SMALL COMPOSITION OPERATORS ON ANALYTIC VECTOR-VALUED FUNCTION SPACES

Peide Liu, Eero Saksman and Hans-Olav Tylli

Let ϕ be an analytic mapping of the unit disk D into itself. We characterize the weak compactness of the composition operator $C_\phi : f \mapsto f \circ \phi$ on the vector-valued Hardy space $H^1(X)$ $(= H^1(D, X))$ and on the Bergman space $B_1(X)$, where X is a Banach space. Reflexivity of X is a necessary condition for the weak compactness of C_ϕ in each case. Assuming this, the operator $C_\phi : H^1(X) \to H^1(X)$ is weakly compact if and only if ϕ satisfies the Shapiro condition: $N_\phi(w) = o(1-|w|)$ as $|w| \to 1^-$, where N_ϕ stands for the Nevanlinna counting function of ϕ. This extends a previous result of Sarason in the scalar case. Similarly, C_ϕ is weakly compact on $B_1(X)$ if and only if the angular derivative condition $\lim_{|w| \to 1^-} (1 - |\phi(w)|)/(1 - |w|) = \infty$ is satisfied. We also characterize the weak compactness of C_ϕ on vector-valued (little and big) Bloch spaces and on $H^\infty(X)$. Finally, we find conditions for weak conditional compactness of C_ϕ on the above mentioned spaces of analytic vector-valued functions.

Introduction

Let $\phi : D \to D$ be analytic, where D is the open unit disk in $\mathbf{C}$. Recently properties of the analytic composition operators

$$(1) \qquad\qquad f \mapsto f \circ \phi$$

have been studied on the Hardy and Bergman spaces in numerous papers. Among the prominent results we mention the characterization [**13**] of compact composition operators on the Hardy space H^p $(1 \le p < \infty)$ in terms of the Nevanlinna counting function N_ϕ of ϕ (see below for the definition of N_ϕ). Thus, $C_\phi : H^p \to H^p$ is compact if and only if N_ϕ satisfies the Shapiro condition: $N_\phi(w) = o(1 - |w|)$ as $|w| \to 1^-$. Likewise, C_ϕ is compact on the Bergman space B_p [**9**] if and only if ϕ satisfies the angular derivative condition: $\lim_{|z| \to 1^-} (1 - |\phi(z)|)/(1 - |z|) = \infty$. Many other important properties of C_ϕ have also been characterized, such as spectral properties, membership in Schatten classes, closedness of the range and weak compactness. In addition, the same questions have been studied for composition operators acting

on analytic functions defined on a domain in $\mathbf{C}^n$. We refer to the recent monographs [**14**] and [**5**] for an overview of the whole spectrum of present knowledge concerning analytic composition operators.

The purpose of this paper is to initiate the study of analytic composition operators on spaces of vector-valued analytic functions. Let X be a Banach space and $1 \leq p < \infty$. The X-valued Hardy spaces $H^p(X)$ on the unit disk $D \subset \mathbf{C}$ are defined as follows

$$(2) \quad H^p(X) = \left\{ f : D \to X \mid f \text{ analytic, } \|f\|_{H^p(X)}^p \right.$$
$$\left. := \sup_{r<1} \frac{1}{2\pi} \int_0^{2\pi} \|f(re^{i\theta})\|_X^p d\theta < \infty \right\},$$

for $p < \infty$ and in addition $\|f\|_{H^\infty(X)} = \sup_{z \in D} \|f(z)\|_X$. The natural definitions of the X-valued Bergman spaces $B_p(X)$ and Bloch spaces will be given later. Important parts of modern harmonic analysis have been extended to the vector-valued case during the last 15 years, and at the same time vector-valued harmonic analysis has had profound applications to classical harmonic analysis (see e.g. the reviews [**4**] and [**11**]). We continue this line of research by asking about basic properties of the vector-valued composition operators (1), where f now belongs to some of the previously mentioned analytic vector-valued spaces.

We shall assume in the sequel that the Banach space X is infinite-dimensional. In this case the operator C_ϕ is never compact. Namely, it fixes the constant functions (if $f(z) \equiv a \in X$, then $C_\phi f = f$) and hence a copy of X itself. However, the question of weak compactness of C_ϕ remains equally interesting as an important 'smallness' property, and in this paper we shall concentrate on it.

A necessary condition for weak compactness of C_ϕ on X-valued spaces of analytic functions is the reflexivity of X. Assuming this, Theorem 3 below states that the operator C_ϕ is weakly compact on the Hardy space $H^1(X)$ if and only if N_ϕ satisfies the Shapiro condition and, moreover, $C_\phi : B_1(X) \to B_1(X)$ is weakly compact if and only if ϕ satisfies the angular derivative condition. The first mentioned criterion extends the result of Sarason [**12**] stating that (in the scalar case) the weak compactness of $C_\phi : H^1 \to H^1$ implies that C_ϕ is compact, i.e. the Shapiro condition holds. We refer to the remarks before Proposition 2 for the fact that the weak compactness of C_ϕ is obvious on $H^p(X)$ or $B_p(X)$ for $1 < p < \infty$ and a reflexive space X.

