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A COMPARISON PRINCIPLE FOR QUASILINEAR ELLIPTIC EQUATIONS AND ITS APPLICATION

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A comparison principle for a class of quasilinear elliptic equations is proved. An application of the comparison principle is given to prove the uniqueness of solutions of Dirichlet problems for a class of elliptic equations with jump discontinuous boundary data. The comparison principle is improved from the one given by Serrin. The uniqueness is proved by reducing the equation to an associated elliptic equation by viewing the graph of the solution from the side.

1. Introduction and Results.

Let $\Omega \subset R^n$, $n \geq 2$, be a bounded domain. We consider the quasilinear elliptic Dirichlet problem

$$(P) \qquad \begin{cases} Qf \equiv \sum_{i,j=1}^n a_{ij}(x, f(x), Df(x)) D_{ij} f(x) = 0 & \text{on} \quad \Omega; \\ f = \phi & \text{on} \quad \partial\Omega, \end{cases}$$

where $(a_{ij}(x,t,p))$ is a positive definite matrix in which each entry is a smooth C^2 function on $\overline{\Omega} \times R \times R^n$.

In this paper, we are mainly interested in a comparison principle in the following form and its applications:

Let D be a set on $\partial\Omega$. If f_1 and f_2 are two functions such that $Qf_1 \leq 0$, $Qf_2 \geq 0$ in Ω, $f_1 \geq f_2$ on $\partial\Omega \setminus \{D\}$, when we can conclude that $f_1 \geq f_2$ in Ω?

The motivation is as follows: When Q is a general quasilinear elliptic operator, it is well known that to solve (P) for $f \in C^2(\Omega) \cap C^0(\overline{\Omega})$ for any ϕ, the domain Ω must satisfy some geometric condition related to the structure of the operator Q (for example, see [**1**] or [**12**]). One typical example is when Q the minimal surface operator. Then (P) is solvable for any ϕ for $f \in C^2(\Omega) \cap C^0(\overline{\Omega})$ if and only if $\partial\Omega$ has non-negative mean curvature ([**6**]). Thus for some domain $\Omega \in R^n$, there are some functions ϕ for which (P) does not have solutions $f \in C^2(\Omega) \cap C^0(\overline{\Omega})$. On the other hand, for any domain $\Omega \in R^n$ and any $\phi \in C^1(\partial\Omega)$, we may be able to produce a function $f \in C^2(\Omega)$ using, for example, the Perron process or a variational process such that f satisfies the equation $Qf = 0$ in Ω and $f = \phi$ on every point on $\partial\Omega$ at which there is a barrier. Naturally we can think of f as a kind of "approximate solution". Then if we want to study the uniqueness of these approximate solutions or the behavior of these approximate solutions near the boundary of the domain, we will need a comparison principle in the form mention above. In general, comparison principles for Q do not hold even if D is empty and one does not have any additional information on the operator Q and the domain Ω. The set D on the boundary $\partial\Omega$ should also play some role in a comparison principle. A quick review of some well known cases may be illuminating. We consider the simplest case in which $D = \{P\}$ is a point on $\partial\Omega$. When Q is the Laplace operator $\triangle$, the Phragmén-Lindelöf maximum principle implies that if a function f satisfies $\triangle f = 0$ in Ω and $f \geq 0$ on $\partial\Omega \setminus \{P\}$, then $f \geq 0$ on $\overline{\Omega} \setminus \{P\}$ if f does not go to negative infinity very quickly as the point approaches P from inside Ω. The comparison principle in this case will not hold if the growth condition is removed (for example, see [**10**]). On the other hand, if Q is the minimal surface operator and a function f satisfies $Qf = 0$ in Ω, $f \geq 0$ on $\partial\Omega\setminus\{P\}$, then $f \geq 0$ in $\overline{\Omega}\setminus\{P\}$ (for example, see [**4**]). Those two typical examples demonstrate that the structure of the operator Q should play a crucial role in a comparison principle for the same D. In this paper, we consider a comparison principle when $D = \{P\}$ is a point on $\partial\Omega$, the operator Q is in the class of "strongly singularly elliptic" operators (see definition below) and one of the functions to be compared is a linear function. We shall then apply the comparison principle to prove a uniqueness result for Dirichlet problems in a two dimensional domain with jump discontinuous boundary data $\phi(x)$.

The class of "strongly singularly elliptic" operators is extended from the class of "singularly elliptic" operator introduced by Serrin in [**12**]. One feature of a singularly elliptic operator Q is that the behavior near a point on $\partial\Omega$ of a solution f to $Qf = 0$ can be controlled by the behavior of f on the rest of the domain. To state the definition of "strongly singularly elliptic" operators, let $p = (p_1, p_2, \cdots, p_n)$ and

$$\varepsilon(x, t, p) = \sum_{i,j=1}^{n} a_{ij}(x, t, p) p_i p_j.$$

Then:

Definition 1. An elliptic equation

$$Qf \equiv \sum_{i,j=1}^{n} a_{ij}(x, f, Df) D_{ij} f = 0 \tag{1}$$

is called **strongly singularly elliptic** if

$$\text{Trace}(a_{ij}(x, t, p)) = 1 \quad \text{for} \quad x \in \Omega, \quad t \in R, \quad p \in R^n, \tag{2}$$

and there is a positive function $\Psi(\rho)$ such that

(3) $$(\varepsilon(x,t,p))^{-1} \geq \Psi(|p|) \quad \text{for} \quad |p| \geq 1, \quad x \in \Omega, \quad t \in R, \quad p \in R^n,$$

and for any positive constant d, if $\psi_d(\rho) = \min_{\rho-d\leq t\leq \rho+d} \Psi(t)$, we have

