements E. The elements F can easily be computed because $\theta = \sigma_u \sigma$ and $\sigma(E) = E$.

So, we have

$$F = \theta(E) = \sigma_u \sigma(E) = \sigma_u(E). \tag{5.2}$$

We recall that $\sigma_u(X_i) = X_{-i}$ for all i.

In order to avoid possible confusion, we have used the same numbering of the nonzero nilpotent orbits as in our papers [**6**, **7**]. That numbering was also used in [**4**] where the trivial orbit is appropriately given the number 0.

Let us now sketch the method used to construct the Tables 4-10. By Theorem 2 we know the characteristics H of the required representatives of nonzero nilpotent G-orbits, or equivalently the neutral elements of the representatives of K-orbits of real Cayley triples. Given such H, we denote by I the set of indices i such that $[H, X_i] = 2X_i$. Then $i > 0$ for each $i \in I$ and the X_i's for $i \in I$ form a basis of the subspace $\mathfrak{g}(2)^c$ (see Section 3). For each subset $J \subset I$ let $\mathfrak{g}(2, J)^c$ be the (complex) subspace of $\mathfrak{g}(2)^c$ spanned by the X_i's with $i \in J$.

We search for all subsets $J \subset I$ satisfying the following three conditions:

(i) $\mathfrak{g}(2, J)^c$ is stable under σ;

(ii) if $i, j \in J$, then $\alpha_i - \alpha_j$ is not a root;

(iii) H belongs to the complex subspace spanned by all H_i with $i \in J$.

(More precisely, we only need to find the representatives of the orbits of the Weyl group of Φ_0 acting on the collection of all subsets $J \subset I$ satisfying the above three conditions.)

Given such J we set

$$E = \Sigma_{i \in J}\ c_i X_i$$

where c_i are nonzero scalars chosen so that $\sigma(E) = E$, i.e., $E \in \mathfrak{g}$. For each such E, there exists $F \in \mathfrak{g}(-2) = \mathfrak{g} \cap \mathfrak{g}(-2)^c$ such that $[F, E] = H$. If there are no subsets $J \subset I$ having properties (i-iii), then one has to construct a suitable $E \in \mathfrak{g}(2)$ by a more elaborate procedure. In most cases the required sets J exist.

After constructing several such Cayley triples (E, H, F), we are faced with the problem of identifying the orbits to which they belong. For that purpose we have to pass to their Cayley transforms (E', H', F'). In fact we only need the neutral elements $H' = i(E + F)$.

Let $\mathcal{O}^c = G^c \cdot E$. If $\mathcal{O}^c \cap \mathfrak{g}$ is a single G-orbit, then the identification problem mentioned above is trivial. Otherwise $\mathcal{O}^c \cap \mathfrak{g}$ is a union of two or three G-orbits. In most cases the invariant

$$\text{inv} := \dim Z_{\mathfrak{k}^c}(H')$$

distinguishes these G-orbits. This invariant is listed in the last column of our Tables 4-10. If this crude invariant fails to distinguish between the various G-orbits in $\mathcal{O}^c \cap \mathfrak{g}$, we use the spectrum of the linear operator ad (H') restricted to $\mathfrak{k}^c$.

When $\mathfrak{g}$ is of type $E_{6(2)}$ or $E_{6(-14)}$, there exist pairs of G-orbits, say $\mathcal{O}'$ and $\mathcal{O}''$ contained in the same nilpotent G-orbit $\mathcal{O}^c$ which cannot be distinguished even by this finer invariant. In these cases we have $\mathcal{O}'' = -\mathcal{O}'$ and there exists an outer automorphism of $\mathfrak{g}$ which interchanges $\mathcal{O}'$ and $\mathcal{O}''$ (see [**8**]). Such pairs $\{\mathcal{O}', \mathcal{O}''\}$ are recorded jointly on the same line in Tables 8 and 9, and the nilpositive element is written as $\pm E$. The two sign choices give two different orbits. This ambiguity is caused by the fact that, in the two cases mentioned above, the automorphism group of the Dynkin diagram of $(\mathfrak{k}^c, \mathfrak{t}^c)$ has order 2. In the other cases, this automorphism group is trivial.