Theorem 4 characterizes weak compactness of C_ϕ on the vector-valued (big and little) Bloch-spaces $\mathcal{B}(X)$ and $\mathcal{B}_0(X)$. This generalizes and extends [**10**, Theorems 1 and 3], where scalar composition operators on the Bloch spaces were studied. As a special case of Theorem 4 we obtain that

[**10**, Theorem 3] also holds for the big Bloch space $\mathcal{B}$: Every weakly compact composition operator on $\mathcal{B}$ is compact. Theorem 6 settles the case $H^\infty(X)$.

Finally, Theorem 7 characterizes weak conditional compactness of C_ϕ on (most of) the vector-valued spaces of analytic functions mentioned above. The conditions for this turn out to be similar to those for weak compactness, the difference being that instead of the reflexivity of X one has to assume that ℓ^1 does not embed into X.

Results.

Throughout this paper ϕ stands for an analytic self-map of D and X is a Banach space. We first recall the definitions of weakly compact operators and the standard vector-valued analytic spaces that we shall consider. If E is a Banach space the bounded linear operator $T : E \to E$ is *weakly compact* if the image TB_E of the unit ball B_E is relatively compact in the weak topology of E. Another way to state this is to demand that every bounded sequence $(x_k) \subset E$ has a subsequence (x_{k_i}) such that the sequence (Tx_{k_i}) converges weakly (the equivalence of the stated conditions follows from the Eberlein-Smulian theorem [**18**, II.c.3]).

The vector-valued Hardy spaces have already been defined in (2). The definition of the X -valued Bergman spaces is analogous:

$$B_p(X) = \left\{ f : D \to X \mid f \text{ analytic}, \|f\|_{B_p(X)}^p := \frac{1}{\pi} \int_D \|f(z)\|_X^p dA(z) < \infty \right\},$$

where dA denotes the two-dimensional Lebesgue measure. The Bloch space $\mathcal{B}(X)$ is defined in a natural manner:

$$\mathcal{B}(X) = \left\{ f : D \to X \mid f \text{ analytic}, \|f\|_{\mathcal{B}(X)} \right.$$
$$\left. := \|f(0)\|_X + \sup_{z \in D}(1 - |z|^2)\|f'(z)\|_X < \infty \right\}.$$

The little Bloch space $\mathcal{B}_0(X)$ is the closed subspace of $\mathcal{B}(X)$ consisting of the analytic functions f with the property $\lim_{|z| \to 1}(1 - |z|^2)\|f'(z)\|_X = 0$. The completeness of these spaces is established exactly as in the scalar case. We shall write, e.g., $H^1 = H^1(\mathbf{C})$.

Consider next the boundedness of the operator C_ϕ on these vector-valued spaces.

It is well known that for analytic $f : D \to X$ the norm $z \mapsto \|f(z)\|_X$ is subharmonic. Namely, it may be expressed as the (continuous) supremum of a family of subharmonic functions:

$$\|f(z)\|_X = \sup_{x' \in B_{X'}} |\langle x', f(z) \rangle|,$$

where $B_{X'}$ stands for the unit ball of the dual space X'. Hence we may apply the Littlewood subordination theorem in the well known manner (see [**5**, Theorem 3.1]) to deduce that C_ϕ is contractive on the spaces $H^p(X)$ and $B_p(X)$ ($p \in [1, \infty[$) in the case that $\phi(0) = 0$. In order to treat the general case we note that it is enough to verify that C_ϕ is bounded whenever ϕ is a conformal automorphism of the disk onto itself. In this case boundedness is established by the simple change of variables $\phi(z) = w$ inside the integral defining the norm of $C_\phi f$. However, in the case of Hardy spaces one first applies the change of variables $\phi(z) = w$ to $C_\phi(f(rz))$ and then lets r increase to 1, using the subharmonicity of the norm $\|f(z)\|_X$. The reason for the extra care is the fact that, contrary to the scalar case, the functions in $H^p(X)$ need not in general have radial boundary values a.e.. In fact, elements in $H^p(X)$ can be identified with X-valued L_p-functions on the boundary with vanishing negative Fourier coefficients if and only if X has the so called analytic Radon-Nikodým property (see for instance [**2**]). The boundedness of C_ϕ on the Bloch spaces reduces to the scalar case treated in [**10**] by noting that $\|f\|_{\mathcal{B}(X)} \leq 2 \sup_{\lambda \in B_{X'}} \|\lambda \circ f\|_{\mathcal{B}}$. Summing up, we have established (the non-trivial parts) of the following:

Proposition 1. *Let X be a Banach space and $\phi : D \to D$ be analytic. The operator C_ϕ is bounded on the spaces $\mathcal{B}(X)$, $H^p(X)$ and $B_p(X)$ for $1 \leq p \leq \infty$. Moreover, C_ϕ maps $\mathcal{B}_0(X)$ into itself only if $\phi \in \mathcal{B}_0$. In the case where $\phi(0) = 0$ the operator C_ϕ is actually a contraction on the Hardy and Bergman spaces.*

From here on we may and shall always assume in the proofs of our theorems that $\phi(0) = 0$, since neither the results nor the conditions on ϕ will change if ϕ is composed with a conformal self-map of D.