(4) $$\int^{\infty} \frac{d\rho}{\rho^2 \psi_d(\rho)} < \infty.$$

The comparison principle obtained in the paper is:

Theorem 1. *Assume that*

(a) *g is a linear function and ω is a subdomain of Ω;*

(b) *$f \in C^2(\Omega) \cap C^0(\overline{\Omega} \setminus \{P\})$ satisfies $Qf = 0$ in Ω;*

(c) *the elliptic equation $Qf = 0$ is strongly singularly elliptic;*

(d) *there are positive constants $\mu > 0$ and $H > 0$ such that for $x \in \Omega$, $t \in R$, $|p| \geq H$,*
$$\epsilon(x,t,p) \leq (1-\mu)|p|^2;$$

(e) *$f \leq\ (\geq)\ g$ on $\partial\omega \setminus \{P\}$.*

Then
$$f \leq\ (\geq)\ g \quad on \quad \overline{\omega} \setminus \{P\}.$$

Theorem 1 can be applied to investigate the uniqueness of solutions and behavior of solutions near a point on the boundary. In this paper, we only give an application of Theorem 1 to the study of the uniqueness of solutions. An application of Theorem 1 to the investigation of the behavior of solutions near a point on the boundary is given in another paper by authors [7]. To apply Theorem 1 to prove a uniqueness result for Dirichlet problems with boundary data $\phi(x)$ which have a jump discontinuity, we need to restrict ourself to a bounded domain Ω in R^2 with $(0,0) \in \partial\Omega$ and an elliptic operator Q given by

(5) $$Qf \equiv a(f,f_x,f_y)f_{xx} + 2b(f,f_x,f_y)f_{xy} + c(f,f_x,f_y)f_{yy}$$

where $a, b, c \in C^1(R \times R^2)$ with $ac - b^2 > 0$ in R^3. We assume throughout the paper that $a(t,p,q)+c(t,p,q) = 1$ for all $(t,p,q) \in R^3$. We shall consider the uniqueness of solutions of the Dirichlet problem (P) when Q is given by (5), ϕ is continuous on $\partial\Omega$ except at $(0,0)$ and ϕ has a jump discontinuity at $(0,0)$.

One typical case is when Q is the minimal surface operator. In this case, when ϕ is continuous on $\partial\Omega$, the uniqueness of solutions to (P) is well known (for example see [**5**], [**12**]). When ϕ is singular on $\partial\Omega$, the uniqueness is obtained in [**3**] (see also [**9**]). The uniqueness for the case that Q is the constant mean curvature operator and ϕ has singularities on $\partial\Omega$ is proved in [**13**]. The proofs of these results exploited the specific structure of the minimal surface operator (or the constant mean curvature operator). When the operator Q takes the general form given in (5), it is not clear how the uniqueness of solutions can be deduced. We shall prove the uniqueness of solutions for a special class of boundary data $\phi(x)$ satisfying the following assumption.

Assumption (A). $\phi \in C^0(\partial\Omega \setminus \{(0,0)\})$ and ϕ a has jump discontinuity at $(0,0)$. If $m < M$ are the two side limits of ϕ at $(0,0)$ along $\partial\Omega$, the set

$$\{(x, y, \phi(x, y)) \mid (x, y) \in \partial\Omega \setminus \{(0,0)\}\} \cup \{(0,0,z) \mid m \leq z \leq M\}$$

can be projected bijectively onto a closed convex curve S on the yz plane.

The uniqueness result is:

Theorem 2. *Assume*
(1) *the elliptic equation $Qf = 0$ is strongly singularly elliptic;*
(2) *there are positive constants $\mu > 0$ and $H > 0$ such that for $x \in \Omega$, $t \in R$, $|p| \geq H$,*
$$\epsilon(x, t, p) \leq (1 - \mu)|p|^2;$$
(3) *the boundary data ϕ satisfying the assumption (A).*
Then the solution of (P) is unique in the class $C^2(\Omega) \cap C^0(\overline{\Omega} \setminus \{(0,0)\})$.

It is natural to ask what kind of operator is in the class of "strongly singularly elliptic" operators. One subclass of strongly singularly elliptic operators is the class of elliptic operators with well defined *genre*, a concept introduced in [**1**] and [**12**].

Definition 2. Q in (1) has **genre** λ if it satisfies (2) and there are positive constants μ_1 and μ_2 such that for $p \in R^n$, $|p| \geq 1$, $t \in R$, $x \in \Omega$,

$$\mu_1|p|^{2-\lambda} \leq \epsilon(x, t, p) \leq \mu_2|p|^{2-\lambda}.$$

From the definition, it is easy to see that the minimal surface operator has *genre* $\lambda = 2$, the Laplace operator has *genre* $\lambda = 0$ and the operator Q satisfies (1) and (2) in Theorem 2 if it has a well defined *genre* greater than 1.

