

II₁-SUBFACTORS ASSOCIATED WITH THE C^* -TENSOR CATEGORY OF A FINITE GROUP

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We determine the subfactors $N \subset R$ of the hyperfinite II₁-factor R with finite index for which the C^* -tensor category of the associated (N, N) -bimodules is equivalent to the C^* -tensor category \mathcal{U}_G of all unitary finite dimensional representations of a given finite group G . It turns out that every subfactor of that kind is isomorphic to a subfactor $R^G \subset (R \otimes \mathbf{L}(\mathbb{C}^r))^H$, where R^G is the fixed point algebra under an outer action α of G , H is a subgroup of G , $\psi : H \rightarrow \mathbf{U}(\mathbb{C}^r)$ is a unitary finite dimensional projective representation of H satisfying a certain additional condition and $(R \otimes \mathbf{L}(\mathbb{C}^r))^H$ is the fixed point algebra under the action $\alpha|_H \otimes \text{Ad } \psi$ of H on $R \otimes \mathbf{L}(\mathbb{C}^r)$.

Introduction.

Recently the concept of C^* -tensor categories appeared in Jones' theory of subfactors. Mainly we are interested in C^* -tensor categories as introduced by J.E. Roberts and R. Longo in [7] (see also [12], [2] and [15]), in which every object has a conjugate. C^* -tensor categories of that kind are called compact in this article. The most natural example of a compact C^* -tensor category is the category \mathcal{U}_G of the unitary finite dimensional representations of a compact group G . Compact C^* -tensor categories may be regarded as a concept to deal with more general symmetries than those described by groups. In particular the unitary finite dimensional corepresentations of a compact matrix pseudogroup (in the sense of S.L. Woronowicz [19]) form a compact C^* -tensor category, but the concept of a compact C^* -tensor category is more comprehensive than that of a compact quantum group.

The (N, N) -bimodules belonging to a II₁-subfactor $N \subset M$ with finite Jones index form a compact C^* -tensor category $\mathcal{B}_{N \subset M}$. The number of the equivalence classes of the irreducible objects of $\mathcal{B}_{N \subset M}$ is finite if and only if the inclusion $N \subset M$ has finite depth. As S. Popa has shown, the subfactors of the hyperfinite II₁-factor with finite depth are completely classified by their standard invariant, which is given by a single commuting square of finite dimensional $*$ -algebras (see [11]). A. Ocneanu found the so-called flatness condition which allows to decide which of the commuting squares

are standard invariants of finite depth subfactors (see [10]). Unfortunately, checking the flatness condition requires very complicated computations in most cases.

But using C^* -tensor categories we get another method for the classification of finite depth subfactors of the hyperfinite factor. It requires the solution of two problems:

- 1) Classify all finite C^* -tensor categories up to equivalence (i.e. all compact C^* -tensor categories for which the number of equivalence classes of irreducible objects is finite).
- 2) Classify all subfactors $N \subset R$ of the hyperfinite II_1 -factor R with finite index, for which the associated C^* -tensor category of the (N, N) -bimodules is equivalent to a given C^* -tensor category \mathcal{C} .

The goal of this paper is to deal with problem 2), if \mathcal{C} is the C^* -tensor category \mathcal{U}_G for a finite group G . The main result of this paper is that every subfactor $N \subset R$ of this kind is isomorphic to a subfactor

$$R^G \subset (R \otimes \mathbf{L}(\mathbb{C}^r))^H.$$

Here R^G denotes the fixed point algebra under an outer action α of G on R , H is a subgroup of G , $\psi : H \rightarrow \mathbf{U}(\mathbb{C}^r)$ is a projective representation of H (satisfying a certain condition), and $(R \otimes \mathbf{L}(\mathbb{C}^r))^H$ denotes the fixed point algebra under an action of H where H acts on R by $\alpha|_H$ and on $\mathbf{L}(\mathbb{C}^r)$ via the conjugation with ψ . In the proof we have to generalize the method used by M. Nakamura and Z. Takeda in order to determine the von Neumann algebras between R^G and R . The proof also uses induced representations and the imprimitivity theorem. Unfortunately, it is possible that different choices of H and ψ yield isomorphic subfactors.

This article is the abridged version of a part of the author's Habilitationsschrift [14].

1. Preliminaries.

1.1. C^* -tensor categories.

We suppose that all sets appearing in this article are small, i.e. belong to a fixed universe. Let \mathcal{C} be a C^* -tensor category. We assume that the objects of \mathcal{C} form a set. For two objects ρ and σ of \mathcal{C} , the (tensor) product of ρ and σ is denoted by $\rho\sigma$ and the space of morphisms with source ρ and target σ by (ρ, σ) . The product of two morphisms $T \in (\rho, \rho')$ and $S \in (\sigma, \sigma')$ is denoted by $T \times S \in (\rho\sigma, \rho'\sigma')$. 1_σ denotes the identity morphism of an object σ of \mathcal{C} .

A compact C^* -tensor category is a strict C^* -tensor category for which every object σ has a conjugate $\bar{\sigma}$ and for which the space (ι, ι) is one di-

mensional (ι unit object). [15] contains a detailed definition. A compact C^* -tensor category \mathcal{C} is called finite if the number of the equivalence classes of the irreducible objects of \mathcal{C} is finite. The statistical dimension of an object σ of a compact C^* -tensor category is denoted by $d(\sigma)$.

Let ρ be an object of a finite C^* -tensor category \mathcal{C} . \mathcal{C}_ρ denotes the following full C^* -tensor subcategory of \mathcal{C} :

The objects of \mathcal{C}_ρ form the smallest subset \mathcal{O}_ρ of the set of objects of \mathcal{C} with the following properties:

- (a) $\iota, \rho \in \mathcal{O}_\rho$.
- (b) If $\tau \in \mathcal{O}_\rho$, then every object equivalent to τ and every subobject of τ belongs to \mathcal{O}_ρ .
- (c) If $\tau, \phi \in \mathcal{O}_\rho$, then $\bar{\tau} \in \mathcal{O}_\rho$ and $\tau\phi \in \mathcal{O}_\rho$.
- (d) Finite direct sums of objects of \mathcal{O}_ρ are objects of \mathcal{O}_ρ .

Let $N \subset M$ be an inclusion of II_1 -factors with finite Jones index $[M : N] < \infty$ and with $N \neq M$ and let

$$N = M_{-1} \subset M = M_0 \subset M_1 \subset M_2 \subset M_3 \subset \dots$$

be the Jones tower for $N \subset M$. An (N, N) -bimodule is a Hilbert space \mathcal{H} endowed with a normal left action λ and a normal right action ρ of N on \mathcal{H} such that $\lambda(N)$ and $\rho(N)$ commute. The (N, N) -bimodules which are equivalent to a subbimodule of ${}_N L^2(M_k)_N$ for some $k \in \mathbb{N}$ form a compact C^* -tensor category $\mathcal{B}_{N \subset M}$, where the product of objects is the N -tensor product \otimes_N (see [15]). $\mathcal{B}_{N \subset M}$ is the smallest C^* -tensor category of (N, N) -bimodules containing the bimodule ${}_N L^2(M)_N$ as an object.