We start by considering composition operators on X-valued Bergman and Hardy spaces. Assuming that C_ϕ is weakly compact on $B_p(X)$ or on $H^p(X)$ we note that X is reflexive. Namely, because C_ϕ fixes constant functions we see that C_ϕ fixes a copy of the space X itself. Hence, for $p \in (1, \infty)$ the question of weak compactness of C_ϕ is trivial since the spaces $B_p(X)$ and $H^p(X)$ are reflexive as closed subspaces of the reflexive Banach spaces $L^p(D, X)$ and $L^p(\partial D, X)$. For these facts, see [**6**, Corollary IV.2] and note that reflexive Banach spaces possess the Radon-Nikodým property [**6**, Corollary III.13] and consequently also the analytic Radon-Nikodým property (cf. [**2**]).

The following simple proposition is based on well-known properties of the de la Vallée-Poussin summability kernels.

Proposition 2. *Let X be a Banach space and let $n \in \mathbf{N}$. Define the*

operator V_n by setting

$$(3) \qquad V_n f(z) = \sum_{k=0}^{n} \hat{f}_k z^k + \sum_{k=n+1}^{2n-1} \frac{2n-k}{n} \hat{f}_k z^k$$

for analytic $f : D \to X$ with the Fourier expansion $f = \sum_{k=0}^{\infty} \hat{f}_k z^k$. Then

$$(4) \qquad \|V_n f\|_{H^1(X)} \leq 2\|f\|_{H^1(X)}$$

for all $f \in H^1(X)$. In addition, given $\varepsilon > 0$ and $r \in (0,1)$ there is $n_0 = n_0(\varepsilon, r) > 0$ such that

$$(5) \quad \|f(z) - V_{n_0} f(z)\|_X \leq \varepsilon \|f\|_{H^1(X)} \quad \text{for all } |z| \leq r \text{ and } f \in H^1(X).$$

Moreover, if X is reflexive, then the operator $V_n : H^1(X) \to H^1(X)$ is weakly compact for all $n \geq 1$. All the statements hold also if $H^1(X)$ is replaced by one of the spaces $H^\infty(X)$ and $B_1(X)$.

Proof. We commence with the Hardy space $H^1(X)$. Let $f \in H^1(X)$ and assume first that f is continuous up to the boundary ∂D. Then V_n (viewed as operating on boundary values) equals convolution with the de la Vallee-Poussin kernel so that $V_n f = (2K_{2n-1} - K_{n-1}) * f$, where K_n denotes the Fejer kernel. Claim (4) follows since K_n is a summability kernel (see [**7**], p. 12). In order to prove (5), let $r \in (0,1)$ and $\varepsilon > 0$. The Poisson integral representation yields the estimate $\|f(z)\|_X \leq C(1-|z|)^{-1}\|f\|_{H^1(X)}$, with C independent of f. Choose n_0 large enough so that $3Cr^{n_0}(1-r)^{-1} \leq \varepsilon$. Given $f \in H^1(X)$, write $f - V_{n_0} f = z^{n_0} g$ with $\|g\|_{H^1(X)} = \|f - V_{n_0} f\|_{H^1(X)}$ and deduce that

$$\begin{aligned}
\|(I - V_{n_0})f(z)\|_X = |z|^{n_0}\|g(z)\|_X &\leq Cr^{n_0}(1-r)^{-1}\|f - V_{n_0} f\|_{H^1(X)} \\
&\leq (\varepsilon/3)\|f - V_{n_0} f\|_{H^1(X)},
\end{aligned}$$

for $|z| \leq r$. This yields (5) in view of (4). In the case that f is not continuous we denote $f_s(z) = f(sz)$ for $0 < s < 1$ and observe that $V_n f_s = (V_n f)_s$. Hence we get, e.g., that $\|(V_n f)_s\|_{H^1(X)} \leq 2\|f_s\|_{H^1(X)}$ and by letting s increase to 1 we obtain (4) and (5) for f. The proof for $H^\infty(X)$ is entirely similar.

We next assume that $f \in B_1(X)$. The uniform boundedness of the operators V_n follows easily by writing

$$(6) \qquad \|f\|_{B_1(X)} = 2\int_0^1 \|f_r\|_{H^1(X)} r \, dr$$

and applying the $H^1(X)$-boundedness together with the relation $V_n f_r = (V_n f)_r$. Towards (5), suppose that $r \in (1/2, 1)$ and $\varepsilon > 0$ are given and

that $f \in B_1(X)$ with $\|f\|_{B_1(X)} \le 1$. Set $s = \sqrt{r}$. Since (5) holds for $H^1(X)$ we may choose n_0 such that $\|g(z) - V_{n_0}g(z)\|_X \le \frac{1}{2}\varepsilon(1-s)\|g\|_{H^1(X)}$ for $|z| \le s$ and all $g \in H^1(X)$. The subharmonicity of $z \mapsto \|f(z)\|_X$ yields that $\|f_r\|_{H^1(X)}$ increases with r, so that (6) yields a radius $r' \in (s, 1)$ with $\|f_{r'}\|_{H^1(X)} \le 2(1-s)^{-1}$. Hence, for $|z| \le r$ we have that $|z/r'| \le |z/s| \le s$ and we may apply our assumption on n_0 to the function $f_{r'}$ and deduce that

$$\|f(z) - V_{n_0}f(z)\|_X = \|f_{r'}(z/r') - V_{n_0}f_{r'}(z/r')\|_X \le \frac{1}{2}\varepsilon(1-s)\|f_{r'}\|_{H^1(X)} \le \varepsilon.$$