The ideas of the proofs: Theorem 1 is proved by modifying the proof of a similar result due to Serrin [**12**]. Theorem 2 is proved by observing that for any solution $f(x, y)$ of (P), we can view the graph of $z = f(x, y)$ from the side and obtain a new function $x = g(y, z)$ for the same graph. Then the discontinuity of ϕ at $(0, 0)$ disappears for the function $g(y, z)$. Furthermore the function $g(y, z)$ satisfies an elliptic equation. Then we apply the classical comparison principle to the function $g(y, z)$ and its elliptic equation.

2. A Comparison Principle in R^n.

In this section, we prove the comparison principle Theorem 1. The proof requires a few lemmas. In [**12**] Serrin defined an elliptic operator Q to be *singularly elliptic* if it satisfies Definition 1 with (4) only needed to hold for $\psi_0(\rho) = \Phi(\rho)$. The first lemma relates the class of strongly singularly elliptic operators to that of singularly elliptic operators introduced by Serrin in [**12**]. Roughly speaking, a strongly singularly elliptic operator is a singularly elliptic operator such that it is still singularly elliptic after f replaced by f plus a linear function.

Lemma 1. *If (1) is strongly singularly elliptic, then for any vector* $\mathbf{b} \in R^n$ *and constant* c*, the equation*

$$\sum_{i,j=1}^{n} a_{ij}(x, f + \mathbf{b} \cdot x + c, Df + \mathbf{b}) D_{ij} f = 0 \tag{6}$$

is also singularly elliptic as defined by Serrin in [**12**].

Proof. By the definition given by Serrin in [**12**], we need to verify that:

1) For all $x \in \Omega$, $t \in R$, $p \in R^n$,

$$\text{Trace}(a_{ij}(x, t + \mathbf{b} \cdot x + c, p + \mathbf{b})) = 1. \tag{7}$$

2) There is a positive function $g(\rho)$ such that for

$$\varepsilon_1(x, t, p) \equiv \sum_{i,j=1}^{n} a_{ij}(x, t + \mathbf{b} \cdot x + c, p + \mathbf{b}) p_i p_j$$

and for all $|p| \geq 1$, $p \in R^n$, $x \in \Omega$, $t \in R$, we have

$$(\varepsilon_1(x, t, p))^{-1} \geq g(|p|) \tag{8}$$

and

$$\int_1^\infty \frac{d\rho}{\rho^2 g(\rho)} < \infty. \tag{9}$$

(7) follows easily from (2). For (8) and (9), we notice that

$$\begin{aligned}\varepsilon_1(x,t,p) = \varepsilon(x,t+\mathbf{b}\cdot x+c,p+\mathbf{b}) \\ - 2\sum_{i,j=1}^{n} a_{ij}(x,t+\mathbf{b}\cdot x+c,p+\mathbf{b})b_i(p_j+b_j) \\ + \sum_{i,j=1}^{n} a_{ij}(x,t+\mathbf{b}\cdot x+c,p+\mathbf{b})b_ib_j.\end{aligned}$$

(7) implies that all eigenvalues of $(a_{ij}(x,t+\mathbf{b}\cdot x+c,p+\mathbf{b}))$ are between 0 and 1. Then by Schwartz inequality, we have

$$\varepsilon_1(x,t,p) \le \varepsilon(x,t+\mathbf{b}\cdot x+c,p+\mathbf{b}) + |\mathbf{b}|(\varepsilon(x,t+\mathbf{b}\cdot x+c,p+\mathbf{b}))^{\frac{1}{2}} + |\mathbf{b}|^2, \tag{10}$$

thus

$$\varepsilon_1(x,t,p) \le 3\varepsilon(x,t+\mathbf{b}\cdot x+c,p+\mathbf{b}) + 3|\mathbf{b}|^2. \tag{11}$$

There are two cases.
Case 1): $\varepsilon(x,t+\mathbf{b}\cdot x+c,p+\mathbf{b}) < |\mathbf{b}|^2$, then $\varepsilon_1(x,t,p) \le 6|\mathbf{b}|^2$;
Case 2): $\varepsilon(x,t+\mathbf{b}\cdot x+c,p+\mathbf{b}) \ge |\mathbf{b}|^2$, then $\varepsilon_1(x,t,p) \le 6\varepsilon(x,t+\mathbf{b}\cdot x+c, p+\mathbf{b})$.
In either case, for $|p+\mathbf{b}| \ge 1$, we have

$$\begin{aligned}\frac{1}{\varepsilon_1(x,t,p)} &\ge \frac{1}{6}\min\left\{\frac{1}{|\mathbf{b}|^2}, \frac{1}{\varepsilon(x,t+\mathbf{b}\cdot x+c,p+\mathbf{b})}\right\} \\ &\ge \frac{1}{6}\min\left\{\frac{1}{|\mathbf{b}|^2}, \Psi(|p+\mathbf{b}|)\right\}.\end{aligned}$$

When $|p+\mathbf{b}| < 1$ and $|p| \ge 1$, it is easy to see that $\varepsilon_1(x,t,p) \le |p|^2$ from (7). Thus

$$\frac{1}{\varepsilon_1(x,t,p)} \ge C_1\min\left\{\frac{1}{|\mathbf{b}|^2}, \Psi(|p+\mathbf{b}|)\right\} \quad \text{for} \quad |p| \ge 1$$

for some constant $C_1 > 0$. If we let $d = |\mathbf{b}|$, then $|p|-|\mathbf{b}| \le |p+\mathbf{b}| \le |p|+|\mathbf{b}|$ and the definition of $\psi_d(\rho)$ implies that for $|p| \ge 1$,

$$(\varepsilon_1(x,t,p))^{-1} \ge g(\rho) \quad \text{where} \quad g(\rho) = C_1\min\left\{\frac{1}{|\mathbf{b}|^2}, \psi_d(\rho)\right\}.$$

That is (8). Finally

$$\frac{1}{\rho^2 g(\rho)} \le \frac{c_2}{\rho^2} + \frac{c_2}{\rho^2\psi_d(\rho)}$$

for some constant c_2. Thus (9) follows from (4). □

For singularly elliptic equations, the behavior of a solution near a point on the boundary can be controlled by the behavior of the solution on the rest of the domain. In [**12**] Serrin proved the following Proposition.