The number ζ that occurs in Table 8 (orbits 12 and 13) is a primitive cube root of 1.

By using the Tables 1-3 and those in the Appendix, one can verify that our Tables 4-10 indeed give real Cayley triples. We give full details for the orbit 36 of Table 8, where $\mathfrak{g}$ is of the type $E_{6(2)}$.

In that case we have

$$E = 4X_2 + \sqrt{12}(X_1 + X_6) + \sqrt{7}(X_3 + X_5) + i\sqrt{15}(X_9 - X_{10}).$$

As $\alpha_3 + \alpha_4 = \alpha_9$, it follows from (4.3) that

$$\xi_9 \, \varepsilon(3,4) = \xi_3\xi_4 \, \varepsilon(\sigma(3), \sigma(4)) = \varepsilon(5,4).$$

By consulting Table 13, we find that $\varepsilon(3,4) = \varepsilon(5,4) = -1$, and so $\xi_9 = 1$. As

$$\sigma(X_1) = X_6, \; \sigma(X_2) = X_2, \; \sigma(X_3) = X_5,$$

and

$$\sigma(X_9) = \xi_9 X_{\sigma(9)} = X_{10},$$

we conclude that $\sigma(E) = E$, i.e., $E \in \mathfrak{g}$. Hence (5.2) gives

$$F = 4X_{-2} + \sqrt{12}(X_{-1} + X_{-6}) + \sqrt{7}(X_{-3} + X_{-5}) - i\sqrt{15}(X_{-9} - X_{-10}).$$

Clearly we have $F \in \mathfrak{g}$. It remains to verify that $[F, E] = H$, where

$$H = 12(H_1 + H_6) + 22(H_3 + H_5) + 16H_2 + 30H_4.$$

A direct computation gives

$$\begin{aligned}[F,E] &= 16[X_{-2},X_2]+12[X_{-1},X_1]+12[X_{-6},X_6]\\ &\quad +7[X_{-3},X_3]+7[X_{-5},X_5]+15[X_{-9},X_9]+15[X_{-10},X_{10}]\\ &\quad +i\sqrt{105}([X_{-3},X_9]-[X_{-5},X_{10}]-[X_{-9},X_3]+[X_{-10},X_5])\\ &= 16H_2+12(H_1+H_6)+7(H_3+H_5)+15(H_9+H_{10})\\ &\quad +i\sqrt{105}[(N(-3,9)-N(-5,10))X_4+(N(-10,5)-N(-9,3))X_{-4}].\end{aligned}$$

To compute these structure constants, we use the fact that $\varepsilon(i,j)=-\varepsilon(-i,k)$ if $\alpha_i+\alpha_j=\alpha_k$, see [**2**, Chapter VIII, §2, Lemma 4]. Hence $\varepsilon(-3,9)=-\varepsilon(3,4)=1$ and $\varepsilon(-5,10)=-\varepsilon(3,4)=1$ (see Tables 3 and 13). We now apply (4.2), using Table 3 in order to compute the integer q, and obtain that $N(-3,9)=N(-5,10)=1$. Since $N(\beta,\alpha)=-N(\alpha,\beta)$ and $N(-\alpha,-\beta)=N(\alpha,\beta)$, see [**2**, Chapter VIII, §2, Proposition 7], $N(-9,3)=N(-10,5)=-1$. Since $H_9=H_3+H_4$, $H_{10}=H_4+H_5$ (see Table 3), we deduce that indeed $[F,E]=H$.

In this example the set I is $\{1,2,3,5,6,8,9,10\}$ and there is no subset $J\subset I$ satisfying all three conditions (i-iii). We used the subset $J=\{1,2,3,5,6,9,10\}$ which satisfies the conditions (i) and (iii), but not (ii).