Finally, assume that X is reflexive. We fix n and denote $G_n = (\bigoplus_0^{2n-1} X)_{\ell^2}$. For any analytic function $f : D \to X$ with Fourier coefficients $\hat{f}(k)$, $k \ge 0$, define $S_n f = (\hat{f}(0), \hat{f}(1), \ldots, \hat{f}(2n-1)) \in G_n$ and for any $x = (x_0, \ldots, x_{2n-1}) \in G_n$ set

(7)
$$(R_n x)(z) = \left(x_0 + x_1 z + \ldots x_n z^n + x_{n+1}\frac{n-1}{n}z^{n+1} + \ldots + x_{2n-1}\frac{1}{n}z^{2n-1} \right)$$

for $z \in D$. It is clear that the operators $S_n : Y \to G_n$ and $R_n : G_n \to Y$ are well-defined and bounded if Y is any of the spaces $H^1(X)$, $H^\infty(X)$ or $B_1(X)$. Hence the operator $V_n = R_n S_n$ is weakly compact since it factorizes through the reflexive space G_n (cf. [18, II.C.5]). □

We recall the Stanton formula [16, Theorem 2] for continuous subharmonic functions $u : D \to \mathbf{R}$,

$$\frac{1}{2\pi}\int_0^{2\pi} u(\phi(re^{i\theta}))d\theta = u(0) + \frac{1}{2\pi}\int_D N_\phi(r, w)d[\Delta u](w),$$

where $r \in (0, 1)$ and the standing assumption $\phi(0) = 0$ has been taken into account. Above $d[\Delta u](w)$ denotes integration with respect to the distributional Laplacian of u, which is a positive measure on D since u is subharmonic. In addition, $N_\phi(r, \cdot)$ denotes the partial Nevanlinna counting function of ϕ :

(8)
$$N_\phi(r, w) = \sum_{z \in \phi^{-1}(w), \, |z| \le r} \log\left(\frac{r}{|z|}\right)$$

for $r \in (0, 1)$. The (standard) Nevanlinna counting function N_ϕ is defined as $N_\phi(w) = N_\phi(1, w) = \sum_{z \in \phi^{-1}(w)} \log(1/|z|)$.

Let $f : D \to X$ be an analytic map. By applying the Stanton formula to the subharmonic function $z \mapsto \|f(z)\|_X$ we obtain

(9) $\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \|f(\phi(re^{i\theta}))\|_X d\theta = \|f(0)\|_X + \frac{1}{2\pi}\int_D N_\phi(r, w)d[\Delta(\|f\|_X)](w).$

Since $N_\phi(r, w)$ increases monotonically to $N_\phi(w)$ as r increases to 1, an application of the monotone convergence theorem yields

$$(10) \qquad \|C_\phi f\|_{H^1(X)} = \|f(0)\|_X + \frac{1}{2\pi} \int_D N_\phi(w) d[\Delta(\|f\|_X)](w).$$

This formula can be viewed as a generalization of the well-known Littlewood-Paley identity. The technique of applying the Stanton formula for the Hardy spaces H^p with $p \neq 2$ was introduced by Shapiro and Sundberg [15], where (10) appears in the scalar case.

We proceed to derive the corresponding formula for the Bergman norm. Note first that $2 \int_{|z|}^1 r \log(r/|z|) dr = B(|z|)$, where $B(u) = \log(1/u) - (1 - u^2)/2$ for $u \in (0, 1)$. Combining this observation with (8) we easily see that

$$\int_0^1 r N_\phi(r, w) dr = \sum_{z \in \phi^{-1}(w)} B(|z|) =: N_\phi^B(w).$$

Multiply (9) by $2r$ and integrate from 0 to 1 with respect to r in order to obtain the exact formula

$$(11) \qquad \|C_\phi f\|_{B_1(X)} = \|f(0)\|_X + \frac{1}{2\pi} \int_D N_\phi^B(w) d[\Delta(\|f\|_X)](w).$$

However, we introduce an equivalent norm that is easier to handle. Note first that $B(1-s) = s^2 + s^3/3 + s^4/4 + \ldots$ and $-s \log(1-s) = s^2 + s^3/2 + s^4/3 + \ldots$ for $s \in (0, 1)$. Consequently $(3/4)(1-u) \log(1/u) \leq B(u) \leq (1-u) \log(1/u)$. Thus, by introducing a modified Nevanlinna counting function

$$(12) \qquad \widetilde{N}_\phi(w) = \sum_{z \in \phi^{-1}(w)} (1 - |z|) \log(1/|z|),$$

we arrive at an equivalent expression for the Bergman norm of $C_\phi f$:

$$(13) \qquad \|C_\phi f\|_{B_1(X)} \sim \|f(0)\|_X + \frac{1}{2\pi} \int_D \widetilde{N}_\phi(w) d[\Delta(\|f\|_X)](w).$$

Important special cases of the previous formulas are obtained if we choose ϕ to be the identity map. In the first place

$$(14) \qquad \|f\|_{H^1(X)} = \|f(0)\|_X + \frac{1}{2\pi} \int_D \log\left(\frac{1}{|w|}\right) d[\Delta(\|f\|_X)](w),$$

and for the Bergman norm there is the equivalence

$$(15) \quad \|f\|_{B_1(X)} \sim \|f(0)\|_X + \frac{1}{2\pi} \int_D (1 - |w|) \log\left(\frac{1}{|w|}\right) d[\Delta(\|f\|_X)](w).$$

The following result extends and complements the scalar results of Mac-
Cluer and Shapiro [**9**], Shapiro [**13**] and Sarason [**12**].