Proposition (Serrin [**12**]). *Let P be a point on $\partial\Omega$ and $f \in C^2(\Omega) \cap C^0(\overline{\Omega}\backslash\{P\})$ be a solution of (1). Assume that (1) is singularly elliptic and there is a positive constant $\mu > 0$ such that*

$$\varepsilon(x,t,p) \le (1-\mu)|p|^2 \quad \text{for} \quad x \in \Omega, \quad t \in R, \quad p \in R^n. \tag{12}$$

Then for any given $\delta > 0$, we have (r denotes the distance from x to P)

$$f \le m = \sup\{f(x) \mid x \in \partial\Omega \cap \{r \ge a\}\} + \delta \quad \text{on} \quad \Omega \cap \{r = a\} \tag{13}$$

for all sufficient small values of a depending only on δ, the diameter of Ω, and the structure of Equation 1.

In application, we usually can only verify that (12) holds for $|p|$ large. A careful inspection of Serrin's proof of the Proposition tells us that (12) is only needed for a bounded range of t if we know that f is bounded on $\overline{\Omega}\backslash\{P\}$ a priori. Thus we obtain the following lemma from Serrin's Proposition with a modification of the proof of the Proposition given by Serrin in [**12**].

Lemma 2. *Let P be a fixed point on $\partial\Omega$ and $f \in C^2(\Omega) \cap C^0(\overline{\Omega}\backslash\{P\})$ be a solution of (1). Assume that (1) is singularly elliptic, $f \in L^\infty(\partial\Omega \setminus \{P\})$ and there are positive constants $\mu > 0$ and $H > 0$ such that*

$$\varepsilon(x,t,p) \le (1-\mu)|p|^2 \quad \text{for} \quad |p| \ge H, \quad x \in \Omega, \quad t \in R, \quad p \in R^n. \tag{14}$$

Then for any given $\delta > 0$, we have (r denotes the distance from x to P)

$$f \le \sup\{f(x) \mid x \in \partial\Omega \cap \{r \ge a\}\} + \delta \quad \text{on} \quad \Omega \cap \{r = a\} \tag{15}$$

for all sufficient small values of a depending only on δ, the diameter of Ω, and the structure of Equation 1.

Proof. For convenience, we let $\phi(x)$ be the restriction of $f(x)$ on $\partial\Omega \setminus \{P\}$.

Step I: There is a constant M such that $|f(x)| \le M$ on $\overline{\Omega} \setminus \{P\}$.

Set $\Psi_1(\rho) = \rho^{-2}$ if $0 < \rho < 1$, $\Psi_1(\rho) = \Psi(\rho)$ if $\rho \ge 1$, where $\Psi(\rho)$ is given in the Definition 1 satisfying by (3) and (4) (with $\psi_0(\rho) = \Phi(\rho)$). Then

$$\int_0^\infty \frac{d\rho}{\rho^3 \Psi_1(\rho)} = \infty.$$

We set

$$\chi(\alpha) = \int_\alpha^\infty \frac{d\rho}{\rho^3 \Psi_1(\rho)}. \tag{16}$$

It is clear that $\chi(\alpha)$ is a monotonically decreasing function with range $(0, \infty)$. Let $\eta(\beta)$ be the inverse function of $\chi(\alpha)$. Then $\eta(\beta)$ is a positive, monotonically decreasing function with range $(0, \infty)$, and

$$\int_0^\infty \chi(\alpha) d\alpha = \int_0^\infty \eta(\beta) d\beta < \infty. \tag{17}$$

We denote by τ the diameter of Ω and define

$$h(r) = \int_r^\tau \eta\left(\mu \log \frac{t}{a}\right) dt \tag{18}$$

where r is the distance from x to the point P, μ is defined in (14) and a is a small positive number to be determined later. Then it is straightforward to verify that $h(r)$ is a positive monotonically decreasing function, $h(\tau) = 0$, $h'(a) = -\infty$, and

$$\frac{h''}{h'^3} = -\frac{\mu \Psi_1(-h')}{r}. \tag{19}$$

L'Hopital's Rule implies that

$$\lim_{a \longrightarrow 0} h(a) = \lim_{a \longrightarrow 0} a \int_1^{\frac{\tau}{a}} \eta(\mu \log t) dt = \lim_{a \longrightarrow 0} \tau \eta\left(\mu \log \frac{\tau}{a}\right) = 0.$$