1.2. II_1 -subfactors defined by finite C^* -tensor categories.

It is possible to construct a finite depth subfactor of the hyperfinite II_1 -factor for a given object σ of a finite C^* -tensor category \mathcal{C} (see [15], Section 3.3, compare also [7]. The construction seems to be known to several people). Let $\bar{\sigma}$ be an object conjugate to σ . If $d(\sigma) > 1$ the tower

$$(1) \quad \begin{array}{ccccccc} (\sigma, \sigma) & \subset & (\bar{\sigma} \sigma, \bar{\sigma} \sigma) & \subset & (\sigma \bar{\sigma} \sigma, \sigma \bar{\sigma} \sigma) & \subset & (\bar{\sigma} \sigma \bar{\sigma} \sigma, \bar{\sigma} \sigma \bar{\sigma} \sigma) & \subset & \dots \\ \cup & & \cup & & \cup & & \cup & & \\ (\iota, \iota) & \subset & (\bar{\sigma}, \bar{\sigma}) & \subset & (\sigma \bar{\sigma}, \sigma \bar{\sigma}) & \subset & (\sigma \bar{\sigma} \sigma, \sigma \bar{\sigma} \sigma) & \subset & \dots \end{array}$$

of finite dimensional $*$ -algebras fulfils the periodicity assumptions used in H. Wenzl's subfactor construction (Theorem 1.5 in [18]). (The inclusion of two successive algebras in the same line is given by $T \mapsto \mathbf{1}_\sigma \times T$ or $T \mapsto \mathbf{1}_{\bar{\sigma}} \times T$ and for two algebras in the same column, it is given by $T \mapsto T \times \mathbf{1}_\sigma$.) Hence the tower defines a II_1 -subfactor $A \subset B$, where the union of the $*$ -algebras

in the upper line (resp. lower line) is ultra-strongly dense in B (resp. A). Obviously, B is the hyperfinite II_1 -factor. We get $[B : A] = d(\sigma)^2$. Let

$$A = B_{-1} \subset B = B_0 \subset B_1 \subset B_2 \subset B_3 \subset \dots$$

be the Jones tower for $A \subset B$. The standard invariant

$$\begin{aligned} \mathbb{C}\mathbf{1} &= A' \cap A \subset A' \cap B \subset A' \cap B_1 \subset A' \cap B_2 \subset \dots \\ &\quad \cup \quad \quad \cup \quad \quad \cup \\ \mathbb{C}\mathbf{1} &= B' \cap B \subset B' \cap B_1 \subset B' \cap B_2 \subset \dots \end{aligned}$$

of $A \subset B$ is equal to

$$(2) \quad \begin{aligned} &(\iota, \iota) \subset (\sigma, \sigma) \subset (\sigma\bar{\sigma}, \sigma\bar{\sigma}) \subset (\sigma\bar{\sigma}\sigma, \sigma\bar{\sigma}\sigma) \subset \dots \\ &\quad \cup \quad \quad \cup \quad \quad \cup \\ &(\iota, \iota) \subset (\bar{\sigma}, \bar{\sigma}) \subset (\bar{\sigma}\sigma, \bar{\sigma}\sigma) \subset \dots \end{aligned}$$

Consequently, the subfactor $A \subset B$ has finite depth. The C^* -tensor category $\mathcal{B}_{A \subset B}$ is equivalent to the full C^* -tensor subcategory $\mathcal{C}_{\sigma\bar{\sigma}}$ of \mathcal{C} (see [15], Theorem 4.1).

1.3. Subfactors of the hyperfinite II_1 -factor belonging to a given finite C^* -tensor category \mathcal{C} .

Let a finite C^* -tensor category \mathcal{C} be given. We assume that $N \subsetneq M = R$ is a subfactor of the hyperfinite II_1 -factor R with finite Jones index such that $\mathcal{B}_{N \subset M}$ is equivalent to \mathcal{C} . Let $A \subset B$ be the subfactor from Section 1.2, where the C^* -tensor category $\mathcal{B}_{N \subset M}$ and the object ${}_N L^2(M)_N$ are used. Example 3.5 in [15] shows that $N \subset M_1$ and $A \subset B$ have the same standard invariant. So they are isomorphic according to S. Popa's result [11] mentioned in the Introduction. If we replace $\mathcal{B}_{N \subset M}$ by the equivalent C^* -tensor category \mathcal{C} and ${}_N L^2(M)_N$ by the corresponding object σ of \mathcal{C} , we obtain the same subfactor $A \subset B$. We observe that $\sigma = \bar{\sigma}$ and that the unit object ι is a subobject of σ . Furthermore we notice $\mathcal{B}_{N \subset M} = \mathcal{B}_{N \subset M_1}$, which follows from $L^2(M) \subset L^2(M_1)$ and $L^2(M_1) \cong L^2(M) \otimes_N L^2(M)$.

If we want to find all subfactors $N \subsetneq R$ of the hyperfinite II_1 -factor R for which the C^* -tensor category $\mathcal{B}_{N \subset M}$ is equivalent to \mathcal{C} , we could proceed as follows:

Regard the objects σ of \mathcal{C} with the following properties:

- (a) $\sigma = \bar{\sigma}$,
- (b) ι is a proper subobject of σ ,
- (c) the C^* -tensor category $\mathcal{C}_\sigma (= \mathcal{C}_{\sigma\bar{\sigma}}$ by Property (a) and (b)) is equal to \mathcal{C} .

Form the subfactors $A \subset B$ for these objects σ according to Section 1.2 and determine the factors C between A and B such that the subfactor $A \subset B$ is isomorphic to the subfactor $A \subset B(A, C)$ where $B(A, C)$ is the basic construction for the subfactor $A \subset C$. For every subfactor $A \subset C$ of that kind the C^* -tensor category is in fact equivalent to \mathcal{C} .

1.4. Notation for groups and their representations.

The group algebra for a finite group G is denoted by $\mathbb{C}[G]$. By a (unitary) projective representation of a finite group G we mean a map π from G into the set $\mathbf{U}(\mathcal{H})$ of the unitary operators of a Hilbert space \mathcal{H} , for which there is a function $c : G \times G \longrightarrow T := \{z \in \mathbb{C} : |z| = 1\}$ such that

$$(3) \quad c(g, h) \pi(g) \pi(h) = \pi(gh)$$

holds for $g, h \in G$. The function $c : G \times G \longrightarrow T$ is called a cocycle of G . A projective representation π of G satisfying (3) is called a c -projective representation of G . The projective kernel $\text{proj ker } \pi$ of π is $\text{proj ker } \pi := \{g \in G : \pi(g) \in \mathbb{C}\mathbf{1}\}$.

