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C^∞ FUNCTIONS ON THE CANTOR SET, AND A SMOOTH m-CONVEX FRÉCHET SUBALGEBRA OF O_2

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We construct a nuclear, spectral invariant, dense Fréchet subalgebra $C^\infty(K)$ of the commutative algebra $C(K)$ of continuous complex valued functions on the Cantor set K. The construction uses the group structure of the 2-adic integers on K.

We then use a smooth crossed product construction to get a dense, nuclear Fréchet subalgebra $\mathcal{O}_2$ of the Cuntz algebra O_2. We prove the general result that a tempered action of a locally compact group on a strongly spectral invariant dense Fréchet subalgebra of a Banach algebra is automatically m-tempered, and obtain the m-convexity of $\mathcal{O}_2$ as a special case.

0. Introduction.

Dense subalgebras of C*-algebras are well-known to be useful in the study of C*-algebras. In the current literature, a dense subalgebra of smooth functions is often viewed as a replacement for C^∞-functions on a manifold, where instead of a manifold we have an underlying "noncommutative space". In this paper, we seek not just a dense sub*algebra* of smooth functions of the C*-algebra O_2, but an algebra with a Fréchet topology, for which the seminorms are submultiplicative. The Fréchet topology is very helpful when it comes to working with a dense subalgebra. It allows the use of functional calculus, and more importantly allows the dense subalgebra to be written as a countable projective limit of Banach algebras. By taking projective limits, interesting results about Fréchet algebras can be deduced from corresponding results on Banach algebras (for example, see [**Da**], [**PhSc**]).

In §1, we define a set of "smooth functions on the Cantor set" $C^\infty_\omega(K)$ for each increasing sequence $\omega = \{c_p\}_{p=1}^\infty$ of positive real numbers c_p. This is done by identifying K with the topological group of 2-adic integers [**Ko**]. Using this group structure on K, we define $C^\infty_\omega(K)$ to be the inverse Fourier transform of the set of ω-rapidly vanishing Schwartz functions $\mathcal{S}^\omega(\widehat{K})$ on the dual group $\widehat{K}$. (The dual group $\widehat{K} = \mathbb{Z}[\frac{1}{2}]/\mathbb{Z}$ is the discrete group of dyadic rationals lying between 0 and 1.) If ω satisfies the summability condition $\sum_{p=1}^\infty 2^p/c_p^q < \infty$ for some $q > 0$, then $C^\infty_\omega(K)$ is nuclear as a Fréchet space, and is a strongly spectral invariant dense Fréchet subalgebra of $C(K)$ (see Definition 1.4). The existence of a subalgebra of $C(K)$ with these properties was a surprise to me, since the Cantor set is totally disconnected, and seems to have no "smooth structure". Note that for different choices of ω, we get apparently different sets of functions $C^\infty_\omega(K)$.

In §2, we restrict to the case $c_p = 2^p$ or $\omega = \{2^p\}_{p=1}^\infty$ (which satisfies the above summability condition for $q > 1$), and show that the action of the free-product of cyclic groups $\mathbb{Z}_2 * \mathbb{Z}_3$ on $C(K)$ in [**Sp**] leaves $C^\infty_\omega(K)$ invariant and is exponentially tempered with respect to the word length function on $\mathbb{Z}_2 * \mathbb{Z}_3$ (Theorem 2.3). Thus the set of "exponentially rapidly vanishing" functions (see Definition 2.2) $\mathcal{S}^e(\mathbb{Z}_2 * \mathbb{Z}_3, C^\infty_\omega(K))$ is a nuclear Fréchet algebra (Corollary 2.4). By [**Sp**] (Theorem 2.1 below), the corresponding reduced $C^\star$ crossed product is isomorphic to the Cuntz algebra O_2, so our Fréchet algebra is a dense subalgebra of O_2, which we will call $\mathcal{O}_2$ for short.

In §3 we show that a tempered action of a locally compact group H, with weight or gauge γ (see §1, §2), on a strongly spectral invariant dense Fréchet subalgebra A of a Banach algebra (Definition 1.4) is automatically m-tempered. Thus the smooth crossed product $\mathcal{S}^\gamma(H, A)$ is m-convex, and as a special case so is the smooth crossed product $\mathcal{O}_2$ we constructed above.

In §4, we show that the action of $\mathbb{Z}_2 * \mathbb{Z}_3$ on $C^\infty_\omega(K)$ is not tempered with respect to the word length function ℓ on $\mathbb{Z}_2 * \mathbb{Z}_3$. Thus it is necessary to use functions which vanish *exponentially* rapidly with respect to ℓ to form the smooth crossed product, as we did above. In §5, we show that the algebra $\mathcal{O}_2$ has an element with unbounded spectrum, from which it follows that $\mathcal{O}_2$ is not spectral invariant in the $C^\star$-algebra O_2. (Hence it follows that $\mathcal{O}_2$ cannot be a set of C^∞-vectors for any group action on O_2 (otherwise it would be spectral invariant by [**Sc 2**], Theorem 2.2), which is why we prefer to use the notation $\mathcal{O}_2$ instead of O_2^∞.)

1. Smooth Functions on the Cantor Set.

We view the Cantor set K as infinite sequences of 0's and 1's (e.g. $.00101\dots$), topologized as the product space $\prod_{\mathbb{N}}\{0,1\}$. One may view this compact topological space as a "disconnection" of the unit circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, where elements of $\mathbb{T}$ are written in their binary expansions. Note, for example, that $.011\overline{1}\dots$ is not equal to $.100\overline{0}\dots$ in K.

The group operation on K is given by "carrying down" instead of up. For example, $.0100\dots + .0100\dots = .0010\dots$, and $.01100\dots + .10100\dots = .11010\dots$. (Note that if $.k_0k_1k_2\dots$ is a binary sequence in K, then $k_0 + 2k_1 + 2^2k_2 + \dots$ defines an infinite power series, with "2" as the indeterminant variable. Then addition is equivalent to "carrying down", since $2^n + 2^n =$

2^{n+1}. In fact, this is precisely the group structure of the 2-adic integers on K [**Ko**].) It is easy to check that this addition is associative, $.000\dots$ acts as the identity, and every element has an inverse. $(K,+)$ is clearly an Abelian, compact topological group, which I will refer to as the *Cantor group*.

Remark. If you define addition by carrying up as you would for usual binary expansions in $\mathbb{T}$, then $.11\bar{1}\dots$ has no inverse. (The inverse would have to be $.000\dots1$, with the 1 "at infinity".)

If s is a finite sequence of 0's and 1's, we let $.s*$ denote the sub*set* of K of all infinite sequences beginning with s. We normalize the Haar measure μ on K so that $\mu(K) = 1$. Then $\mu(.0*) = \mu(.1*)$ so they both equal $1/2$. Similarly, $\mu(.00*) = \mu(.01*) = \mu(.10*) = \mu(.11*) = 1/4$, etc. Haar measure is the natural product measure on K.

We denote the dual group of the Cantor group by $G = \widehat{K}$. Then G is the discrete group of dyadic rationals from 0 to 1 (i.e. $G = \mathbb{Z}[\frac{1}{2}]/\mathbb{Z}$). G can be represented pictorially as a "pyramid" - see Figure 1.1.