Since

$$|h'(r)| = \eta\left(\mu \log \frac{r}{a}\right)$$

and $\eta(\beta) \longrightarrow \infty$ as $\beta \longrightarrow 0^+$, for H given in the condition (14), there is a constant $c(H, \eta)$, such that $\eta(\beta) \geq H$ for $0 < \beta < c(H, \eta)$. Thus $\eta(\mu \log \frac{r}{a}) \geq H$ if $1 \leq \frac{r}{a} \leq e^d$ with $d = c(H, \eta)/\mu$. That is, for any number $a > 0$,

$$|h'(r)| = \eta\left(\mu \log \frac{r}{a}\right) \geq H \quad \text{for} \quad a \leq r \leq ae^d. \tag{20}$$

Now we choose a number a_1 such that $a_1 e^d \leq \tau/2$ and define $a_{k+1} = a_k e^{-d} = a_1 e^{-kd}$ for $k = 1, 2, 3, \cdots$. We set

$$\begin{aligned} h_1(r) &= \int_r^\tau \eta\left(\mu \log \frac{t}{a_2}\right) dt; \\ M &= \sup_{r(x)=a_1} \{f(x)\} + \sup_{x \in \partial\Omega \setminus \{P\}} \{\phi(x)\}; \\ w_1(x) &= h_1(r(x)) + M. \end{aligned} \tag{21}$$

Then

$$Dw_1(x) = h_1'(r(x))\frac{x}{|x|}. \tag{22}$$

From (20), (22) and the definition of a_1, a_2, we have

$$|Dw_1(x)| = |h_1'(r(x))| \geq H \quad \text{for} \quad a_2 \leq r \leq a_1. \tag{23}$$

Then for any constant b, we have that on $a_2 \leq r \leq a_1$

(24)

$$\begin{aligned} L\{w_1 + b\} &\equiv \sum_{i,j=1}^{n} a_{ij}(x, w_1(x) + b, Dw_1(x)) D_{ij} w_1(x) \\ &\leq \epsilon_1(x, h_1 + M + b, Dh_1)\frac{h_1''}{{h_1'}^2} + \left(1 - \frac{\epsilon_1(x, h_1 + M + b, Dh_1)}{{h_1'}^2}\right)\frac{h_1'}{r} \\ &\leq \epsilon_1(x, h_1 + M + b, Dh_1)\frac{h_1''}{{h_1'}^2} + \mu\frac{h_1'}{r} \leq \frac{1}{\Psi_1(-h_1')}\frac{h_1''}{{h_1'}^2} + \mu\frac{h_1'}{r} = 0. \end{aligned}$$

Here we have used (14), (19) and (23). Then Theorem 15.1 (on page 459) in [**12**] implies that

$$f(x) \leq w_1(x) \quad \text{on} \quad a_2 \leq r \leq a_1.$$

In particular, since $h_1(r)$ is monotonically decreasing,

$$f(x) \leq h_1(a_2) + M \quad \text{on} \quad a_2 \leq r \leq a_1. \tag{25}$$

Now for $a_3 \leq r \leq a_2$, we set

$$h_2(r) = \int_r^{\tau} \eta\left(\mu \log \frac{t}{a_3}\right) dt \tag{26}$$

and

$$w_2(x) = h_2(r(x)) + M + h_1(a_2). \tag{27}$$

Similar to the argument with w_1, since $|Dw_2(x)| = |h_2'(r(x))| \geq H$ for $a_3 \leq r \leq a_2$, we have that for any constant b, on $a_3 \leq r \leq a_2$

$$L\{w_2 + b\} \equiv \sum_{i,j=1}^{n} a_{ij}(x, w_2(x) + b, Dw_2(x)) D_{ij} w_2(x) \leq 0.$$

Once again Theorem 15.1 (on page 459) in [**12**] implies that

$$f(x) \leq w_2(x) \quad \text{on} \quad a_3 \leq r \leq a_2.$$

In particular, since $h_2(r)$ is monotonically decreasing,

$$f(x) \le M + h_1(a_2) + h_2(a_3) \quad \text{on} \quad a_3 \le r \le a_2. \tag{28}$$

Combine (25) and (28), we get

$$f(x) \le M + h_1(a_2) + h_2(a_3) \quad \text{on} \quad a_3 \le r \le a_1.$$

Repeating this process, we arrive at

$$f(x) \le M + \sum_{i=1}^{\infty} h_i(a_{i+1}) \quad \text{on} \quad 0 < r \le a_1,$$

where

$$h_i(t) = \int_t^{\tau} \eta\left(\mu \log \frac{t}{a_{i+1}}\right) dt, \quad i = 1,\, 2,\, 3, \cdots. \tag{29}$$

If we can show

$$\sum_{i=1}^{\infty} h_i(a_{i+1}) < \infty, \tag{30}$$

$f(x)$ is bounded from above on $0 < r(x) \le a_1$. Since $f(x)$ is bounded on $a_1 \le r(x) \le \tau$, $f(x)$ is bounded from above on $\overline{\Omega} \setminus \{P\}$. In a similar manner, we can show that $f(x)$ is bounded from below on $\overline{\Omega} \setminus \{P\}$.