$\mathbf{L}(\mathcal{H})$ denotes the algebra of all continuous linear operators on \mathcal{H} . We denote the action $x \in \mathbf{L}(\mathcal{H}) \longmapsto \pi(g) x \pi(g^{-1})$ of G on $\mathbf{L}(\mathcal{H})$ by $\text{Ad } \pi$.

Two projective representations $\pi_1 : G \longrightarrow \mathbf{U}(\mathcal{H}_1)$ and $\pi_2 : G \longrightarrow \mathbf{U}(\mathcal{H}_2)$ are called equivalent, if there are a unitary operator $U : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ and a function $\mu : G \longrightarrow T$ such that $U \pi_1(g) U^* = \mu(g) \pi_2(g)$ for every $g \in G$.

Let H be a subgroup of G . We put $N(H) := \bigcap_{g \in G} gHg^{-1}$. We use the notation G/H for the space of the left cosets kH in G . We assume that a set \mathcal{V} of representatives for the left cosets kH , $k \in G$, is fixed such that the neutral element e belongs to \mathcal{V} . For every $g \in G$, $g = k(g)h(g)$ denotes the unique decomposition of g such that $k(g) \in \mathcal{V}$ and $h(g) \in H$.

If $\pi : H \longrightarrow \mathbf{U}(\mathcal{K})$ is a finite dimensional representation of H we define the induced representation $\text{ind } \pi : G \longrightarrow \mathbf{U}(\mathcal{K} \otimes \ell^2(G/H))$ by

$$(\text{ind } \pi)(g) (\xi \otimes \delta_{kH}) = \pi(h(gk)) \xi \otimes \delta_{gkH}$$

for $\xi \in \mathcal{K}$, $k \in \mathcal{V}$ and $g \in G$.

1.5. The subfactor $A \subset B$ for a finite group G .

We consider the C^* -tensor category \mathcal{U}_G of the unitary finite dimensional representations of a finite group G . The product of objects is the usual tensor product of representations, the object $\bar{\rho}$ conjugate to an object $\rho : G \longrightarrow \mathbf{U}(\mathcal{H})$ is the contragredient representation, and (ρ, ρ) is the fixed point algebra $\mathbf{L}(\mathcal{H})$ under the action $\text{Ad } \rho$.

There exists an outer action of G on the hyperfinite II_1 -factor R . V.F.R. Jones proved that two outer actions of G on R are conjugate (see [5]). The

fixed point algebra R^G under an outer action α of G is a II_1 -factor. If $\sigma : G \longrightarrow \mathbf{U}(\mathcal{H})$ is a finite dimensional representation of G , $\alpha \otimes \text{Ad } \sigma$ is an action of G on $R \otimes \mathbf{L}(\mathcal{H})$ and the fixed point algebra $(R \otimes \mathbf{L}(\mathcal{H}))^G$ is a II_1 -factor. The subfactor $R^G \subset (R \otimes \mathbf{L}(\mathcal{H}))^G$ has index $(\dim \mathcal{H})^2$. If $\dim \mathcal{H} > 1$ the standard invariant of this subfactor is equal to the standard invariant of the subfactor $A \subset B$ from Section 1.2 where the C^* -tensor category \mathcal{U}_G and the object σ is used (see [17] or [13]). According to S. Popa's result [11] these subfactors are isomorphic.

2. The Theorem.

2.1. The subfactors $R^G \subset (R \otimes \mathbf{L}(\mathbb{C}^r))^H$.

As before we assume an outer action α of G on R . Moreover, let $\psi : H \longrightarrow \mathbf{U}(\mathbb{C}^r)$ be a projective representation of a subgroup H of G . $\alpha|_H \otimes \text{Ad } \psi$ is an action of H on $R \otimes \mathbf{L}(\mathbb{C}^r)$, let $(R \otimes \mathbf{L}(\mathbb{C}^r))^H$ be the fixed point algebra under this action. Then $R^G \subset (R \otimes \mathbf{L}(\mathbb{C}^r))^H$ is a II_1 -subfactor with index $[G : H] \cdot r^2$. In [13] these subfactors are investigated and their principal and dual principal graphs are computed. In [13] only ordinary representations ψ are used, but the transfer to projective representations is easy.

We review some facts from [13]. The tensor product $\bar{\psi} \otimes \psi$ of the conjugate representation $\bar{\psi}$ and of ψ is an ordinary representation of H and the induced representation $\text{ind}(\bar{\psi} \otimes \psi)$ of G is defined on the Hilbert space $\mathcal{V} := \overline{\mathbb{C}^r} \otimes \mathbb{C}^r \otimes \ell^2(G/H)$, where $\overline{\mathbb{C}^r}$ is the Hilbert space dual to \mathbb{C}^r . The $*$ -algebra $\ell^\infty(G/H)$ acts on $\ell^2(G/H)$ by multiplication, the $*$ -algebra

$$C := \mathbb{C} \text{id}_{\overline{\mathbb{C}^r}} \otimes \mathbf{L}(\mathbb{C}^r) \otimes \ell^\infty(G/H) \cong \mathbf{L}(\mathbb{C}^r) \otimes \ell^\infty(G/H)$$

of linear operators on \mathcal{V} is invariant under the action $\text{Ad } \text{ind}(\bar{\psi} \otimes \psi)$ of G . Let $e_{kH} \in \ell^\infty(G/H)$ be the linear operator on $\ell^2(G/H)$ determined by $e_{kH} \delta_{lH} = \delta_{kH, lH} \delta_{kH}$ for $l \in \mathcal{V}$. Every element x of $(R \otimes C)^G$ can be written uniquely as $x = \sum_{k \in \mathcal{V}} x_k \otimes e_{kH}$ where $x_k \in R \otimes \mathbf{L}(\mathbb{C}^r)$. $x \in (R \otimes C)^G \longmapsto x_e$ defines an isomorphism ι from $(R \otimes C)^G$ onto $(R \otimes \mathbf{L}(\mathbb{C}^r))^H$, hence the subfactors $R^G \subset (R \otimes C)^G$ and $R^G \subset (R \otimes \mathbf{L}(\mathbb{C}^r))^H$ are isomorphic. By using this result and an invariance principle due to A. Wassermann, one finds that the subfactor obtained from $R^G \subset (R \otimes \mathbf{L}(\mathbb{C}^r))^H$ by an application of the basic construction is isomorphic to $R^G \subset (R \otimes \mathbf{L}(\mathcal{V}))^G$ where $\text{Ad } \text{ind}(\bar{\psi} \otimes \psi)$ acts on $\mathbf{L}(\mathcal{V})$. So we obtain $\mathcal{B}_{R^G \subset (R \otimes \mathbf{L}(\mathbb{C}^r))^H} = \mathcal{B}_{R^G \subset (R \otimes \mathbf{L}(\mathcal{V}))^G}$. According to the Sections 1.2, 1.3 and 1.5, the C^* -tensor category $\mathcal{B}_{R^G \subset (R \otimes \mathbf{L}(\mathbb{C}^r))^H}$ is equivalent to $(\mathcal{U}_G)_{\text{ind}(\bar{\psi} \otimes \psi)}$. The kernel K of $\text{ind}(\bar{\psi} \otimes \psi)$ is equal to the projective kernel $\text{proj } \ker \psi|_{N(H)}$ of the restriction of ψ onto $N(H)$.