0							1	$lev = 0$
				1/2				$lev = 1$
		1/4				3/4		$lev = 2$
	1/8		3/8		5/8		7/8	$lev = 3$
	$\vdots$		$\vdots$		$\vdots$		$\vdots$	

Figure 1.1. $G = \widehat{K}$, dual of Cantor group.

If $g \in G$ can be written $g = l/2^p$, l odd, then we define $lev(g) = p$, the *level* of g. (This function is the same as the base 2 logarithm of the non-archimedian 2-norm on $\mathbb{Q}$ [**Ko**].) Note that there are 2^{p-1} g's in G with $lev(g) = p$. When two elements of G are added, the sum lies at the same level of the pyramid or above. Thus $lev(g+h) \le \max(lev(g), lev(h))$.

The pairing between G and K is given by

$$(g, \vec{k}) \mapsto e^{2\pi i \langle g, \vec{k} \rangle},$$

where

$$\langle g, \vec{k} \rangle = g(k_0 + 2k_1 + 2^2 k_2 + \dots) = (l/2^p)(k_0 + 2k_1 + 2^2 k_2 + \dots) \pmod{\mathbb{Z}}.$$

This makes sense since $g2^n$ is an integer if $n \ge p$. (To see that G is really the dual group, let K_n be the subgroup $\{.0\dots0*\}$ of K, where there are n zeros. Then $lev(g) \le n$ if and only if g defines a trivial character on K_n, i.e. g factors to a character of $K/K_n \cong \mathbb{Z}_{2^n}$. Since there are 2^n g's with

$lev(g) \le n$, this gives all of $\widehat{\mathbb{Z}}_{2^n}$. Exercise: show that every character of K factors through some K_n.)

A *gauge* or *length function* ℓ on a locally compact group H is a Borel measurable function $\ell\colon H \to [0,\infty)$ which satisfies $\ell(\mathrm{id}_H) = 0$, $\ell(g^{-1}) = \ell(g)$, and $\ell(gh) \le \ell(g) + \ell(h)$ for $g, h \in H$. If $\omega = \{c_p\}_{p=1}^{\infty}$ is an increasing sequence of positive numbers, then we may define a gauge (also denoted by ω) on G by

$$\omega(g) = \begin{cases} c_{lev(g)} & \text{if } \ lev(g) \ge 1, \\ 0 & \text{if } \ lev(g) = 0. \end{cases}$$

(Note $lev(g) = 0$ iff $g = 0$.)

Lemma 1.2. *The group G has polynomial growth, and $\omega = \{c_p\}_{p=1}^{\infty}$ satisfies the summability condition*

$$\sum_{p=1}^{\infty} \frac{2^p}{c_p^q} < \infty,$$

for some $q > 0$, if and only if $\sum_{g \in G - \{0\}} \frac{1}{\omega(g)^q} < \infty$.

Proof. Note that if $U \subset G$ is finite, then $\bigcup_{n=0}^{\infty} U^n$ is contained in a "finite sub-pyramid", so $|U^n| \le M < \infty$ for all n, and G has polynomial growth. For the second part,

$$\sum_{g \ne 0} \frac{1}{\omega(g)^q} = \sum_{p=1}^{\infty} \left(\sum_{lev(g)=p} \frac{1}{\omega(g)^q} \right) = \frac{1}{2} \sum_{p=1}^{\infty} \frac{2^p}{c_p^q},$$

where we used $|\{g \in G \mid lev(g) = p\}| = 2^{p-1}$. □

Define $\mathcal{S}^{\omega}(G)$, the *ω-rapidly vanishing functions on G*, by

$$\mathcal{S}^{\omega}(G) = \{\varphi\colon G \to \mathbb{C} \,|\, \|\varphi\|_d < \infty, \quad d = 0, 1, 2, \dots\},$$

where

$$\|\varphi\|_d = \|\omega^d \varphi\|_1 = \sum_{g \in G} \omega(g)^d |\varphi(g)|. \tag{1.3}$$

Then $\mathcal{S}^{\omega}(G)$ is a Fréchet $\star$-algebra under convolution [**Sc 1**], Theorem 1.3.2. (In fact, it is a strongly spectral invariant (see below) dense Fréchet subalgebra of the Banach algebra $L^1(G)$ by [**Sc 2**], Theorem 6.7.)

Definition 1.4. We say that a Fréchet algebra A is *m-convex* if there exists a family $\{\| \quad \|_n\}_{n=0}^{\infty}$ of topologizing seminorms for A which are submultiplicative:

$$\|ab\|_n \le \|a\|_n \|b\|_n, \qquad a, b \in A.$$

A is a *Fréchet* $\star$*-algebra* if A has a continuous involution defined on it. Let A be a dense Fréchet subalgebra of a Banach algebra B [**Sc 2**], Definition 1.1. If A is nonunital, let $\tilde{A}$ and $\tilde{B}$ denote the respective unitizations. Then A is *spectral invariant* (SI) in B if every $a \in \tilde{A}$ is invertible in $\tilde{B}$ if and only if a is invertible in $\tilde{A}$. A is *strongly spectral invariant* (SSI) in B if there is some $C > 0$ such that for every m, there is some $p_m \geq m$ and $D_m > 0$ such that

$$\|a_1 \dots a_n\|_m \leq C^n D_m \sum \|a_1\|_{k_1} \dots \|a_n\|_{k_n},$$

for all n-tuples $a_1, \dots a_n \in A$ and all n, where the sum is over those k_i's such that $\sum_{i=1}^n k_i \leq p_m$, and $\| \quad \|_0$ denotes the norm on B. Then $SSI \Rightarrow SI$ by [**Sc 2**], Proposition 1.7, Theorem 1.17. If a Fréchet algebra A is strongly spectral invariant in some Banach algebra, then A is automatically m-convex [**Sc 2**], Proposition 1.7.

Recall that by basic locally compact Abelian group theory [**Ru**], we have an isomorphism of $C^\star$-algebras $C(K) \cong C^\star(G)$ given by the Fourier transform

$$\varphi \in C(K) \mapsto \widehat{\varphi}(g) = \int_K \varphi(\vec{k}) e^{2\pi i \langle g, \vec{k} \rangle} d\vec{k}.$$

Thus we may define $C^\infty_\omega(K)$, the *ω-C^∞-functions on the Cantor set*, to be the inverse image of $\mathcal{S}^\omega(G)$ in $C(K)$. Then $C^\infty_\omega(K)$ is automatically a Fréchet algebra under pointwise multiplication, since $\mathcal{S}^\omega(G)$ is a Fréchet algebra under convolution.

We give an alternative way of describing $C^\infty_\omega(K)$, which is closer to the usual definition of a C^∞-function on the circle $\mathbb{T}$, for example.

Definition 1.5. Let $\omega = \{c_p\}_{p=1}^\infty$ be an increasing sequence of positive numbers. For $\varphi \in C(K)$ and $p \in \mathbb{N}$, define the *pth approximate derivative* $\delta_p(\varphi) \in C(K)$ by

$$\delta_p(\varphi)(\vec{k}) = \frac{\varphi(\vec{k} + 1/2^p) - \varphi(\vec{k})}{1/c_p}.$$

Here $1/2^p = .0 \dots 01 \in K$ is the binary expansion for $1/2^p$, so the 1 is in the pth spot. (If we were using 2-adic notation, $1/2^p = .0 \dots 01$ would be written 2^p. In keeping with the analogy of K with the circle group $\mathbb{T}$, we will stick to the binary notation.) Note that $1/2^0 = 1 = 0$ in K, so $\varphi(\vec{k} + 1/2^0) = \varphi(\vec{k})$ and it is appropriate to define $\delta_0(\varphi) = 0$ for all φ. It is easy to check from the definition that

$$\delta_p(\varphi\psi) = \delta_p(\varphi)\psi + \varphi\delta_p(\psi) + \delta_p(\varphi)\delta_p(\psi)/c_p,$$

so δ_p is not quite a derivation.