It remains to show (30). From (29), we have

$$\begin{aligned}\sum_{i=1}^{\infty} h_i(a_{i+1}) &= \sum_{i=1}^{\infty} \int_{a_{i+1}}^{\tau} \eta\left(\mu \log \frac{t}{a_{i+1}}\right) dt \\ &= \sum_{i=1}^{\infty} a_{i+1} \int_1^{\tau/a_{i+1}} \eta(\mu \log y) dy \\ &= \sum_{i=1}^{\infty} a_1 e^{-id} \int_1^{\tau e^{id}/a_1} \eta(\mu \log y) dy.\end{aligned}$$

But

$$\begin{aligned}&\int_1^{\infty} e^{-xd} \left(\int_1^{\tau e^{xd}/a_1} \eta(\mu \log y) dy\right) dx \\ &= \sum_1^{\infty} \int_i^{i+1} e^{-xd} \left(\int_1^{\tau e^{xd}/a_1} \eta(\mu \log y) dy\right) dx \\ &\ge \sum_1^{\infty} e^{-(i+1)d} \int_1^{\tau e^{id}/a_1} \eta(\mu \log y) dy \\ &= e^{-d} \sum_1^{\infty} e^{-id} \int_1^{\tau e^{id}/a_1} \eta(\mu \log y) dy.\end{aligned}$$

Thus we need only to show

$$\int_1^\infty e^{-xd}\left(\int_1^{\tau e^{xd}/a_1}\eta(\mu\log y)dy\right)dx<\infty. \tag{31}$$

By Fubini's theorem, we exchange the order of integration to get

$$\begin{aligned}
&\int_1^\infty e^{-xd}\left(\int_1^{\tau e^{xd}/a_1}\eta(\mu\log y)dy\right)dx\\
&=\int_1^{\tau e^d/a_1}\eta(\mu\log y)\left(\int_1^\infty e^{-xd}dx\right)dy\\
&\quad+\int_{\tau e^d/a_1}^\infty \eta(\mu\log y)\left(\int_{\frac{1}{d}\log\frac{a_1 y}{\tau}}^\infty e^{-xd}dx\right)dy\\
&=\int_1^{\tau e^d/a_1}\frac{1}{d}e^{-d}\eta(\mu\log y)dy+\int_{\tau e^d/a_1}^\infty \frac{\tau}{da_1}\eta(\mu\log y)\frac{1}{y}dy\\
&\le\int_1^{\tau e^d/a_1}\frac{\tau}{da_1 y}\eta(\mu\log y)dy+\int_{\tau e^d/a_1}^\infty \frac{\tau}{da_1}\eta(\mu\log y)\frac{1}{y}dy\\
&\le\int_0^\infty \frac{\tau}{da_1\mu}\eta(\beta)d\beta<\infty
\end{aligned}$$

here we have used (17). Thus (30) is true.

Step II: Proof of (15) from (14).

Let M be the number given in Step I. (2) implies all eigenvalues of $(a_{ij}(x,t,p))$ are between 0 and 1. Thus for $|p|\le H$, $x\in\overline{\Omega}$, $t\le 4M$, there is a positive constant $\mu_1>0$ such that:

$$\text{All eigenvalues of the matrix } (a_{ij}(x,t,p))\le 1-\mu_1 \qquad \text{for} \quad |p|\le H, x\in\overline{\Omega}, t\le 4M.$$

Combining this with condition (14), we have (for $\mu_2=\min\{\mu,\mu_1\}$)

$$\epsilon(x,t,p)\le(1-\mu_2)|p|^2 \quad \text{for} \quad x\in\Omega,\ |t|\le 4M,\ p\in R^n. \tag{32}$$

Let $\Psi_1(\rho)$, $\chi(\alpha)$ and $\eta(\beta)$ be the same as those defined in the proof of Step I. Let $h(r)$ be the function defined by formula (18) with the constant μ replaced by μ_2. Then $h(r)$ has all the properties verified in Step I and satisfies the equation (19) with the constant μ replaced by μ_2. Now we choose a small a, such that $h(a)\le M/4$, then $0\le h(r)\le M/4$ for $a\le r\le\tau$. Set

$$w(x)=h(r(x))+M_1, \quad M_1=\sup\{\phi(x)\,|\,x\in\partial\Omega,\ r(x)\ge a\}. \tag{33}$$

Then

$$|w(x)| \le h(r(x)) + M \le M/4 + M = 5M/4,$$

and if b is a constant such that $|b| \le 5M/2$,

$$|w(x) + b| \le 5M/4 + 5M/2 < 4M. \tag{34}$$

Thus from (19), (32) and (34), as we did in (24), we have

$$L\{w+b\} \equiv \sum_{i,j=1}^{n} a_{ij}(x, w(x)+b, Dw(x))D_{ij}w(x) \le 0. \tag{35}$$

Now we fix a point x_0 in Ω and set $b = f(x_0) - w(x_0)$. Then $|b| \le |f(x_0)| + |w(x_0)| \le M + 5M/4 < 5M/2$. We can substitute this b into (35) to get

$$\sum_{i,j=1}^{n} a_{ij}(x_0, f(x_0), Dw(x_0))D_{ij}w(x_0) \le 0.$$

Since x_0 is arbitrary, we further have

$$\sum_{i,j=1}^{n} a_{ij}(x, f(x), Dw(x))D_{ij}w(x) \le 0.$$

Then the proof of Theorem 1 on page 459 in [**12**] yields

$$f(x) \le w(x) \quad \text{on} \quad \overline{\Omega} \cap \{r \ge a\}.$$

That is

$$f(x) \le h(r(x)) + \sup\{\phi(x) \,|\, x \in \partial\Omega,\ r(x) \ge a\} \quad \text{on} \quad \overline{\Omega} \cap \{r \ge a\}.$$

Since $h(a) \longrightarrow 0$ as $a \longrightarrow 0$ and $h(r)$ is monotonically decreasing, for any given $\delta > 0$, when a is small,

$$f(x) \le \sup_{r(x)\ge a}\{\phi(x)\} + \delta \quad \textit{on} \quad \overline{\Omega} \cap \{r \ge a\}.$$

That is (15). □

Since the numbers a and δ in Lemma 2 can be made arbitrary small, we obtain the following conclusion.