Table 4.

Cayley triples in $G_{2(2)}$.

	$\alpha_i(H)$	k_i	E	inv
1	01	1,2	X_6	2
2	10	2,3	X_4	2
3	02	2,4	X_2+X_4	2
4	02	2,4	X_2-X_4	4
5	22	6,10	X_1+X_2	2

Table 5.

Cayley triples in $F_{4(4)}$.

	$\alpha_i(H)$	k_i	E	inv
1	1000	2,3,2,1	X_{24}	10
2	0001	2,4,3,2	$X_{16} + X_{24}$	12
3	0001	2,4,3,2	X_{21}	10
4	0100	3,6,4,2	$X_{14} + X_{20} + X_{22}$	10
5	0100	3,6,4,2	$X_{14} + X_{21}$	6
6	2000	4,6,4,2	$X_1 + X_{14} + X_{20} + X_{22}$	22
7	2000	4,6,4,2	$X_8 + X_{18} + X_{23}$	14
8	2000	4,6,4,2	$\sqrt{2}(X_1 + X_{23})$	10
9	0002	4,8,6,4	$\sqrt{2}(X_4 + X_{19})$	10
10	0010	4,8,6,3	$X_{15} + \sqrt{2}(X_9 + X_{20})$	6
11	2001	6,10,7,4	$\sqrt{3}(X_1 + X_{14}) + 2X_{16}$	6
12	2001	6,10,7,4	$\sqrt{3}X_8 + 2X_{16}$	6
13	0101	5,10,7,4	$\sqrt{2}(X_{10} + X_{15}) + X_{14}$	4
14	1010	6,11,8,4	$\sqrt{3}X_8 + X_9 - 2X_{16}$	6
15	1010	6,11,8,4	$\sqrt{3}X_8 + X_9 + 2X_{16}$	4

Table 5.
(continued)

	$\alpha_i(H)$	k_i	E	inv
16	0200	6,12,8,4	$X_2+\sqrt{3}X_8+X_9-2X_{16}$	12
17	0200	6,12,8,4	$X_2+\sqrt{3}X_8+X_9+2X_{16}$	8
18	0200	6,12,8,4	$X_2+\sqrt{3}X_8-X_9+2X_{16}$	6
19	2200	10,18,12,6	$\sqrt{10}X_1+\sqrt{6}(X_2+X_9+X_{16})$	10
20	2200	10,18,12,6	$\sqrt{10}X_1+\sqrt{6}(X_2+X_{13})$	6
21	1012	10,19,14,8	$2\sqrt{2}X_4+\sqrt{5}X_8+3X_9$	4
22	0202	10,20,14,8	$X_2+2\sqrt{2}X_4+\sqrt{5}X_8+3X_9$	8
23	0202	10,20,14,8	$X_2+2\sqrt{2}X_4+\sqrt{5}X_8-3X_9$	4
24	2202	14,26,18,10	$\sqrt{14}X_1+2\sqrt{2}X_2+\sqrt{10}X_4-3\sqrt{2}X_9$	4
25	2202	14,26,18,10	$\sqrt{14}X_1+2\sqrt{2}X_2+\sqrt{10}X_4+3\sqrt{2}X_9$	6
26	2222	22,42,30,16	$\sqrt{22}X_1+\sqrt{42}X_2+\sqrt{30}X_3+4X_4$	4

Table 6.

Cayley triples in $F_{4(-20)}$.

	$\alpha_i(H)$	k_i	E	inv
1	0001	2,4,3,2	X_{21}	16
2	0002	4,8,6,4	$\sqrt{2}(X_4+X_{19})$	22

Table 7.

Cayley triples in $E_{6(6)}$.