It is well known that the $\star$-subalgebra of linear combinations of "cylinder functions"

$$\mathcal{A} = \text{span}\{\chi_{.s*\ldots} | s \text{ a finite sequence of } 0's \text{ and } 1's\}$$

is dense in $C(K)$. (Use the Stone-Weierstrass theorem.) It is also easy to check that under the Fourier transform, $\mathcal{A}$ corresponds to the finite support functions $c_f(G) \subseteq \mathcal{S}^\omega(G)$.

Theorem 1.6. *Assume that $\omega = \{c_p\}_{p=1}^\infty$ satisfies the summability condition of Lemma 1.2. Then the Fréchet algebra $C_\omega^\infty(K)$ is the completion of $\mathcal{A}$ in the seminorms*

$$\|\varphi\|_d = \sum_{p=1}^{\infty} 2^p \|\delta_p^d(\varphi)\|_\infty. \tag{1.7}$$

Moreover, $\mathcal{S}^\omega(G) \cong C_\omega^\infty(K)$ are nuclear as Fréchet spaces. They are m-convex Fréchet $\star$-algebras, strongly spectral invariant in their respective $C^\star$-algebras.

Remark 1.8. 1) It is interesting that $\lim_{p\to\infty} \delta_p(\varphi) = 0$ for $\varphi \in \mathcal{A}$, so the "derivative" in the usual sense is always zero. (Note that for fixed $g \in G$, $\delta_p(e^{-2\pi i\langle g,\vec{k}\rangle})$ is the zero function in $C(K)$ if and only if $lev(g) < p$.) Hence it would have been impractical, for example, to take sup norms of derivatives in (1.7), since the seminorms would all be zero on $\mathcal{A}$, and we could not have got a topology equivalent to the one from $\mathcal{S}^\omega(G)$.

2) In the terminology of [**JiSc**], (G,ω) is a *rapid decay* group. In fact $\mathcal{S}_2^\omega(G) = \mathcal{S}_1^\omega(G) \subseteq L^1(G)$ by the summability condition of Lemma 1.2, and [**Sc 1**], Theorem 6.8. It follows that $\mathcal{S}^\omega(G)$ is a SSI dense Fréchet subalgebra of $C^\star(G) = C_r^\star(G)$ [**JiSc**], Definition 1.5, Lemma 3.11, Proof of Theorem 2.6(b).

Question 1.9. What is the relationship between $C_\omega^\infty(K)$ and the subalgebra of $C(K)$ that you get by identifying K with $[0,1]-$"the middle thirds", and then restricting the C^∞-functions $C^\infty[0,1]$ to K? Note that in our difference quotients, $(\varphi(\vec{k}+1/2^p) - \varphi(\vec{k}))/(1/c_p)$, we are using a nonstandard addition, so one might expect them to be different.

Proof of Theorem 1.6. By Lemma 1.2 and [**Sc 1**], Theorem 6.24, $\mathcal{S}^\omega(G)$ is a nuclear Fréchet space, and is SSI in $C^\star(G)$ as noted in Remark 1.8 (2) above. The strong spectral invariance property implies m-convexity [**Sc 1**], Proposition 1.7, or apply [**Sc 1**], Theorem 1.3.2 to get both the m-convexity and $\star$-algebra statements for $\mathcal{S}^\omega(G)$. Since $C_\omega^\infty(K) \cong \mathcal{S}^\omega(G)$ as Fréchet algebras (by definition of $C^\infty_\omega(K)$), it remains only to prove the first assertion of the theorem. Let $g \in G$ have $lev(g) = p$ for $p \geq 1$. Then $\langle g, 1/2^p\rangle = \frac{\pm 1}{2^p}2^{p-1} \pmod{\mathbb{Z}} = \pm 1/2 \pmod{\mathbb{Z}}$. Since $e^{\pm\pi i} = -1$, we have $(e^{-2\pi i\langle g,1/2^p\rangle} - 1) = -2$. Let $\varphi \in C(K)$. Then

$$\begin{aligned}
\omega(g)^d\widehat{\varphi}(g) &= \omega(g)^d \int_K \varphi(\vec{k})e^{2\pi i\langle g,\vec{k}\rangle}d\vec{k} \\
&= c_p^d \int_K \varphi(\vec{k})e^{2\pi i\langle g,\vec{k}\rangle}d\vec{k} \\
&= \frac{c_p^{d-1}}{-2}\int_K \varphi(\vec{k})\left(\frac{e^{2\pi i\langle g,\vec{k}-1/2^p\rangle} - e^{2\pi i\langle g,\vec{k}\rangle}}{1/c_p}\right)d\vec{k} \\
&= \frac{c_p^{d-1}}{-2}\int_K \frac{\varphi(\vec{k}+1/2^p) - \varphi(\vec{k})}{1/c_p}e^{2\pi i\langle g,\vec{k}\rangle}d\vec{k} \\
&= \frac{c_p^{d-1}}{-2}\widehat{\delta_p\varphi}(g) = \cdots = \frac{\widehat{\delta_p^d\varphi}(g)}{(-2)^d}.
\end{aligned}$$

The same equality clearly holds if $lev(g) = p = 0$, since both sides are zero. So multiplication by $\omega(g)^d$ in $\mathcal{S}^\omega(G)$ is the same as applying $\delta_p^d/(-2)^d$ in $C^\infty_\omega(K)$, and then evaluating the Fourier transform at g. (Note that p depends on g, though.) Using this, we have

$$\begin{aligned}
\|\omega^d\widehat{\varphi}\|_1 &= \sum_{p=0}^\infty\left(\sum_{lev(g)=p}|\omega(g)^d\widehat{\varphi}(g)|\right) = \sum_{p=0}^\infty\left(\sum_{lev(g)=p}\left|\frac{\widehat{(\delta_p^d\varphi)}(g)}{(-2)^d}\right|\right) \\
&\leq \sum_{p=0}^\infty \frac{2^p}{2^{d+1}}\int_K |(\delta_p^d\varphi)(\vec{k})|d\vec{k} \leq \frac{1}{2^{d+1}}\left[\sum_{p=0}^\infty 2^p\|\delta_p^d\varphi\|_\infty\right] \\
&= \frac{1}{2^{d+1}}\|\varphi\|_d.
\end{aligned}$$

Thus the topology on $\mathcal{A}$, given by (1.7), is at least as strong as the topology induced from $\mathcal{S}^\omega(G)$. We show that the topologies are equivalent. By the Fourier inversion formula [**Ru**],

$$\varphi(\vec{k}) = \sum_{g\in G} e^{-2\pi i\langle g,\vec{k}\rangle}\widehat{\varphi}(g) = \sum_{lev(g)<p} e^{-2\pi i\langle g,\vec{k}\rangle}\widehat{\varphi}(g) + \sum_{lev(g)\geq p} e^{-2\pi i\langle g,\vec{k}\rangle}\widehat{\varphi}(g).$$