By applying Theorem 27.39 in [3] and observing $\overline{\text{ind}(\bar{\psi} \otimes \psi)} = \text{ind}(\bar{\psi} \otimes \psi)$, we obtain that every irreducible representation of G is contained in the n -

fold tensor product $(\text{ind}(\bar{\psi} \otimes \psi))^{\otimes n}$ for some $n \in \mathbb{N}$ if and only if $K = \{e\}$. So $\mathcal{B}_{R^G \subset (R \otimes \mathbf{L}(\mathbb{C}^r))^H}$ is equivalent to \mathcal{U}_G if and only if $\text{proj ker } \psi|N(H) = \{e\}$.

The subfactor $R^G \subset (R \otimes \mathbf{L}(\mathbb{C}^r))^H$ is irreducible if and only if ψ is irreducible. We even have the more general result

$$(R^G)' \cap (R \otimes \mathbf{L}(\mathbb{C}^r))^H = \mathbb{C}\mathbf{1} \otimes \psi(H)'.$$

The inclusion $' \supset'$ is obvious, the other inclusion follows from

$$(R^G)' \cap \left((R \otimes \mathbf{L}(\mathbb{C}^r))^H \right) \subset \left(((R^G)' \cap R) \otimes \mathbf{L}(\mathbb{C}^r) \right)^H.$$

Now let us formulate the main result of this article:

Theorem 2.2. *Let G be a finite group and let $N \subset M = R$ be a subfactor of the hyperfinite II_1 -factor R with finite Jones index such that the C^* -tensor category $\mathcal{B}_{N \subset M}$ is equivalent to \mathcal{U}_G . Then there are a subgroup H of G , a projective finite dimensional representation $\psi : H \rightarrow \mathbf{U}(\mathbb{C}^r)$ ($r \in \mathbb{N}$) of H satisfying $\text{proj ker } \psi|N(H) = \{e\}$ and an outer action α of G on the hyperfinite II_1 -factor R such that $N \subset M$ is isomorphic to*

$$(4) \quad R^G \subset (R \otimes \mathbf{L}(\mathbb{C}^r))^H.$$

2.3. Remarks.

(1) Let $R^G \subset (R \otimes \mathbf{L}(\mathbb{C}^r))^H$ and $R^{\tilde{G}} \subset (R \otimes \mathbf{L}(\mathbb{C}^{\tilde{r}}))^{\tilde{H}}$ be two subfactors as in Theorem 2.2 and let $\tilde{\psi} : \tilde{H} \rightarrow \mathbf{U}(\mathbb{C}^{\tilde{r}})$ denote the projective representation used in the second subfactor.

If there is an isomorphism γ from G onto \tilde{G} such that $\gamma(H) = \tilde{H}$ and such that the projective representation $\tilde{\psi} \circ \gamma$ of H is equivalent to the representation ψ , then the subfactors are isomorphic.

The converse is not true, as the following easy example shows:

Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, $G = H$ and $\psi : \mathbb{Z}_2 \times \mathbb{Z}_2 \rightarrow \mathbf{U}(\mathbb{C}^2)$ be the projective representation of $\mathbb{Z}_2 \times \mathbb{Z}_2$ given by the Pauli matrices:

$$\begin{aligned} \psi(0,0) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \psi(1,0) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \psi(0,1) &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, & \psi(1,1) &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned}$$

The subfactor $R^G \subset (R \otimes \mathbf{L}(\mathbb{C}^2))^G$ given by these data is an irreducible subfactor with index 4 and depth 2. (The principle graph can be computed

as in [13]. Observe that every irreducible projective representation with the same cocycle is equivalent to ψ .) All irreducible subfactors of the hyperfinite II_1 -factor with index 4 and depth 2 are isomorphic to $R^G \subset R$ or $R^{\mathbb{Z}_4} \subset R$. The C^* -tensor category $\mathcal{B}_{R^G \subset (R \otimes \mathbf{L}(\mathbb{C}^2))^G}$ is equivalent to \mathcal{U}_G and $\mathcal{B}_{R^{\mathbb{Z}_4} \subset R}$ to $\mathcal{U}_{\mathbb{Z}_4}$. Moreover, \mathcal{U}_G and $\mathcal{U}_{\mathbb{Z}_4}$ are not equivalent. So the subfactor $R^G \subset (R \otimes \mathbf{L}(\mathbb{C}^2))^G$ is isomorphic to $R^G \subset R$.

(2) The principal graph and the dual principal graph of the subfactors (4) can be computed as in [13]. A slight modification is necessary as only ordinary representations are used there.

The C^* -tensor categories associated with the subfactors dual to the subfactors (4) are investigated in [14]. If ψ is an ordinary representation they can be described by the (H, H) -vector bundles over G introduced in [6] by H. Kosaki and S. Yamagami. The general case requires a slight generalization.

2.4. Examples.

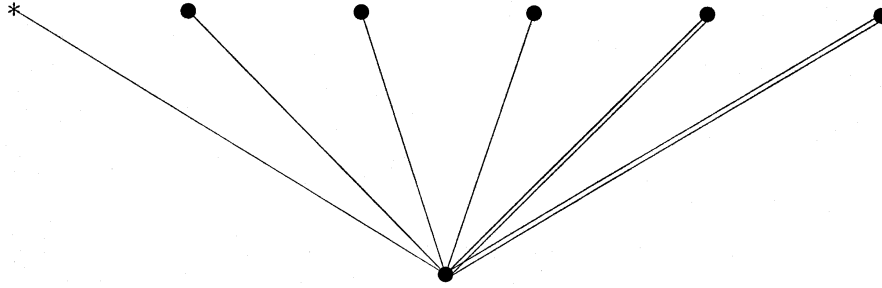
(1) Let G be the group \mathcal{S}_3 . Theorem 2.2 (and Remark 2.3 (1)) imply that there are three irreducible subfactors of the hyperfinite II_1 -factor such that the associated C^* -tensor category is equivalent to \mathcal{U}_G :

- $R^{\mathcal{S}_3} \subset R$, $[R : R^{\mathcal{S}_3}] = 6$, depth 2.
- $R^{\mathcal{S}_3} \subset R^{\mathbb{Z}_2}$, $[R^{\mathbb{Z}_2} : R^{\mathcal{S}_3}] = 3$, unique subfactor of the hyperfinite II_1 -factor with principal graph A_5 .
- $R^{\mathcal{S}_3} \subset (R \otimes \mathbf{L}(\mathbb{C}^2))^{\mathcal{S}_3} =: S$ (with the unique irreducible two dimensional representation $\psi : \mathcal{S}_3 \rightarrow \mathbf{U}(\mathbb{C}^2)$ of \mathcal{S}_3),
 $[S : R^{\mathcal{S}_3}] = 4$, principal graph $D_5^{(1)}$.