Theorem 3.

(i) $C_\phi : B_1(X) \to B_1(X)$ *is weakly compact if and only if X is reflexive
  and*

(16) $$\lim_{|w| \to 1^-} (1 - |\phi(w)|)/(1 - |w|) = \infty.$$

(ii) $C_\phi : H^1(X) \to H^1(X)$ *is weakly compact if and only if X is reflexive
  and*

(17) $$N_\phi(w) = o(1 - |w|) \quad as \ |w| \to 1^-.$$

Proof. (i)   If $C_\phi : B_1(X) \to B_1(X)$ is weakly compact we have already
seen that X must be reflexive. Fix a vector $x_0 \in X$ with $\|x_0\|_X = 1$ and
consider the subspace $M \subset B_1(X)$, where $M = \{x_0 f(z) \mid f(z) \in B_1\}$.
Clearly $C_\phi : M \to M$ and the map $C_\phi : M \to M$ is 'isomorphic' to the scalar
composition operator $C_\phi : B_1 \to B_1$. Hence the scalar operator $C_\phi : B_1 \to B_1$
is weakly compact. Now B_1 is isomorphic to ℓ^1 (see [**18**, III.A.11]) and every
weakly compact operator on ℓ^1 is actually compact, since ℓ^1 has the Schur
property [**18**, III.C.9]. Hence $C_\phi : B_1 \to B_1$ is compact and this forces ϕ to
satisfy the angular derivative criterion (16) according to [**9**, Theorem 3.5].

In order to prove the other direction, we modify ideas due to Shapiro
and Sundberg [**15**]. We assume the conditions of part (i) and consider the
operators T_n:

$$T_n = C_\phi - C_\phi V_n \ : B_1(X) \to B_1(X),$$

where $n \geq 1$ and V_n is the weakly compact operator provided by Proposition
2. It is enough to establish that the operator norm of T_n tends to zero as
$n \to \infty$, since the weakly compact operators form a closed ideal of $L(B_1(X))$
(see [**18**, II.c.6]). Suppose thus that $\varepsilon > 0$ is given. Condition (16) yields
$r \in (1/2, 1)$ so that $1 - |w| \leq \varepsilon(1 - |\phi(w)|)$ for $|w| \in (r, 1)$. For these values
of w we may combine the Littlewood inequality $N_\phi(w) \leq \log(1/|w|)$ (see
[**14**, Corollary 10.4(b)]) with definition (12) and deduce that

(18) $$\widetilde{N}_\phi(w) \leq \varepsilon(1 - |w|)N_\phi(w) \leq \varepsilon(1 - |w|) \log(1/|w|)$$

for $|w| \geq r$. We next fix a cut-off function ψ that will be needed later in the
argument: Let $\psi \in C_0^\infty(D)$ satisfy $0 \leq \psi(z) \leq 1$, $\psi(z) = 1$ for $|z| \leq r$ and
$\psi(z) = 0$ for $|z| \geq (1 + r)/2$. Proposition 2 yields $n \in \mathbf{N}$ such that

(19) $$\|f(z) - V_n f(z)\|_X \leq \min \left\{ \varepsilon, 1/ \sup_{w \in D} |\Delta \psi(w)| \right\} \|f\|_{B_1(X)}$$

for $z \in \operatorname{supp}(\psi)$ and all $f \in B_1(X)$.

Assume next that $f \in B_1(X)$ is arbitrary. As $(I - V_n)f(0) = 0$ we may invoke formula (13) in the form

$$\|T_n f\|_{B_1(X)} \sim \frac{1}{2\pi} \int_{r < |w| < 1} \widetilde{N}_\phi(w) d[\Delta(\|(I - V_n)f\|_X)](w)$$
$$+ \frac{1}{2\pi} \int_{|w| \leq r} \widetilde{N}_\phi(w) d[\Delta(\|(I - V_n)f\|_X)](w) =: A + B,$$

say. The first term is easily estimated using (18) and (15):

$$A \leq \frac{1}{2\pi} \int_{r < |w| < 1} \varepsilon(1 - |w|) \log\left(\frac{1}{|w|}\right) d[\Delta(\|(I - V_n)f\|_X)](w)$$
$$\leq 2\varepsilon \|(I - V_n)f\|_{B_1(X)} \leq 6\varepsilon \|f\|_{B_1(X)},$$

where the extra constant 2 comes from the fact that we used the equivalent norm (15). In order to bound the second term we use the crude estimate $\widetilde{N}_\phi(w) \leq \log(1/|w|)$, which follows from (12) and the Littlewood inequality, and apply the substitution $w = rw'$:

$$B \leq \frac{1}{2\pi} \int_{|w| \leq r} \log\left(\frac{1}{|w|}\right) d[\Delta(\|(I - V_n)f\|_X)](w)$$
$$\leq \frac{1}{2\pi} \int_D \log(1/|w'|) d[\Delta(\|(I - V_n)f_r\|_X)](w') +$$
$$+ \frac{1}{2\pi} r^2 \log\left(\frac{1}{r}\right) \int_D d[\Delta(\|(I - V_n)f_r\|_X)](w') =: E + F.$$