Since $\delta_p(e^{-2\pi i\langle g,\vec{k}\rangle}) = 0$ if (and only if) $lev(g) < p$, we have

$$|(\delta_p^d\varphi)(\vec{k})| = \left|\delta_p^d\left(\sum_{lev(g)\geq p} e^{-2\pi i\langle g,\vec{k}\rangle}\widehat{\varphi}(g)\right)\right| = \left|\sum_{lev(g)\geq p}\delta_p^d\left(e^{-2\pi i\langle g,\vec{k}\rangle}\right)\widehat{\varphi}(g)\right|,$$

since the series converges absolutely. Since $\delta_p^d(e^{-2\pi i\langle g,\vec{k}\rangle}) = \sigma_p(g)^d e^{-2\pi i\langle g,\vec{k}\rangle}$, where $\sigma_p(g) = (e^{-2\pi i\langle g,1/2^p\rangle} - 1)c_p$, it follows that

$$|(\delta_p^d\varphi)(\vec{k})| \leq \sum_{lev(g)\geq p} (2c_p)^d|\widehat{\varphi}(g)|.$$

Thus

$$\begin{aligned}\|\varphi\|_d &= \sum_{p=1}^{\infty} 2^p\|\delta_p^d(\varphi)\|_\infty \\ &\leq \sum_{p=1}^{\infty} 2^p \left(\sum_{lev(g)\geq p}(2c_p)^d|\widehat{\varphi}(g)|\right) = 2^d\sum_{g\neq 0}\tilde{\omega}_d(g)|\widehat{\varphi}(g)| \leq 2^d\|\tilde{\omega}_d\widehat{\varphi}\|_1,\end{aligned}$$

where $\tilde{\omega}_d(g) = \sum_{p=1}^{lev(g)} 2^p c_p^d$. But

$$\begin{aligned}\tilde{\omega}_d(g) &\leq \left(\sum_{p=1}^{lev(g)} \frac{2^p}{c_p^q}\right) c_{lev(g)}^{d+q} \qquad \text{since the } c_p\text{'s are increasing} \\ &\leq \left(\sum_{p=1}^{\infty}\frac{2^p}{c_p^q}\right)\omega(g)^{d+q} = C\omega(g)^{d+q}, \qquad \text{by definition of } \omega\end{aligned}$$

where $C < \infty$ by our summability assumption on ω. Thus $\|\varphi\|_d \leq 2^d C\|\omega^{d+q}\widehat{\varphi}\|_1$, completing the proof of Theorem 1.6. □

Remark 1.10. Theorem 1.6 is analogous to the well-known isomorphism $C^\infty(\mathbb{T}) \cong \mathcal{S}(\mathbb{Z})$. Note that the formula $(-2\omega(g))^d\widehat{\varphi}(g) = \widehat{\delta_p^d\varphi}(g)$ for $\varphi \in C_\omega^\infty(K)$ obtained in the proof is similar to the formula $(2\pi i n)^d\widehat{\varphi}(n) = \widehat{\varphi^{(d)}}(n)$, $n \in \mathbb{Z}$, for $\varphi \in C^\infty(\mathbb{T})$, obtained using integration by parts.

2. A tempered action of $\mathbb{Z}_2 * \mathbb{Z}_3$.

Recall that the free product $\mathbb{Z}_2 * \mathbb{Z}_3$ is $\{a^{\epsilon_1}b^{j_1}ab^{j_2}\dots b^{j_n}a^{\epsilon_2} | j_i = \pm 1, \epsilon_i = 0 \text{ or } 1\}$, with the obvious group multiplication. Here a, b are the generators of the cyclic groups $\mathbb{Z}_2$, $\mathbb{Z}_3$ respectively. The word length function ℓ corresponding to the generating set $U = \{a, 0, b, b^{-1}\}$ is $\ell(a^{\epsilon_1}b^{j_1}\dots b^{j_n}a^{\epsilon_2}) = 2n - 1 + \epsilon_1 + \epsilon_2$. Then ℓ is a gauge on $\mathbb{Z}_2 * \mathbb{Z}_3$ (§1).

A *weight* γ on a locally compact group H is a Borel measurable function $\gamma\colon H \to [1,\infty)$ which satisfies $\gamma(\mathrm{id}_H) = 1$, $\gamma(g^{-1}) = \gamma(g)$, and $\gamma(gh) \leq \gamma(g)\gamma(h)$ for all $g, h \in H$. We let e^ℓ denote the *exponentiated word weight* on $\mathbb{Z}_2 * \mathbb{Z}_3$ [**Sc 1**], Example 1.1.17. Then e^ℓ is easily seen to be a weight on $\mathbb{Z}_2 * \mathbb{Z}_3$.

We may think of the Cantor set K as infinite words in a and b as follows:

$$K = \{a^{k_0} b^{\tilde{k}_1} a b^{\tilde{k}_2} a \dots | \vec{k} \in K\},$$

where $\tilde{k}_i = -1$ if $k_i = 0$, and $\tilde{k}_i = +1$ if $k_i = 1$. Then $\mathbb{Z}_2 * \mathbb{Z}_3$ acts on K on the left, and we may form the reduced $C^\star$-crossed product $C^\star_r(\mathbb{Z}_2 * \mathbb{Z}_3, C(K))$.

Theorem 2.1 [**Sp**], [**Ch**]. *$C^\star_r(\mathbb{Z}_2 * \mathbb{Z}_3, C(K)) \cong M_2(O_2) \cong O_2$, where O_2 denotes the $n = 2$ Cuntz $C^\star$-algebra.*

Definition 2.2. If a locally compact group H with weight or gauge γ acts on a Fréchet algebra A, we say that the action is *γ-tempered* if for every $m \in \mathbb{N}$, there exists $C > 0$, $p, d \in \mathbb{N}$ such that

$$\|\alpha_h(a)\|_m \leq C(1 + \gamma(h))^d \|a\|_p, \qquad a \in A, h \in H.$$

By [**Sc 1**], Theorem 2.2.6, $\mathcal{S}^\gamma(H, A)$ is a Fréchet algebra under convolution if the action is γ-tempered. $\mathcal{S}^\gamma(H, A)$ is also dense in $L^1(H, B)$ and in the reduced $C^\star$-crossed product $C^\star_r(H, B)$, if A is a dense subalgebra of a $C^\star$-algebra B. Throughout this paper, the smooth crossed product $\mathcal{S}^\gamma(H, A)$ will denote L^1-rapidly vanishing functions from H to A [**Sc 1**], §2.1. We will abbreviate $\mathcal{S}^{e^\ell}$ by $\mathcal{S}^e$. The following theorem is the main result of this paper.