(2) Let G be the group $\mathbb{Z}_2 \times \mathcal{S}_3$. We denote the subgroup $\mathbb{Z}_2 \times \langle (1, 2) \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ of $\mathbb{Z}_2 \times \mathcal{S}_3$ by K . Using the results of [4], V., §25, we obtain that $|H^2(K, T)| = 2$ and that $|H^2(H, T)| = 1$ if H is a subgroup of G not conjugate to K . Similarly as before, we find that there are three irreducible subfactors of the hyperfinite II_1 -factor such that the associated C^* -tensor category is equivalent to \mathcal{U}_G :

- $R^G \subset R$, $[R : R^G] = 12$, depth 2.
- $R^G \subset R^{\{e\} \times \langle (1, 2) \rangle} =: T$, $[T : R] = 6$.
- $R^G \subset (R \otimes L(\mathbb{C}^2))^K =: F$ (with the irreducible projective representation $\psi : K \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \rightarrow \mathbf{U}(\mathbb{C}^2)$ from Remark 2.3 (1)),
 $[F : R] = 12$.

We use the methods from [13] for computing the principal and dual principal graph of the subfactor $R^G \subset F$. Every irreducible representation of H with the same cocycle as ψ is equivalent to ψ ; hence the subfactor has depth 2.



is the principal graph as well as the dual principal graph of $R^G \subset F$. According to [16] there is a 12 dimensional Hopf- $*$ -algebra A such that $R^G \subset F$ is equivalent to $R^A \subset R$ where A acts on R by an outer action. A is neither commutative nor cocommutative as in the principal and in the dual principal graph there are even vertices which are connected with the unique odd vertex by more than one edge. In particular, $R^G \subset F$ is not equivalent to $R^G \subset R$.

The C^* -tensor category $\mathcal{B}_{R^A \subset R}$ is equivalent to the C^* -tensor category $\mathcal{U}_{A^{cop}}$ of the unitary corepresentations of A^{cop} , where A^{cop} is the Hopf- $*$ -algebra A endowed with the reversed comultiplication (see [14] for details). So the C^* -tensor category \mathcal{U}_G is equivalent to $\mathcal{U}_{A^{cop}}$ although A^{cop} is not commutative.

It would be interesting to find an answer for the following question:

Question 2.5. Does there exist non-isomorphic finite groups G_1 and G_2 for which the C^* -tensor categories \mathcal{U}_{G_1} and \mathcal{U}_{G_2} are equivalent?

Observe that neither the Tannaka-Krein theorem nor S. Doplicher's and J.E. Roberts' results concerning compact groups and C^* -tensor categories with a symmetry (see [1]) imply the answer no.

3. The Proof of the Theorem.

Let $G \neq \{e\}$ and let $N \subset M = R$ be a subfactor of the hyperfinite II_1 -factor R with finite index such that the C^* -tensor category $\mathcal{B}_{N \subset M}$ is equivalent to \mathcal{U}_G . Section 1.3 and Section 1.5 show that there is a finite dimensional representation $\sigma : G \longrightarrow \mathbf{U}(\mathcal{H})$ of G such that the subfactor $N \subset M_1$ is isomorphic to the subfactor

$$R^G \subset (R \otimes \mathbf{L}(\mathcal{H}))^G$$

where G acts on R by an outer action α and on $R \otimes \mathbf{L}(\mathcal{H})$ by $\alpha \otimes \text{Ad } \sigma$.

As M is a factor between N and M_1 , there is a factor P between R^G and $(R \otimes \mathbf{L}(\mathcal{H}))^G$ such that $N \subset M$ is isomorphic to $R^G \subset P$. We intend to determine the factors P between R^G and $(R \otimes \mathbf{L}(\mathcal{H}))^G$.

In [8] and [9] M. Nakamura and Z. Takeda showed that every von Neumann algebra between R^G and R is equal to R^H for some subgroup H of G . Some parts of our subsequent considerations may be regarded as a generalization of the methods used there. We will investigate the von Neumann algebras between $((R \otimes \mathbf{L}(\mathcal{H}))^G)'$ and $(R^G)'$ where the commutants are formed in $L^2(R) \otimes \mathcal{H}$. $R \otimes \mathbf{L}(\mathcal{H})$ acts on $L^2(R) \otimes \mathcal{H}$ by $(r \otimes x) \bar{s} \otimes \xi = \bar{r} \bar{s} \otimes x \xi$ for $r \in R$, $x \in \mathbf{L}(\mathcal{H})$, $s \in R$ and $\xi \in \mathcal{H}$. (\bar{s} denotes the element $s \in R$, if it is regarded as an element of $L^2(R)$.)

The commutant R' of R in $L^2(R)$ is $S = \{\rho(r) : r \in R\}$ (ρ right multiplication). $\beta(g) \rho(r) = \rho(\alpha(g)r)$ for $g \in G$ and $r \in R$ defines an outer action β of G on S .

The commutant $(R^G)'$ in $L^2(R)$ is generated by S and $\{u_g : g \in G\}$ where u_g is the unitary operator in $L^2(R)$ defined by $u_g \bar{r} = \overline{\alpha(g)r}$ for $r \in R$, so the commutant can be identified with the crossed product $S \rtimes_\beta G$, which is a II_1 -factor. It follows that the commutant $(R^G)'$ in $L^2(R) \otimes \mathcal{H}$ is equal to $(S \rtimes G) \otimes \mathbf{L}(\mathcal{H})$.

We determine the commutant $((R \otimes \mathbf{L}(\mathcal{H}))^G)'$ in $L^2(R) \otimes \mathcal{H}$: Obviously there is a homomorphism j from $S \rtimes G$ into $((R \otimes \mathbf{L}(\mathcal{H}))^G)'$ given by $j(x) = x \otimes \mathbf{1}$ for $x \in S$ and $j(u_g) = u_g \otimes \sigma(g)$ for $g \in G$. j is injective and normal and the image $T := j(S \rtimes G)$ is a II_1 -factor contained in $((R \otimes \mathbf{L}(\mathcal{H}))^G)'$. In fact we have

$$T = ((R \otimes \mathbf{L}(\mathcal{H}))^G)'.$$

In order to show the last equation it suffices to verify $[T : j(S)] = [((R \otimes \mathbf{L}(\mathcal{H}))^G)' : j(S)]$. We have $[T : j(S)] = [S \rtimes G : S] = |G|$ as well as

$$\begin{aligned} [((R \otimes \mathbf{L}(\mathcal{H}))^G)' : j(S)] &= [((R \otimes \mathbf{L}(\mathcal{H}))^G)' : (R \otimes \mathbf{L}(\mathcal{H}))'] \\ &= [R \otimes \mathbf{L}(\mathcal{H}) : (R \otimes \mathbf{L}(\mathcal{H}))^G] = |G|. \end{aligned}$$

It is possible to derive the last equation from

$$\begin{aligned} (\dim \mathcal{H})^2 \cdot |G| &= [R \otimes \mathbf{L}(\mathcal{H}) : R] \cdot [R : R^G] = [R \otimes \mathbf{L}(\mathcal{H}) : R^G] \\ &= [R \otimes \mathbf{L}(\mathcal{H}) : (R \otimes \mathbf{L}(\mathcal{H}))^G] \cdot [(R \otimes \mathbf{L}(\mathcal{H}))^G : R^G] \end{aligned}$$

and

$$(5) \quad [(R \otimes \mathbf{L}(\mathcal{H}))^G : R^G] = (\dim \mathcal{H})^2$$

(see [13] or [17] as to a proof of equation (5)).