	$\alpha_i(H)$	k_i	E	inv
1	010000	1,2,2,3,2,1	X_{36}	18
2	100001	2,2,3,4,3,2	$X_{23}+X_{36}$	16
3	000100	2,3,4,6,4,2	$X_{24}+X_{30}+X_{34}$	12
4	020000	2,4,4,6,4,2	$\sqrt{2}(X_2+X_{35})$	18
5	020000	2,4,4,6,4,2	$X_2+X_{24}+X_{30}-X_{34}$	24
6	200002	4,4,6,8,6,4	$\sqrt{2}(X_1+X_6+X_{29}+X_{31})$	16
7	120001	4,6,7,10,7,4	$\sqrt{3}(X_2+X_{24})+2X_{23}$	10
8	110001	3,4,5,7,5,3	$\sqrt{2}(X_{17}+X_{31})+X_{23}$	10
9	220002	6,8,10,14,10,6	$\sqrt{6}(X_1+X_{21})+2(X_2+X_{24})$	10
10	001010	3,4,6,8,6,3	$X_{15}+X_{23}+\sqrt{2}(X_{22}+X_{28})$	10
11	100101	4,5,7,10,7,4	$X_{24}+\sqrt{2}(X_{12}+X_{16}+X_{22}+X_{25})$	8
12	000200	4,6,8,12,8,4	$X_4+X_{15}+2X_{23}+\sqrt{3}(X_{13}+X_{14})$	12
13	020200	6,10,12,18,12,6	$\sqrt{10}X_2+\sqrt{6}(X_4+X_{15}+X_{23})$	12
14	211012	8,10,14,19,14,8	$2\sqrt{2}(X_1+X_6)+\sqrt{5}(X_{13}+X_{14})$ $+3X_{15}$	6

Table 7.
(continued)

	$\alpha_i(H)$	k_i	E	inv
15	011010	4,6,8,11,8,4	$X_{15}+2X_{23}+\sqrt{3}(X_{13}+X_{14})$	8
16	111011	6,8,11,15,11,6	$\sqrt{6}(X_7+X_{16})+X_{15}+2(X_{13}+X_{14})$	6
17	121011	7,10,13,18,13,7	$\frac{1}{\sqrt{7}}(2\sqrt{15}X_2+2\sqrt{6}X_7+\sqrt{10}X_8$ $-5X_{12}+\sqrt{42}X_{15}+7X_{16})$	6
18	222022	12,16,22,30,22,12	$\sqrt{12}(X_1+X_6)+\sqrt{22}X_5$ $+\frac{1}{\sqrt{11}}(2\sqrt{14}X_2+4\sqrt{2}X_3$ $+2\sqrt{30}X_8-\sqrt{210}X_9)$	6
19	200202	8,10,14,20,14,8	$X_4+3X_{15}+2\sqrt{2}(X_1+X_6)$ $+\sqrt{5}(X_{13}+X_{14})$	8
20	222222	16, 22, 30, 42, 30, 16	$4(X_1+X_6)+\sqrt{30}(X_3+X_5)$ $+\sqrt{22}X_2+\sqrt{42}X_4$	6
21	220202	10,14,18,26,18,10	$\sqrt{10}(X_1+X_6)+\sqrt{14}X_2$ $+\sqrt{2}(2X_4+3X_{15})$	8
22	200202	8,10,14,20,14,8	$X_4-3X_{15}+2\sqrt{2}(X_1+X_6)$ $+\sqrt{5}(X_{13}+X_{14})$	10
23	000200	4,6,8,12,8,4	$X_4-X_{15}+2X_{23}+\sqrt{3}(X_{13}+X_{14})$	14

Table 8.

Cayley triples in $E_{6(2)}$.