Theorem 2.3. *Let $\omega = \{c_p\}_{p=1}^\infty$, where $c_p = 2^p$. Then the action of $\mathbb{Z}_2 * \mathbb{Z}_3$ on $C(K)$ defined above leaves $C^\infty_\omega(K)$ invariant and is e^ℓ-tempered.*

Corollary 2.4. *$\mathcal{S}^e(\mathbb{Z}_2 * \mathbb{Z}_3, C^\infty_\omega(K))$ is a dense Fréchet $\star$-subalgebra of $C^\star_r(\mathbb{Z}_2 * \mathbb{Z}_3, C(K)) \cong O_2$, and is also nuclear as a Fréchet space.*

Proof of Corollary 2.4 from Theorem 2.3. Apply remarks preceding Theorem 2.3 to see that it is a Fréchet algebra. By [**Sc 1**], Corollary 4.9 it is also a Fréchet $\star$-algebra. Since $\omega = \{2^p\}$ satisfies the summability condition of Lemma 1.2, $C^\infty_\omega(K)$ is nuclear by Theorem 1.6. Then by [**Sc 1**], Proposition 6.34, Theorem 6.24, Proposition 6.13 (1), so is the smooth crossed product. □

Proof of Theorem 2.3. This is equivalent to showing that the induced action of $\mathbb{Z}_2 * \mathbb{Z}_3$ on $\psi \in \mathcal{S}^\omega(G)$ is e^ℓ-tempered:

$$\alpha_\eta(\psi)(g) = \int_K (\alpha_\eta \varphi)(\vec{k}) e^{2\pi i \langle g, \vec{k} \rangle} d\vec{k},$$

where $\varphi \in C^\infty_\omega(K)$, $\psi = \widehat{\varphi}$. See (1.3) for the seminorms we will be using for $\mathcal{S}^\omega(G)$.

Lemma 2.5 (**Truncation of lower levels**). *For any specified level* $p \in \mathbb{N}$, *and for all* $d \in \mathbb{N}$ *and* $\psi \in \mathcal{S}^\omega(G)$,

$$\|\omega^d \psi\|_1 \leq 2^{(d+1)p} \|\varphi\|_\infty + \sum_{lev(g) \geq p} |\omega^d \psi(g)|,$$

where $\varphi \in C^\infty_\omega(K)$ *is the inverse Fourier transform of* ψ.

Proof.

$$\begin{aligned}
&\sum_{lev(g)<p} |\omega(g)^d \psi(g)| \\
&= \sum_{r=1}^{p-1} \sum_{lev(g)=r} 2^{rd} |\psi(g)| \leq \sum_{r=1}^{p-1} \sum_{lev(g)=r} 2^{rd} \|\varphi\|_\infty \\
&= \sum_{r=1}^{p-1} 2^{r-1} 2^{rd} \|\varphi\|_\infty = (2^{p(d+1)-1} - 1)\|\varphi\|_\infty \leq 2^{p(d+1)} \|\varphi\|_\infty.
\end{aligned}$$

□

For $\xi \in \mathbb{Z}_2 * \mathbb{Z}_3$, I will let $\xi*$ denote the set of words in K which begin with ξ. Written in terms of 0's and 1's, we have for example $a* = \{.1*\}$. I will let $\hat{a}*$ denote the complement of this set. Here are a few more examples: $ba* = b* = \{.01*\}$. Note that since $b*$ indicates a *reduced* word beginning with b, there must be an a after the b. Similarly $abab^{-1}a* = abab^{-1}* = \{.110*\}$.

The action of a on K is measure preserving, and in particular $\mu(a\xi*) = \mu(\xi*)$ for all ξ. However, the action of b does not leave the Haar measure on K invariant: If ξ begins in b or b^{-1}, or $\xi = 0$, we have $\mu(b^{-j}a\xi*) = 1/2\mu(a\xi*)$, $\mu(b^{-j}\{b^{-j}a\xi*\}) = \mu(b^{-j}a\xi*)$, $\quad \mu(b^{-j}\{b^j a\xi*\}) = 2\mu(b^j a\xi*)$. So we have the following integral formula's (where χ is any function in $L^\infty(K)$):

$$\int_{a\xi*} \varphi(b^{-j}\vec{k})\chi(\vec{k})d\vec{k} = 2\int_{b^{-j}a\xi*} \varphi(\vec{k})\chi(b^j\vec{k})d\vec{k} \tag{2.6}$$

$$\int_{b^j a\xi*} \varphi(b^{-j}\vec{k})\chi(\vec{k})d\vec{k} = \frac{1}{2}\int_{a\xi*} \varphi(\vec{k})\chi(b^j\vec{k})d\vec{k} \tag{2.7}$$

$$\int_{b^{-j}a\xi*} \varphi(b^{-j}\vec{k})\chi(\vec{k})d\vec{k} = \int_{b^j a\xi*} \varphi(\vec{k})\chi(b^j\vec{k})d\vec{k}. \tag{2.8}$$

If S is any measurable set, then finally:

$$\int_S \varphi(a\vec{k})\chi(\vec{k})d\vec{k} = \int_{aS} \varphi(\vec{k})\chi(a\vec{k})d\vec{k}. \tag{2.9}$$

Lemma 2.10. *If* $\eta = b^{j_1}ab^{j_2}a \dots b^{j_n}a \in \mathbb{Z}_2 * \mathbb{Z}_3$ *and* $\chi(\vec{k}) = e^{2\pi i\langle g,\vec{k}\rangle}$, *then*

$$\alpha_\eta(\psi) = \int_K \varphi(\eta^{-1}\vec{k})\chi(\vec{k})d\vec{k} = 2^n \int_{\eta^{-1}a*} \varphi(\vec{k})\chi(\eta\vec{k})d\vec{k}$$

$$+\frac{1}{2^n}\int_{\hat{a}*}\varphi(\vec{k})\chi(\eta\vec{k})d\vec{k} \tag{2.11}$$
$$+\sum_{i=1}^{n}\frac{2^{(n-i)}}{2^{(i-1)}}\int_{ab^{-j_n}a\dots b^{-j_{i+1}}ab^{j_i}*}\varphi(\vec{k})\chi(\eta\vec{k})d\vec{k}.$$

Proof. Use induction on n, and the formulas (2.6)-(2.9). ☐

From now on, we let $\eta = b^{j_1}ab^{j_2}a\dots b^{j_n}a$. Note that $n = \ell(\eta)/2$. We will estimate $\|\omega^d\alpha_\eta(\psi)\|_1$ by breaking $\alpha_\eta(\psi)$ up as in Lemma 2.10. We proceed by applying $\sum_{g\in G}\omega(g)^d$ to the absolute value of each of the $n+2$ terms in (2.11).

For the first term, note that if $\vec{k} = .\eta^{-1}ak_{n+1}k_{n+2}\dots$, then $\langle g,\eta\vec{k}\rangle = g(1+2k_{n+1}+2^2k_{n+2}+\dots) = g/2^n(2^n+2^{n+1}k_{n+1}+2^{n+2}k_{n+2}+\dots) = \langle g/2^n,\vec{k}\rangle+$ terms depending only on η, but not $\vec{k}$. Thus we have

$$\begin{aligned}
&2^n\sum_{g\in G}\omega(g)^d\left|\int_{\eta^{-1}a*}\varphi(\vec{k})e^{2\pi i\langle g,\eta\vec{k}\rangle}d\vec{k}\right|\\
&= 2^n\sum_{g\in G}\omega(g)^d\left|\int_{\eta^{-1}a*}\varphi(\vec{k})e^{2\pi i\langle g/2^n,\vec{k}\rangle}d\vec{k}\right|\\
&\le 2^n\sum_{lev(g)>n}\omega(2^ng)^d\left|\int_{\eta^{-1}a*}\varphi(\vec{k})e^{2\pi i\langle g,\vec{k}\rangle}d\vec{k}\right|\\
&= 2^{-dn+n}\sum_{lev(g)>n}\omega(g)^d\left|\int_{\eta^{-1}a*}\varphi(\vec{k})e^{2\pi i\langle g,\vec{k}\rangle}d\vec{k}\right|\\
&\le 2^{(d+1)n}\sum_{g\in G}\omega(g)^d\left|\int_{\eta^{-1}a*}\varphi(\vec{k})e^{2\pi i\langle g,\vec{k}\rangle}d\vec{k}\right|,
\end{aligned} \tag{2.12}$$

where the second step used the fact that $g\mapsto g/2^n$ is a bijection between G and $\{g\in G\mid lev(g)>n, g<\frac{1}{2^n}\}\cup\{0\}$, the third step used $\omega(2^ng) = 2^{-n}\omega(g)$, and the last step is to make the estimate (2.16) below work more smoothly.