Lemma 3.1. (i) Let Q be the finite dimensional $*$ -subalgebra of $S \rtimes G \otimes \mathbf{L}(\mathcal{H})$ generated by $\{u_g \otimes x : g \in G, x \in \mathbf{L}(\mathcal{H})\}$, which obviously is isomorphic

to $\mathbb{C}[G] \otimes \mathbf{L}(\mathcal{H})$. Let \mathcal{D} denote the set of all $*$ -subalgebras D of Q with the same unit like Q and with the following property: There exist \mathbb{C} -vector spaces $B_g \subset \mathbf{L}(\mathcal{H})$, $g \in G$, such that D is the direct sum $\bigoplus_{g \in G} u_g \otimes B_g$ as a \mathbb{C} -vector space ($B_g = 0$ is permitted).

The map $F \mapsto F \cap Q$ is a bijection between the set

$$\mathcal{F} := \{F \text{ a von Neumann algebra: } S \subset F \subset S \rtimes G \otimes \mathbf{L}(\mathcal{H})\}$$

and \mathcal{D} .

(ii) If F is a von Neumann algebra belonging to \mathcal{F} and $E_F : S \rtimes G \otimes \mathbf{L}(\mathcal{H}) \rightarrow F$ denotes the conditional expectation onto F corresponding to the unique normalized trace tr of $S \rtimes G \otimes \mathbf{L}(\mathcal{H})$, then for any $s \in S$, $g \in G$ and $x \in \mathbf{L}(\mathcal{H})$ there exists an operator $a \in \mathbf{L}(\mathcal{H})$ such that $E_F(su_g \otimes x) = su_g \otimes a$.

Proof. Let F be a von Neumann algebra in \mathcal{F} , let (v_1, \dots, v_m) be an orthonormal base of \mathcal{H} and let $\epsilon_{ij} \in \mathbf{L}(\mathcal{H})$ ($i, j = 1, \dots, m$) be defined by $\epsilon_{ij}v_k = \delta_{j,k}v_i$ for $k = 1, \dots, m$. For $x \in \mathbf{L}(\mathcal{H})$ and $g \in G$ there are unique elements $s_{h,ij} \in S$ ($h \in G$, $i, j = 1, \dots, m$) such that

$$(6) \quad E_F(u_g \otimes x) = \sum_{h \in G} \sum_{i,j=1}^m s_{h,ij} u_h \otimes \epsilon_{ij}.$$

$$\begin{aligned} E_F(u_g \otimes x) \cdot (n \otimes \mathbf{1}) &= E_F(u_g n \otimes x) = E_F(u_g n u_g^{-1} u_g \otimes x) \\ &= E_F(\alpha(g)n \cdot u_g \otimes x) = (\alpha(g)n \otimes \mathbf{1}) \cdot E_F(u_g \otimes x) \end{aligned}$$

holds for every $n \in S$, hence

$$\sum_{h,i,j} s_{h,ij} u_h n \otimes \epsilon_{ij} = (\alpha(g)n \otimes \mathbf{1}) \sum_{h,i,j} s_{h,ij} u_h \otimes \epsilon_{ij}.$$

Using $u_h \cdot n = \alpha(h)n \cdot u_h$ we obtain $s_{h,ij} \cdot \alpha(h)n = \alpha(g)n \cdot s_{h,ij}$ for $h \in G$, $n \in S$ and $i, j = 1, \dots, m$. Replacing n by $\alpha(g^{-1})n$ we get

$$s_{h,ij} \cdot \alpha(hg^{-1})n = n \cdot s_{h,ij}$$

for every $n \in S$. If $h = g$ this relation means that $s_{g,ij}$ belongs to $S \cap S' = \mathbb{C}\mathbf{1}$. Since the action α is outer, for $h \in G \setminus \{e\}$ there is no element $s \in S \setminus \{0\}$ such that $s \cdot \alpha(h)n = n \cdot s$ holds for every $n \in S$. This implies $s_{h,ij} = 0$ for $h \neq g$. By summarizing the considerations from above we obtain $E_F(u_g \otimes x) = u_g \otimes a$ for a suitable operator $a \in \mathbf{L}(\mathcal{H})$. Now (ii) follows immediately.

Moreover, we get $E_F(Q) \subset Q$ and $E_F(Q) = F \cap Q$. If $\sum_{g \in G} u_g \otimes x_g \in F \cap Q$, then

$$\sum_{g \in G} u_g \otimes x_g = E_F \left(\sum_{g \in G} u_g \otimes x_g \right) = \sum_{g \in G} E_F(u_g \otimes x_g).$$

$E_F(u_g \otimes x_g) \in u_g \otimes \mathbf{L}(\mathcal{H})$ implies $u_g \otimes x_g = E_F(u_g \otimes x_g) \in F \cap Q$ for every $g \in G$. Hence $F \cap Q$ is a direct sum $\bigoplus_{g \in G} u_g \otimes B_g$.

The assignment $E_F \mapsto E_F|_Q$ is injective, furthermore $E_F|_Q$ is the conditional expectation from Q onto $F \cap Q$ with respect to the trace $\text{tr}|_Q$ such that $E_F|_Q$ is determined by its image $F \cap Q$. Hence the map $F \in \mathcal{F} \mapsto F \cap Q \in \mathcal{D}$ is injective.