	$\alpha_i(H)$	k_i	E	inv
1	010000	1,2,2,3,2,1	X_{36}	18
2	100001	2,2,3,4,3,2	$X_{23}+X_{36}$	18
3	100001	2,2,3,4,3,2	$X_{32}+X_{33}$	14
4	000100	2,3,4,6,4,2	$X_{24}+X_{30}+X_{34}$	18
5	000100	2,3,4,6,4,2	$X_{24}+X_{30}-X_{34}$	10
6	020000	2,4,4,6,4,2	$-X_2+X_{24}+X_{30}+X_{34}$	36
7	020000	2,4,4,6,4,2	$X_2+X_{24}+X_{30}-X_{34}$	20
8	020000	2,4,4,6,4,2	$\sqrt{2}(X_2+X_{35})$	18
9,10	110001	3,4,5,7,5,3	$\pm[X_{22}+X_{23}+X_{25}+i(X_{26}-X_{28})]$	12
11	200002	4,4,6,8,6,4	$\sqrt{2}(X_1+X_6+X_{29}+X_{31})$	14
12,13	001010	3,4,6,8,6,3	$\pm\frac{1}{\sqrt{3}}[X_{15}+X_{19}-X_{23}+X_{24}-X_{27}$ $-X_{30}+\sqrt{2}(X_{18}+X_{21}+\zeta X_{22}$ $+\bar{\zeta}X_{25}+\bar{\zeta}X_{26}+\zeta X_{28})]$	16
14	001010	3,4,6,8,6,3	$\sqrt{2}(X_{15}+X_{30})+X_{22}+X_{25}$	8
15	120001	4,6,7,10,7,4	$\sqrt{3}(X_{13}+X_{14})-X_{15}+2X_{23}$	10
16	120001	4,6,7,10,7,4	$\sqrt{3}(X_{13}+X_{14})+X_{15}+2X_{23}$	10

Table 8.
(continued)

	$\alpha_i(H)$	k_i	E	inv
17	100101	4,5,7,10,7,4	$\sqrt{2}(X_{12}+X_{16}+X_{22}+X_{25})+X_{24}$	6
18	011010	4,6,8,11,8,4	$\sqrt{3}(X_{13}+X_{14})-X_{15}+2X_{23}$	10
19	011010	4,6,8,11,8,4	$\sqrt{3}(X_{13}+X_{14})+X_{15}+2X_{23}$	6
20	000200	4,6,8,12,8,4	$X_4+\sqrt{3}(X_{13}+X_{14})-X_{15}+2X_{23}$	20
21	000200	4,6,8,12,8,4	$-X_4+\sqrt{3}(X_{13}+X_{14})+X_{15}+2X_{23}$	12
22	000200	4,6,8,12,8,4	$X_4+\sqrt{3}(X_{13}+X_{14})+X_{15}+2X_{23}$	10
23	020200	6,10,12,18,12,6	$\sqrt{10}X_2+\sqrt{6}(-X_4+X_{15}+X_{23})$	18
24	020200	6,10,12,18,12,6	$\sqrt{10}X_2+\sqrt{6}(X_4+X_{15}+X_{23})$	10
25	220002	6,8,10,14,10,6	$\sqrt{3}(iX_1-iX_6+X_{18}+X_{21})$ $+\sqrt{2}(X_2+X_8+X_{19}-X_{24})$	18
26	220002	6,8,10,14,10,6	$\sqrt{3}(iX_7-iX_{11}+X_{12}+X_{16})$ $+2(X_{13}+X_{14})$	10
27, 28	111011	6,8,11,15,11,6	$\pm[\sqrt{3}(X_7+X_{11})+i\sqrt{3}(X_{12}-X_{16})$ $2(X_{13}+X_{14})+X_{15}]$	6
29,30	121011	7,10,13,18,13,7	$\pm[X_7+X_{11}+i\sqrt{6}(X_{12}-X_{16})$ $+\sqrt{10}X_2+\sqrt{6}X_{15}]$	8
31	211012	8, 10, 14, 19, 14, 8	$3X_{15}+2\sqrt{2}(X_1+X_6)$ $+\sqrt{5}(X_{13}+X_{14})$	6

Table 8.
(continued)