Next we estimate the second term in the expression (2.11). Note that if $\vec{k} = .0k_1k_2\dots$, then $\eta\vec{k} = .0b^{j_1}\dots b^{j_n}k_1k_2\dots$. Hence $\langle g,\eta\vec{k}\rangle = g(2^{n+1}k_1+2^{n+2}k_2+\dots) = \langle 2^ng,\vec{k}\rangle+$ terms depending only on η, but not $\vec{k}$. Thus we have

$$\begin{aligned}
&\frac{1}{2^n}\sum_{lev(g)\ge n}\omega(g)^d\left|\int_{\hat{a}*}\varphi(\vec{k})e^{2\pi i\langle g,\eta\vec{k}\rangle}d\vec{k}\right|\\
&= \frac{1}{2^n}\sum_{p=n}^{\infty}\sum_{lev(g)=p}\omega(g)^d\left|\int_{\hat{a}*}\varphi(\vec{k})e^{2\pi i\langle 2^ng,\vec{k}\rangle}d\vec{k}\right|
\end{aligned} \tag{2.13}$$

$$= \frac{1}{2^n} \sum_{p=n}^{\infty} \sum_{lev(g)=p-n} 2^n \omega(g/2^n)^d \left| \int_{\hat{a}*} \varphi(\vec{k}) e^{2\pi i \langle g, \vec{k} \rangle} d\vec{k} \right|$$
$$\leq 2^{n(d+1)} \sum_{g \in G} \omega(g)^d \left| \int_{\hat{a}*} \varphi(\vec{k}) e^{2\pi i \langle g, \vec{k} \rangle} d\vec{k} \right|,$$

where the 2^n in the third expression takes into account the repeats in the 1st and 2nd expression, and the last step used $\omega(g/2^n) = 2^n \omega(g)$. (By using the "truncation of lower levels" Lemma 2.5 above, we will be able to get by with only summing over $lev(g) \geq n$ in the left hand side of (2.13).)

Next we estimate terms in the last expression of (2.11). Fix some i between 1 and n. Any $\vec{k} \in ab^{-j_n}a \ldots b^{-j_{i+1}}ab^{j_i}*$ is of the form $\vec{k} = ab^{-j_n}a \ldots b^{-j_{i+1}}ab^{j_i}k_{n-i+2}k_{n-i+3} \ldots$. Since $\eta = b^{j_1}a \ldots b^{j_n}a$, we have $\eta\vec{k} = b^{j_1}a \ldots b^{j_{i-1}}ab^{-j_i}k_{n-i+2}k_{n-i+3} \ldots$. Thus $\langle g, \eta\vec{k} \rangle = g(2^{i+1}k_{n-i+2} + 2^{i+2}k_{n-i+3} + \ldots) = g2^{2i-1-n}(2^{n-i+2}k_{n-i+2} + \ldots) +$ terms depending only on η, but not $\vec{k}$. We split into two types of terms - those like (2.12) and those like (2.13). We have:

Case 1. $2i - 1 - n \leq 0$. Imitating (2.12), we get

(2.14)
$$\frac{2^{(n-i)}}{2^{(i-1)}} \sum_{g \in G} \omega(g)^d \left| \int_{ab^{-j_n} \ldots b^{-j_2}ab^{j_1}a*} \varphi e^{\langle g, \eta\vec{k} \rangle} d\vec{k} \right|$$
$$= 2^{n+1-2i} \sum_{g \in G} \omega(g)^d \left| \int_{ab^{-j_n}a \ldots b^{-j_2}ab^{j_1}*} \varphi(\vec{k}) e^{2\pi i \langle g/2^{n+1-2i}, \vec{k} \rangle} d\vec{k} \right|$$
$$\leq 2^{n+1-2i} \sum_{lev(g) > n+1-2i} \omega(2^{n+1-2i}g)^d \left| \int_{ab^{-j_n}a \ldots b^{-j_2}ab^{j_1}*} \varphi(\vec{k}) e^{2\pi i \langle g, \vec{k} \rangle} d\vec{k} \right|$$
$$= 2^{(1-d)(n+1-2i)} \sum_{lev(g) > n+1-2i} \omega(g)^d \left| \int_{ab^{-j_n}a \ldots b^{-j_2}ab^{j_1}*} \varphi(\vec{k}) e^{2\pi i \langle g, \vec{k} \rangle} d\vec{k} \right|$$
$$\leq 2^{(d+1)n} \sum_{g \in G} \omega(g)^d \left| \int_{ab^{-j_n}a \ldots b^{-j_2}ab^{j_1}*} \varphi(\vec{k}) e^{2\pi i \langle g, \vec{k} \rangle} d\vec{k} \right|.$$

Case 2. $2i - 1 - n > 0$. Note that since $i \leq n$, $2i - 1 - n < n$. Imitating (2.13), we get

(2.15)
$$\frac{2^{(n-i)}}{2^{(i-1)}} \sum_{lev(g) \geq n} \omega(g)^d \left| \int_{ab^{-j_n} \ldots b^{-j_2}ab^{j_1}*} \varphi e^{\langle g, \eta\vec{k} \rangle} \right|$$
$$\leq \frac{1}{2^{2i-1-n}} \sum_{lev(g) \geq 2i-n-1} \omega(g)^d \left| \int_{ab^{-j_n} \ldots b^{-j_2}ab^{j_1}*} \varphi e^{\langle g2^{2i-1-n}, \vec{k} \rangle} \right|$$

$$= \frac{1}{2^{2i-1-n}} \sum_{g\in G} 2^{(2i-1-n)} \omega(g/2^{2i-1-n})^d \left| \int_{ab^{-j_n}a\ldots b^{-j_2}ab^{j_1}*} \varphi(\vec{k}) e^{2\pi i\langle g,\vec{k}\rangle} d\vec{k} \right|$$
$$= 2^{(2i-1-n)d} \sum_{g\in G} \omega(g)^d \left| \int_{ab^{-j_n}a\ldots b^{-j_2}ab^{j_1}*} \varphi(\vec{k}) e^{2\pi i\langle g,\vec{k}\rangle} d\vec{k} \right|$$
$$\leq 2^{(d+1)n} \sum_{g\in G} \omega(g)^d \left| \int_{ab^{-j_n}a\ldots b^{-j_2}ab^{j_1}*} \varphi(\vec{k}) e^{2\pi i\langle g,\vec{k}\rangle} d\vec{k} \right|.$$