According to (14) and (19) we obtain that $E = \|(I - V_n)f_r\|_{H^1(X)} \leq 2\varepsilon \|f\|_{B_1(X)}$. The remaining term can be taken care of as follows:

$$F = \frac{1}{2\pi} \log\left(\frac{1}{r}\right) \int_{rD} d[\Delta(\|(I - V_n)f\|_X)](w)$$
$$\leq \frac{1}{2\pi} \log(1/r) \int_D \psi(w) d[\Delta(\|(I - V_n)f\|_X)](w)$$
$$= \frac{1}{2\pi} \log(1/r) \int_D \Delta\psi(w)\|(I - V_n)f(w)\|_X dw \leq 2\varepsilon \|f\|_{B_1(X)},$$

according to (19). We deduce that $\|T_n f\|_{B_1(X)} \leq C\varepsilon \|f\|_{B_1(X)}$ with C independent of ε and f, which finishes the proof.

(ii) For the necessity of the stated conditions we argue as above in part (i). Here we apply the result of Sarason [12] stating that weakly compact (scalar) composition operators $C_\phi : H^1 \to H^1$ are compact. Condition (17) follows when this is combined with the Shapiro characterization [13, Theorem 2.3] of compact composition operators on H^2 and the

well-known fact that compactness of C_ϕ on H^p does not depend on p (cf. [5, Theorem 3.12]).

The proof of the sufficiency follows the same outline as the one of part (i), albeit being slightly simpler, as it now suffices to use the standard Nevanlinna counting function. Hence we omit the details. □

Recall that the Banach space E has the Dunford-Pettis property if for every pair of weakly null sequences $(x'_n) \in E'$ and $(x_n) \in E$ we have $\lim_{n\to\infty}\langle x'_n, x_n\rangle = 0$. As a consequence, it follows that if $T : E \to E$ and $S : E \to E$ are weakly compact linear operators, then TS is compact. Moreover, T maps weakly null sequences to norm null sequences. We refer to [18, III.D.33-34] for these facts. We will apply this useful concept when dealing with vector-valued Bloch spaces in the proof of the following theorem, which extends [10, Theorem 3 (see also Theorems 1 and 2)].

Theorem 4.

(i) $C_\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ *is weakly compact if and only if X is reflexive and*

(20) $$\lim_{a\to 1^-} \sup_{\{z\in D:|\phi(z)|\geq a\}} \frac{(1-|z|^2)|\phi'(z)|}{1-|\phi(z)|^2} = 0.$$

(ii) $C_\phi : \mathcal{B}_0(X) \to \mathcal{B}_0(X)$ *is weakly compact if and only if X is reflexive and*

(21) $$\lim_{|z|\to 1^-} \frac{(1-|z|^2)|\phi'(z)|}{1-|\phi(z)|^2} = 0.$$

Proof. (i) We first prove the necessity. Exactly as in the proof of Theorem 3.(i) it suffices to show that condition (20) follows if the scalar operator $C_\phi : \mathcal{B} \to \mathcal{B}$ is weakly compact. For this purpose we produce a weakly compact counterpart of the necessity argument of [10, Theorem 2]. Assume thus that $C_\phi : \mathcal{B} \to \mathcal{B}$ is weakly compact, but (20) does not hold. Then there is a sequence (z_n) in D and a constant $\varepsilon > 0$ so that $|\phi(z_n)| \to 1$ as $n \to \infty$ together with

(22) $$\frac{(1-|z_n|^2)|\phi'(z_n)|}{1-|\phi(z_n)|^2} \geq \varepsilon \quad \text{for each } n \geq 1.$$

Denote $w_n = \phi(z_n)$. By multiplying ϕ by a unimodular constant and extracting a subsequence, if needed, we may also assume that $w_n \to 1$ as $n \to \infty$ and $|(w_{2n} - w_{2n+1})/(1 - \bar{w}_{2n}w_{2n+1})| \geq 1/2$ for all $n \geq 1$. Define

$$f_n(z) = \log\left(\frac{1 - \bar{w}_{2n}z}{1 - \bar{w}_{2n+1}z}\right)$$

for $n \geq 1$ and $z \in D$. Clearly $f_n \in \mathcal{B}_0$ and a simple computation shows that $\|f_n\|_{\mathcal{B}_0} \leq 4$ for each $n \geq 1$. Observe that for each $k \geq 1$ it holds that $\hat{f}_n(k) \to 0$ as $n \to \infty$. Hence, if g is a polynomial one obtains

$$\lim_{n \to \infty} \langle g, f_n \rangle = \lim_{n \to \infty} \int_D \overline{g(z)} f_n(z) dA(z) = 0,$$

where the brackets refer to the duality $\mathcal{B}_0' = B_1$ (see [**19**, Theorem 5.2.8]). Since polynomials are dense in B_1 and the sequence (f_n) is bounded we conclude that $f_n \to 0$ weakly in $\mathcal{B}_0$, and hence in $\mathcal{B}$. However, $\mathcal{B}_0$ certainly has the Dunford-Pettis property as $\mathcal{B}_0' = B_1 \approx \ell_1$ has the Schur property. This fact and the weak compactness of C_ϕ yield that the sequence $(C_\phi f_n)$ is norm null in $\mathcal{B}$. But this is impossible since using (22) we may estimate that for each $n \geq 1$