Lemma 3. *Under the same assumptions as in Lemma 2, for $\phi(x) = f(x)$ on $x \in \partial\Omega \setminus \{P\}$, we have*

$$\inf_{\partial\Omega\setminus\{P\}}\{\phi\} \le f(x) \le \sup_{\partial\Omega\setminus\{P\}}\{\phi\} \quad \textit{for} \quad x \in \Omega.$$

We now can prove Theorem 1.

Proof of Theorem 1. Set $f(x) = v(x) + g(x)$, $g(x) = \mathbf{b} \cdot x + c$, then $v(x)$ satisfies $v(x) \leq 0$ on $\partial\omega \setminus \{P\}$ and

$$\sum_{i,j=1}^{n} a_{ij}(x, v + \mathbf{b} \cdot x + c, Dv + \mathbf{b}) D_{ij} v = 0. \tag{36}$$

Hence we only need to prove that $v(x) \leq 0$ on $\overline{\omega} \setminus \{P\}$.

From (10) in the proof of Lemma 1 and assumption (d), for $|p| > H$, we have

$$\epsilon_1(x, t, p) \leq (1 - \mu)|p|^2 + (3 - 2\mu)|p||\mathbf{b}| + (3 - 2\mu)|\mathbf{b}|^2.$$

Thus there is a $T > 0$, such that for some $\mu_1 > 0$,

$$\epsilon_1(x, t, p) \leq (1 - \mu_1)|p|^2 \quad \text{for} \quad |p| > T, x \in \overline{\omega},\ t \in R.$$

Finally the assumption (c) and Lemma 1 imply (36) is singularly elliptic. Then we can apply Lemma 3 to $v(x)$ on the subdomain ω to get $v(x) \leq 0$ on $\overline{\omega} \setminus \{P\}$ (note: Since $v(x) \leq 0$ on $\partial\omega \setminus \{P\}$, we do not need $v \in L^\infty(\partial\omega \setminus \{P\})$ in Lemma 3). □

3. Uniqueness of Solutions.

In this section, we prove the uniqueness result Theorem 2. First we need the following lemma. When Q is the minimal surface operator and ϕ is continuous, the idea in the proof was used in [**11**].

Lemma 4. *Suppose ϕ satisfies the assumption (A) and Q given in (5) is strongly singularly elliptic. Then for any solution f of (P) in $C^2(\Omega) \cap C^0(\overline{\Omega} \setminus \{(0, 0)\})$, $f_x(x, y) \neq 0$ in Ω.*

Proof. Suppose that for some $(x_0, y_0) \in \Omega$, $f_x(x_0, y_0) = 0$. Then the tangent plane at (x_0, y_0) of the surface $z = f(x, y)$ is given by

$$z = f_y(x_0, y_0)(y - y_0) + f(x_0, y_0).$$

This plane is perpendicular to the yz plane. Let

$$v(x, y) = f_y(x_0, y_0)(y - y_0) + f(x_0, y_0).$$

Then $v(x_0, y_0) = f(x_0, y_0)$. Since the plane is perpendicular to the yz plane, the assumption (A) implies that the tangent plane intersects the convex curve S defined in the assumption (A) at exactly two distinct points. Thus

$v = \psi$ on $\partial\Omega \setminus \{(0,0)\}$ at most at two points, say q_1, q_2 if they exist. Since the graph of v is the tangent plane to the graph of f at $(x_0, y_0, f(x_0, y_0))$, there are at least two curves γ_1 and γ_2 which intersect at (x_0, y_0), divide the neighborhood of (x_0, y_0) into four distinct, disjoint, open, connected sectors $\omega_1, \omega_2, \omega_3, \omega_4$ and $f - v = 0$ on γ_1 and γ_2 (e.g. see Lemma 1 in [**8**]). Then there must exist four (possibly nondistinct) points p_1, p_2, p_3, p_4 arranged in order around $\partial\Omega$ such that p_1 and p_3 are endpoints of (an extension of) γ_1 and p_2 and p_4 are endpoints of (an extension of) γ_2. On these extensions of γ_1 and γ_2, we have $f = v$. Since f is discontinuous at $(0,0)$, we see that $\{p_1, p_2, p_3, p_4\} \subseteq \{(0,0), q_1, q_2\}$. Hence there must exist an open set $U \subset \Omega$ such that $f = v$ on $\partial U \setminus \{(0,0)\}$. By Theorem 1, $f = v$ in U. Thus $f = v$ in Ω by the unique continuation property for solutions of elliptic equation ([**2**]). That is, the graph of f is part of the tangent plane. This contradicts the assumption (A). Thus $f_x(x,y) \neq 0$ in Ω. □

Now we prove Theorem 2.