Now we are ready to collect terms and do the final estimate. By Lemma 2.5, Lemma 2.10 and (2.12)-(2.15), we have

(2.16)
$$\begin{aligned}
&\|\omega^d \alpha_\eta(\psi)\|_1 \\
&\leq 2^{(d+1)n}\|\varphi\|_\infty + \sum_{lev(g)\geq n} |\omega^d\alpha_\eta(\psi)(g)| \\
&\leq 2^{(d+1)n}\Bigg(\|\varphi\|_\infty + \sum_{g\in G}\omega(g)^d\Bigg[\left|\int_{\eta^{-1}a*}\varphi e^{\langle g,\vec{k}\rangle}\right| + \left|\int_{\hat{a}*}\varphi e^{\langle g,\vec{k}\rangle}\right| \\
&\qquad + \sum_{i=1}^{n}\left|\int_{ab^{-j_n}\ldots b^{j_1}a*}\varphi e^{\langle g,\vec{k}\rangle}\right|\Bigg]\Bigg) \\
&\leq 2^{(d+1)n}\Bigg(\|\varphi\|_\infty + \sum_{p=0}^{\infty}\frac{1}{2^d}\sum_{lev(g)=p}\Bigg[\left|\int_{\eta^{-1}a*}(\delta_p^d\varphi)e^{--}\right| \\
&\qquad + \left|\int_{\hat{a}*}(\delta_p^d\varphi)e^{--}\right| + \sum_{i=1}^{n}\left|\int_{ab^{-j_n}a\ldots b^{j_1}*}(\delta_p^d\varphi)e^{--}\right|\Bigg]\Bigg) \\
&\leq 2^{(d+1)n}\left(\|\varphi\|_\infty + \sum_{p=1}^{\infty}\frac{2^p}{2^{d+1}}\|\delta_p^d\varphi\|_\infty\right) \leq 2^{(d+1)n}\left(\|\varphi\|_\infty + \|\varphi\|_d\right) \\
&= 2^{(d+1)\ell(\eta)/2}C_d\big(\|\psi\|_1 + \|\omega^{d+q}\psi\|_1\big) \leq C_d e^{rd\ell(\eta)}\big(\|\psi\|_1 + \|\psi\|_{d+q}\big),
\end{aligned}$$

where the third step used the integration by parts formula $(-2\omega(g))^d\widehat{\varphi}(g) = \widehat{\delta_p^d\varphi}(g)$ of Theorem 1.6, and the fourth step used the fact that there are 2^{p-1} elements of G at level p, and $\mu(K) = 1$. This proves Theorem 2.3 for all η beginning in b or b^{-1}, and ending in a. Now show that a by itself acts temperedly (i.e. $\|\alpha_a(\psi)\|_d \leq C_d'\|\psi\|_{d+q'}$), and then get the result for general η by composing a on either side of η. □

3. m-convexity of the smooth crossed product.

By Corollary 2.4, we know that $\mathcal{S}^e(\mathbb{Z}_2 * \mathbb{Z}_3, C_\omega^\infty(K))$ is a Fréchet algebra under convolution. We say that an action of a group H on a Fréchet algebra

A is *m-γ-tempered* for some weight or gauge γ on H, if there exists a family of submultiplicative seminorms $\{\| \ \ \|_n\}$ (Definition 1.4), such that for every n, there exists $C > 0$, and $d \in \mathbb{N}$ such that

$$\|\alpha_h(a)\|_n \leq C(1+\gamma(h))^d \|a\|_n, \qquad a \in A, h \in H.$$

If the action is m-γ-tempered, then the smooth crossed product $\mathcal{S}^\gamma(H, A)$ is an m-convex Fréchet algebra [**Sc 1**], Theorem 3.1.7.

Theorem 3.1. *Let A be a dense Fréchet subalgebra of a Banach algebra B. Assume that a locally compact group H acts isometrically on B and γ-temperedly on A, where γ is a weight or gauge on H. Then if A is strongly spectral invariant in B, it follows that H acts m-γ-temperedly on A.*

Proof. If γ is a gauge, we may replace γ with the equivalent weight $1+\gamma$ so that $\gamma \geq 1$ [**Sc 1**], §1. Let $\{\| \ \ \|_n\}_{n=0}^\infty$ be some increasing family of seminorms for A, where $\| \ \ \|_0$ is the Banach algebra norm on B. Recall that by [**Sc 1**], Theorem 3.1.18, it suffices to show that for every $m \in \mathbb{N}$, there exists $q, d \in \mathbb{N}$ and $D > 0$ such that

$$\begin{aligned}(3.2)\qquad &\|\alpha_{h_1}(a_1)\dots\alpha_{h_n}(a_n)\|_m \\ &\leq D^n\big(\gamma(h_1)\gamma(h_1^{-1}h_2)\dots\gamma(h_{n-1}^{-1}h_n)\big)^d \|a_1\|_q \dots \|a_n\|_q,\end{aligned}$$

for all n-tuples $h_1, \dots h_n \in N$, $a_1, \dots a_n \in A$, and all n. By the strong spectral invariance,

$$(3.3)\qquad \|\alpha_{h_1}(a_1)\dots\alpha_{h_n}(a_n)\|_m \leq C^n D_m \sum_{k_1+\dots k_n \leq p} \|\alpha_{h_1}(a_1)\|_{k_1} \dots \|\alpha_{h_n}(a_n)\|_{k_n}.$$

In the sum, if $k_i = 0$, then $\|\alpha_{h_i}(a_i)\|_{k_i} = \|a_i\|_{k_i}$ since the action is assumed isometric on B. If $k_i \neq 0$, then $\|\alpha_{h_i}(a_i)\|_{k_i} \leq C'\gamma(h_i)^{d'}\|a_i\|_{p'}$ since the action is γ-tempered on A. Since the k_i's are $\leq p$, there is some upper bound for the C', d', and p''s - call it q. (Note we consider m fixed as we let n run, so p is also fixed.) Then by (3.3), and since the seminorms are increasing, we have

$$\begin{aligned}(3.4)\qquad &\|\alpha_{h_1}(a_1)\dots\alpha_{h_n}(a_n)\|_m \\ &\leq C^n D_m q^p \|a_1\|_q \dots \|a_n\|_q \sum_{k_1+\dots k_n \leq p} \big(\gamma(h_{i_1})\dots\gamma(h_{i_p})\big)^q,\end{aligned}$$

where $i_1, \dots i_p$ include all indices for which $k_i \neq 0$.