$$
\begin{aligned}
(23) \qquad \|C_\phi f_n\|_{\mathcal{B}} &\geq (1 - |z_{2n}|^2)|(f_n \circ \phi)'(z_{2n})| \\
&\geq \varepsilon(1 - |w_{2n}|^2) \left| \frac{-\bar{w}_{2n}}{1 - |w_{2n}|^2} + \frac{\bar{w}_{2n+1}}{1 - \bar{w}_{2n+1}w_{2n}} \right| \\
&= \varepsilon|(\bar{w}_{2n} - \bar{w}_{2n+1})/(1 - w_{2n}\bar{w}_{2n+1})| \geq \varepsilon/2,
\end{aligned}
$$

and we are done.

Towards the converse direction, assume that X is reflexive and (20) holds. Define the operator $\widetilde{V}_n$ by setting $\widetilde{V}_n f(z) = f(0) + \int_0^z V_n f'(w)dw$, where V_n is the operator provided by Proposition 1. Clearly $\widetilde{V}_n f$ is analytic and by applying Proposition 1 and the maximum principle we see that

$$
\begin{aligned}
\sup_{|z|=r} \|(\widetilde{V}_n f)'(z)\|_X = \sup_{|z|=r} \|V_n f'(z)\|_X &= \sup_{|w|=1} \|V_n f_r'(w)\|_X \leq 2\|f_r'\|_{H^\infty(X)} \\
&= 2 \sup_{|z|=r} \|f'(z)\|_X.
\end{aligned}
$$

This yields that $\|\widetilde{V}_n\|_{\mathcal{B}(X) \to \mathcal{B}(X)} \leq 2$ for all n. It suffices to show that the norm $\|C_\phi - C_\phi \widetilde{V}_n\|$ can be made arbitrarily small by choosing n large enough. Namely, the weak compactness of the boundedly supported Fourier multiplier $\widetilde{V}_n$ is established exactly as in the proof of Proposition 1. Condition (20) yields $r < 1$ such that $|\phi'(z)|(1 - |z|^2) \leq \varepsilon(1 - |\phi(z)|^2)/3$ if $|\phi(z)| \geq r$. Proposition 1 in turn enables us to choose an integer n such that for all $h \in H^\infty(X)$ it holds that $\|(h - V_n h)(z)\|_X \leq \varepsilon(1 - r)\|h\|_{H^\infty(X)}$ whenever $|z| \leq \sqrt{r}$. Hence for $|z| \leq r$ we arrive at the estimate

$$
\begin{aligned}
(24) \qquad \|(f - \widetilde{V}_n f)'(z)\|_X &\leq \sup_{|z| \leq \sqrt{r}} \|(f'_{\sqrt{r}} - V_n f'_{\sqrt{r}})(z)\|_X \\
&\leq \varepsilon(1 - r)\|f'_{\sqrt{r}}\|_{H^\infty(X)} \leq \varepsilon\|f\|_{\mathcal{B}(X)}
\end{aligned}
$$

according to the definition of the Bloch norm. Suppose next that $f \in \mathcal{B}(X)$ and $z \in D$ are arbitrary and consider

$$A := (1 - |z|^2)\|(C_\phi g)'(z)\|_X = \frac{|\phi'(z)(1 - |z|^2)|}{1 - |\phi(z)|^2}\|g'(\phi(z))\|_X(1 - |\phi(z)|^2),$$

where we have denoted $g := f - \widetilde{V}_n f$. If $|\phi(z)| > r$, the choice of r yields that $A \leq (\varepsilon/3)\|g\|_{B(X)} \leq \varepsilon\|f\|_{B(X)}$. On the other hand, for $|\phi(z)| \leq r$ an application of the Schwarz lemma and (24) shows that $A \leq \varepsilon\|f\|_{B(X)}$. Hence $\|C_\phi - C_\phi \widetilde{V}_n\| \leq \varepsilon$ and the proof of part (i) is finished.

(ii)   The proof of the sufficiency of the stated conditions follows from that of part (i) since (21) clearly implies (20). One only has to check that condition (21) yields that C_ϕ maps $\mathcal{B}_0(X)$ inside $\mathcal{B}_0(X)$, which is straightforward. For the necessity one notes that weak compactness of C_ϕ on $\mathcal{B}_0$ implies (21) according to [**10**, Theorems 1 and 3].   □

Madigan and Matheson [**10**, Theorem 3] observed that every weakly compact composition operator on B_0 is compact. This actually follows directly from the general fact that every weakly compact operator on $\mathcal{B}_0$ is compact, which is a consequence of the Gantmacher theorem [**18**, II.c.6.b] and the Schur property of $\mathcal{B}_0'$. We record separately a special case of our Theorem 4 that complements [**10**, Theorem 3]. Recall that, contrary to the case of $\mathcal{B}_0$, there are weakly compact operators on $\mathcal{B}$ that are not compact, since $\mathcal{B} \approx \ell^\infty$ (see [**18**, Remark on p. 45]) .

Corollary 5. $C_\phi : \mathcal{B} \to \mathcal{B}$ *is weakly compact only if it is compact.*

In our last result about weak compactness we consider the operator C_ϕ on $H^\infty(X)$.