	$\alpha_i(H)$	k_i	E	inv
32	200202	8,10,14,20,14,8	$X_4 + 3X_{15} + 2\sqrt{2}(X_1 + X_6)$ $+\sqrt{5}(X_{13} + X_{14})$	6
33	200202	8,10,14,20,14,8	$-X_4 + 3X_{15} + 2\sqrt{2}(X_1 + X_6)$ $+\sqrt{5}(X_{13} + X_{14})$	12
34	220202	10,14,18,26,18,10	$\sqrt{10}(X_1 + X_6) + \sqrt{14}X_2$ $+\sqrt{2}(2X_4 + 3X_{15})$	6
35	220202	10,14,18,26,18,10	$\sqrt{10}(X_1 + X_6) + \sqrt{14}X_2$ $+\sqrt{2}(2X_4 - 3X_{15})$	10
36	222022	12,16,22,30,22,12	$4X_2 + \sqrt{12}(X_1 + X_6)$ $+\sqrt{7}(X_3 + X_5) + i\sqrt{15}(X_9 - X_{10})$	8
37	222222	16,22,30,42,30,16	$4(X_1 + X_6) + \sqrt{30}(X_3 + X_5)$ $+\sqrt{22}X_2 + \sqrt{42}X_4$	6

Table 9.
Cayley triples in $E_{6(-14)}$.

	$\alpha_i(H)$	k_i	E	inv
1,2	010000	1,2,2,3,2,1	$\pm iX_{36}$	26
3,4	100001	2,2,3,4,3,2	$\pm i(X_{23}+X_{36})$	30
5	100001	2,2,3,4,3,2	$X_{27}-X_{35}$	18
6	020000	2,4,4,6,4,2	$\sqrt{2}(X_{17}-X_{31})$	20
7,8	110001	3,4,5,7,5,3	$\pm[\sqrt{2}(X_{17}-X_{31})+iX_{23}]$	12
9	200002	4,4,6,8,6,4	$\sqrt{2}(X_1+X_{20}+X_{21}+X_{29})$	30
10,11	120001	4,6,7,10,7,4	$\pm[\sqrt{3}(X_2+X_{24})+2iX_{23}]$	18
12	220002	6,8,10,14,10,6	$\sqrt{6}(X_1+X_{21})+2(X_2+X_{24})$	10

Table 10.
Cayley triples in $E_{6(-26)}$.

	$\alpha_i(H)$	k_i	E	inv
1	100001	2,2,3,4,3,2	$X_{23}+X_{36}$	24
2	200002	4,4,6,8,6,4	$\sqrt{2}(X_1+X_6+X_{29}+X_{31})$	24

6. Appendix.

In this appendix we give the tables of the signs $\varepsilon(i,j)$ of the structure constants $N(i,j)$ for $1 \le i,j \le N$ and for the complex simple Lie algebras G_2, F_4, and E_6. The sign $\varepsilon(i,j)$ occurs in the row i and column j. The column numbers are abbreviated: Only the last digit is shown. When $\alpha_i+\alpha_j$ is not a root, then $\varepsilon(i,j)$ is not defined and we have indicated this by writing a zero entry at such positions. In the Tables 12 and 13 we write "p" instead of "+" and "n" instead of "−" (for typographical reasons).

Table 11.
$\varepsilon(i,j)$ for G_2.

	1	2	3	4	5	6
1	0	+	+	+	0	0
2	−	0	0	0	+	0
3	−	0	0	−	0	0
4	−	0	+	0	0	0
5	0	−	0	0	0	0
6	0	0	0	0	0	0

Table 12.
$\varepsilon(i,j)$ for F_4.