We have to show that this map is surjective. Let $C = \bigoplus_{g \in G} u_g \otimes B_g$ be a $*$ -algebra in \mathcal{D} . For $g \in G$ let $(b_g^1, \dots, b_g^{n(g)})$ ($n(g) \in \{0, 1, \dots, m^2\}$) be a base of the \mathbb{C} -vector space B_g .

$$F := \left\{ \sum_{g \in G} \sum_{j=1}^{n(g)} s_{g,j} u_g \otimes b_g^j : s_{g,j} \in S \text{ for } g \in G \text{ and } j = 1, \dots, n(g) \right\}$$

is a $*$ -algebra satisfying $S \subset F \subset S \rtimes G \otimes \mathbf{L}(\mathcal{H})$ and $F \cap Q = C$. It is not difficult to prove that F really is a von Neumann algebra. One easily sees that the proof of this fact is not necessary for the proof of Theorem 2.2, so we omit it. \square

Lemma 3.2. *Via $F \mapsto F \cap (\mathbb{C}\mathbf{1} \otimes \mathbf{L}(\mathcal{H}))$, we have a bijective correspondence between the von Neumann algebras F satisfying $T \subset F \subset S \rtimes G \otimes \mathbf{L}(\mathcal{H})$ and the $*$ -subalgebras B of $\mathbb{C}\mathbf{1} \otimes \mathbf{L}(\mathcal{H}) \cong \mathbf{L}(\mathcal{H})$, which have the same unit as $\mathbf{L}(\mathcal{H})$ and are invariant under the action $\text{Ad } \sigma$ of G . The inverse map is*

$$B \mapsto F_B := \left\{ \sum_{g \in G} \sum_{j=1}^k s_{g,j} u_g \otimes \sigma(g) b^j : s_{g,j} \in S \text{ for } g \in G, j = 1, \dots, k \right\},$$

where (b^1, \dots, b^k) is a base of B .

Proof. Let F be a von Neumann algebra between T and $S \rtimes G \otimes \mathbf{L}(\mathcal{H})$ and $F \cap Q = \bigoplus_{g \in G} u_g \otimes B_g$ be the decomposition of $F \cap Q$ as in Lemma 3.1. We put $B := B_e$. Obviously B is a $*$ -algebra. We get

$$(7) \quad \sigma(g)B = B_g \quad \text{for every } g \in G,$$

as the following conclusion shows:

$$\begin{aligned} x \in B_g &\iff E_F(u_g \otimes x) = u_g \otimes x &\iff & \quad (\text{note } u_g \otimes \sigma(g) \in T) \\ &\iff (u_g \otimes \sigma(g)) \cdot E_F(\mathbf{1} \otimes \sigma(g^{-1})x) = (u_g \otimes \sigma(g)) \cdot (\mathbf{1} \otimes \sigma(g^{-1})x) \\ &\iff E_F(\mathbf{1} \otimes \sigma(g^{-1})x) = \mathbf{1} \otimes \sigma(g^{-1})x &\iff & \\ &\iff \sigma(g^{-1})x \in B. \end{aligned}$$

The injectivity of $F \mapsto F \cap Q$ implies that the assignment $F \mapsto B$ is injective, too.

If we multiply $u_g \otimes \sigma(g)$ from right instead of from left we get

$$(8) \quad B \sigma(g) = B_g \quad \text{for } g \in G.$$

The relations (7) and (8) imply $\sigma(g) B \sigma(g^{-1}) = B$ for every $g \in G$.

Conversely, let B be a $*$ -subalgebra of $\mathbf{L}(\mathcal{H})$ with the same unit invariant under the action $\text{Ad } \sigma$. It is easy to check that $D = \bigoplus_{g \in G} u_g \otimes \sigma(g)B$ is a $*$ -subalgebra of Q . According to the proof of Lemma 3.1, F_B is the von Neumann algebra in $L^2(R) \otimes \mathcal{H}$ associated with D . Clearly, $F_B \cap (\mathbb{C}\mathbf{1} \otimes \mathbf{L}(\mathcal{H})) = B$ and $T \subset F_B \subset S \rtimes G \otimes \mathbf{L}(\mathcal{H})$ hold. \square

Lemma 3.3. *The center Z_F of a von Neumann algebra F between T and $S \rtimes G \otimes \mathbf{L}(\mathcal{H})$ consists of those elements of the center Z_B of $B := F \cap (\mathbb{C}\mathbf{1} \otimes \mathbf{L}(\mathcal{H}))$ which are invariant under $\text{Ad } \sigma$.*

Proof. $Z_F = F' \cap F \subset (S' \otimes \mathbf{L}(\mathcal{H})) \cap (S \rtimes G \otimes \mathbf{L}(\mathcal{H})) = \mathbb{C}\mathbf{1} \otimes \mathbf{L}(\mathcal{H})$ (observe $S' \cap S \rtimes G = \mathbb{C}\mathbf{1}$) and consequently $Z_F \subset B$. $B \subset F$ implies $Z_F \subset Z_B$. Since the elements of Z_F commute with $u_g \otimes \sigma(g)$ for $g \in G$, we obtain $Z_F \subset \{x \in Z_B : \sigma(g)x\sigma(g^{-1}) = x\}$.

Conversely, every element x of Z_B invariant under $\text{Ad } \sigma$ commutes with $S \otimes \mathbb{C}\mathbf{1}$, B and $u_g \otimes \sigma(g)$ for $g \in G$. Since the von Neumann algebra F is generated by these elements, x belongs to Z_F . \square

Lemma 3.4. *Let F be a factor between T and $S \rtimes G \otimes \mathbf{L}(\mathcal{H})$ and let $B := F \cap (\mathbb{C}\mathbf{1} \otimes \mathbf{L}(\mathcal{H}))$. There are a subgroup H of G , projective representations $\rho : H \rightarrow \mathbf{U}(\mathbb{C}^d)$ and $\psi : H \rightarrow \mathbf{U}(\mathbb{C}^r)$ and a unitary operator $U : \mathcal{H} \rightarrow \mathbb{C}^d \otimes \mathbb{C}^r \otimes \ell^2(G/H)$ such that $\rho \otimes \psi$ is an ordinary representation of H ,*

$$(9) \quad U \sigma(g) U^* = \text{ind}(\rho \otimes \psi)(g) \quad \text{for } g \in G \quad \text{and}$$

$$(10) \quad U B U^* = \mathbf{L}(\mathbb{C}^d) \otimes \mathbb{C} \text{id}_{\mathbb{C}^r} \otimes \ell^\infty(G/H).$$

(In particular, if $c : H \times H \rightarrow T$ is the cocycle for ψ , so $\bar{c}(h_1, h_2) = \overline{c(h_1, h_2)}$ ($h_1, h_2 \in H$) is the cocycle for ρ .)

Proof. The proof of Lemma 3.2 showed $\sigma(g) B \sigma(g^{-1}) = B$ for every $g \in G$. Since every automorphism of an algebra maps the center of this algebra onto itself, $\text{Ad } \sigma$ leaves the center Z_B of B invariant. So the group G acts on the set $\{p_1, \dots, p_l\}$ of the minimal projections of the center Z_B by $g.p_j = \sigma(g)p_j\sigma(g)^{-1}$. This action is transitive, otherwise there would exist a projection $p \neq 0, \mathbf{1}$ of Z_B satisfying $p = g.p$ for every $g \in G$. Hence p

would belong to the center Z_F of F by Lemma 3.3, which is a contradiction to the assumption that F is a factor.