Theorem 6. $C_\phi : H^\infty(X) \to H^\infty(X)$ *is weakly compact if and only if X is reflexive and*

(25) $\|\phi\|_\infty < 1.$

Proof. (i)   Assume first that X is reflexive and let $\phi \in H^\infty$ with $\|\phi\|_\infty < 1$. We again show that $\|C_\phi - C_\phi V_n\|_{H^\infty(X) \to H^\infty(X)}$ can be made arbitrarily small once n is large enough. Given $\varepsilon > 0$, Proposition 2 yields an integer n such that $\|f(w) - V_n f(w)\|_X \leq \varepsilon\|f\|_{H^\infty(X)}$ whenever $|w| \leq \|\phi\|_\infty$ and $f \in H^\infty(X)$. Hence we get the desired estimate

$$\|(C_\phi f - C_\phi V_n f)(z)\|_X = \|(f - V_n f)(\phi(z))\|_X \leq \varepsilon\|f\|_{H^\infty(X)}$$

for all $z \in D$. For the converse, assume that $C_\phi : H^\infty(X) \to H^\infty(X)$ is weakly compact. Deduce as before that X is reflexive and the scalar operator $C_\phi : H^\infty \to H^\infty$ is weakly compact. We invoke the beautiful result, due to Ülger [**17**] and independently to Aron, Galindo and Lindström [**1**], stating that every weakly compact homomorphism on H^∞ is compact. In our special cituation it follows that C_ϕ is compact on H^∞, and hence that $\|\phi\|_\infty < 1$, see [**5**, Exercise 3.2.2]. Alternatively, if one wants to avoid the general result for homomorphisms, a direct argument can be obtained by specializing to the case $E = \mathbf{C}$ in the proof of the implication $(2) \Rightarrow (3)$ of [**1**, Proposition 3]. $\qquad\square$

Remark. If ϕ belongs to the disk algebra A_0, then the above theorem holds with the same proof also for the vector-valued 'disk algebra' $A_0(X)$, which is the subspace of $H^\infty(X)$ containing those functions that extend continuously to the closed disk $\overline{D}$.

We finally briefly consider another, somewhat related 'smallness' property of the composition operator C_ϕ. Given a Banach space E, recall that an operator $T \in L(E)$ is weakly conditionally compact (abbreviated T is w.c.c) if any bounded sequence (x_n) in E contains a subsequence (x_{n_k}) such that the sequence (Tx_{n_k}) is weakly Cauchy. Rosenthal's ℓ^1-criterion (see [**8**, 2.e.5]) implies that T is not w.c.c if and only if T fixes a copy of ℓ^1 in E. Note that if ℓ^1 does not embed into E, then every $T \in L(E)$ is w.c.c. Hence the set of w.c.c. operators on E is, in general, strictly larger than the set of weakly compact operators on E. Our previous results have the following counterparts.

Theorem 7. *Assume that ℓ^1 does not embed in X. Then C_ϕ is w.c.c on $H^1(X)$ if and only if ϕ satisfies the Shapiro condition* (17). *Analogously, the conditions* (16), (20) *and* (25) *characterize when C_ϕ is w.c.c on $B_1(X)$, $\mathcal{B}(X)$ and $H^\infty(X)$, respectively.*

Proof. The sufficiency of the stated conditions is established in each case exactly as in the proofs of Theorems 3, 4 and 6. Namely, the operators V_n or $\widetilde{V}_n$ are w.c.c in the case that ℓ^1 does not embed in X, since each summand in (7) clearly factorizes through X. Towards the other direction, assume that C_ϕ is w.c.c. on one of the spaces considered in the theorem. Clearly ℓ^1 cannot embed in X while C_ϕ fixes a copy of X. Moreover, since the vector-valued C_ϕ always carries a 'copy' of the corresponding scalar operator, in order to deduce the appropriate condition on ϕ, it is enough according to Theorems 3, 4 and 7.(i) to show that all w.c.c. operators on the spaces $H^1, B_1, \mathcal{B}$ and H^∞ are weakly compact. In the cases of H^1 and B_1 this follows immediately

from the weak sequential completeness of the spaces $L_1(\mu)$ (see [**18**, III.C.14] and recall that H^1 is a closed subspace of $L^1(\partial D)$ and $B_1 \approx \ell^1$). For H^∞ (resp. $\mathcal{B}$) one uses the deep result of Bourgain [**3**, Remark p. 45] (resp. of Rosenthal [**8**, 2.f.4]) stating that every non-weakly compact operator on H^∞ (resp. ℓ^∞) acts isomorphically on a copy of ℓ^∞, and hence on a copy of ℓ^1. Recall here that $\mathcal{B} \approx \ell^\infty$ (see e.g. [**18**, Remark on p. 45]). □

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WUHAN UNIVERSITY
WUHAN, HUBEI 430072
P. R. CHINA
E-mail address: pdliu@whu.edu.cn

P.O. BOX 4 (YLIOPISTONKATU 5)
FIN–00014 UNIVERSITY OF HELSINKI
FINLAND
E-mail address: saksman@cc.helsinki.fi

P.O. BOX 4 (YLIOPISTONKATU 5)
FIN–00014 UNIVERSITY OF HELSINKI
FINLAND
E-mail address: hojtylli@cc.helsinki.fi