Proof of Theorem 2. By Lemma 1, $f_x(x,y) \neq 0$ in Ω. Then either $f_x(x,y) < 0$ in Ω or $f_x(x,y) > 0$ in Ω. Thus there is a function $x = g(y,z)$ such that:

1) $x = g(y,z)$ is defined on the domain D on yz plane where D is the region bounded by the convex curve S;

2) $x = g(y, f(x,y))$ for $(x,y) \in \Omega$.

We claim the function $x = g(y,z)$ has the following properties:

Property 1: $x = g(y,z)$ is continuous on $\overline{D}$; $g(0,z) = 0$ for $m \leq z \leq M$.

Property 2: $x = g(y,z)$ is C^2 on D, $g_z \neq 0$;

Property 3: $g(y,z)$ satisfies on D the equation

(37)
$$\begin{aligned} Lg \equiv\; & c\left(z, \frac{1}{g_z}, -\frac{g_y}{g_z}\right) g_{yy} + 2\left(\frac{1}{g_z} b\left(z, \frac{1}{g_z}, -\frac{g_y}{g_z}\right) - \frac{g_y}{g_z} c\left(z, \frac{1}{g_z}, -\frac{g_y}{g_z}\right)\right) g_{yz} \\ & + \left(\frac{1}{g_z^2} a\left(z, \frac{1}{g_z}, -\frac{g_y}{g_z}\right) - 2\frac{g_y}{g_z^2} b\left(z, \frac{1}{g_z}, -\frac{g_y}{g_z}\right) + \frac{g_y^2}{g_z^2} c\left(z, \frac{1}{g_z}, -\frac{g_y}{g_z}\right)\right) g_{zz} \\ & = 0 \end{aligned}$$

and Equation 37 is elliptic where $g_z \neq 0$.

Assuming the properties for the moment, if there are two solutions $f_1(x,y)$, $f_2(x,y)$ to the problem (P), we have two functions $g_1(y,z)$ and $g_2(y,z)$ with above properties. By Property 1 and the definitions of $g_1(y,z)$ and $g_2(y,z)$, we have $g_1(y,z) = g_2(y,z)$ on $S = \partial D$. Since $f_{1x}(x,y) \neq 0$ and $f_{2x}(x,y) \neq 0$, $g_{1z}(y,z) \neq 0$ and $g_{2z}(y,z) \neq 0$ on D. We also note that $f_{1x}(x,y)$ and $f_{2x}(x,y)$ have the same sign which is determined = by $\phi(x)$, thus $g_{1z}(y,z)$ and $g_{2z}(y,z)$ have the same sign. Now since the coefficients of the elliptic operator Lg do not contain the variable g, an application of a classical comparison principle (e.g. [5]) shows that $g_1(y,z) = g_2(y,z)$ on D. Thus $f_1(x,y) = f_2(x,y)$ on $=\Omega$. This is the uniqueness desired.

Now we still need to prove the properties:

Property 1. First of all we observe that the assumption (A) implies that the region D enclosed by the convex curve S is either in $\{(y,z) \mid y \geq 0\}$ or $\{(y,z) \mid y \leq 0\}$. We may assume that

$$D \subset \{(y,z) \mid y \geq 0\}. \tag{38}$$

We define $g(0,z) = 0$ for $m \leq z \leq M$. Since the graph of $f(x,y)$ is continuous except at $(0,0)$, we only need to show that $g(y,z)$ is continuous at each point $(0,z)$ for $m \leq z \leq M$. If this is not true, there is a sequence of (y_k, z_k) such that $y_k \longrightarrow 0$, $z_k \longrightarrow z_1$ as $k \longrightarrow \infty$ such that $m \leq z_1 \leq M$ and $|x_k| = |g(y_k, z_k)| \geq c_1 > 0$. We may assume $x_k \longrightarrow x_1 \neq 0$. By (38), any point of the form $(x,0)$ is in $\partial\Omega$. Then $(x_1, 0) \in \partial\Omega \setminus \{(0,0)\}$ and $z_k = f(x_k, y_k) \longrightarrow z_1 = f(x_1, 0)$. That is, $(x_1, 0, f(x_1, 0))$ is projected to $(0, z_1)$. But the assumption (A) implies that $(0, z_1)$ is projected from $(0,0,z_1)$. It contradicts to fact that the projection in the assumption (A) is bijective. Thus $x = g(y,z)$ is continuous on $\overline{D}$ and $g(0,z) = 0$ for $m \leq z \leq M$.

Property 2. This part follows directly from the smooth assumption that $f \in C^2(\Omega)$, the fact that $f_x \neq 0$ and the implicit function theorem.

Property 3. This part follows from a direct computation: Indeed from the formula $x = g(y, f(x,y))$ and chain rule, we have $f_y = -\frac{g_y}{g_z}$, $f_x = \frac{1}{g_z}$, and

$$\begin{aligned}
f_{yy} &= -\frac{1}{g_z}\left\{g_{yy} - 2\frac{g_y}{g_z}g_{yz} + \frac{g_y^2}{g_z^2}g_{zz}\right\};\\
f_{xy} &= -\frac{1}{g_z}\left\{\frac{1}{g_z}g_{zy} - \frac{g_y}{g_z^2}g_{zz}\right\};\\
f_{xx} &= -\frac{1}{g_z^3}g_{zz}.
\end{aligned}$$

Substituting those into $Qf = 0$, we get (37). Finally it is easy to check directly that (37) is an elliptic equation. □

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