	12345	67890	12345	67890	1234
1	0p000	p00pp	00p00	p0000	00p0
2	n0n00	0p000	p000p	00p00	0p00
3	0p0pp	p0p0p	0p000	0p00p	p000
4	00n00	p0pp0	p0ppp	0p0p0	0000
5	00n00	0p0n0	00n00	n0000	0p00
6	n0nn0	0pn00	0n00p	00p00	n000
7	0n00n	n0n0n	0n0p0	0p0p0	0000
8	00nn0	pp00p	00n00	n0000	n000
9	n00np	00000	0n000	00p0p	0000
10	n0n00	0pn00	nn00n	000p0	0000
11	0n0n0	0000p	00000	n000p	0000
12	00n00	pp0pp	00p00	000p0	0000
13	n00np	00p00	0n00n	0n000	0000
14	000n0	0n000	00000	n0n00	0000
15	0n0n0	n000p	00p00	0n000	0000
16	n000p	00p00	p00p0	00000	0000
17	00nn0	0n000	00p0p	00000	0000
18	0n000	n00n0	000p0	00000	0000
19	000n0	0n00n	0n000	00000	0000
20	00n00	000n0	n0000	00000	0000
21	00n00	p0p00	00000	00000	0000
22	0n00n	00000	00000	00000	0000
23	n0000	00000	00000	00000	0000
24	00000	00000	00000	00000	0000

Table 13.
$\varepsilon(i,j)$ for E_6.

	12345	67890	12345	67890	12345	67890	123456
1	00p00	000p0	00p0p	000p0	p00pp	00p00	p00000
2	000n0	000pp	0p00p	p0p00	p0p00	00000	0000p0
3	n00n0	00p0p	000p0	p000p	00000	p000p	00p000
4	0pp0p	0p000	p0000	000p0	0p00p	0p000	000p00
5	000n0	n0pp0	0pp00	0p000	00000	00p0p	0p0000
6	0000p	0000p	000pp	00pp0	0p0p0	p00p0	000000
7	000n0	00p0p	000p0	p000p	000n0	00n00	n00000
8	00n0n	0n000	n000p	00p00	p0p00	00000	000n00
9	nn00n	00000	n00p0	0000p	0n000	0n000	00p000
10	0nn00	nn000	00p00	0p000	0000n	0n000	0p0000
11	000n0	00pp0	0pp00	0p000	000n0	n00n0	000000
12	0n00n	00000	n00p0	000pp	0000p	00000	n00000
13	n000n	0000n	n0000	n0p00	00p00	00000	00p000
14	00n00	nn0n0	0n000	00000	p0p00	00000	0p0000
15	nn000	n0n00	00000	0p00p	00000	0n00n	000000
16	0nn00	0n000	00p00	0p0p0	0p000	000n0	000000
17	0000n	0000n	n000n	n0000	n0000	00000	n00000
18	0n000	n0n00	00n00	0000p	0000p	00p00	000000
19	n00n0	n0000	0n000	n0000	00p00	0000n	000000
20	00n00	0n0n0	0n00n	00n00	00000	000n0	000000
21	nn000	00n00	000n0	0p000	0p000	p0000	000000
22	000n0	n00p0	00000	n0000	n0000	00p00	000000
23	0n000	00n00	00nn0	000n0	000n0	00000	000000
24	n0000	np000	p0000	00000	00p00	0p000	000000
25	n00n0	0000p	0n000	00n00	00000	p0000	000000
26	00n00	n0000	p0000	00000	n000n	00000	000000
27	000n0	000pp	0000p	00000	000n0	00000	000000
28	n000n	0p000	00000	00n00	0n000	00000	000000
29	00000	n0000	p0000	p000p	00000	00000	000000
30	00n0n	00000	0000p	000p0	00000	00000	000000
31	n0000	0p000	0p000	0p000	00000	00000	000000
32	0000n	0000n	000n0	00000	00000	00000	000000
33	00n00	000n0	00n00	00000	00000	00000	000000
34	000n0	00p00	00000	00000	00000	00000	000000
35	0n000	00000	00000	00000	00000	00000	000000
36	00000	00000	00000	00000	00000	00000	000000

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