Note that

$$\gamma(h_i) \leq \gamma(h_{i-1}^{-1}h_i)\gamma(h_{i-2}^{-1}h_{i-1})\dots\gamma(h_1^{-1}h_2)\gamma(h_1)$$

$$\leq \big(\gamma(h_1)\gamma(h_1^{-1}h_2)\dots\gamma(h_{n-1}^{-1}h_n)\big)$$

since $\gamma \geq 1$ and γ is submultiplicative. Also $\sum_{k_1+\dots k_n\leq p}(1) \leq p^n$ so by (3.4), we have

$$\begin{aligned}&\|\alpha_{h_1}(a_1)\dots\alpha_{h_n}(a_n)\|_m\\ &\leq (Cp)^n D_m q^p\big(\gamma(h_1)\gamma(h_1^{-1}h_2)\dots\gamma(h_{n-1}^{-1}h_n)\big)^q\|a_1\|_q\dots\|a_n\|_q.\end{aligned}$$

This clearly gives (3.2). □

Corollary 3.5. *Let A, B, H, and γ be as in Theorem 3.1. Then the set of γ-rapidly vanishing L^1-functions from H to A forms an m-convex dense Fréchet subalgebra $\mathcal{S}^\gamma(H,A)$ of $L^1(H,B)$. In particular, $\mathcal{S}^e(\mathbb{Z}_2*\mathbb{Z}_3, C^\infty_\omega(K))$ is an m-convex Fréchet algebra.*

Proof. The first statement follows from the m-γ-temperedness of the action, and [**Sc 1**], Theorem 3.1.7. For the second statement, the action of $\mathbb{Z}_2*\mathbb{Z}_3$ is e^ℓ-tempered by Theorem 2.3, and $C^\infty_\omega(K)$ is SSI in $C(K)$ as we noted in Theorem 1.6 (see [**JiSc**], Definition 1.5, Lemma 3.11, proof of Theorem 2.6(b)). Hence $\mathbb{Z}_2*\mathbb{Z}_3$ acts m-e^ℓ-temperedly by Theorem 3.1 and this smooth crossed product is m-convex. □

4. Action of $\mathbb{Z}_2*\mathbb{Z}_3$ is not ℓ-tempered.

We show that the action of $\mathbb{Z}_2*\mathbb{Z}_3$ on $C^\infty_\omega(K)$ is not tempered with respect to the word length ℓ on $\mathbb{Z}_2*\mathbb{Z}_3$, so that it is necessary to use the larger weight e^ℓ to get a tempered action in Theorem 2.3. Let $\varphi \in C^\infty_\omega(K)$ be the characteristic function $\chi_{\hat{a}*}$ of the set of infinite words in K beginning in b or b^{-1}. Let $\eta = b^{j_1}a\dots b^{j_n}a \in \mathbb{Z}_2*\mathbb{Z}_3$. Then by Lemma 2.10,

$$\begin{aligned}\|\omega^d\alpha_\eta(\widehat{\varphi})\|_1 &= \sum_{g\in G}|\omega^d\alpha_\eta(\widehat{\varphi})(g)| = \frac{1}{2^n}\sum_{g\in G}\omega(g)^d\left|\int_{\hat{a}*}\varphi(\vec{k})e^{2\pi i\langle g,\eta\vec{k}\rangle}d\vec{k}\right|\\ &= \frac{1}{2^n}\sum_{g\in G}\omega(g)^d\left|\int_{\hat{a}*}e^{2\pi i\langle g,\eta\vec{k}\rangle}d\vec{k}\right|\\ &= \frac{1}{2^n}\sum_{g\in G}\omega(g)^d\left|\int_{\hat{a}*}e^{2\pi i\langle 2^n g,\vec{k}\rangle}d\vec{k}\right|\\ &\geq \frac{1}{2^n}\sum_{lev(g)=n+1}\omega(g)^d\left|\int_{\hat{a}*}e^{2\pi i\langle 2^n g,\vec{k}\rangle}d\vec{k}\right|\\ &= 2^{(n+1)d}\left|\int_{\hat{a}*}e^{2\pi i(k_0+2k_1+\dots)}d\vec{k}\right| = 2^{(n+1)d}\mu(\hat{a}*) = 2^{\ell(\eta)d/2}C_d,\end{aligned}$$

where the $\frac{1}{2^n}$ went away because there are 2^n elements of G at level $n+1$, and we used the fact that $\omega(g)^d = (2^{n+1})^d$ for g at level $n+1$, and that $\ell(\eta) = 2n$. We have proved:

Lemma 4.1. *Let $\eta = b^{j_1}a \dots b^{j_n}a$. Then $\|\alpha_\eta(\chi_{\hat{a}*})\|_d$ is bounded below by something directly proportional to $(2^{d/2})^{\ell(\eta)}$, and thus grows exponentially fast with respect to the word length $\ell(\eta)$. Hence the action of $\mathbb{Z}_2 * \mathbb{Z}_3$ on $C^\infty_\omega(K)$ in Theorem 2.3 is not ℓ-tempered.*

5. $\mathcal{O}_2$ is not spectral invariant in O_2.

Theorem 5.1. *The smooth Cuntz algebra $\mathcal{O}_2 = \mathcal{S}^e(\mathbb{Z}_2 * \mathbb{Z}_3, C^\infty(K))$ has an element whose spectrum contains the unbounded interval $(3, \infty)$. Hence $\mathcal{O}_2$ is not spectral invariant in the $C^\star$-algebra O_2.*

Proof. Let $g = ab \in \mathbb{Z}_2 * \mathbb{Z}_3$. Then $\mathbb{Z} \cong < g > \subseteq \mathbb{Z}_2 * \mathbb{Z}_3$ and $C^\star(\mathbb{Z}) \cong C^\star(\langle g \rangle) \subseteq C^\star_r(\mathbb{Z}_2 * \mathbb{Z}_3) \subseteq O_2$. Also $\ell|_{\mathbb{Z}}(g^n) = |2n|$. Thus $C^\star(\mathbb{Z}) \cap \mathcal{O}_2 = C^\star(\mathbb{Z}) \cap \mathcal{S}^e(\mathbb{Z}_2 * \mathbb{Z}_3) = \mathcal{S}^e(\mathbb{Z})$. Assume that $\varphi \in \mathcal{S}^e(\mathbb{Z})$ is not invertible in $\mathcal{S}^e(\mathbb{Z})$, but that φ is invertible in the bigger algebra $C^\star_r(\mathbb{Z})$. Then if φ were invertible in $\mathcal{O}_2$, it would be invertible in O_2 and so in the smaller $C^\star$-algebra $C^\star_r(\mathbb{Z})$[**Dix**]. Thus φ is not invertible in $\mathcal{O}_2$. It follows that for any $\varphi \in \mathcal{S}^e(\mathbb{Z})$, φ is invertible in $\mathcal{S}^e(\mathbb{Z})$ if and only if φ is invertible in $\mathcal{O}_2$. So the spectrum is the same in either of these smooth algebras.

Lemma 5.2. *The function e^{-n^2} in the Fréchet algebra $\mathcal{S}^e(\mathbb{Z})$ has the unbounded interval $(3, \infty)$ contained in its spectrum.*

Proof. Represent $\mathbb{Z}$ on $\mathbb{C}$ via $n \mapsto e^{rn}$, $r \in \mathbb{R}$. Then $\varphi_r(\xi) = \sum_{n \in \mathbb{Z}} \xi(n) e^{rn}$ gives a family of simple $\mathcal{S}^e(\mathbb{Z})$-modules. The value of $\varphi_r(e^{-n^2}) = \sum_{n \in \mathbb{Z}} e^{-n^2} e^{rn}$ ranges continuously between

$$\varphi_0(e^{-n^2}) = \sum_{n=-\infty}^{+\infty} e^{-n^2} \leq \int_{-\infty}^{+\infty} e^{-x^2} dx + 1 = \sqrt{\pi} + 1 \leq 3$$

and $+\infty$ as r ranges between 0 and $+\infty$. □

Since the spectrum of e^{-n^2} is the same in $\mathcal{O}_2$ and $\mathcal{S}^e(\mathbb{Z})$ by our preceding remarks, we have proved Theorem 5.1. □

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