So we are able to apply the imprimitivity theorem. There are a subgroup H , an ordinary representation $\pi : H \longrightarrow \mathbf{U}(\mathcal{K})$ of H and a unitary operator $U : \mathcal{H} \longrightarrow \mathcal{K} \otimes \ell^2(G/H)$ such that

$$\begin{aligned} U \sigma(g) U^* &= \text{ind } \pi(g) \quad \text{for } g \in G \text{ and} \\ \{U p_j U^* : j = 1, \dots, l\} &= \{id_{\mathcal{K}} \otimes e_{kH} : k \in \mathcal{V}\} \end{aligned}$$

(see Section 2.1 for the definition of e_{kH}). Without loss of generality we assume $U p_1 U^* = id_{\mathcal{K}} \otimes e_H$.

From now on we will identify \mathcal{H} and $\mathcal{K} \otimes \ell^2(G/H)$ as well as σ and $\text{ind } \pi$. The finite dimensional factor Bp_1 may be identified with a subfactor of $\mathbf{L}(\mathcal{K}) \otimes \mathbb{C} e_H \cong \mathbf{L}(\mathcal{K})$. There are finite dimensional Hilbert spaces \mathbb{C}^d and \mathbb{C}^r and a unitary operator $V : \mathcal{K} \longrightarrow \mathbb{C}^d \otimes \mathbb{C}^r$ such that

$$V Bp_1 V^* = \mathbf{L}(\mathbb{C}^d) \otimes \mathbb{C} id_{\mathbb{C}^r}$$

holds.

Using V we identify \mathcal{K} and $\mathbb{C}^d \otimes \mathbb{C}^r$. We fix an element $h \in H$. From $\sigma(h) p_1 \sigma(h)^* = p_1$ we conclude that

$$\begin{aligned} x = y \otimes \mathbf{1} \otimes e_H \in Bp_1 &= \mathbf{L}(\mathbb{C}^d) \otimes \mathbb{C} \mathbf{1} \otimes \mathbb{C} e_H \longmapsto \sigma(h) x \sigma(h)^* \\ &= \pi(h)(y \otimes \mathbf{1}) \pi(h)^* \otimes e_H \end{aligned}$$

is an automorphism of Bp_1 . This automorphism is inner, hence there is an operator $\rho(h) \in \mathbf{U}(\mathbb{C}^d)$ such that

$$(11) \quad \pi(h)(y \otimes id_{\mathbb{C}^r}) \pi(h)^* = \rho(h) y \rho(h)^* \otimes id_{\mathbb{C}^r}$$

for $y \in \mathbf{L}(\mathbb{C}^d)$. Let h_1 and h_2 be elements of H . For $y \in \mathbf{L}(\mathbb{C}^d)$ we have

$$\begin{aligned} \rho(h_1 h_2) y \rho(h_1 h_2)^* \otimes \mathbf{1} &= \pi(h_1 h_2) (y \otimes \mathbf{1}) \pi(h_1 h_2)^* \\ &= \pi(h_1) (\pi(h_2) (y \otimes \mathbf{1}) \pi(h_2)^*) \pi(h_1)^* = \rho(h_1) \rho(h_2) y \rho(h_2)^* \rho(h_1)^* \otimes \mathbf{1}. \end{aligned}$$

Hence $\rho(h_1 h_2)^{-1} \rho(h_1) \rho(h_2) \in \mathbf{L}(\mathbb{C}^d)' = \mathbb{C} \mathbf{1}$. So there is a $c(h_1, h_2) \in T$ such that $c(h_1, h_2) \rho(h_1) \rho(h_2) = \rho(h_1 h_2)$, hence $\rho : H \longrightarrow \mathbf{U}(\mathbb{C}^d)$, $h \longmapsto \rho(h)$, is a projective representation. Relation (11) implies that $(\rho(h) \otimes \mathbf{1})^{-1} \pi(h)$ is a unitary operator in $(\mathbf{L}(\mathbb{C}^d) \otimes \mathbb{C} id_{\mathbb{C}^r})' = \mathbb{C} id_{\mathbb{C}^d} \otimes \mathbf{L}(\mathbb{C}^r)$. Therefore there is a unique unitary operator $\psi(h) \in \mathbf{U}(\mathbb{C}^r)$ such that $\pi(h) = \rho(h) \otimes \psi(h)$. Since π is an ordinary representation of H , $\psi : H \longrightarrow \mathbf{U}(\mathbb{C}^r)$, $h \longmapsto \psi(h)$, is a projective representation of H with cocycle c .

For every $j = 1, \dots, l$ there is a $g \in G$ and a $k \in \mathcal{V}$ such that $p_j = g \cdot p_1 = id_{\mathcal{K}} \otimes e_{kH}$. Hence

$$\begin{aligned} Bp_j &= \sigma(g) Bp_1 \sigma(g)^{-1} \\ &= \text{ind}(\rho \otimes \psi)(g) \left(\mathbf{L}(\mathbb{C}^d) \otimes \mathbb{C} \mathbf{1} \otimes \mathbb{C} e_H \right) \text{ind}(\rho \otimes \psi)(g)^{-1} \\ &= \mathbf{L}(\mathbb{C}^d) \otimes \mathbb{C} \mathbf{1} \otimes \mathbb{C} e_{kH}, \end{aligned}$$

and Equation (10) follows. \square

Every factor between R^G and $(R \otimes \mathbf{L}(\mathcal{H}))^G$ is the commutant F' of a factor F between $T = \left((R \otimes \mathbf{L}(\mathcal{H}))^G \right)'$ and $(R^G)' = S \rtimes G \otimes \mathbf{L}(\mathcal{H})$. Let F be defined as in Lemma 3.4. By applying U we identify \mathcal{H} and $\mathbb{C}^d \otimes \mathbb{C}^r \otimes \ell^2(G/H)$ again.

Let C be the $*$ -algebra $\mathbb{C} id_{\mathbb{C}^d} \otimes \mathbf{L}(\mathbb{C}^r) \otimes \ell^\infty(G/H)$. We get

$$(S \otimes B)' = S' \otimes B' = R \otimes (\mathbf{L}(\mathbb{C}^d) \otimes \mathbb{C} \mathbf{1} \otimes \ell^\infty(G/H))' = R \otimes C$$

and

$$F' = \{x \in R \otimes C : x \text{ and } u_g \otimes \sigma(g) \text{ commute for every } g \in G\} = (R \otimes C)^G.$$

So every subfactor $N \subset M = R$ with finite index for which the C^* -tensor category $\mathcal{B}_{N \subset M}$ is equivalent to \mathcal{U}_G is isomorphic to a subfactor $R^G \subset (R \otimes C)^G$. The arguments from Section 2.1 show that this subfactor is equivalent to $R^G \subset (R \otimes \mathbf{L}(\mathbb{C}^r))^H$ and that the restriction of the projective kernel of ψ to $N(H)$ has to be $\{e\}$.

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